ESTIMATION OF PROBABILITY CHARACTERISTICS BY GENERALIZED BERNSTEIN POLYNOMIALS

JAE WON LEE, GEORGY L. SHEVLYAKOV, NIKITA O. VILCHEVSKY

Kumi, Korea; St.Petersburg, Russia

Summary. Approximations based on the Bernstein polynomials are used for smoothing a sample quantile function and estimating the underlying distribution and its characteristics. The generalized Bernstein-type polynomials are introduced to reduce the bias of estimation under various types of distributions including finite distributions. The asymptotic behavior of the expectations of these estimators is studied.

1. INTRODUCTION

We begin with some definitions. Let X be a random variable with the distribution function $F(x) = P(X \leq x)$. The *quantile function* of X is

$$Q(t) = F^{-1}(t) = \inf\{x|\, F(x) \geq t\}, \qquad 0 < t < 1. \tag{1}$$

Given a sample $x_1, x_2, \dots, x_n$ from F, a natural estimator of the quantile function is the sample quantile function

$$\widehat{Q}_n(t) = \widetilde{F}_n^{-1}(t) = \inf\{x|\, \widetilde{F}_n(x) \geq t\}, \qquad 0 < t < 1, \tag{2}$$

where

$$\widetilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x)$$

is the sample distribution function.

Several smooth quantile function estimators have been proposed, Parzen's difference kernel estimators [1]

$$\widehat{Q}_n^K(t) = \int_0^1 \frac{1}{h_n} k\left(\frac{u-t}{h_n}\right) \widehat{Q}_n(u)\, du$$

among them, where $k(\cdot)$ is a symmetric distribution density and h_n is a bandwidth parameter. These estimators were investigated in [2-4]. Vilchevskiy and Shevlyakov [5] introduced the Bernstein polynomial estimator for the quantile function

$$\widehat{Q}_n^B(t) = \sum_{i=1}^{n} \left[\binom{n-1}{i} t^{i-1}(1-t)^{n-i}\right] x_{(i)}, \qquad 0 \leq t \leq 1. \tag{3}$$

Now we dwell on the general properties of the Bernstein polynomials. They are widely used in the theoretical studies on convergence processes of approximations to continuous on $[0,1]$ functions. The reason is that, first, these polynomials have a rather simple structure, and, second, they provide the uniform convergence in the Chebyshev norm $\|f\| = \max_{t\in[0,1]} |f(t)|$.

The Bernstein polynomial being a uniform approximation to a continuous function $f(t)$ on a closed interval $[0,1]$ (the Weierstrass approximation theorem), is defined as

$$B_n(f,t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i} t^i (1-t)^{n-i}. \tag{4}$$

The Bernstein polynomials have the following basic properties:

(BP1) if a function $f(t)$ has k continuous derivatives
then $B_n^{(k)}(f,t) \to f^{(k)}(t)$ uniformly on $[0,1]$;

(BP2) if a bounded function $f(t)$ has a jump discontinuity at $t=c$ then

$$B_n(f,c) \to \frac{f(c_-)+f(c_+)}{2};$$

(BP3) for a twice continuously differentiable function $f(t)$, there holds a following asymptotic representation

$$B_n(f,t) = f(t) + \frac{f''(t)t(1-t)}{2n} + o\left(\frac{1}{n}\right); \tag{5}$$

(BP4) if $f(t)$ is a monotonic function on $[0,1]$ then $B_n(f,t)$ is also monotonic on $[0,1]$.

These properties allow to use the Bernstein polynomials for estimating distribution laws and their characteristics [5-10].

2. THE BERNSTEIN-TYPE POLYNOMIALS FOR ORDER STATISTICS

Consider a sample $x_1, x_2, \ldots, x_n$ from the distribution $F(x)$ defined on $[a,b]$ and the corresponding set of order statistics $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Introduce the polynomial of the Bernstein type as

$$\mathcal{B}_n^{[a,b]}(t) = a(1-t)^{n+1} + \sum_{i=1}^{n} x_{(i)} \binom{n+1}{i} t^i (1-t)^{n+1-i} + bt^{n+1}. \tag{6}$$

Let $Q(t)$ be the quantile function (1), the inverse to $F(x)$. Take the function $q_{n+1}(t)$ as

$$q_{n+1}(t) = \begin{cases} a, & \text{for} \quad 0 \le t < 1/(n+1), \\ x_{(i)}, & \text{for} \quad i/(n+1) \le t < (i+1)/(n+1), \qquad 1 \le i \le n, \\ b, & \text{for} \quad t \ge 1. \end{cases}$$

Hence we have

$$\mathcal{B}_n^{[a,b]}(t) = B_{n+1}(q_{n+1}, t).$$

Evidently, the introduced function $q_{n+1}(t)$ is the inverse to the sample distribution function, and $q_{n+1}(t) \to Q(t)$ as $n \to \infty$, that is, for finite n, $q_{n+1}(t)$ is an estimator for the quantile function $Q(t)$.

It is known that the expectation of the sample distribution function coincides with its true shape. Approximating the inverse to the sample distribution function, we may foresee the analogous assertion.

Now we introduce a more general construction than (6). We should distinguish the situations with finite and infinite bounds for the underlying distribution. Taking into account this remark, we define the following Bernstein-type polynomials:

$$\mathcal{B}_n^{[a,b]}(\Phi,t) = \Phi(a)(1-t)^{n+1} + \sum_{i=1}^{n} \Phi(x_{(i)}) \binom{n+1}{i} t^i (1-t)^{n+1-i} + \Phi(b)t^{n+1}, \tag{7}$$

$$\mathcal{B}_n^{[a,\infty]}(\Phi,t) = \Phi(a)(1-t)^{n} + \sum_{i=1}^{n} \Phi(x_{(i)}) \binom{n}{i} t^i (1-t)^{n-i}, \tag{8}$$

$$\mathcal{B}_n^{[-\infty,b]}(\Phi,t) = \sum_{i=0}^{n-1} \Phi(x_{(i)}) \binom{n}{i} t^i (1-t)^{n-i} + \Phi(b)t^{n}, \tag{9}$$

$$\mathcal{B}_n^{[-\infty,\infty]}(\Phi,t)=\sum_{i=0}^{n-1}\Phi(x_{(i)})\binom{n-1}{i}t^i(1-t)^{n-1-i}. \tag{10}$$

Consider now the asymptotic behavior of the expectations for these polynomials. The expectation $E_F\{\mathcal{B}_n^{[a,b]}(\Phi,t)\}\equiv I(\Phi,Q,t)$ can be written as

$$I(\Phi,Q,t)=\Phi(a)(1-t)^{n+1}+\Phi(b)t^{n+1}$$

$$+\sum_{i=1}^{n}E_F\{\Phi(x_{(i)})\}\binom{n+1}{i}t^i(1-t)^{n+1-i}=\Phi(a)(1-t)^{n+1}+\Phi(b)t^{n+1}$$

$$+\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\int_a^b n\Phi(x)\binom{n-1}{i-1}(F(x))^{i-1}(1-F(x))^{n-i}\,dF$$

$$=\Phi(a)(1-t)^{n+1}+\Phi(b)t^{n+1}$$

$$+n\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\int_0^1\Phi(Q(u))\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du.$$

The behavior of $E_F\{\mathcal{B}_n^{[a,b]}(\Phi,t)\}$ as $n\to\infty$ is given by the following result.

Theorem 1. Let $\Psi(u)\equiv\Phi(Q(u))$ have a fourth bounded derivative on $[0,1]$. Then the expectation of the Bernstein-type polynomial (7) has the form

$$I^{[a,b]}(\Phi,Q,t)=E_F\{\mathcal{B}_n^{[a,b]}(\Phi,t)\}=\Psi(t)+\frac{t(1-t)}{n+2}\Psi''(t)+o\left(\frac{1}{n}\right).$$

Proof. First write out the Taylor series for $\Psi(u)$ at $u=\dfrac{i}{n+1}$ with the Lagrange form of the remainder:

$$\Psi(u)=\Psi\left(\frac{i}{n+1}\right)+\Psi'\left(\frac{i}{n+1}\right)\left(u-\frac{i}{n+1}\right)+\frac{1}{2}\Psi^{(2)}\left(\frac{i}{n+1}\right)\left(u-\frac{i}{n+1}\right)^2$$

$$+\frac{1}{6}\Psi^{(3)}\left(\frac{i}{n+1}\right)\left(u-\frac{i}{n+1}\right)^3+\frac{1}{24}\Psi^{(4)}\left(\xi_i\right)\left(u-\frac{i}{n+1}\right)^4,$$

where

$$\xi_i=\frac{i}{n+1}+\theta\left(u-\frac{i}{n+1}\right),\qquad 0<\theta<1.$$

In this case we have

$$I^{[a,b]}(\Phi,Q,t)=\Phi(a)(1-t)^{n+1}+\Phi(b)t^{n+1}+n\sum_{s=0}^{3}\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}$$

$$\times\frac{1}{s!}\Psi^{(s)}\left(\frac{i}{n+1}\right)\int_0^1\left(u-\frac{i}{n+1}\right)^s\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du \tag{11}$$

$$+\frac{n}{24}\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\int_0^1\Psi^{(4)}(\xi_i)\left(u-\frac{i}{n+1}\right)^4\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du.$$

Now we continue with the following notations:

$$A_0 = \Phi(a)(1-t)^{n+1} + \Phi(b)t^{n+1}$$

$$+n\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\Psi\left(\frac{i}{n+1}\right)\int_0^1\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du,$$

$$A_s = \frac{n}{s!}\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\Psi^{(s)}\left(\frac{i}{n+1}\right)\int_0^1\left(u-\frac{i}{n+1}\right)^s\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du,$$

$$s = 1, 2, 3;$$

$$A_4 = \frac{n}{24}\sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\int_0^1\Psi^{(4)}(\xi_i)\left(u-\frac{i}{n+1}\right)^4\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du.$$

To evaluate A_0, A_1, A_2, A_3 and A_4, we use the following relations:

- $\Phi(a) = \Phi(Q(0))$, $\Phi(b) = \Phi(Q(1))$, where $a = Q(0)$, $b = Q(1)$.
- The integrals in the formulas for A_0, A_1, A_2 A_3, and A_4 are expressed through the B-function and the central moments of the order statistics of a uniform on $(0,1)$ random variable:

$$\int_0^1 u^k u^{i-1}(1-u)^{n-i}\,du = \frac{(k+i-1)!(n-i)!}{(n+k)!},$$

$$n\int_0^1\left(u-\frac{i}{n+1}\right)\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du = 0,$$

$$n\int_0^1\left(u-\frac{i}{n+1}\right)^2\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du = \frac{i(n+1-i)}{(n+1)^2(n+2)},$$

$$n\int_0^1\left(u-\frac{i}{n+1}\right)^3\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du = \frac{2i(n+1-i)(n+1-2i)}{(n+1)^3(n+2)(n+3)},$$

$$n\int_0^1\left(u-\frac{i}{n+1}\right)^4\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i}\,du$$

$$= \frac{3i(n+1-i)(in^2-i^2n+2n^2+5i^2-4in-5i+4n+2)}{(n+1)^4(n+2)(n+3)(n+4)}.$$

Taking into account the above relations, we get

$$A_0 = \Phi(a)(1-t)^{n+1} + \Phi(b)t^{n+1} + \sum_{i=1}^{n}\binom{n+1}{i}t^i(1-t)^{n+1-i}\Psi\left(\frac{i}{n+1}\right)$$

$$= \sum_{i=0}^{n+1}\binom{n+1}{i}t^i(1-t)^{n+1-i}\Phi\left(\frac{i}{n+1}\right) = B_{n+1}(\Phi(Q), t),$$

$$A_1 = 0,$$

$$A_2 = \frac{nt(1-t)}{2(n+1)(n+2)}\sum_{i=0}^{n-1}\Psi''\left(\frac{i+1}{n+1}\right)\binom{n-1}{i}t^i(1-t)^{n-1-i},$$

or, by the expansion

$$\Psi''\left(\frac{i+1}{n+1}\right) = \Psi''\left(\frac{i}{n-1}+\frac{n-2i-1}{n^2-1}\right) = \Psi''\left(\frac{i}{n-1}\right) + \frac{n-2i-1}{n^2-1}\Psi'''(\zeta_i),$$

where

$$\zeta_i = \frac{i}{n-1} + \theta\left(u - \frac{i}{n-1}\right), \qquad 0 < \theta < 1.$$

Further, using the definition of the Bernstein polynomials (4), we obtain

$$A_2 = \frac{nt(1-t)}{2(n+1)(n+2)} \sum_{i=0}^{n-1} \Psi''\left(\frac{i}{n-1}\right) \binom{n-1}{i} t^i (1-t)^{n-1-i} + R_n$$
$$= \frac{nt(1-t)}{2(n+1)(n+2)} B_{n-1}\left(\Psi'', t)\right) + R_n = \frac{nt(1-t)}{2(n+1)(n+2)} \left[\Psi''(t) + O\left(\frac{1}{n}\right)\right] + R_n.$$

By splitting the sum into the positive and negative parts with the subsequent use of the binomial identity, the remainder R_n can be transformed as follows:

$$R_n = \frac{nt(1-t)}{2(n+1)(n+2)} \sum_{i=0}^{n-1} \Psi'''(\zeta_i) \frac{[(n-1-i)-i]}{n^2-1} \binom{n-1}{i} t^i (1-t)^{n-1-i} = \frac{nt(1-t)}{2(n+1)^2(n+2)}$$

$$\times \left[(1-t) \sum_{i=0}^{n-2} \Psi'''(\zeta_i) \binom{n-2}{i} t^i (1-t)^{n-2-i} - t \sum_{i=0}^{n-2} \Psi'''(\zeta_{i+1}) \binom{n-2}{i} t^i (1-t)^{n-2-i}\right].$$

Hence

$$|R_n| < \frac{nt(1-t)}{2(n+1)^2(n+2)} \max_{t\in[0,1]} |\Psi'''(t)|,$$

and taking into account the latter relation, we get

$$A_2 = \frac{nt(1-t)}{2(n+1)(n+2)} \Psi''(t) + o\left(\frac{1}{n}\right).$$

Further consider the expressions for A_3 and A_4. Using the above formulas for the central moments of order statistics, it can be shown that the values of A_3 and A_4 are of order $1/n^2$. First, split the term A_3 into two parts and change the variable i to $i+1$ and then to $i+2$, in the positive and negative parts respectively:

$$A_3 = \frac{n}{6} \sum_{i=1}^{n} \binom{n+1}{i} t^i (1-t)^{n+1-i} \frac{2i(n+1-i)(n+1-2i)}{n(n+1)^3(n+2)(n+3)} \Psi^{(3)}\left(\frac{i}{n+1}\right)$$
$$= \frac{1}{3(n+1)^2(n+2)(n+3)} \sum_{i=1}^{n} \frac{n![(n-i)+(1-i)]}{(i-1)!(n-i)!} t^i (1-t)^{n+1-i} \Psi^{(3)}\left(\frac{i}{n+1}\right)$$
$$= \frac{1}{3(n+1)^2(n+2)(n+3)} \left[\sum_{i=1}^{n-1} \frac{n!}{(i-1)!(n-1-i)!} t^i (1-t)^{n+1-i} \Psi^{(3)}\left(\frac{i}{n+1}\right)\right.$$
$$\left. - \sum_{i=2}^{n} \frac{n!}{(i-2)!(n-1-i)!} t^i (1-t)^{n+1-i} \Psi^{(3)}\left(\frac{i}{n+1}\right)\right]$$
$$= \frac{n(n-1)t(1-t)}{3(n+1)^2(n+2)(n+3)} \left[(1-t) \sum_{i=0}^{n-2} \binom{n-2}{i} t^i (1-t)^{n-2-i} \Psi^{(3)}\left(\frac{i+1}{n+1}\right)\right.$$
$$\left. - t \sum_{i=0}^{n-2} \binom{n-2}{i} t^i (1-t)^{n-2-i} \Psi^{(3)}\left(\frac{i+2}{n+1}\right)\right].$$

Hence we have a following estimate:

$$|A_3| < \frac{t(1-t)}{3n^2} \max_{t\in[0,1]} |\Psi^{(3)}(t)|.$$

The similar but more tedious calculations yield an estimate for A_4:

$$|A_4| < \frac{t^2(1-t)^2(1+t)}{8n^2} \max_{t\in[0,1]} |\Psi^{(4)}(t)|.$$

By substituting the relations for A_0, A_1, and A_2 into (11) and taking into account asymptotic representation (BP3) for the Bernstein polynomials and the estimates for the terms A_3 and A_4, we obtain

$$E_F\{\mathcal{B}_{n+1}(\Phi,t)\} = \Psi(t) + \frac{t(1-t)}{2(n+1)}\Psi''(t) + \frac{nt(1-t)}{2(n+1)(n+2)}\Psi''(t) + o\left(\frac{1}{n}\right)$$

$$= \Psi(t) + \frac{t(1-t)}{(n+2)}\Psi''(t) + o\left(\frac{1}{n}\right),$$

and this concludes the proof of Theorem 1.

Example 1. Put

$$\Phi(u) = u, \qquad F(x) = \begin{cases} 0, & \text{for} \quad x < a, \\ (x-a)/(b-a), & \text{for} \quad a \le x < b, \\ 1, & \text{for} \quad x \ge b. \end{cases}$$

Then the direct evaluation of (7) yields $E_F\{\mathcal{B}_{n+1}(u,t)\} = a + (b-a)t$, that is the precise expression for the inverse function to the uniform distribution on $[a,b]$. If we take $\Phi(u) = u^3$, then the corresponding calculations yield

$$E_F\{\mathcal{B}_{n+1}(u^3,t)\} = [a+(b-a)t]^3 + \frac{t(1-t)}{n+2}6[a+(b-a)t](b-a)^2 + \frac{6t(1-t)(1-2t)(b-a)^3}{(n+2)(n+3)},$$

and this completely corresponds to the assertion of Theorem 1.

The proofs of the assertions describing the asymptotic behavior of the Bernstein-type polynomials (8), (9), and (10) practically coincide with the proof of the latter theorem. All the differences refer to the choice of the points, at which the expansion of the function $\Psi(u) = \Phi(Q(u))$ is made.

Now we formulate the corresponding theorems.

Theorem 2. Let $\Psi(u) \equiv \Phi(Q(u))$ have a bounded fourth derivative on $[0,1]$. Then the expectation of the Bernstein-type polynomial (8) has the form

$$I^{[a,\infty]}(\Phi,Q,t) = E_F\{\mathcal{B}_n^{[a,\infty]}(\Phi,t)\} = \Psi(t) - \frac{t}{n+1}\Psi'(t) + \frac{t(1-t)}{n+2}\Psi''(t) + o\left(\frac{1}{n}\right).$$

Theorem 3. Let $\Psi(u) \equiv \Phi(Q(u))$ have a bounded fourth derivative on $[0,1]$. Then the expectation of the Bernstein-type polynomial (9) has the form

$$I^{[-\infty,b]}(\Phi,Q,t) = E_F\{\mathcal{B}_n^{[-\infty,b]}(\Phi,t)\} = \Psi(t) + \frac{1-t}{n+1}\Psi'(t) + \frac{t(1-t)}{n+2}\Psi''(t) + o\left(\frac{1}{n}\right).$$

Theorem 4. Let $\Psi(u) \equiv \Phi(Q(u))$ have a bounded fourth derivative on $[0,1]$. Then the expectation of the Bernstein-type polynomial (10) has the form

$$I^{[-\infty,\infty]}(\Phi,Q,t) = E_F\{\mathcal{B}_n^{[-\infty,\infty]}(\Phi,t)\} = \Psi(t) + \frac{1-2t}{n+1}\Psi'(t) + \frac{t(1-t)}{n+2}\Psi''(t) + o\left(\frac{1}{n}\right).$$

Example 2. Let the Bernstein-type polynomial (10) be used under the conditions of Example 1. Then we have for the expectation $E_F\{\mathcal{B}_n^{[-\infty,\infty]}(\Phi,t)\}$ at $\Phi(u)=u$

$$E_F\{\mathcal{B}_n^{[-\infty,\infty]}(u,t)\} = a + (b-a)t + \frac{(b-a)(1-2t)}{n+1}.$$

Comparing this result with the similar of Example 1, you can see that the bias of approximation becomes greater if an inappropriate type of the Bernstein polynomial estimator is used. Further, if $\Phi(u)=u^3$ then we get

$$E_F\{\mathcal{B}_n^{[-\infty,\infty]}(u^3,t)\} = [a+(b-a)t]^3 + 3\frac{(1-2t)}{n+1}[a+(b-a)t]^2(b-a)$$

$$+6\frac{t(1-t)}{n+2}[a+(b-a)t](b-a)^2 + 6\frac{(5t^2-5t+1)a+(9t^2-11t+3)t(b-a)}{(n+1)(n+2)}(b-a)^2$$

$$+6\frac{(1-2t)(10t^2-10t+1)}{(n+1)(n+2)(n+3)}(b-a)^3,$$

and this result also completely agrees with the assertion of Theorem 4.

3. FINAL REMARKS

It is rather convenient to use the Bernstein approximations to the quantile function, as through them, it is easy to derive the moments' characteristics of random variables. In particular, we have for the estimate of expectation

$$\widehat{\mu} = \int_a^b x\, d\widehat{F}(x) = \int_0^1 \widehat{Q}(t)\, dt.$$

Hence, using the Bernstein-type polynomials (7), we obtain

$$\widehat{\mu} = \int_0^1 \mathcal{B}_n^{[a,b]}(u,t)\, dt = \frac{a + \sum_1^n x_{(i)} + b}{n+2}.$$

For approximation (10) we have

$$\widehat{\mu} = \int_0^1 \mathcal{B}_n^{[-\infty,\infty]}(u,t)\, dt = \frac{\sum_1^n x_{(i)}}{n}.$$

The estimate for the distribution density is expressed through the quantile function in the parametric form, namely regarding t as a parameter

$$f(x) = F'(x) \cong 1/B_n'(t), \qquad x = B_n(t).$$

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