CATEGORIES, FUNCTORS, AND CLASSIFYING SPACES

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Abstract. Category theory is a useful tool in mathematics. It unifies different areas of mathematics by looking at sets and spaces from a more general perspective.

In this paper, I will give the basic definitions of a category and give examples through abstract algebra and algebraic topology. Next, I will define covariant and contravariant functors. Then, I will introduce the classifying space associated to categories. Finally, I will give an example of a classifying space, $K(G,1)$.

1. Categories

We begin by defining classes and proper classes as to not run into trouble when defining categories. Proper classes give a more concrete way of describing classes too large to be called a set, and they let us disregard Russell's paradox.

1.1 Definition. A *class* is defined as a collection of elements. A *set* is a class with a cardinality associated to it. A *proper class* is a class which does not have a cardinality associated with it. In other words, a proper class is a class that is not a set.

1.2 Remark. Collections with a finite number of elements, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are classes with an associated cardinality, so they are sets. The collection of all sets, however, is a proper class.

1.3 Definition. A *category* $\mathcal{C}$ has three components:

(i) A class $\mathrm{Obj}(\mathcal{C})$ whose elements we call *objects*.
(ii) A set $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ of *morphisms* for any $X, Y \in \mathrm{Obj}(\mathcal{C})$. In particular, for every $X \in \mathrm{Obj}(\mathcal{C})$, there exists an *identity morphism* $\mathrm{Id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$.
(iii) A *composition* function $\circ : \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) \to \mathrm{Mor}_{\mathcal{C}}(X,Z)$ for any $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, satisfying

$$(f \circ g) \circ h = f \circ (g \circ h),$$

and for every $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we have $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$.

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1.4 Examples. Throughout these examples, the composition functions and identities are the usual and obvious ones.

(a) $\mathcal{C} = \mathbf{Set}$. The class $\text{Obj}(\mathcal{C})$ consists of all sets, so $\text{Obj}(\mathcal{C})$ is a proper class. Any arbitrary functions between sets are the morphisms of **Set**.
(b) $\mathcal{C} = \mathbf{Grp}$. Groups are the objects of **Grp** and group homomorphisms are the morphisms.
(c) $\mathcal{C} = \mathbf{Vct}_{\mathbb{F}}$. Vector spaces over a field $\mathbb{F}$ are the objects and linear transformations are the morphisms.
(d) $\mathcal{C} = \mathbf{Top}$. Topological spaces are the objects and continuous maps are the morphisms.
(e) $\mathcal{C} = \mathbf{Top}_*$. Topological spaces with a fixed basepoint are the objects and basepoint preserving continuous maps are the morphisms.
(f) $\mathcal{C} = \mathbf{Toph}$. Like **Top**, the objects of **Toph** are topological spaces, but the morphisms are homotopy classes of continuous maps.
(g) $\mathcal{C} = \mathbf{Chx}$. Chain complexes are the objects and chain maps are the morphisms.

1.5 Definition. A *subcategory* of $\mathcal{C}$ is a category $\mathcal{D}$ such that

(i) $\text{Obj}(\mathcal{D}) \subset \text{Obj}(\mathcal{C})$;
(ii) $\text{Mor}_{\mathcal{D}}(X, Y) \subseteq \text{Mor}_{\mathcal{C}}(X, Y)$ whenever $X, Y \in \text{Obj}(\mathcal{D})$; and
(iii) the composition function and identities are inherited from $\mathcal{C}$.

1.6 Remark. If $\text{Mor}_{\mathcal{D}}(X, Y) = \text{Mor}_{\mathcal{C}}(X, Y)$ whenever $X, Y \in \text{Obj}(\mathcal{D})$, then we say $\mathcal{D}$ is a *full subcategory*.

1.7 Examples.

(a) The category **Grp** is a subcategory of **Set**.
(b) The category **Ab** whose objects are abelian groups and whose morphisms are group homomorphisms is a full subcategory of **Grp**.
(c) The category **Haus** whose objects are Hausdorff spaces and whose morphisms are continuous maps is a full subcategory of **Top**.

1.8 Definition. Let $\mathcal{C}$ be a category. A morphism $f \in \text{Mor}_{\mathcal{C}}(X, Y)$ is an *equivalence* if there exists a morphism $g \in \text{Mor}_{\mathcal{C}}(Y, X)$ such that

$$g \circ f = \text{Id}_X \quad \text{and} \quad f \circ g = \text{Id}_Y.$$

The morphism g is called the *inverse* of f.

1.9 Examples.

(a) Bijections are equivalences in **Set**.
(b) Group isomorphisms are equivalences in **Grp**.
(c) Homeomorphisms are equivalences in **Top**.
(d) Homotopy classes of homotopy equivalences are equivalences in **Toph**.

1.10 Remark. A *groupoid* is a category in which every morphism is an equivalence. If a groupoid has only one object, its morphisms form a group.

2. Functors

The usefulness of category theory is only truly revealed when we consider relationships between categories. Throughout this section, let $\mathcal{C}$ and $\mathcal{D}$ be categories.

2.1 Definition. A *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ is a function such that

(i) if $X \in \mathrm{Obj}(\mathcal{C})$, then $T(X) \in \mathrm{Obj}(\mathcal{D})$;
(ii) if $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, then $T(f) \in \mathrm{Mor}_{\mathcal{D}}(T(X), T(Y))$;
(iii) if $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y), g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$, then

$$T(g \circ f) = T(g) \circ T(f); \text{ and}$$

(iv) for every $X \in \mathrm{Obj}(\mathcal{C})$, $T(\mathrm{Id}_X) = \mathrm{Id}_{T(X)}$.

2.2 Remark. If $T : \mathcal{C} \to \mathcal{D}$ is a covariant functor and $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, then we call $T(f) \in \mathrm{Mor}_{\mathcal{D}}(T(X), T(Y))$ the *induced morphism.* When the functor is understood, we will write the induced morphism $T(f)$ as f_*.

2.3 Examples. (Covariant Functors)

(a) The covariant *identity functor* $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined by $\mathrm{Id}_{\mathcal{C}}(X) = X$ for all $X \in \mathrm{Obj}(\mathcal{C})$ and $\mathrm{Id}_{\mathcal{C}}(f) = f$ for all morphisms f.
(b) The covariant *power set functor* $P : \mathbf{Set} \to \mathbf{Set}$ assigns to each set X its power set $P(X)$, and to each function $f : X \to Y$, $f_* : P(X) \to P(Y)$ sends each subset $U \subseteq X$ to its image $f(U) \subseteq Y$.
(c) For any group G, the commutator subgroup $[G, G] = \{xyx^{-1}y^{-1} : x, y \in G\}$ is a normal subgroup of G. Since any group homomorphism $G \to H$ maps commutators to commutators, then $T : \mathbf{Grp} \to \mathbf{Grp}$ defined by $G \mapsto [G, G]$ is a covariant functor.
(d) The abelianization of a group G defines the *factor-commutator functor*, a covariant functor $T : \mathbf{Grp} \to \mathbf{Ab}$ defined by $G \mapsto G/[G, G]$.
(e) The *forgetful functor* (or *underlying functor*) $U : \mathbf{Grp} \to \mathbf{Set}$ sends each group to its underlying set and each group homomorphism to its underlying set function. Similarly, U can be also be defined for $\mathbf{Top} \to \mathbf{Set}$.
(f) The *fundamental group functor* $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ assigns to each topological space (X, x_0) its fundamental group based at x_0, and to each basepoint preserving continuous map $f : X \to X$ its induced group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$.
(g) The *singular chain complex functor* $C_n : \mathbf{Top} \to \mathbf{Chx}$ assigns to a space X the chain complex of singular chains in X and to a continuous map $f : X \to Y$ its induced chain map $f_\sharp : C_n(X) \to C_n(Y)$.
(h) The *algebraic homology functor* assigns to a chain complex its sequence of homology groups and to a chain map its induced homomorphisms on homology. This is a covariant functor from $\mathbf{Chx}$ to the category whose objects are sequences of abelian groups and whose morphisms are sequences of homomorphisms.
(i) The *singular homology functor* H_n is the composition of the singular chain complex functor and the algebraic homology functor, so each topological space is assigned to its sequence of singular homology groups.

2.4 Remark. The map $T : \mathbf{Grp} \to \mathbf{Ab}$ defined by $G \mapsto Z(G)$, the centre of G, is not a functor, since group homomorphisms may send an element in the centre to an element outside the centre.

2.5 Definition. A *contravariant functor* $S : \mathcal{C} \to \mathcal{D}$ is a function such that

(i) if $X \in \mathrm{Obj}(\mathcal{C})$, then $S(X) \in \mathrm{Obj}(\mathcal{D})$;
(ii) if $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, then $S(f) \in \mathrm{Mor}_{\mathcal{D}}(S(Y), S(X))$;
(iii) if $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y), g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$, then

$$S(g \circ f) = S(f) \circ S(g); \text{ and}$$

(iv) for every $X \in \mathrm{Obj}(\mathcal{C})$, $S(\mathrm{Id}_X) = \mathrm{Id}_{S(X)}$.

2.6 Remark. The difference between covariant and contravariant functors is that the induced morphisms of contravariant functors go in the reverse direction. If $S : \mathcal{C} \to \mathcal{D}$ is a contravariant functor and $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, then when the functor is understood, we will write the induced morphism $S(f)$ as f^*. Note that for induced morphisms of covariant functors, we use a lower star, and for induced morphisms of contravariant functors, we use an upper star.

2.7 Examples. (Contravariant Functors)

(a) The contravariant *power set functor* $P : \mathbf{Set} \to \mathbf{Set}$ assigns to each set X its power set $P(X)$, and to each function $f : X \to Y$, $f^* : P(Y) \to P(X)$ sends each subset $U \subseteq Y$ to its inverse image $f^{-1}(U) \subseteq X$.
(b) The *dual space functor* $\mathbf{Vct}_{\mathbb{F}} \to \mathbf{Vct}_{\mathbb{F}}$ assigns to each vector space V its dual space V^*, the space of linear maps $V \to \mathbb{F}$, and to each linear map $\varphi : V \to W$ the dual map $\varphi^* : W^* \to V^*$ defined by $\varphi^*(f)x = f(\varphi x)$ for any $f \in W^*$, $x \in V$.

2.8 Proposition. *Functors of either variance send equivalences to equivalences.*
Proof.
Let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ be an equivalence. Then by definition, there exists a morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. If $T : \mathcal{C} \to \mathcal{D}$ is a covariant functor, then we have

$$T(g) \circ T(f) = T(g \circ f) = T(\mathrm{Id}_X) = \mathrm{Id}_{T(X)}$$

and

$$T(f) \circ T(g) = T(f \circ g) = T(\mathrm{Id}_Y) = \mathrm{Id}_{T(Y)}$$

by 2.1(iii),(iv), which implies $T(f)$ is also an equivalence. Similarly, if $S : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then

$$S(f) \circ S(g) = S(g \circ f) = S(\mathrm{Id}_X) = \mathrm{Id}_{S(X)}$$

and

$$S(g) \circ S(f) = S(f \circ g) = S(\mathrm{Id}_Y) = \mathrm{Id}_{S(Y)}$$

by 2.4(iii),(iv), so $S(f)$ is an equivalence. □

2.9 Definition. Let $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C} \to \mathcal{D}$ be covariant functors. A *natural transformation* $\tau : S \to T$ is a function that assigns to each $X \in \mathrm{Obj}(\mathcal{C})$ a morphism $\tau_X \in \mathrm{Mor}_{\mathcal{D}}(S(X), T(X))$ so that for all $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, the diagram

$$\begin{array}{ccc} S(X) & \xrightarrow{S(f)} & S(Y) \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ T(X) & \xrightarrow{T(f)} & T(Y) \end{array}$$

commutes. If τ_X is an equivalence for every $X \in \mathrm{Obj}(\mathcal{C})$, then τ is called a *natural isomorphism* or *natural equivalence*.

2.10 Definition. Let $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{C} \to \mathcal{D}$ be contravariant functors. A *natural transformation* $\sigma : S \to T$ is a function that assigns to each $X \in \mathrm{Obj}(\mathcal{C})$ a morphism $\sigma_X \in \mathrm{Mor}_{\mathcal{D}}(S(X), T(X))$ so that for all $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, the diagram

$$\begin{array}{ccc} S(Y) & \xrightarrow{S(f)} & S(X) \\ {\scriptstyle \sigma_Y}\downarrow & & \downarrow{\scriptstyle \sigma_X} \\ T(Y) & \xrightarrow{T(f)} & T(X) \end{array}$$

commutes. If σ_X is an equivalence for every $X \in \mathrm{Obj}(\mathcal{C})$, then σ is called a *natural isomorphism* or *natural equivalence*.

2.11 Remark. If there exists a natural equivalence between functors S and T, then we say S and T are *naturally equivalent*.

2.12 Definition. A functor $T : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there is a functor $S : \mathcal{D} \to \mathcal{C}$ such that $S \circ T$ and $T \circ S$ are naturally equivalent to $\mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Id}_{\mathcal{D}}$, respectively. We say then that the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

3. Classifying Spaces

So far, we have been viewing topology from a categorical and functorial viewpoint. Conversely, we can induce topological structure on categories through classifying spaces.

3.1 Definition. Let $\mathcal{C}$ be a category. The *classifying space* of $\mathcal{C}$, denoted $B\mathcal{C}$, is a Δ-complex constructed as follows: Consider each object of $\mathcal{C}$ as a 0-simplex. Attach a 1-simplex for every morphism of $\mathcal{C}$. Then attach a 2-simplex for every commutative triangle created by the 1-simplices. Then attach a 3-simplex for every commutative tetrahedron created by 2-simplices. Continuing in this fashion will create a (possibly infinite dimensional) Δ-complex $B\mathcal{C}$.

3.2 Remark. The underlying simplicial set of $B\mathcal{C}$ is called the *nerve* of $\mathcal{C}$.

3.3 Definition. An *Eilenberg-MacLane space* $K(G, n)$ is a topological space X having only one nontrivial homotopy group $\pi_n(X) \cong G$.

3.4 Remark.[5] In general, we cannot characterize categories by their classifying spaces. However, if the category is a groupoid, we can characterize it by its classifying space, up to equivalence of categories. For a groupoid $\mathcal{C}$, the classifying space $B\mathcal{C}$ is homotopy equivalent to a disjoint union of spaces $K(\mathrm{Mor}_{\mathcal{C}}(X, X), 1)$, where X ranges over the representatives of equivalence classes of objects in $\mathcal{C}$; $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a group since each morphism in a groupoid has an inverse (in other words, $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is the group of automorphisms of X).

4. $K(G,1)$ Spaces

In the previous section, we saw how $K(G,1)$ spaces arose in classifying spaces of groupoids. Now we will focus on the case when our groupoid has one object. In this section, let $\mathcal{C}$ be a groupoid with one object X and let G be any group.

4.1 Remark. We can deduce from Remark 3.4 that since $\mathcal{C}$ has one object, then the classifying space $B\mathcal{C}$ is homotopy equivalent to $K(G,1)$, where $G = \mathrm{Mor}_{\mathcal{C}}(X,X)$.

4.2 Definition. A path-connected space whose fundamental group is isomorphic to G and which has a contractible universal covering space is called a $K(G,1)$ *space*. This definition is actually equivalent to Definition 3.3 in the case $n = 1$. [1]

4.3 Examples.

(a) Recall that $\pi_1(S^1) = \mathbb{Z}$. Since $\mathbb{R}$ is the universal covering space of S^1 and $\mathbb{R}$ is contractible, then S^1 is a $K(\mathbb{Z},1)$.

(b) A product $K(G,1) \times K(H,1)$ is a $K(G \times H,1)$ since its universal cover is the product of the universal covers of $K(G,1)$ and $K(H,1)$. In particular, this implies $\mathbb{R}^n$ is the universal cover of the n-dimensional torus T^n, so T^n is a $K(\mathbb{Z}^n,1)$.

4.4 Proposition.[1] *Let X be a connected CW complex and let Y be a $K(G,1)$ space. Then every homomorphism $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is induced by a map $f : (X,x_0) \to (Y,y_0)$ that is unique up to homotopy fixing x_0.*

4.5 Theorem.[1] *The homotopy type of a CW complex $K(G,1)$ is uniquely determined by G.*

4.6 Remark. Theorem 4.5 connects homotopy type to groups. Thus instead of dealing with CW complexes or other constructions, we can look to the group G to decide the homotopy type of $K(G,1)$. In terms of categories, if we think of the groupoid $\mathcal{C}$ with one object X, we only need to look at the morphisms of X to itself to determine the homotopy type of the classifying space $B\mathcal{C} = K(\mathrm{Mor}_{\mathcal{C}}(X,X),1)$.

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