GROUP EXTENSIONS

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Abstract. One of the largest tasks in group theory is to classify and describe all possible groups. Simple groups have only recently been completely classified. Group extensions is a way to describe for given finite groups K and G all possible groups H which have a normal subgroup isomorphic to K and whose factor group is isomorphic to G.

In this paper, I will construct different ways of looking at group extensions. I will then recreate proofs of theorems to help in computing group extensions. Next, I plan to discuss a specific type of group extension called central extensions. Finally, I will compute some concrete examples.

The approach taken in this paper is based on the lecture notes [1] from a graduate level course I took with Robert Boltje on finite groups in Fall 2003. The section on central extensions is based on researching further into group extensions and group cohomology [2].

1. Group Extensions

Throughout the paper, let G, K be groups. We do not assume G or K is finite.

1.1 Definition. A *group extension* of G by K is a *short exact sequence*

$$1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1,$$

such that H is a group, and the image of each incoming map of K, H, G is equal to the kernel of the outgoing map. This means ε is injective, $\mathrm{im}(\varepsilon) = \ker(\nu)$, and ν is surjective.

The above group extension is *equivalent* to $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ if and only if there exists an isomorphism $\varphi : H \to \tilde{H}$ such that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\varepsilon} & H & \xrightarrow{\nu} & G & \longrightarrow & 1 \\ & & {\scriptstyle 1_K}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle 1_G} & & \\ 1 & \longrightarrow & K & \xrightarrow{\tilde{\varepsilon}} & \tilde{H} & \xrightarrow{\tilde{\nu}} & G & \longrightarrow & 1 \end{array}$$

commutes. This definition of equivalence creates an equivalence relation on $\mathrm{ext}(G, K)$, the set of extensions of G by K. The set of equivalence classes of $\mathrm{ext}(G, K)$ is denoted by $\mathrm{Ext}(G, K)$.

1.2 Remark. By the First Isomorphism Theorem, if $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is a group extension of G by K, then H has a normal subgroup $\varepsilon(K)$ with factor group $H/\varepsilon(K) = H/\ker(\nu) \cong G$.

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The converse is also true. If H is a group with a normal subgroup N such that $N \cong K$ and $H/N \cong G$, then there exists a group extension $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$, with $\varepsilon : K \overset{\cong}{\to} N \overset{inj}{\to} H$, the composition of the isomorphism $K \cong N$ and the inclusion $N \leq H$, and $\nu : H \overset{surj}{\to} H/N \overset{\cong}{\to} G$, the composition of the natural epimorphism $H \twoheadrightarrow H/N$ and the isomorphism $H/N \cong G$.
If $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ are equivalent, then by the definition of equivalent extensions, there exists an isomorphism $\varphi : H \to \tilde{H}$, so $H \cong \tilde{H}$. The converse is not true. If $H \cong \tilde{H}$, then there may be extensions of K by G which are not equivalent.

1.3 Proposition. *If there exists a homomorphism $\varphi : H \to \tilde{H}$ which makes the above diagram commute for $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$, two group extensions of G by K, then φ has to be an isomorphism, and so the group extensions are equivalent.*
Proof.
Let $h \in H$ such that $\varphi(h) = 1$. Then $\tilde{\nu}(\varphi(h)) = 1$, but since the diagram commutes, this implies $\nu(h) = 1$. Since $\ker(\nu) = \operatorname{im}(\varepsilon)$, then there exists an $x \in K$ such that $\varepsilon(x) = h$. So $\varphi(\varepsilon(x)) = \varphi(h) = 1$. Again, since the diagram commutes, we have $\tilde{\varepsilon}(x) = \varphi(\varepsilon(x)) = 1$. Because $\tilde{\varepsilon}$ is injective, then $x = 1$, so $h = \varepsilon(x) = \varepsilon(1) = 1$. Since $h = 1$, then φ is injective.
Now let $\tilde{h} \in \tilde{H}$. We want to show there exists $h \in H$ such that $\varphi(h) = \tilde{h}$. Consider $\tilde{\nu}(\tilde{h}) \in G$. Since both ν and $\tilde{\nu}$ are surjective, then there exists some $x \in H$ such that $\nu(x) = \tilde{\nu}(\tilde{h})$. Since the diagram commutes, then $\nu(x) = \tilde{\nu}(\varphi(x)) = \tilde{\nu}(\tilde{h})$. Considering $\tilde{\nu}(\tilde{h}^{-1}\varphi(x))$, we have $\tilde{\nu}(\tilde{h}^{-1}\varphi(x)) = \tilde{\nu}(\tilde{h})^{-1}\tilde{\nu}(\varphi(\tilde{h})) = 1$. So $\tilde{h}^{-1}\varphi(x) \in \ker(\tilde{\nu}) = \operatorname{im}(\tilde{\varepsilon})$. This implies there exists $y \in K$ such that $\tilde{\varepsilon}(y) = \tilde{h}^{-1}\varphi(x)$. Then $\tilde{h} = \varphi(x)(\tilde{\varepsilon}(y))^{-1}$. But since the diagram commutes, then $\tilde{\varepsilon}(y) = \varphi(\varepsilon(y))$, so we have $\tilde{h} = \varphi(x)\varphi(\varepsilon(y))^{-1} = \varphi(x \cdot \varepsilon(y)^{-1})$. Set $h := x \cdot \varepsilon(y)^{-1}$. Thus φ is surjective and hence an isomorphism. $\square$

1.4 Proposition. *Let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ be a group extension of G by K. For all $x \in G$, let $h_x \in H$ be $\nu(h_x) = x$. Then we have:*
(a) For any $h \in H$, there exist unique elements $x \in G$, $a \in K$ such that $h = h_x\varepsilon(a)$.
(b) For any $x \in G$ and $a \in K$ there exists a unique $\alpha_x(a) \in K$ such that $\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)h_x^{-1}$. Also, $\alpha_x \in \operatorname{Aut}(K)$.
(c) For any $x, y \in G$, there exists a unique $\kappa(x, y) \in K$ such that $h_x h_y = \varepsilon(\kappa(x, y))h_{xy}$. In particular, $h_1 = \varepsilon(\kappa(1, 1))$. Also, $\alpha_x \circ \alpha_y = c_{\kappa(x,y)}\alpha_{xy}$, where $c_a \in \operatorname{Aut}(K)$ is the conjugation automorphism $k \mapsto aka^{-1}$ for $a \in K$.
(d) For any $x, y, z \in G$, $\kappa(x, y)\kappa(xy, z) = \alpha_x(\kappa(y, z))\kappa(x, yz)$.
(e) Let $h'_x \in H$ be such that $\nu(h'_x) = x = \nu(h_x)$ for all $x \in G$. Then there exists a unique function $g : K \to G$ such that $h'_x = h_x\varepsilon(g(x))$ for all $x \in G$. If $\alpha' : G \to \operatorname{Aut}(K)$ and $\kappa' : G \times G \to K$ are constructed as above from h'_x, $x \in G$, then

$$\alpha'_x = c_{f(x)} \circ \alpha_x \text{ and } \kappa'(x, y) = f(x) \cdot \alpha_x(f(y)) \cdot \kappa(x, y) \cdot f(xy)^{-1}$$

for all $x, y \in G$, where $f : G \to K$ is defined by $f(x) := \alpha_x(g(x))$ for all $x \in G$.
Proof.
(a) Let $h \in H$ and $x := \nu(h)$. Then $\nu(h_x^{-1}h) = \nu(h_x)^{-1}\nu(h) = x^{-1}x = 1$. Since

$\operatorname{im}(\varepsilon) = \ker(v)$, then there exists $a \in K$ such that $\varepsilon(a) = h_x^{-1}h$. So, $h = h_x\varepsilon(a)$. To show uniqueness, assume there exists a $y \in G, b \in K$ such that $h = h_y\varepsilon(b)$. Then $\nu(h) = \nu(h_y\varepsilon(b)) = \nu(h_y)\nu(\varepsilon(b))$. By definition, $\nu(h_y) = y$, and since $\operatorname{im}(\varepsilon) = \ker(\nu)$, then $\nu(\varepsilon(b)) = 1$. So $\nu(h_y)\nu(\varepsilon(b)) = y \cdot 1 = y$. So $x = \nu(h) = y$, and thus $\nu(h_y^{-1}h) = 1$. This implies $\varepsilon(a) = \varepsilon(b)$, as above. But by definition, ε is injective, so $a = b$. Therefore uniqueness is shown, as desired.
(b) Since $\operatorname{im}(\varepsilon) = \ker(\nu)$, then for all $x \in G, a \in K$, we have $\nu(h_x\varepsilon(a)h_x^{-1}) = \nu(h_x)\nu(\varepsilon(a))\nu(h_x)^{-1} = \nu(h_x)\nu(h_x)^{-1} = 1$, which implies $h_x\varepsilon(a)h_x^{-1} \in \ker(\nu) = \operatorname{im}(\varepsilon)$. By the definition of image, there exists a $b \in K$ such that $\varepsilon(b) = h_x\varepsilon(a)h_x^{-1}$, and since ε is injective, b is unique. Let $\alpha_x(a) := b$.
Now we want to show α_x is an automorphism of K. Let $x \in G, a, b \in K$. Then $\alpha_x(a), \alpha_x(b) \in K$, so $\alpha_x(a)\alpha_x(b) \in K$. This implies

$$\begin{aligned}\varepsilon(\alpha_x(a)\alpha_x(b)) &= \varepsilon(\alpha_x(a))\varepsilon(\alpha_x(b)) = h_x\varepsilon(a)h_x^{-1}h_x\varepsilon(b)h_x^{-1}\\ &= h_x\varepsilon(a)\varepsilon(b)h_x^{-1} = h_x\varepsilon(ab)h_x^{-1} = \varepsilon(\alpha_x(ab)).\end{aligned}$$

Since ε is injective, then $\alpha_x(a)\alpha_x(b) = \alpha_x(ab)$, so α_x is a homomorphism from K to K. To show injectivity, let $\alpha_x(a) = 1$. Then $\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)h_x^{-1} = 1$. So $h_x\varepsilon(a) = h_x$, which means $\varepsilon(a) = 1$. But ε is injective, so $a = 1$, and thus α_x is also injective. Now consider $b \in K$. Then from above, $h_x\varepsilon(b)h_x^{-1} \in \ker(\nu) = \operatorname{im}(\varepsilon)$. So there exists $a \in K$ such that $\varepsilon(a) = h_x\varepsilon(b)h_x^{-1}$. Moving h_x^{-1} and h_x, we have $h_x\varepsilon(a)h_x^{-1} = \varepsilon(b)$, which by definition implies $b = \alpha_x(a)$. So α_x is surjective and hence an automorphism.
(c) Let $x, y \in G$. Consider $h_xh_yh_{xy}^{-1} \in H$. Then $\nu(h_xh_yh_{xy}^{-1} = xy(xy)^{-1} = 1$. Since $\ker(\nu) = \operatorname{im}(\varepsilon)$, then there exists $a \in K$ such that $\varepsilon(a) = h_xh_yh_{xy}^{-1}$. Since ε is injective, a is unique. Set $\kappa(x, y) := a$.
For $x, y \in G, a \in K$, consider $\varepsilon(\alpha_x(\alpha_y(a)))$. Then

$$\begin{aligned}\varepsilon(\alpha_x(\alpha_y(a))) &= h_x\varepsilon(\alpha_y(a))h_x^{-1} = h_xh_y\varepsilon(a)h_y^{-1}h_x^{-1}\\ &= h_xh_yh_{xy}h_{xy}^{-1}\varepsilon(a)h_{xy}^{-1}h_{xy}h_y^{-1}h_x^{-1}\\ &= \varepsilon(\kappa(x,y))h_{xy}\varepsilon(a)h_{xy}^{-1}\varepsilon(\kappa(x,y))^{-1}\\ &= \varepsilon(\kappa(x,y))\varepsilon(\alpha_{xy}(a))\varepsilon(\kappa(x,y))^{-1}\\ &= \varepsilon(\kappa(x,y)\alpha_{xy}(a)\kappa(x,y)^{-1}).\end{aligned}$$

Since ε is injective, this implies $(\alpha_x \circ \alpha_y)(a) = (c_{\kappa(x,y)} \circ \alpha_{xy})(a)$, as desired.
(d) Let $x, y, z \in G$. Then by (c),

$$\begin{aligned}\varepsilon(\kappa(x,y)\kappa(xy,z))h_{xyz} &= \varepsilon(\kappa(x,y))\varepsilon(\kappa(xy,z))h_{(xy)z}\\ &= \varepsilon(\kappa(x,y))h_{xy}h_z\\ &= (h_xh_y)h_z\end{aligned}$$

and by (b) and (c), we have

$$\begin{aligned}\varepsilon(\alpha_x(\kappa(y,z))\kappa(x,yz))h_{xyz} &= \varepsilon(\alpha_x(\kappa(y,z)))\varepsilon(\kappa(x,yz))h_{x(yz)}\\ &= \varepsilon(\alpha_x(\kappa(y,z)))h_xh_{yz}\\ &= h_x\varepsilon(\kappa(y,z))h_x^{-1}h_xh_{yz} = h_x\varepsilon(\kappa(y,z))h_{yz}\\ &= h_x(h_yh_z).\end{aligned}$$

Since H is associative, then $(h_xh_y)h_z = h_x(h_yh_z)$. Thus, ε injective implies $\kappa(x,y)\kappa(xy,z) = \alpha_x(\kappa(y,z))\kappa(x,yz)$.

(e) Let $x \in G$. Since $\nu(h_x^{-1}h'_x) = x^{-1}x = 1$, and $\ker(\nu) = \operatorname{im}(\varepsilon)$, then there exists a unique $g(x) \in K$ such that $\varepsilon(g(x)) = h_x^{-1}h'_x$. From (b), for all $x \in G, a \in K$ we have

$$\begin{aligned}
\varepsilon(\alpha'_x(a)) &= h'_x\varepsilon(a)h_x'^{-1} \\
&= h_x h_x^{-1} h'_x \varepsilon(a) h_x'^{-1} h_x h_x^{-1} \\
&= h_x \varepsilon(g(x))\varepsilon(a)\varepsilon(g(x))^{-1} h_x^{-1} \\
&= h_x \varepsilon(g(x) a g(x)^{-1}) h_x^{-1},
\end{aligned}$$

so $\alpha'_x(a) = \alpha_x(g(x)ag(x)^{-1})$. Let $f(x) := \alpha_x(g(x))$. Then $\alpha'_x = c_{\alpha_x(g(x))} \circ \alpha_x = c_{f(x)} \circ \alpha_x$. Finally, for all $x, y \in G$ we have

$$\begin{aligned}
\varepsilon(\kappa'(x,y)) &= h'_x \cdot h'_y \cdot h_{xy}'^{-1} \\
&= h_x\varepsilon(g(x)) \cdot h_y\varepsilon(g(y)) \cdot \varepsilon(g(xy))^{-1} h_{xy}^{-1} \\
&= h_x\varepsilon(g(x)) \cdot h_x^{-1}h_x \cdot h_y \cdot h_{xy}^{-1}h_{xy} \cdot \varepsilon(g(y)g(xy)^{-1})h_{xy}^{-1} \\
&= (h_x\varepsilon(g(x))h_x^{-1}) \cdot (h_x h_y h_{xy}^{-1}) \cdot (h_{xy}\varepsilon(g(y)g(xy)^{-1})h_{xy}^{-1}) \\
&= \varepsilon(\alpha_x(g(x))) \cdot \varepsilon(\kappa(x,y)) \cdot \varepsilon(\alpha_{xy}(g(y)g(xy)^{-1}) \\
&= \varepsilon[\alpha_x(g(x)) \cdot \kappa(x,y) \cdot \alpha_{xy}(g(y)) \cdot \alpha_{xy}(g(xy))^{-1}] \\
&= \varepsilon[f(x) \cdot \kappa(x,y) \cdot \alpha_{xy}(g(y)) \cdot \kappa(x,y)^{-1}\kappa(x,y) \cdot f(xy)^{-1}] \\
&= \varepsilon[f(x) \cdot \kappa(x,y)\alpha_{xy}(g(y))\kappa(x,y)^{-1} \cdot \kappa(x,y) \cdot f(xy)^{-1}] \\
&= \varepsilon[f(x) \cdot \alpha_x(\alpha_y(g(y))) \cdot \kappa(x,y) \cdot f(xy)^{-1}] \\
&= \varepsilon[f(x) \cdot \alpha_x(f(y)) \cdot \kappa(x,y) \cdot f(xy)^{-1}].
\end{aligned}$$

Since ε is injective, then $\kappa'(x,y) = f(x) \cdot \alpha_x(f(y)) \cdot \kappa(x,y) \cdot f(xy)^{-1}$. □

2. Parameter Systems

At first, the α and κ functions in Proposition 1.4 seem very arbitrary, but looking more closely at them can actually give more insight into group extensions. These functions, together called a parameter system, can construct a group structure that makes computing group extensions easier.

2.1 Definition. A *parameter system* of G in K is a pair (α, κ) of maps $\alpha : G \to \operatorname{Aut}(K)$, $x \mapsto \alpha_x$, and $\kappa : G \times G \to K$ with the following properties:
(1) For all $x, y \in G$, $\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}$.
(2) For all $x, y, z \in G$, $\kappa(x,y)\kappa(xy,z) = \alpha_x(\kappa(y,z))\kappa(x,yz)$.
We call α the *automorphism system* and κ the $factorsystem$ of (α, κ), and we denote the set of parameter systems of G in K by $\operatorname{par}(G, K)$.

2.2 Lemma. *Let* $(\alpha, \kappa) \in \operatorname{par}(G, K)$. *Then* $\alpha_1 = c_{\kappa(1,1)}, \kappa(1,1) = \kappa(1,z)$, *and* $\kappa(x,1) = \alpha_x(\kappa(1,1))$ *for all* $x, z \in G$.
Proof.
By Proposition 1.4(c), $\alpha_1 \circ \alpha_1 = c_{\kappa(1,1)} \circ \alpha_1$. In particular, $\alpha_1 \circ \alpha_1(\alpha_1^{-1}(b)) = c_{\kappa(1,1)} \circ \alpha_1(\alpha_1^{-1}(b))$, which implies $\alpha_1 = c_{\kappa(1,1)}$. For $z \in G$, this and Proposition

1.4(d) imply

$$\begin{aligned}\kappa(1,1)\kappa(1,z) &= \kappa(1,1)\kappa(1,1\cdot z) = \alpha_1(\kappa(1,z))\kappa(1,1\cdot z)\\ &= \kappa(1,1)\kappa(1,z)\kappa(1,1)^{-1}\kappa(1,z).\end{aligned}$$

So $\kappa(1,1)=\kappa(1,z)$. For $x\in G$, Proposition 1.4(d) implies

$$\kappa(x,1\cdot 1)\kappa(x\cdot 1,1)=\alpha_x(\kappa(1,1))\kappa(x,1\cdot 1).$$

So $\kappa(x,1)=\alpha_x(\kappa(1,1))$, as desired. □

The set of functions from $G\to K$ is a group under the multiplication $(fg)(x):=f(x)g(x)$ for $f,g:G\to K, x\in G$ denoted by $F(G,K)$.

2.3 Lemma. *If $(\alpha,\kappa)\in \mathrm{par}(G,K)$ and $f:G\to K$, set ${}^f(\alpha,\kappa):=(\alpha',\kappa')$ with*

$$\alpha'_x := c_{f(x)}\circ\alpha_x,\ \textit{and}\ \kappa'(x,y):=f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1},$$

for $x,y\in G$. This defines a group action of $F(G,K)$ on $\mathrm{par}(G,K)$.
Proof.
To show these properties define a group action, we must show ${}^1(\alpha,\kappa)=(\alpha,\kappa)$ for all $(\alpha,\kappa)\in\mathrm{par}(G,K)$, and ${}^g({}^f(\alpha,\kappa))={}^{(gf)}(\alpha,\kappa)$ for all $f,g\in \mathrm{F}(G,K),(\alpha,\kappa)\in \mathrm{par}(G,K)$. Set ${}^f(\alpha,\kappa):=(\alpha',\kappa')$ and ${}^g(\alpha',\kappa'):=(\alpha'',\kappa'')$. Then for all $x,y\in G$, we have

$$\alpha''_x=c_{g(x)}\circ\alpha'_x=c_{g(x)}\circ c_{f(x)}\circ\alpha_x=c_{g(x)f(x)}\circ\alpha_x=c_{(gf)(x)}\circ\alpha_x$$

and

$$\begin{aligned}\kappa''(x,y) &= g(x)\alpha'_x(g(y))\kappa'(x,y)g(xy)^{-1}\\ &= g(x)f(x)\alpha_x(g(y))f(x)^{-1}f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1}g(xy)^{-1}\\ &= (gf)(x)\cdot\alpha_x(g(y))\alpha_x(f(y))\cdot\kappa(x,y)\cdot(gf)(xy)^{-1}\\ &= (gf)(x)\cdot\alpha_x((gf)(y))\cdot\kappa(x,y)\cdot(gf)(xy)^{-1},\end{aligned}$$

so $(\alpha'',\kappa'')={}^{gf}(\alpha,\kappa)$. If $f=1$, then $\alpha'_x=c_1\circ\alpha_x=\alpha_x$, and $\kappa'(x,y)=\alpha_x(1)\kappa(x,y)=\kappa(x,y)$, for all $x,y\in G$, so ${}^1(\alpha,\kappa)=(\alpha,\kappa)$. Now we must verify that (α',κ') is again a parameter system. For $x,y,z\in G$, we have

$$\begin{aligned}\alpha'_x\circ\alpha'_y &= c_{f(x)}\circ\alpha_x\circ c_{f(y)}\circ\alpha_y\\ &= c_{f(x)}\circ\alpha_x\circ c_{f(y)}\circ\alpha_x^{-1}\circ\alpha_x\circ\alpha_y\\ &= c_{f(x)}\circ c_{\alpha_x(f(y))}\circ c_{\kappa(x,y)}\circ\alpha_{xy}\\ &= c_{f(x)\alpha_x(f(y))\kappa(x,y)}\circ c^{-1}_{f(xy)}\circ c_{f(xy)}\circ\alpha_{xy}\\ &= c_{f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1}}\circ c_{f(xy)}\circ\alpha_{xy}\\ &= c_{\kappa'(x,y)}\circ\alpha'_{xy}\end{aligned}$$

and

$$\begin{aligned}
&\kappa'(x,y)\kappa'(xy,z)\\
&= f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1}f(xy)\alpha_{xy}(f(z))\kappa(xy,z)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y))\kappa(x,y)\alpha_{xy}(f(z))\kappa(xy,z)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y))\kappa(x,y)\alpha_{xy}(f(z))\kappa(x,y)^{-1}\kappa(x,y)\kappa(xy,z)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y))\alpha_x(\alpha_y(f(z)))\kappa(x,y)\kappa(xy,z)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y))\alpha_x(\alpha_y(f(z)))\alpha_x(\kappa(y,z))\kappa(x,yz)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y)\alpha_y(f(z))\kappa(y,z))\alpha_x(f(yz)^{-1}\alpha_x(f(yz))\kappa(x,yz)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y)\alpha_y(f(z))\kappa(y,z)f(yz)^{-1})\alpha_x(f(yz))\kappa(x,yz)f(xyz)^{-1}\\
&= f(x)\alpha_x(f(y)\alpha_y(f(z))\kappa(y,z)f(yz)^{-1})f(x)^{-1}f(x)\alpha_x(f(yz))\kappa(x,yz)f(xyz)^{-1}\\
&= f(x)\alpha_x(\kappa'(y,z))f(x)^{-1}f(x)\alpha_x(f(yz))\kappa(x,yz)f(xyz)^{-1}\\
&= f(x)\alpha_x(\kappa'(y,z))f(x)^{-1}\kappa'(x,yz)\\
&= \alpha'_x(\kappa'(y,z))\kappa'(x,yz).
\end{aligned}$$

Thus $(\alpha',\kappa') \in \mathrm{par}(G,K)$. □

We call two parameter systems of G in K *equivalent* if they belong to the same $F(G,K)$-orbit. We denote the set of equivalence classes by $\mathrm{Par}(G,K)$.

2.4 Proposition. *Let $(\alpha,\kappa) \in \mathrm{par}(G,K)$. Then the set $H := K \times G$ together with multiplication defined by*

$$(a,x)(b,y) := (a\cdot\alpha_x(b)\cdot\kappa(x,y), xy), \;\; for\ a,b \in K, x,y \in G$$

is a group with the identity element $(\kappa(1,1)^{-1},1)$ and inverse element $(a,x)^{-1} = (\kappa(1,1)^{-1}\kappa(x^{-1},x)^{-1}\alpha_{x^{-1}}(a)^{-1},x^{-1})$. Furthermore, the functions $\varepsilon : K \to K\times G$, $a \mapsto (\kappa(1,1)^{-1}a,1)$, and $\nu : K\times G \to G, (a,x) \mapsto x$, are group homomorphisms such that $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is a group extension of G by K.

Proof.

First we show associativity holds. Let $a,b,c \in K, x,y,z \in G$. Then

$$\begin{aligned}
[(a,x)(b,y)](c,z) &= (a\alpha_x(b)\kappa(x,y),xy)(c,z)\\
&= (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(xy,z),xyz)
\end{aligned}$$

and

$$\begin{aligned}
(a,x)[(b,y)(c,z)] &= (a,x)(b\alpha_y(c)\kappa(y,z),yz)\\
&= (a\alpha_x(b\alpha_y(c)\kappa(y,z))\kappa(x,yz),xyz)\\
&= (a\alpha_x(b)\alpha_x(\alpha_y(c))\alpha_x(\kappa(y,z))\kappa(x,yz),xyz)\\
&= (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(x,y)^{-1}\alpha_x(\kappa(y,z))\kappa(x,yz),xyz)\\
&= (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(x,y)^{-1}\kappa(x,y)\kappa(xy,z),xyz)\\
&= (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(xy,z),xyz).
\end{aligned}$$

Now consider $(\kappa(1,1)^{-1},1)$. Since $\alpha_1 = c_{\kappa(1,1)}$ and $\kappa(1,1) = \kappa(1,z)$ for all $z \in G$, then for $a \in K, x \in G$, we have

$$\begin{aligned}
(\kappa(1,1)^{-1},1)(a,x) &= (\kappa(1,1)^{-1}\alpha_1(a)\kappa(1,x),1\cdot x)\\
&= (\kappa(1,1)^{-1}\kappa(1,1)a\kappa(1,1)^{-1}\kappa(1,x),x) = (a,x).
\end{aligned}$$

So $(\kappa(1,1)^{-1},1)$ is the identity element. Next, for $a \in K, x \in G$ we see that

$$\begin{aligned}
&(\kappa(1,1)^{-1}\kappa(x^{-1},x)^{-1}\alpha_{x^{-1}}(a)^{-1},x^{-1})(a,x)\\
&= (\kappa(1,1)^{-1}\kappa(x^{-1},x)^{-1}\alpha_{x^{-1}}(a)^{-1}\alpha_{x^{-1}}(a)\kappa(x^{-1},x),x^{-1}x)\\
&= (\kappa(1,1)^{-1}\kappa(x^{-1},x)^{-1}\kappa(x^{-1},x),1)\\
&= (\kappa(1,1)^{-1},1),
\end{aligned}$$

which implies $(\kappa(1,1)^{-1}\kappa(x^{-1},x)^{-1}\alpha_{x^{-1}}(a)^{-1},x^{-1})$ is the inverse element of (a,x). Thus, H is a group under the given multiplication.
Now we show ε and ν are homomorphisms. For $a,b \in K$ we have

$$\begin{aligned}
\varepsilon(a)\varepsilon(b) &= (\kappa(1,1)^{-1}a,1)(\kappa(1,1)^{-1}b,1)\\
&= (\kappa(1,1)^{-1}a\alpha_1(\kappa(1,1)^{-1}b)\kappa(1,1),1\cdot 1)\\
&= (\kappa(1,1)^{-1}a\kappa(1,1)\kappa(1,1)^{-1}b\kappa(1,1)^{-1}\kappa(1,1),1)\\
&= (\kappa(1,1)^{-1}ab,1) = \varepsilon(ab),
\end{aligned}$$

so ε is a homomorphism. If $\varepsilon(a) = \varepsilon(b)$, then $(\kappa(1,1)^{-1}a,1) = (\kappa(1,1)^{-1}b,1)$, which immediately implies $a = b$, so ε is injective. For any $a,b \in K$, and $x,y \in G$ we have

$$\nu((a,x)(b,y)) = \nu(a\alpha_x(b)\kappa(x,y),xy) = xy = \nu(a,x)\nu(b,y),$$

so ν is a homomorphism. Since for any $a \in K, x \in G$ we have $\nu(a,x) = x$, then ν is surjective. Lastly, $(a,x) \in \varepsilon(K)$ if and only if $x = 1$ if and only if $(a,x) \in \ker(\nu)$. Thus $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is a group extension of G by K. $\square$

The following theorem was originally proven by Schreier (1901-1929). The bijections between $\mathrm{Ext}(G,K)$ and $\mathrm{Par}(G,K)$ show that given groups G and K, we can compute group extensions by looking explicitly at the group elements instead of only abstractly at the groups themselves.

2.5 Theorem. (Schreier) *The constructions in Proposition 1.4 and Proposition 2.4 induce mutually inverse bijections between* $\mathrm{Ext}(G,K)$ *and* $\mathrm{Par}(G,K)$.
Proof.
Assume $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ are equivalent group extensions of G by K. Then by definition, there exists an isomorphism $\varphi : H \to \tilde{H}$ such that

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "\varepsilon"] \arrow[d, "1_K"'] & H \arrow[r, "\nu"] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, "1_G"] & 1\\
1 \arrow[r] & K \arrow[r, "\tilde{\varepsilon}"] & \tilde{H} \arrow[r, "\tilde{\nu}"] & G \arrow[r] & 1
\end{tikzcd}$$

commutes. For all $x \in G$, let $h_x \in H$ such that $\nu(h_x) = x$ and let $\alpha : G \to \mathrm{Aut}(K)$, $\kappa : G \times G \to K$ be constructed as in Proposition 1.4, so

$$\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)h_x^{-1} \text{ and } h_xh_y = \varepsilon(\kappa(x,y))h_{xy}$$

for all $x,y \in G, a \in K$. Also let $\tilde{h}_x := \varphi(h_x)$ for every $x \in G$. Then we have $\tilde{\nu}(\tilde{h}_x) = \tilde{\nu}(\varphi(h_x)) = \nu(h_x) = x$ for every x. We can therefore use the $\tilde{h}_x$ to construct a parameter system $(\tilde{\alpha},\tilde{\kappa})$ associated to the group extension $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$. Since the above diagram commutes, then applying φ to the two above equations implies

$$\tilde{\varepsilon}(\alpha_x(a)) = \tilde{h}_x\tilde{\varepsilon}(a)\tilde{h}_x^{-1} \text{ and } \tilde{h}_x\tilde{h}_y = \tilde{\varepsilon}(\kappa(x,y))\tilde{h}_{xy}^{-1}.$$

So $\tilde{\alpha} = \alpha$ and $\tilde{\kappa} = \kappa$, which implies that equivalent extensions have equivalent parameter systems. Thus the construction in Proposition 1.4 induces a map

$$\Phi : \mathrm{Ext}(G,K) \longrightarrow \mathrm{Par}(G,K).$$

Now let $(\alpha,\kappa) \in \mathrm{Par}(G,K), f \in F(G,K)$, and let $(\tilde{\alpha},\tilde{\kappa}) := {}^f(\alpha,\kappa)$. Also let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ be the group extensions associated to (α,κ) and $(\tilde{\alpha},\tilde{\kappa})$, respectively. (Note: H is defined here as in Proposition 2.4, i.e. $H = K \times G$ with the operation given) Define $\varphi : H \to \tilde{H}$ by

$$\varphi(a,x) := (da\alpha_x(d)^{-1}f(x)^{-1},x) \text{ with } d := \kappa(1,1)^{-1}f(1)^{-1}\kappa(1,1).$$

For all $a,b \in K$, and $x,y \in G$ we have

$$\begin{aligned}
&\varphi(a,x)\varphi(b,y) = (da\alpha_x(d)^{-1}f(x)^{-1},x)\cdot(db\alpha_y(d)^{-1}f(y)^{-1},y)\\
&= (da\alpha_x(d)^{-1}f(x)^{-1}\tilde{\alpha}_x(db\alpha_y(d)^{-1}f(y)^{-1})\tilde{\kappa}(x,y),xy)\\
&= (da\alpha_x(d)^{-1}f(x)^{-1}f(x)\alpha_x(db\alpha_y(d)^{-1}f(y)^{-1})f(x)^{-1}\cdot\\
&\qquad\qquad\qquad\qquad\cdot f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1},xy)\\
&= (da\alpha_x(d)^{-1}\alpha_x(d)\alpha_x(b\alpha_y(d)^{-1})\alpha_x(f(y)^{-1})\alpha_x(f(y))\kappa(x,y)f(xy)^{-1},xy)\\
&= (da\alpha_x(b)\alpha_x(\alpha_y(d))^{-1}\kappa(x,y)f(xy)^{-1},xy)
\end{aligned}$$

and

$$\begin{aligned}
\varphi((a,x)(b,y)) &= \varphi(a\alpha_x(b)\kappa(x,y),xy)\\
&= (da\alpha_x(b)\kappa(x,y)\alpha_{xy}(d)^{-1}f(xy)^{-1},xy)\\
&= (da\alpha_x(b)\kappa(x,y)\alpha_{xy}(d)^{-1}\kappa(x,y)^{-1}\kappa(x,y)f(xy)^{-1},xy)\\
&= (da\alpha_x(b)\alpha_x(\alpha_y(d))^{-1}\kappa(x,y)f(xy)^{-1},xy).
\end{aligned}$$

So $\varphi(a,x)(b,y) = \varphi((a,x)(b,y))$, which implies φ is a homomorphism. For $a \in K$ and $x \in G$ we also have

$$\begin{aligned}
\varphi(\varepsilon(a)) &= \varphi(\kappa(1,1)^{-1}a,1) = (d\kappa(1,1)^{-1}a\alpha_1(d)^{-1}f(1)^{-1},1)\\
&= (\kappa(1,1)^{-1}f(1)^{-1}\kappa(1,1)\kappa(1,1)^{-1}a\alpha_1(d)^{-1}f(1)^{-1},1)\\
&= (\kappa(1,1)^{-1}f(1)^{-1}a\kappa(1,1)d^{-1}\kappa(1,1)^{-1}f(1)^{-1},1)\\
&= (\kappa(1,1)^{-1}f(1)^{-1}a\kappa(1,1)\kappa(1,1)^{-1}f(1)\kappa(1,1)\kappa(1,1)^{-1}f(1)^{-1},1)\\
&= (\kappa(1,1)^{-1}f(1)^{-1}a,1) = (f(1)a_1(f(1))\kappa(1,1)f(1\cdot 1)^{-1}a,1)\\
&= (\tilde{\kappa}(1,1)^{-1}a,1) = \tilde{\varepsilon}(a)
\end{aligned}$$

and

$$\tilde{\nu}(\varphi(a,x)) = \tilde{\nu}(da\alpha_x(d)^{-1}f(x)^{-1},x) = x = \nu(x).$$

This implies the diagram from above commutes, so by Proposition 1.3, since φ is a homomorphism and the diagram commutes, then the two group extensions $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ are equivalent. Thus the construction in Proposition 2.4 induces a map

$$\Psi : \mathrm{Par}(G,K) \longrightarrow \mathrm{Ext}(G,K).$$

Finally, we must show that Φ and Ψ are mutually inverse bijections. Let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ be a group extension and, for every $x \in G$, let $h_x \in H$

be such that $\nu(h_x) = x$. Also let (α, κ) be the parameter system constructed in Proposition 1.4 from h_x, for $x \in G$, and let $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ be the group extension constructed from (α, κ) in Proposition 2.4. Let $\varphi : \tilde{H} \to H$ be defined by

$$\varphi(a, x) := \varepsilon(\kappa(1,1)a\kappa(x,1)^{-1})h_x,$$

for all $a, b \in K$ and $x, y \in G$. Then

$$\begin{aligned}
\varphi((a,x)(b,y)) &= \varphi(a\alpha_x(b)\kappa(x,y), xy) \\
&= \varepsilon(\kappa(1,1)a\alpha_x(b)\kappa(x,y)\kappa(xy,1)^{-1})h_{xy} \\
&= \varepsilon(\kappa(1,1)a\alpha_x(b)\alpha_x(\kappa(y,1))^{-1}\kappa(x,y))h_{xy} \\
&= \varepsilon(\kappa(1,1)a\alpha_x(b)\alpha_x(\kappa(y,1))^{-1})\varepsilon(\kappa(x,y))h_{xy} \\
&= \varepsilon(\kappa(1,1)a\alpha_x(b)\alpha_x(\kappa(y,1))^{-1})h_x h_y
\end{aligned}$$

and

$$\begin{aligned}
\varphi(a,x)\varphi(b,y) &= \varepsilon(\kappa(1,1)a\kappa(x,1)^{-1})h_x\varepsilon(\kappa(1,1)b\kappa(y,1)^{-1})h_y \\
&= \varepsilon(\kappa(1,1)a\kappa(x,1)^{-1})\varepsilon(\alpha_x(\kappa(1,1)b\kappa(y,1)^{-1}))h_x h_y \\
&= \varepsilon(\kappa(1,1)a\kappa(x,1)^{-1}\alpha_x(\kappa(1,1))\alpha_x(b)\alpha_x(\kappa(y,1)^{-1}))h_x h_y \\
&= \varepsilon(\kappa(1,1)a\alpha_x(b)\alpha_x(\kappa(y,1)^{-1}))h_x h_y,
\end{aligned}$$

by Lemma 2.2, so φ is a homomorphism. Moreover, for $a \in K$ and $x \in G$ we have

$$\begin{aligned}
\varphi(\tilde{\varepsilon}(a)) &= \varphi(\kappa(1,1)^{-1}a, 1) = \varepsilon(\kappa(1,1)\kappa(1,1)^{-1}a\kappa(1,1)^{-1})h_1 \\
&= \varepsilon(a)\varepsilon(\kappa(1,1))^{-1}h_1 = \varepsilon(a),
\end{aligned}$$

by Proposition 1.4(c), and

$$\nu(\varphi(a,x)) = \nu(\varepsilon(\kappa(1,1)a\kappa(x,1)^{-1})h_x) = \nu(h_x) = x = \tilde{\nu}(a,x).$$

Hence, the group extensions $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{\varepsilon}} \tilde{H} \xrightarrow{\tilde{\nu}} G \longrightarrow 1$ are equivalent, so $\Psi \circ \Phi = \mathrm{id}$.
Now let $(\alpha, \kappa) \in \mathrm{par}(G, K)$ and let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ be the group extension constructed in Proposition 2.4. Set

$$h_x := (\kappa(1,1)^{-1}\kappa(x,1), x) \in H,$$

for $x \in G$. Immediately we have $\nu(h_x) = x$. So for $x \in G$ and $a \in K$, then

$$\begin{aligned}
h_x\varepsilon(a) &= (\kappa(1,1)^{-1}\kappa(x,1), x) \cdot (\kappa(1,1)^{-1}a, 1) \\
&= (\kappa(1,1)^{-1}\kappa(x,1)\alpha_x(\kappa(1,1))^{-1}\alpha_x(a)\kappa(x,1), x) \\
&= (\kappa(1,1)^{-1}\alpha_x(a)\kappa(x,1), x)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon(\alpha_x(a))h_x &= (\kappa(1,1)^{-1}\alpha_x(a), 1) \cdot (\kappa(1,1)^{-1}\kappa(x,1), x) \\
&= (\kappa(1,1)^{-1}\alpha_x(a)\alpha_1(\kappa(1,1)^{-1}\kappa(x,1))\kappa(1,x), x) \\
&= (\kappa(1,1)^{-1}\alpha_x(a)\kappa(1,1)\kappa(1,1)^{-1}\kappa(x,1)\kappa(1,1)^{-1}\kappa(1,x), x) \\
&= (\kappa(1,1)^{-1}\alpha_x(a)\kappa(x,1)\kappa(1,1)^{-1}\kappa(1,x), x) \\
&= (\kappa(1,1)^{-1}\alpha_x(a)\kappa(x,1), x).
\end{aligned}$$

Furthermore, for all $x, y \in G$ we have

$$\begin{aligned} h_x h_y &= (\kappa(1,1)^{-1}\kappa(x,1), x) \cdot (\kappa(1,1)^{-1}\kappa(y,1), y) \\ &= (\kappa(1,1)^{-1}\kappa(x,1)\alpha_x(\kappa(1,1))^{-1}\alpha_x(\kappa(y,1))\kappa(x,y), xy) \\ &= (\kappa(1,1)^{-1}\alpha_x(\kappa(y,1))\kappa(x,y), xy) \\ &= (\kappa(1,1)^{-1}\kappa(x,y)\kappa(xy,1), xy) \end{aligned}$$

and

$$\begin{aligned} \varepsilon(\kappa(x,y))h_{xy} &= (\kappa(1,1)^{-1}\kappa(x,y), 1) \cdot (\kappa(1,1)^{-1}\kappa(xy,1), xy) \\ &= (\kappa(1,1)^{-1}\kappa(x,y)\alpha_1(\kappa(1,1)^{-1}\kappa(xy,1))\kappa(1,xy), xy) \\ &= (\kappa(1,1)^{-1}\kappa(x,y)\kappa(1,1)\kappa(1,1)^{-1}\kappa(xy,1)\kappa(1,1)^{-1}\kappa(1,xy), xy) \\ &= (\kappa(1,1)^{-1}\kappa(x,y)\kappa(xy,1)\kappa(1,1)^{-1}\kappa(1,xy), xy) \\ &= (\kappa(1,1)^{-1}\kappa(x,y)\kappa(xy,1), xy). \end{aligned}$$

This shows that the parameter system constructed from $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ equals (α, κ). Therefore $\Phi \circ \Psi = \mathrm{id}$, and so the proof is complete. □

The next proposition is helpful in simplifying group extensions. If the conditions hold for a particular group extension, then the group extension becomes a well known product of groups that is much easier to work with.

2.6 Proposition. *Let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ be a group extension of G by K. Then the following are equivalent:*

(i) *There exists a homomorphism $\sigma : G \to H$ such that $\nu \circ \sigma = \mathrm{id}_G$.*
(ii) *$\varepsilon(K)$ has a complement in H.*

Proof.
(i)⇒(ii): Let $\sigma : G \to H$ be a homomorphism such that $\nu \circ \sigma = \mathrm{id}_G$. We will show that $\sigma(G)$ is a complement of $\varepsilon(K) = \ker(\nu)$ in H. Let $h \in \ker(\nu) \cap \sigma(G)$. Then $h = \sigma(x)$ for some $x \in G$, so we have $x = \nu(\sigma(x)) = \nu(h) = 1$ and thus $h = \sigma(x) = 1$. So for an arbitrary $h \in H$, then $h = h\sigma(\nu(h))^{-1} \cdot \sigma(\nu(h))$ with $h\sigma(\nu(h))^{-1} \in \ker(\nu)$ (since $\nu(h\sigma(\nu(h))^{-1}) = \nu(h)\nu(\sigma(\nu(h))^{-1}) = \nu(h)\nu(h)^{-1} = 1$) and $\sigma(\nu(h)) \in \sigma(G)$.
(ii)⇒(i): Let C be a complement of $\varepsilon(K) = \ker(\nu)$ in H. Then the map $\gamma : C \to H/\varepsilon(K)$, $c \mapsto c\varepsilon(K)$ is an isomorphism. By the homomorphism theorem, the map $\bar{\nu} : H/\varepsilon(K) \to G$, $h\varepsilon(K) \mapsto \nu(h)$ is also an isomorphism. So now the map

$$\sigma : G \xrightarrow{\bar{\nu}^{-1}} H/\varepsilon(K) \xrightarrow{\gamma^{-1}} C \xrightarrow{\iota} H$$

satisfies $\nu(\sigma(x)) = (\nu \circ \iota \circ \gamma^{-1} \circ \bar{\nu}^{-1})(x) = x$, with ι being the inclusion map. Hence there exists a unique $c \in C$ such that $x = \bar{\nu}(\gamma(c))$, so we only need to show $\nu(\iota(c)) = \bar{\nu}(\gamma(c))$. But $\bar{\nu}(\gamma(c)) = \bar{\nu}(c \ker(v)) = \nu(c) = \nu(\iota(c))$, so we are done. □

2.7 Definition. If the conditions in Proposition 2.6 hold, then we say the group extension $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ *splits* and σ is a *splitting map.*

2.8 Remark. If $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ splits and $\sigma : G \to H$ satisfies $\nu \circ \sigma = \mathrm{id}_G$, then we can use the elements $h_x := \sigma(x)$, $x \in G$, to construct a corresponding parameter system. Since σ is a homomorphism, then $h_x h_y = h_{xy}$

for all $x, y \in G$, so $\kappa(x, y) = 1$ for all $x, y \in G$. It follows that $\alpha : G \to \mathrm{Aut}(K)$ is a homomorphism.
Conversely, if $\alpha : G \to \mathrm{Aut(K)}$ is a homomorphism and $\kappa(x, y) = 1$ for all $x, y \in G$, then (α, κ) is a parameter system of G in K and the corresponding group extension is given by the semidirect product of G with K under the action defined by α.

2.9 Definition. Let $\alpha : G \to \mathrm{Aut}(K)$, $x \mapsto \alpha_x$ be a homomorphism. We can write the corresponding left action exponentially: $\alpha_x(k) = {}^x k$ for $x \in G$ and $k \in K$. Define a multiplication on the set $K \times G$ by

$$(k_1, g_1) \cdot (k_2, g_2) := (k_1 {}^{g_1} k_2, g_1 g_2).$$

This defines a group structure with identity element $(1_K, 1_G)$ and $(k, g)^{-1} = ({}^{g^{-1}}(k^{-1}), g^{-1})$. We call this group the *(external) semidirect product* of G with K with respect to the action α and denote it $K \rtimes G$ (or $K \rtimes_\alpha G$).

3. Group Cohomology and Group Extensions with Abelian Kernel

If $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is a group extension, then $K \cong \mathrm{im}(\varepsilon) = \ker(\nu)$, so we say $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is a group extension with abelian kernel if K is abelian. So in this section, we replace K with A and we assume A is an abelian group.

Although group cohomology has a complex definition that is difficult to understand at first, cohomology appears everywhere in mathematics. In particular, the connection to group extensions in Example 3.4 is very interesting to note. The relationship is clean and gives insight into looking at both group extensions and group cohomology. When dealing with finite groups of relatively prime orders (if A abelian), cohomology implies there is only one group extension in each equivalence class of group extensions.

3.1 Definition. Let $\alpha : G \to \mathrm{Aut}(A)$, $x \mapsto \alpha_x$ be a homomorphism and write the corresponding action such that $\alpha_x(a) = {}^x a$ for $x \in G$ and $a \in A$. For $n \in \mathbb{N}_0$, let $F(G^n, A)$ denote the abelian group of functions $f : G^n \to A$ under the multiplication $(fg)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) g(x_1, \ldots, x_n)$, for $f, g \in F(G^n, A)$ and $x_1, \ldots, x_n \in G$. If $n = 0$, then set $G^n := \{1\}$. For each $n \in \mathbb{N}_0$, there exists a homomorphism

$$\partial^n := \partial^n_\alpha : F(G^n, A) \longrightarrow F(G^{n+1}, A)$$

given by

$$\begin{aligned}(\partial^n_\alpha(f))(x_0, \ldots, x_n) \quad := \quad & {}^{x_0} f(x_1, \ldots, x_n) \cdot \\ & \cdot (\prod_{i=1}^{n} f(x_0, \ldots, x_{i-1}x_i, \ldots, x_n)^{(-1)^i}) \cdot \\ & \cdot f(x_0, \ldots, x_{n-1})^{(-1)^{n+1}},\end{aligned}$$

for $f \in F(G^n, A)$ and $(x_0, \ldots, x_n) \in G^{n+1}$. In particular, if $n = 0$, then we have $(\partial^0(f))(x) := {}^x f(1) \cdot f(1)^{-1}$. In the next subsection, we will show that $\partial^{n+1} \circ \partial^n = 1$, for all $n \in \mathbb{N}_0$. Assuming this, we have $\operatorname{im}(\partial^n) \leq \ker(\partial^{n+1}) \leq F(G^{n+1}, A)$, for all $n \in \mathbb{N}_0$. We write

$$B^n(G, A) := B^n_\alpha(G, A) := \operatorname{im}(\partial^{n-1}_\alpha)$$

and

$$Z^n(G, A) := Z^n_\alpha(G, A) := \ker(\partial^n_\alpha),$$

for $n \in \mathbb{N}_0$, and we set $B^0(G, A) := B^0_\alpha(G, A) := 1$. The elements of $B^n_\alpha(G, A)$ are called *n-coboundaries* and the elements of $Z^n_\alpha(G, A)$ are called *n-cocycles* of G with coefficients in A (under the action α). Finally, we set

$$H^n(G, A) := H^n_\alpha(G, A) := Z^n_\alpha(G, A)/B^n_a(G, A).$$

The group $H^n_\alpha(G, A)$ is called the *n-th cohomology group* of G with coefficients in A (under the action α) and the elements are called *cohomology classes*. If $f \in Z^n(G, A)$, we denote its cohomology class by $[f] \in H^n(G, A)$.

3.2 Proposition. *For each $n \in \mathbb{N}_0$, let ∂^n_α be defined as in Definition 3.1. To simplify notation, denote ∂^n_α by*

$$(\partial^n_\alpha(f))(x_0, x_1, \ldots, x_n) := \prod_{i=0}^{n+1} (d^i_{n,\alpha}(f))^{(-1)^i}(x_0, \ldots, x_n),$$

where $d^i_{n,\alpha} : F(G^n, A) \longrightarrow F(G^{n+1}, A)$ is defined as

$$d^i_{n,\alpha}(f)(x_0, \ldots, x_n) := \begin{cases} i = 0 : & {}^{x_0}f(x_1, \ldots, x_n) \\ 1 \leq i \leq n : & f(x_0, \ldots, x_{i-1}x_i, \ldots, x_n) \\ i = n+1 : & f(x_0, \ldots, x_{n-1}) \end{cases}.$$

Then $\partial^{n+1}_\alpha \circ \partial^n_\alpha = 1$ for all $n \in \mathbb{N}_0$.

Proof.

From the notation, we have

$$(\partial^{n+1}_\alpha \circ \partial^n_\alpha)(f)(x_0, \ldots, x_{n+1}) = \prod_{j=0}^{n+2} \prod_{i=0}^{n+1} (d^j_{n+1,\alpha} \circ d^i_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(i+j)}}.$$

Let (i, j) denote the index for the ij-th term

$$x_{(i,j)} := (d^j_{n+1,\alpha} \circ d^i_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(i+j)}}$$

and let $S := \{(i, j) : 0 \leq i \leq n+1, 0 \leq j \leq n+2\}$. Then

$$(\partial^{n+1}_\alpha \circ \partial^n_\alpha)(f)(x_0, \ldots, x_{n+1}) = \prod_{s \in S} x_s.$$

Let $A := \{(i, j) \in S : i \geq j\}$ and $B := \{(i, j) \in S : i < j\}$ be subsets of S. Clearly $A \cap B = \emptyset$ and $A \cup B = S$.

Define a function $f : A \to B$, $(i, j) \mapsto (j, i+1)$. Consider $a_1 = (i_1, j_1), a_2 = (i_2, j_2)$, $i_1 \neq i_2$ or $j_1 \neq j_2$. So $a_1 \neq a_2$. Then $f(a_1) = (j_1, i_1+1), f(a_2) = (j_2, i_2+1)$. Since $i_1 \neq i_2$ or $j_1 \neq j_2$, then either $i_1 + 1 \neq i_2 + 1$ or $j_1 \neq j_2$, so $f(a_1) \neq f(a_2)$. This implies f is injective.

Now let $b = (i', j') \in B$. By definition, $i' < j'$, so $i' - 1 < j' - 1$ and thus $i' \leq j' - 1$. So $j' - 1 \geq i'$, which means $(j' - 1, i') \in A$. So we see that for every $b = (i', j') \in B$, there exists $(j' - 1, i') \in A$ such that $f(j' - 1, i') = (i', j')$, so f is surjective. Thus we have shown f is bijective.

Consider x_a, for $a = (i, j) \in A$. We want to show that $x_a = (x_{f(a)})^{-1}$. We have

four cases:
(1) $i = 0, j = 0$:

$$\begin{aligned} x_{(0,0)} &= (d^0_{n+1,\alpha} \circ d^0_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(0+0)}} \\ &= {}^{x_0}(d^0_{n,\alpha}(f)(x_1, \ldots, x_{n+1})^{(-1)^{(0)}}) \\ &= {}^{x_0 x_1} f(x_2, \ldots, x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} x_{(0,1)} &= (d^1_{n+1,\alpha} \circ d^0_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(0+1)}} \\ &= d^0_{n,\alpha}(f)(x_0 x_1, \ldots, x_{n+1})^{(-1)^1} \\ &= {}^{x_0 x_1} f(x_2, \ldots, x_{n+1})^{-1}, \end{aligned}$$

so $x_{(0,0)} = (x_{(0,1)})^{-1}$.
(2) $i > 0, j = 0$:

$$\begin{aligned} x_{(i,0)} &= (d^0_{n+1,\alpha} \circ d^i_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(i+0)}} \\ &= {}^{x_0}(d^i_{n,\alpha}(f)(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})^{(-1)^i}) \\ &= {}^{x_0} f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \end{aligned}$$

with $x_i x_{i+1}$ in the i-th position, and

$$\begin{aligned} x_{(0,i+1)} &= (d^{i+1}_{n+1,\alpha} \circ d^0_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(0+i+1)}} \\ &= d^0_{n,\alpha}(f)(x_0, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^{i+1}} \\ &= {}^{x_0} f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^{i+1}}, \end{aligned}$$

so $x_{(i,0)} = (x_{(0,i+1)})^{-1}$.
(3) $i \geq j$, with $i, j \neq 0$ for $0 < i \leq n$:

$$\begin{aligned} x_{(i,j)} &= (d^j_{n+1,\alpha} \circ d^i_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(i+j)}} \\ &= d^i_{n,\alpha}(f)(x_0, \ldots, x_{j-1} x_j, \ldots, x_{n+1})^{(-1)^{i+j}} \\ &= f(x_0, \ldots, x_{j-1} x_j, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^{i+j}} \end{aligned}$$

and

$$\begin{aligned} x_{(j,i+1)} &= (d^{i+1}_{n+1,\alpha} \circ d^j_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(j+(i+1))}} \\ &= d^j_{n,\alpha}(f)(x_0, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^{i+j+1}} \\ &= f(x_0, \ldots, x_{j-1} x_j, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^{i+j+1}}, \end{aligned}$$

so $x_{(i,j)} = (x_{(j,i+1)})^{-1}$.
(4) $i = n + 1$:

$$\begin{aligned} x_{(n+1,j)} &= (d^j_{n+1,\alpha} \circ d^{n+1}_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(n+1+j)}} \\ &= d^{n+1}_{n,\alpha}(f)(x_0, \ldots, x_{j-1} x_j, \ldots, x_{n+1})^{(-1)^{n+j+1}} \\ &= f(x_0, \ldots, x_{j-1} x_j, \ldots, x_n)^{(-1)^{n+j+1}} \end{aligned}$$

and

$$\begin{aligned} x_{(j,n+2)} &= (d^{n+2}_{n+1,\alpha} \circ d^{j}_{n,\alpha})(f)(x_0, \ldots, x_{n+1})^{(-1)^{(j+n+2)}} \\ &= d^{j}_{n,\alpha}(f)(x_0, \ldots, , x_n)^{(-1)^{(n+j+1)+1}} \\ &= f(x_0, \ldots, x_{j-1}x_j, \ldots, x_n)^{(-1)^{(n+j+1)+1}}, \end{aligned}$$

so $x_{(n+1,j)} = (x_{(j,n+2)})^{-1}$.
Thus for any $a \in A$ we have $x_a = (x_{f(a)})^{-1}$. Since there is a bijection between A and B, then we can cancel every x_s for all $s \in S$. So $\partial^{n+1}_\alpha \circ \partial^n_\alpha = 1$, as desired. □

3.3 Remark. Let $\alpha : G \to \mathrm{Aut}(A)$ be a homomorphism.
(a) We can identify $F(G^0, A)$ with A under the map $f \mapsto f(1)$. This implies

$$Z^0(G, A) = A^G := \{a \in A : {}^x a = a \text{ for all } x \in G\},$$

the subgroup of G-fixed points of A. Since $B^0(G, A) = 1$ by definition, then $H^0(G, A) \cong A^G$.
(b) A function $f : G \to A$ is in $Z^1(G, A)$ if and only if

$$f(xy) = {}^x f(y) \cdot f(x)$$

for all $x, y \in G$. The 1-cocycles of G with coefficients in A are called the *crossed homomorphisms* from G to A. If the action of G on A is trivial, then the crossed homomorphisms are exactly the homomorphisms. A function $f : G \to A$ is a 1-coboundary if and only if there exists an element $a \in A$ such that

$$f(x) = {}^x a \cdot a^{-1},$$

for all $x \in G$. These 1-coboundaries are called the *principal* crossed homomorphisms. If G acts trivially on A, then the principal crossed homomorphisms are also trivial, so $H^0(G, A) \cong \mathrm{Hom}(G, A)$.
(c) A function $f : G^2 \to A$ is a 2-cocycle if and only if

$${}^x f(y, z) f(x, yz) = f(xy, z) f(x, y),$$

for all $x, y, z \in G$, and f is a 2-coboundary if and only if there exists a function $g : G \to A$ such that

$$f(x, y) = {}^x g(y) g(x) g(xy)^{-1},$$

for all $x, y \in G$.
(d) A function $f : G^3 \to A$ is a 3-cocycle if and only if

$${}^w f(x, y, z) f(w, xy, z) f(w, x, y) = f(wx, y, z) f(w, x, yz)$$

for all $w, x, y, z \in G$. Moreover, f is a 3-coboundary if and only if there exists a function $g : G^2 \to A$ such that

$$f(x, y, z) = {}^x g(y, z) g(xy, z)^{-1} f(x, yz) g(x, y)^{-1}$$

for all $x, y, z \in G$.
(e) If there exists $e \in \mathbb{N}$ such that $a^e = 1$ for all $a \in A$ (we say A is of *finite exponent*), then $f^e = 1$ for all $f \in F(G^n, A)$ and all $n \in \mathbb{N}_0$. In particular, each coboundary and each cohomology class has an order which divides e.

3.4 Example. Let $1 \longrightarrow K \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ be a group extension, let

$h_x \in H$ with $\nu(h_x) = x$ for all $x \in G$, and let $(\alpha, \kappa) \in \mathrm{par}(G, A)$ be the parameter system as defined in Proposition 1.4. Then

$$\varepsilon(\alpha_x(a)) = h_x \varepsilon(a) h_x^{-1}, \; h_x h_y = \varepsilon(\kappa(x, y)) h_{xy},$$

$$\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}, \text{ and } \alpha_x(\kappa(y, z))\kappa(x, yz) = \kappa(xy, z)\kappa(x, y),$$

for all $a \in A$, and $x, y, z \in G$. Since A is abelian, the map $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism. We also see that κ is a 2-cocycle of G with coefficients in A under the action defined by α. If $(\alpha', \kappa') \in \mathrm{par}(G, A)$ is equivalent to (α, κ), then by Lemma 2.3 there exists a function $f : G \to A$ such that

$$\alpha'_x = c_{\alpha_x(f(x))} \circ \alpha_x \text{ and } \kappa'(x, y) = f(x)\alpha_x(f(y))\kappa(x, y)f(xy)^{-1},$$

for all $x, y \in G$. Again, since A is abelian, then $\alpha' = \alpha$. By rearranging the terms of $\kappa'(x, y)$, we see κ and κ' belong to the same cohomology class. Altogether this implies (α, κ) and (α', κ') are equivalent parameter systems if and only if $\alpha = \alpha'$ and $[\kappa] = [\kappa'] \in H^2_\alpha(G, A)$. Therefore, we have another bijection:

$$\mathrm{Ext}(G, A) \cong \mathrm{Par}(G, A) \cong \dot{\bigcup_{\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(A))}} H^2_\alpha(G, A).$$

We can see from this that for every $\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, the set $\mathrm{Ext}_\alpha(G, A)$ of equivalence classes of extensions of G by A which have α as their automorphism system is in bijection to $H^2_\alpha(G, A)$, and thus it also forms an abelian group. In this notation, we have

$$\mathrm{Ext}(G, A) = \dot{\bigcup_{\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(A))}} \mathrm{Ext}_\alpha(G, A).$$

For every $\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, the identity element in $\mathrm{Ext}_\alpha(G, A)$ is given by the semidirect product of G by A under the action α.

3.5 Proposition. *Let $\alpha : G \to \mathrm{Aut}(A)$ be a homomorphism and assume G is finite. Then $[f]^{|G|} = 1$ for all n-cocycles $f \in Z^n_\alpha(G, A)$ and all $n \in \mathbb{N}$.*

Proof.

Let $n \in \mathbb{N}$, let $f \in Z^n_\alpha(G, A)$, and let $x_0, \ldots, x_n \in G$. Then we have

$$\begin{aligned} &f(x_0, \ldots, x_{n-1})^{(-1)^n} \\ &= {}^{x_0}f(x_1, \ldots, x_n) \cdot (\prod_{i=1}^{n} f(x_0, \ldots, x_{i-1}x_i, \ldots, x_n)^{(-1)^i}). \end{aligned}$$

If we fix $x_0, \ldots, x_{n-1} \in G$ and multiply the above equations for the different elements $x_n \in G$, then we have

$$\begin{aligned} &f(x_0, \ldots, x_{n-1})^{(-1)^n |G|} \\ &= {}^{x_0}(\prod_{x_n \in G} f(x_1, \ldots, x_n)) \cdot \prod_{i=1}^{n} (\prod_{x_n \in G} f(x_0, \ldots, x_{i-1}x_i, \ldots, x_n))^{(-1)^i}. \end{aligned}$$

Let $g : G^{n-1} \to A$ by $g(x_1, \ldots, x_{n-1}) := \prod_{x \in G} f(x_1, \ldots, x_{n-1}, x)$. Then the above equation implies

$$f^{|G|} = \partial^{n-1}(g^{(-1)^n}),$$

and $[f]^{|G|} = 1$ in $H^n(G, A)$. $\square$

3.6 Corollary. *Let G and A be finite groups of relatively prime orders. Then $H^n_\alpha(G,A) = 1$ for all $\alpha \in \operatorname{Hom}(G, \operatorname{Aut}(A))$ and all $n \in \mathbb{N}$.*
Proof.
Let $k := |G|$ and $l := |A|$. Since k and l are relatively prime, then there exist elements $r, s \in \mathbb{Z}$ such that $rk + sl = 1$. From Remark 3.3(e) and Proposition 3.5 we know $[f]^k = 1$ and $[f]^l = 1$ for all $f \in Z^n_\alpha(G,A)$ and all $n \in \mathbb{N}$. Then we also have $[f] = [f]^1 = [f]^{rk+sl} = ([f]^k)^r([f]^l)^s = 1$. □

3.7 Remark. The above corollary together with Proposition 2.6 and Remark 2.8 implies that if G and A have relatively prime orders, then every group extension of G by A splits. In particular, every group extension of G by A is a semidirect product.

4. Central Extensions

As in the last section, let A be an abelian group.

4.1 Proposition. *Let G and A be groups, with A abelian, and let (α, κ) be a parameter system with corresponding extension $1 \longrightarrow A \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$. Then the following are equivalent:*

(i) $\varepsilon(A) \subseteq Z(H)$
(ii) $\alpha(g) = \mathrm{id}_A$ *for all* $g \in G$ *(α is trivial).*

Proof.
(i)⇒(ii): By Proposition 1.4(b), for any $x \in G, a \in A$ there exists a unique $\alpha_x(a) \in A$ such that $\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)h_x^{-1}$. Since $\varepsilon(A) \subseteq Z(H)$, then we have $\varepsilon(\alpha_x(a))h_x = h_x\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)$, which implies $\varepsilon(\alpha_x(a)) = \varepsilon(a)$. The injectivity of ε implies $\alpha_x(a) = a$, so α is trivial.
(ii)⇒(i): Again by Proposition 1.4(b), for any $x \in G, a \in A$ there exists a unique $\alpha_x(a) \in A$ such that $\varepsilon(\alpha_x(a)) = h_x\varepsilon(a)h_x^{-1}$. Since α is trivial, then $\varepsilon(\alpha_x(a)) = \varepsilon(a) = h_x\varepsilon(a)h_x^{-1}$. By Proposition 1.4(a), for all $h \in H$ there exist unique elements $x \in G, a \in A$ such that $h = h_x\varepsilon(a)$, which means $h_x^{-1} = \varepsilon(a)h^{-1}$. Thus

$$\varepsilon(a) = h_x\varepsilon(a)h_x^{-1} = (h_x\varepsilon(a))(h_x^{-1}) = (h)(\varepsilon(a)h^{-1}) = h\varepsilon(a)h^{-1}.$$

Therefore for all $h \in H$, $\varepsilon(a)h = h\varepsilon(a)$, so $\varepsilon(A) \subseteq Z(H)$. □

4.2 Definition. An extension $1 \longrightarrow A \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$ is called *central* if (i) or (ii) in Proposition 4.1 holds.

4.3 Definition. Let $f : G^2 \to A$ be a 2-cocycle (i.e. $f \in Z^2(G,A)$). We say f is *symmetric* if $f(x,y) = f(y,x)$ for all $x, y \in G$.

4.4 Proposition. *Let $\alpha = 1$ and $\kappa \in Z^2(G,A)$. If $(1,\kappa) \in \operatorname{par}(G,A)$ with corresponding extension $1 \longrightarrow A \xrightarrow{\varepsilon} H \xrightarrow{\nu} G \longrightarrow 1$. Then H is abelian if and only if κ is symmetric and G is abelian.*
Proof.
Assume H is abelian. Since $\varepsilon(A)$ is normal in H, then $H/\varepsilon(A) \cong G$ is also abelian. By Proposition 1.4(c), for all $x, y \in G$ there exists a unique $\kappa(x,y) \in A$ such that $h_xh_y = \varepsilon(\kappa(x,y))h_{xy}$. Similarly, we have $h_yh_x = \varepsilon(\kappa(y,x))h_{yx}$. By assumption H

is abelian, so $h_x h_y = h_y h_x$. This implies $\varepsilon(\kappa(x,y))h_{xy} = \varepsilon(\kappa(y,x))h_{yx}$. But since G is abelian, then $xy = yx$, so $h_{xy} = h_{yx}$. That leaves $\varepsilon(\kappa(x,y)) = \varepsilon(\kappa(y,x))$. Thus since ε is injective, then $\kappa(x,y) = \kappa(y,x)$.
Now assume κ is symmetric and G is abelian. Since α is trivial, then using the multiplication defined in Proposition 2.4 yields

$$(a,x)(b,y) = (ab \cdot \kappa(x,y), xy), \text{ for } a,b \in A, x,y \in G.$$

Since A and G are abelian, then $ab = ba$, $xy = yx$ for all $a,b \in A, x,y \in G$. By definition, κ is symmetric means $\kappa(x,y) = \kappa(y,x)$. Thus we have

$$(a,x)(b,y) = (ab \cdot \kappa(x,y), xy) = (ba \cdot \kappa(y,x), yx) = (b,y)(a,x),$$

for all $a,b \in A, x,y \in G$, which implies H is abelian. □

5. Group Extensions with Non-abelian Kernel

In this section, we go back to using K, which we do not assume is abelian. We will write $[\alpha,\kappa]$ for the equivalence class of any element $(\alpha,\kappa) \in \mathrm{par}(G,K)$.

5.1 Definition. An automorphism $f \in \mathrm{Aut}(K)$ is called an *inner* automorphism if $f = c_a$ for some $a \in K$. The set $\mathrm{Inn}(K)$ of inner automorphisms is the image of the homomorphism $c : K \to \mathrm{Aut}(K)$, $a \mapsto c_a$, so $\mathrm{Inn}(K)$ is a subgroup of $\mathrm{Aut}(K)$. Since $f \circ c_a \circ f^{-1} = c_{f(a)}$ for all $f \in \mathrm{Aut}(K)$ and all $a \in K$, then $\mathrm{Inn}(K)$ is also a normal subgroup. Note that $\mathrm{Inn}(K) \cong K/Z(K)$. We call the quotient $\mathrm{Out}(K) := \mathrm{Aut}(K)/\mathrm{Inn}(K)$ the group of *outer* automorphisms of K.

5.2 Remark. For each $(\alpha,\kappa) \in \mathrm{par}(G,K)$ one has $\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}$ for all $x,y \in G$. This shows that the function $\omega : G \to \mathrm{Out}(K)$, $x \mapsto \alpha_x \mathrm{Inn}(K)$ is a group homomorphism. We call ω the *pairing* induced by the automorphism system α. If (α',κ') is an equivalent parameter system, then $\alpha'_x = c_{f(x)} \circ \alpha_x$ for some function $f : G \to K$, which shows that the pairing ω' induced by α' is equal to ω. So, each element in $\mathrm{Par}(G,K)$ defines a pairing $\omega : G \to \mathrm{Out}(K)$. By Schreier's Theorem (2.5), every element in $\mathrm{Ext}(G,K)$ also defines a pairing. If K is abelian, then $\mathrm{Inn}(K) = 1$, so $\mathrm{Out}(K) \cong \mathrm{Aut}(K)$. Therefore, in the abelian case, there is no difference between pairings and automorphism systems.
For each $\omega \in \mathrm{Hom}(G, \mathrm{Out}(K))$, we denote by $\mathrm{ext}_\omega(G,K)$ (respectively for $\mathrm{par}_\omega(G,K)$) the set of extensions of G by K (resp. parameter systems of G in K) which induce the pairing ω. Similarly, we denote by $\mathrm{Ext}_\omega(G,K)$ (resp. $\mathrm{Par}_\omega(G,K)$) the set of equivalence classes of extensions of G by K (resp. parameter systems of G in K) which induce the pairing ω. Thus we have

$$\mathrm{Ext}(G,K) = \dot{\bigcup_{\omega \in \mathrm{Hom}(G,\mathrm{Out}(K))}} \mathrm{Ext}_\omega(G,K)$$

and

$$\mathrm{Par}(G,K) = \dot{\bigcup_{\omega \in \mathrm{Hom}(G,\mathrm{Out}(K))}} \mathrm{Par}_\omega(G,K).$$

Again by Schreier's Theorem (2.5), this implies $\mathrm{Ext}_\omega(G,K) \cong \mathrm{Par}_\omega(G,K)$ for each $\omega \in \mathrm{Hom}(G, \mathrm{Out}(K))$. It is possible that $\mathrm{Ext}_\omega(G,K)$ is empty. Later in this section we will see when this happens, and we will also describe $\mathrm{Ext}_\omega(G,K)$ when it is non-empty.

5.3 Definition. A subgroup $N \leq G$ is called a *characteristic* subgroup of G if $f(N) = N$ for all $f \in \mathrm{Aut}(K)$.

5.4 Remark. For each automorphism $f \in \mathrm{Aut}(K)$, the restriction $f|_{Z(K)}$ defines an automorphism of $Z(K)$, the center of K, since $Z(K)$ is characteristic in K. This defines a group homomorphism $\mathrm{res}^K_{Z(K)} : \mathrm{Aut}(K) \to \mathrm{Aut}(Z(K))$ whose kernel contains $\mathrm{Inn}(K)$. By the homomorphism theorem, we obtain a homomorphism $\mathrm{Out}(K) \to \mathrm{Aut}(K)$, $f\mathrm{Inn}(K) \mapsto f|_{Z(K)}$, which we also denote by $\mathrm{res}^K_{Z(K)}$.
If $\omega \in \mathrm{Hom}(G, \mathrm{Out}(K))$, then its composition with $\mathrm{res}^K_{Z(K)}$ gives a homomorphism $\zeta := \mathrm{res}^K_{Z(K)} \circ \omega : G \to \mathrm{Aut}(Z(K))$.We will show that $\mathrm{Par}_\omega(G,K)$ is already determined by ζ.

5.5 Theorem. *Let* $\omega \in \mathrm{Hom}(G, \mathrm{Out}(K))$ *with* $\mathrm{Par}(G,K) \neq \emptyset$ *and let* $\zeta := \mathrm{res}^K_{Z(K)} \circ \omega \in \mathrm{Hom}(G, \mathrm{Aut}(Z(K)))$. *Then the function*

$$Z^2_\zeta(G, Z(K)) \times \mathrm{par}_\omega(G,K) \to \mathrm{par}_\omega(G,K),\ (\gamma, (\alpha, \kappa)) \mapsto (\alpha, \gamma\kappa),$$

with

$$(\gamma\kappa)(x,y) := \gamma(x,y)\kappa(x,y),$$

for $x, y \in G$, *defines an action of the group* $Z^2_\zeta(G, Z(K))$ *on the set* $\mathrm{par}_\omega(G,K)$. *Moreover, this action induces an action of* $H^2_\zeta(G, Z(K))$ *on* $\mathrm{Par}_\omega(G,K)$ *which is transitive and free. In particular, for any element* $(\alpha,\kappa) \in \mathrm{par}_\omega(G,K)$, *the map*

$$H^2_\zeta(G, Z(K)) \longrightarrow \mathrm{Par}_\omega(G,K),\ [\gamma] \longmapsto {}^{[\gamma]}[\alpha,\kappa] = [\alpha, \gamma\kappa]$$

is a bijection.

Proof.
We first show that for $\gamma \in Z^2_\zeta(G, Z(K))$ and $(\alpha,\kappa) \in \mathrm{par}_\omega(G,K)$ we also have $(\alpha, \gamma\kappa) \in \mathrm{par}_\omega(G,K)$. For all $x, y, z \in G$ we have

$$\begin{aligned}
(\gamma\kappa)(x,y) \cdot (\gamma\kappa)(xy,z) &= \gamma(x,y)\kappa(x,y)\gamma(xy,z)\kappa(xy,z) \\
&= \gamma(x,y)\gamma(xy,z)\kappa(x,y)\kappa(xy,z) \\
&= \zeta_x(\gamma(y,z))\gamma(x,yz)\alpha_x(\kappa(y,z))\kappa(x,yz) \\
&= \alpha_x(\gamma(y,z)\kappa(y,z))\gamma(x,yz)\kappa(x,yz) \\
&= \alpha_x((\gamma\kappa)(y,z))(\gamma\kappa)(x,yz),
\end{aligned}$$

since $\alpha(z) = \zeta(z)$ for each $z \in Z(K)$, and

$$\begin{aligned}
c_{(\gamma\kappa)(x,y)} \circ \alpha_{xy} &= c_{\gamma(x,y)\kappa(x,y)} \circ \alpha_{xy} \\
&= c_\gamma(x,y) \circ c_{\kappa(x,y)} \circ \alpha_{xy} \\
&= c_{\kappa(x,y)} \circ \alpha_{xy} = \alpha_x \circ \alpha_y,
\end{aligned}$$

since $\gamma(x,y) \in Z(K)$. Moreover, for all $(\alpha,\kappa) \in \mathrm{par}_\omega(G,K)$ and $\gamma, \delta \in Z^2_\zeta(G, Z(K))$ we have

$${}^\delta({}^\gamma(\alpha,\kappa)) = {}^\delta(\alpha, \gamma\kappa) = (\alpha, \delta\gamma\kappa) = {}^{\delta\gamma}(\alpha,\kappa)$$

and ${}^1(\alpha,\kappa) = (\alpha,\kappa)$, so we have thus established an action of $Z^2_\zeta(G, Z(K))$ on $\mathrm{par}_\omega(G,K)$.

Now let $(\alpha,\kappa),(\alpha',\kappa')\in \mathrm{par}_\omega(G,K)$ be equivalent and let $\gamma\in Z^2_\zeta(G,Z(K))$. Then there exists a function $f:G\to K$ such that

$$\alpha'_x = c_{f(x)}\circ\alpha_x \text{ and } \kappa'(x,y)=f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1},$$

for all $x,y\in G$. Multiplying the last equation with $\gamma(x,y)$ yields

$$\gamma(x,y)\kappa'(x,y)=f(x)\alpha_x(f(y))\gamma(x,y)\kappa(x,y)f(xy)^{-1},$$

again since $\gamma(x,y)\in Z(K)$. This shows that also ${}^\gamma(\alpha,\kappa)=(\alpha,\gamma\kappa)$ and ${}^\gamma(\alpha',\kappa')=(\alpha',\gamma\kappa')$ are equivalent. Hence we obtain an action of $Z^2_\zeta(G,Z(K))$ on $\mathrm{Par}_\omega(G,K)$. Next, let $(\alpha,\kappa)\in\mathrm{par}_\omega(G,K)$ and let $\gamma\in B^2_\zeta(G,Z(K))$. We will show that ${}^\gamma(\alpha,\kappa)$ is equivalent to (α,κ). There exists a function $f:G\to Z(K)$ such that $\gamma(x,y)=\zeta_x(f(y))f(xy)^{-1}f(x)=\alpha_x(f(y))f(xy)^{-1}f(x)$ for all $x,y\in G$. With this f we have

$$\alpha_x=c_{f(x)}\circ\alpha_x$$

and

$$(\gamma\kappa)(x,y)=\gamma(x,y)\kappa(x,y)=f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1},$$

for all $x,y\in G$, so ${}^\gamma(\alpha,\kappa)$ is equivalent to (α,κ), and thus we have an action of $H^2_\zeta(G,Z(K))$ on $\mathrm{Par}_\omega(G,K)$.

Next we show the γ action is free. Let $\gamma_1,\gamma_2\in Z^2_\zeta(G,Z(K))$ and $(\alpha,\kappa)\in\mathrm{par}_\omega(G,K)$ such that ${}^{\gamma_1}(\alpha,\kappa)$ and ${}^{\gamma_2}(\alpha,\kappa)$ are equivalent. Let $\gamma:=\gamma_1^{-1}\gamma_2$. Then ${}^\gamma(\alpha,\kappa)=(\alpha,\kappa)$ is equivalent to (α,κ). Thus there exists a function $f:G\to K$ such that $\alpha_x=c_{f(x)}\circ\alpha_x$ and $\gamma(x,y)\kappa(x,y)=f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1}$ for all $x,y\in G$. This implies that $c_{f(x)}=\mathrm{id}_K$ for all $x\in K$ so that $f(x)\in Z(K)$ for all $x\in K$. So we have $\gamma(x,y)=f(x)\alpha_x(f(y))f(xy)^{-1}=f(x)\zeta_x(f(y))f(xy)^{-1}$. Therefore, $\gamma\in B^2_\zeta(G,Z(K))$ and $[\gamma_1]=[\gamma_2]\in H^2_\zeta(G,Z(K))$.

Finally, we show that the action of $H^2_\zeta(G,Z(K))$ on $\mathrm{Par}_\omega(G,K)$ is transitive. Let $(\alpha,\kappa),(\beta,\lambda)\in\mathrm{par}_\omega(G,K)$. We will show that there exists $\gamma\in Z^2_\zeta(G,Z(K))$ such that (α,κ) and ${}^\gamma(\beta,\lambda)$ are equivalent. For each $x\in G$ we have $\alpha_x\mathrm{Inn}(K)=\omega(x)=\beta_x\mathrm{Inn}(K)$. Thus there exists an element $f(x)\in K$ such that $c_{f(x)}\circ\alpha_x=\beta_x$. We set $\kappa'(x,y):=f(x)\alpha_x(f(y))\kappa(x,y)f(xy)^{-1}$ for all $x,y\in G$. Then $(\beta,\kappa')\in\mathrm{par}_\omega(G,K)$ and (α,κ) is equivalent to (β,κ'). Since $(\beta,\lambda)\in\mathrm{par}_\omega(G,K)$, then we have $c_{\kappa'(x,y)}\circ\beta_{xy}=\beta_x\circ\beta_y=c_{\lambda(x,y)}\circ\beta_{xy}$ and $c_{\kappa'(x,y)}=c_{\lambda(x,y)}$ for all $x,y\in K$. This means $\gamma(x,y):=\kappa'(x,y)\lambda(x,y)^{-1}\in Z(K)$ for all $x,y\in G$. We show that $\gamma\in Z^2_\zeta(G,Z(K))$. For $x,y,z\in G$ we have

$$\begin{aligned}
\gamma(x,y)\gamma(xy,z) &= \kappa'(x,y)\lambda(x,y)^{-1}\gamma(xy,z)\\
&= \kappa'(x,y)\gamma(xy,z)\lambda(x,y)^{-1}\\
&= \kappa'(x,y)\kappa'(xy,z)\lambda(xy,z)^{-1}\lambda(x,y)^{-1}\\
&= \beta_x(\kappa'(y,z))\kappa'(x,yz)\lambda(x,yz)^{-1}\beta_x(\lambda(y,z))^{-1}\\
&= \beta_x(\kappa'(y,z))\gamma(x,yz)\beta_x(\lambda(y,z))^{-1}\\
&= \beta_x(\kappa'(y,z)\lambda(y,z)^{-1})\gamma(x,yz)\\
&= \zeta_x(\gamma(y,z))\gamma(x,yz).
\end{aligned}$$

This implies that $(\beta,\kappa')={}^\gamma(\beta,\lambda)$ and that (α,κ) is equivalent to $(\beta,\kappa')={}^\gamma(\beta,\lambda)$. Thus, the proof is complete. $\square$

5.6 Theorem. *Assume that $Z(K)=1$. Then $|\mathrm{Par}_\omega(G,K)|=1$ for every $\omega:G\to\mathrm{Out}(K)$.*

Proof.
For each $x \in G$, choose $\alpha_x \in \mathrm{Aut}(K)$ such that $\omega(x) = \alpha_x \mathrm{Inn}(K)$. For all $x, y \in G$ we have $\alpha_x \alpha_y \mathrm{Inn}(K) = \omega(x)\omega(y) = \omega(xy) = \alpha_{xy}\mathrm{Inn}(K)$. Thus there exist elements $\kappa(x,y) \in K$ such that $\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}$ for all $x, y \in G$. For all $x, y, z \in G$ we have

$$\begin{aligned} c_{\kappa(x,y)\kappa(xy,z)} \circ \alpha_{xyz} &= c_{\kappa(x,y)} \circ c_{\kappa(xy,z)} \circ \alpha_{xyz} \\ &= c_{\kappa(x,y)} \circ \alpha_{xy} \circ \alpha_z \\ &= \alpha_x \circ \alpha_y \circ \alpha_z \\ &= \alpha_x \circ c_{\kappa(y,z)} \circ a_{yz} \\ &= \alpha_x \circ c_{\kappa(y,z)} \circ \alpha_x^{-1} \circ \alpha_x \circ \alpha_{yz} \\ &= c_{\alpha_x(\kappa(y,z))} \circ c_{\kappa(x,yz)} \circ \alpha_{x(yz)} \\ &= c_{\alpha_x(\kappa(y,z))\kappa(x,yz)} \circ \alpha_{xyz}, \end{aligned}$$

and therefore $c_{\kappa(x,y)\kappa(xy,z)} = c_{\alpha_x(\kappa(y,z))\kappa(x,yz)}$. Since $Z(K) = 1$, then this implies $\kappa(x,y)\kappa(xy,z) = \alpha_x(\kappa(y,z))\kappa(x,yz)$ for all $x, y, z \in G$. Hence, $(\alpha, \kappa) \in \mathrm{par}_\omega(G, K)$, and $\mathrm{Par}_\omega(G, K)$ is not empty. On the other hand, by Theorem 5.5, $\mathrm{Par}_\omega(G, K)$ is in bijection to $H^2_\zeta(G, Z(K))$, where $\zeta := \mathrm{res}^K_{Z(K)} \circ \omega$. But since $Z(K) = 1$, we have $F(G^2, Z(K)) = 1$ and $H^2_\zeta(G, Z(K)) = 1$. $\square$

5.7 Theorem. *Let $\omega : G \to \mathrm{Out}(K)$ be a group homomorphism and let $\zeta := \mathrm{res}^K_{Z(K)} \circ \omega \in \mathrm{Hom}(G, \mathrm{Aut}(Z(K)))$. Moreover, for each $x \in G$, let $\alpha_x \in \mathrm{Aut}(K)$ be an automorphism with $\omega(x) = \alpha_x \mathrm{Inn}(K)$. Then the following hold:*
(a) For all $x, y \in G$ there exists an element $\chi(x,y) \in K$ such that $\alpha_x \circ \alpha_y = c_{\chi(x,y)} \circ \alpha_{xy}$.
(b) Let $\chi(x,y) \in K$ be chosen as in (a). Then for all $x, y, z \in G$ the element $\vartheta(x,y,z) := \alpha_x(\chi(y,z))\chi(x,yz)\chi(xy,z)^{-1}$ lies in $Z(K)$, and the function $\vartheta : G^3 \to Z(K)$ is an element of $Z^3_\zeta(G, Z(K))$.
(c) The cohomology class $[\vartheta] \in H^3_\zeta(G, Z(K))$ of the element $\vartheta \in Z^3_\zeta(G, Z(K))$ defined in (b) does not depend on the choices of $\alpha_x \in \mathrm{Aut}(K)$ and $\chi(x,y) \in K$ for $x, y \in G$.
Proof.
(a) For all $x, y \in G$ we have

$$\alpha_x \alpha_y \mathrm{Inn}(K) = \omega(x)\omega(y) = \omega(xy) = \alpha_{xy}\mathrm{Inn}(K),$$

which implies that $\alpha_x \alpha_y \alpha_{xy}^{-1} \in \mathrm{Inn}(K)$.
(b) For all $x, y, z \in G$ we have

$$\begin{aligned} &c_{\vartheta(x,y,z)} \\ &= c_{\alpha_{x(\chi(y,z))}} \circ c_{\chi(x,yz)} \circ c^{-1}_{\chi(xy,z)} \circ c^{-1}_{\chi(x,y)} \\ &= \alpha_x \circ c_{\chi(y,z)} \circ \alpha_x^{-1} \circ \alpha_x \circ \alpha_{yz} \circ \alpha_{xyz}^{-1} \circ \alpha_{xyz} \circ \alpha_z^{-1} \circ \alpha_{xy}^{-1} \circ \alpha_{xy} \circ \alpha_y^{-1} \circ \alpha_x^{-1} \\ &= \alpha_x \circ \alpha_y \circ \alpha_z \circ \alpha_{yz}^{-1} \circ \alpha_{yz} \circ \alpha_z^{-1} \circ \alpha_y^{-1} \circ \alpha_x^{-1} \\ &= \mathrm{id}_K, \end{aligned}$$

so $\vartheta(x,y,z) \in Z(K)$. Now we show that $\vartheta \in Z^3_\zeta(G, Z(K))$. Let $x,y,z,w \in G$. Then

$$\begin{aligned}
&\zeta_x(\vartheta(y,z,w))\vartheta(x,yz,w)\vartheta(x,y,z) \\
&= \alpha_x(\alpha_y(\chi(z,w)))\alpha_x(\chi(y,zw))\alpha_x(\chi(yz,w))^{-1}\alpha_x(\chi(y,z))^{-1}\vartheta(x,yz,w)\cdot \\
&\qquad\qquad \cdot\vartheta(x,y,z) \\
&= \alpha_x(\alpha_y(\chi(z,w)))\alpha_x(\chi(y,zw))\alpha_x(\chi(yz,w))^{-1}\vartheta(x,yz,w)\alpha_x(\chi(y,z))^{-1}\cdot \\
&\qquad\qquad \cdot\vartheta(x,y,z) \\
&= \alpha_x(\alpha_y(\chi(z,w)))\alpha_x(\chi(y,zw))\alpha_x(\chi(yz,w))^{-1}\cdot \\
&\qquad\qquad \cdot\alpha_x(\chi(yz,w))\chi(x,yzw)\chi(xyz,w)^{-1}\chi(x,yz)^{-1}\alpha_x(\chi(y,z))^{-1}\cdot \\
&\qquad\qquad \cdot\alpha_x(\chi(y,z))\chi(x,yz)\chi(yx,z)^{-1}\chi(x,y)^{-1} \\
&= \alpha_x(\alpha_y(\chi(z,w)))\alpha_x(\chi(y,zw))\chi(x,yzw)\chi(xyz,w)^{-1}\chi(xy,z)^{-1}\chi(x,y)^{-1} \\
&= \alpha_x(\alpha_y(\chi(z,w)))\alpha_x(\chi(y,zw))\chi(x,yzw)\chi(xy,zw)^{-1}\chi(x,y)^{-1}\cdot \\
&\qquad\qquad \cdot\chi(x,y)\chi(xy,zw)\chi(xyz,w)^{-1}\chi(xy,z)^{-1}\chi(x,y)^{-1} \\
&= \alpha_x(\alpha_y(\chi(z,w)))\vartheta(x,y,zw)\chi(x,y)\chi(xy,zw)\chi(xyz,w)^{-1}\chi(xy,z)^{-1}\chi(x,y)^{-1} \\
&= \chi(x,y)\alpha_{xy}(\chi(z,w))\chi(xy,zw)\chi(xyz,w)^{-1}\cdot \\
&\qquad\qquad \cdot\chi(xy,z)^{-1}\chi(x,y)^{-1}\vartheta(x,y,zw) \\
&= \chi(x,y)\vartheta(xy,z,w)\chi(x,y)^{-1}\vartheta(x,y,zw) \\
&= \vartheta(xy,z,w)\vartheta(x,y,zw),
\end{aligned}$$

as desired.

(c) If, for each $x \in G$ we choose $\alpha'_x \in \mathrm{Aut}(K)$ such that $\alpha'_x\mathrm{Inn}(K) = \omega(x)$ and for each $x,y \in G$ we choose $\chi'(x,y) \in K$ such that $\alpha'_x \circ \alpha'_y = c_{\chi'(x,y)} \circ \alpha'_{xy}$, then there exists a function $f : G \to K$ such that $\alpha'_x = c_{f(x)} \circ \alpha_x$. This implies

$$\begin{aligned}
\alpha'_x \circ \alpha'_y &= c_{f(x)} \circ \alpha_x \circ c_{f(y)} \circ \alpha_y \\
&= c_{f(x)} \circ \alpha_x \circ c_{f(y)} \circ \alpha_x^{-1} \circ \alpha_x \circ \alpha_y \\
&= c_{f(x)} \circ c_{\alpha_x(f(y))} \circ c_{\chi(x,y)} \circ \alpha_{xy} \\
&= c_{f(x)\alpha_x(f(y))\chi(x,y)} \circ c^{-1}_{f(xy)} \circ \alpha'_{xy} \\
&= c_{f(x)\alpha_x(f(y))\chi(x,y)f(xy)^{-1}} \circ \alpha'_{xy},
\end{aligned}$$

and so we have

$$\chi'(x,y) = f(x)\alpha_x(f(y))\chi(x,y)f(xy)^{-1}g(x,y)$$

for all $x,y \in G$ with a function $g : G \times G \to Z(K)$. For all $x,y,z \in G$, the corresponding function

$$\vartheta'(x,y,z) := \alpha'_x(\chi'(y,z))\chi'(x,yz)\chi'(xy,z)^{-1}\chi'(x,y)^{-1}$$

satisfies

$$
\begin{aligned}
&\vartheta'(x,y,z)\\
&= f(x)\alpha_x(f(y)\alpha_y(f(z))\chi(y,z)f(yz)^{-1}g(y,z))f(x)^{-1}\cdot\\
&\qquad\qquad \cdot f(x)\alpha_x(f(yz))\chi(x,yz)f(xyz)^{-1}g(x,yz)\cdot\\
&\qquad\qquad \cdot g(xy,z)^{-1}f(xyz)\chi(xy,z)^{-1}\alpha_{xy}(f(z))^{-1}f(xy)^{-1}\cdot\\
&\qquad\qquad \cdot g(x,y)^{-1}f(xy)\chi(x,y)^{-1}\alpha_x(f(y))^{-1}f(x)^{-1}\\
&= f(x)\alpha_x(f(y))\alpha_x(\alpha_y(f(z)))\alpha_x(\chi(y,z))\cdot\\
&\qquad\qquad \cdot\chi(x,yz)\chi(xy,z)^{-1}\alpha_{xy}(f(z)^{-1})\chi(x,y)^{-1}\alpha_x(f(y)^{-1})f(x)^{-1}\cdot\\
&\qquad\qquad \cdot\alpha_x(g(y,z))g(x,yz)g(xy,z)^{-1}g(x,y)^{-1}\\
&= f(x)\alpha_x(f(y))\alpha_x(\alpha_y(f(z)))\vartheta(x,y,z)\chi(x,y)\alpha_{xy}(f(z)^{-1})\cdot\\
&\qquad\qquad \cdot\chi(x,y)^{-1}\alpha_x(f(y)^{-1})f(x)^{-1}(\partial^2_\zeta(g))(x,y,z)\\
&= f(x)\alpha_x(f(y))\alpha_x(\alpha_y(f(z)))\alpha_x(\alpha_y(f(z)^{-1}))\cdot\\
&\qquad\qquad \cdot\alpha_x(f(y)^{-1})f(x)^{-1}\vartheta(x,y,z)(\partial^2_\zeta(g))(x,y,z)\\
&= \vartheta(x,y,z)(\partial^2_\zeta(g))(x,y,z).
\end{aligned}
$$

Thus the cohomology classes $[\vartheta]$ and $[\vartheta']$ coincide. □

5.8 Definition. Let $\omega : G \to \mathrm{Out}(K)$ be a homomorphism and let $\zeta := \mathrm{res}^K_{Z(K)} \circ \omega \in \mathrm{Hom}(G, \mathrm{Aut}(Z(K)))$. The element $[\vartheta] \in H^3_\zeta(G, Z(K))$ defined in Theorem 5.7 is called the *obstruction* of ω.

5.9 Theorem. *Let $\omega : G \to \mathrm{Out}(K)$ be a group homomorphism and let $\zeta := \mathrm{res}^K_{Z(K)} \in \mathrm{Hom}(G, \mathrm{Aut}(Z(K)))$. Then $\mathrm{Par}_\omega(G,K) \neq \emptyset$ if and only if the obstruction $[\vartheta] \in H^3_\zeta(G, Z(K))$ of ω is trivial.*

Proof.

Assume $\mathrm{Par}_\omega(G,K) \neq \emptyset$ and let $(\alpha,\kappa) \in \mathrm{par}_\omega(G,K)$. Then we have $\omega(x) = \alpha_x\mathrm{Inn}(K)$, $\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}$, and $\alpha_x(\kappa(y,z))\kappa(x,yz)\kappa(xy,z)^{-1}\kappa(x,y)^{-1} = 1$, for all $x,y,z \in G$. This implies that we can define the obstruction $[\vartheta]$ of ω using the elements $\alpha_x \in \mathrm{Aut}(K)$ and $\kappa(x,y) \in K$ for $x,y \in G$, and that $[\vartheta] = 1$.

Conversely, if we choose elements $\alpha_x \in \mathrm{Aut}(K)$ such that $\omega(x) = \alpha_x\mathrm{Inn}(K)$ for all $x \in G$, and elements $\chi(x,y) \in K$ such that $\alpha_x \circ \alpha_y = c_{\chi(x,y)} \circ \alpha_{xy}$ for all $x,y \in G$, then we obtain the obstruction $[\vartheta] \in H^3_\zeta(G,Z(K))$ of ω from the 3-cocycle $\vartheta(x,y,z) := \alpha_x(\chi(y,z))\chi(x,yz)\chi(xy,z)^{-1}\chi(x,y)^{-1} \in Z(K)$, for $x,y,z \in G$. Since $[\vartheta] = 1$, there exists an element $\varphi : G \times G \to Z(K)$ such that $\vartheta = \partial^2_\zeta(\varphi)$. We set $\kappa(x,y) := \varphi(x,y)^{-1}\chi(x,y)$ for $x,y \in G$ and we show that $(\alpha,\kappa) \in \mathrm{par}_\omega(G,K)$. For all $x,y,z \in G$ we have

$$\alpha_x \circ \alpha_y = c_{\kappa(x,y)} \circ \alpha_{xy}$$

and

$$\begin{aligned}
\kappa(x,y)\kappa(xy,z) &= \varphi(x,y)^{-1}\chi(x,y)\varphi(xy,z)^{-1}\chi(xy,z)\\
&= \varphi(x,y)^{-1}\varphi(xy,z)^{-1}\chi(x,y)\chi(xy,z)\\
&= \varphi(x,yz)^{-1}\alpha_x(\varphi(y,z))^{-1}(\partial^2_\varsigma(\varphi))(x,y,z)\chi(x,y)\chi(xy,z)\\
&= \varphi(x,yz)^{-1}\alpha_x(\varphi(y,z))^{-1}\vartheta(x,y,z)\chi(x,y)\chi(xy,z)\\
&= \varphi(x,yz)^{-1}\alpha_x(\varphi(y,z))^{-1}\alpha_x(\chi(y,z))\chi(x,yz)\\
&= \alpha_x(\kappa(y,z))\kappa(x,yz),
\end{aligned}$$

which thus completes the proof. □

6. Examples

To end this paper, we will compute some examples to give greater insight into how group extensions are structured. In this section, let C_n denote the cyclic group with n elements and $\mathrm{Sym}(n)$ denote the symmetric group on n elements.

6.1 Example. Let $K = C_2 =< x >$, $G = C_2 =< y >$, and $C_4 =< a >$. Then

$$1 \longrightarrow C_2 \longrightarrow C_2 \times C_2 \longrightarrow C_2 \longrightarrow 1$$

$$\begin{array}{rcl}
x \longmapsto & (x,1) & \\
 & (x,y) & \longmapsto y
\end{array}$$

and

$$1 \longrightarrow C_2 \longrightarrow C_4 \longrightarrow C_2 \longrightarrow 1$$

$$\begin{array}{rcl}
x \longmapsto & a^2 & \\
 & a & \longmapsto y
\end{array}$$

are extensions of G by K.

6.2 Example. Let $K = C_3 =< x >$, $G = C_2 =< y >$, and $\mathrm{Sym}(3) =< \sigma, \tau >$. Then

$$1 \longrightarrow C_3 \longrightarrow C_3 \times C_2 \longrightarrow C_2 \longrightarrow 1$$

$$\begin{array}{rcl}
x \longmapsto & (x,1) & \\
 & (x,y) & \longmapsto y
\end{array}$$

and

$$1 \longrightarrow C_3 \longrightarrow \mathrm{Sym}(3) \longrightarrow C_2 \longrightarrow 1$$

$$\begin{array}{rcl}
x \longmapsto & \sigma & \\
 & \sigma & \longmapsto 1\\
 & \tau & \longmapsto y
\end{array}$$

are extensions of G by K.

6.3 Example. Let $K = \mathrm{Sym}(3)$ and $G = C_2$. We want to find all groups H such that $1 \longrightarrow \mathrm{Sym}(3) \longrightarrow H \longrightarrow C_2 \longrightarrow 1$ is a group extension of G by K.

Since $|Z(\mathrm{Sym}(3))| = 1$, then by Theorem 5.6, we have $|\mathrm{Par}_\omega(C_2, \mathrm{Sym}(3))| = 1$ for every $\omega : C_2 \to \mathrm{Out}(\mathrm{Sym}(3))$. From Remark 5.2, we have

$$\mathrm{Par}(C_2, \mathrm{Sym}(3)) = \dot{\bigcup_{\omega \in \mathrm{Hom}(C_2, \mathrm{Out}(\mathrm{Sym}(3)))}} \mathrm{Par}_\omega(C_2, \mathrm{Sym}(3)),$$

which implies that $|\mathrm{Par}(C_2, \mathrm{Sym}(3))| = |\mathrm{Hom}(C_2, \mathrm{Out}(\mathrm{Sym}(3)))|$. By definition, $\mathrm{Out}(\mathrm{Sym}(3)) = \mathrm{Aut}(\mathrm{Sym}(3))/\mathrm{Inn}(\mathrm{Sym}(3))$, so we first want to find $\mathrm{Aut}(\mathrm{Sym}(3))$. Since automorphisms take 2-cycles to 2-cycles and 3-cycles to 3-cycles, then there are six possible automorphisms:

$f_1 : \tau \mapsto \tau, \sigma \mapsto \sigma,$ $\quad f_2 : \tau \mapsto \tau\sigma, \sigma \mapsto \sigma,$ $\quad f_3 : \tau \mapsto \tau\sigma^2, \sigma \mapsto \sigma,$
$f_4 : \tau \mapsto \tau, \sigma \mapsto \sigma^2,$ $\quad f_5 : \tau \mapsto \tau\sigma, \sigma \mapsto \sigma^2,$ $\quad f_6 : \tau \mapsto \tau\sigma^2, \sigma \mapsto \sigma^2.$

Since $f_2(f_4(\tau)) = f_2(\tau) = \tau\sigma$ and $f_4(f_2(\tau)) = f_4(\tau\sigma) = f_4(\tau)f_4(\sigma) = \tau\sigma^2$, then $\mathrm{Aut}(\mathrm{Sym}(3))$ is not abelian. So $|\mathrm{Aut}(\mathrm{Sym}(3))| = 6$ and $\mathrm{Aut}(\mathrm{Sym}(3))$ is not abelian, which implies $\mathrm{Aut}(\mathrm{Sym}(3)) \cong \mathrm{Sym}(3)$.

From Definition 5.1, we have $\mathrm{Inn}(G) \cong G/Z(G)$ for all groups G. Since $Z(\mathrm{Sym}(3))$ is trivial, then $\mathrm{Inn}(\mathrm{Sym}(3)) \cong \mathrm{Sym}(3)$. So we have

$$\mathrm{Out}(\mathrm{Sym}(3)) \cong \mathrm{Aut}(\mathrm{Sym}(3))/\mathrm{Inn}(\mathrm{Sym}(3)) \cong \mathrm{Sym}(3)/\mathrm{Sym}(3) = 1,$$

which means $|\mathrm{Hom}(C_2, \mathrm{Out}(\mathrm{Sym}(3)))| = 1$. Thus the only extension of C_2 by $\mathrm{Sym}(3)$ is the direct product $H = \mathrm{Sym}(3) \times C_2$.

References

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