Mathematics 440 & 508

Homework #10

XI.5-1. Let $\{f_n(z)\}$ be a uniformly bounded sequence of analytic functions on a domain D and let $z_0 \in D$. Suppose that for each $m \geq 0$, $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ normally on D.

Ans: Let $D(z_0, r)$ be the largest open disk with centre z_0 contained in D. We first show that f_n converges uniformly to 0 on any closed subdisk $\bar{D}(z_0, r_1) \subset D(z_0, r)$. Let M be the uniform bound on $f_n(z)$ on D. Then by homework exercise V.2.12, $|f_n^{(m)}(z_0)| \leq m! M / r^m$ and hence

$$|f_n(z) - \sum_{m=0}^{N} \frac{f_n^{(m)}(z_0)}{m!}(z - z_0)^m| \leq \sum_{m=N+1}^{\infty} M\Big(\frac{|z - z_0|}{r}\Big)^m.$$

For $|z - z_0|/r \leq r_1/r < 1$ the bound on the RHS may be made smaller than $\epsilon/2$ for sufficiently large N, uniformly in n. And then since $f_n^{(m)}(z_0) \to 0$ as $n \to \infty$, the finite sum $\sum_{m=0}^{N} \frac{f_n^{(m)}(z_0)}{m!}(z - z_0)^m$ may be made smaller than $\epsilon/2$ for n sufficiently large uniformly in z for $z \in D(z_0, r_1)$. So $f_n(z) \to 0$ uniformly on $D(z_0, r_1)$.

By Montel's theorem, p.308, f_n has a subsequence f_{n_k} that converges normally on D to some f analytic in D. By what we have just shown, f is identically zero on $D(z_0, r)$ and hence 0 in all of D by the uniqueness principle (p.156). Since this limit is independent of the subsequence, it is clear that this means that the full sequence f_n converges normally to 0 on D. Otherwise, it would have a subsequence that does not converge to 0 for all z and, by Montel's theorem, this would in turn have a normally convergent subsequence that converges to a limit that is not identically zero But this would be a contradiction of what we have just proved.

(One does not really need to use Montel's theorem to prove this result. To give a direct argument, notice that one can connect z_0 to any $z_1 \in D$ by a finite chain of overlapping disks, with the centre of each being in the previous disk. Then one inductively proves that $f_n(z) \to 0$ uniformly on each of these disks. It is then easy to conclude that f_n converges uniformly to 0 on each closed subdisk of D and hence normally to 0 on D.)

XI.5–8. Let $\{f_n(z)\}$ be a sequence of analytic functions on a domain D. Suppose that $\int\int_D |f_n(z)|\, dx\, dy \leq 1$ for $n \geq 1$.

(a) Show that $\{f_n(z)\}$ has a subsequence that converges normally to an analytic function $f(z)$ on D. *Hint.* To estimate $f(z)$ use the mean value property with respect to area (see Exercise III.4.1).

(b) Show that $\int\int_D |f(z)|\, dx\, dy \leq 1$.

(c) Show that if $\int\int_D |f_n(z) - f_m(z)|\, dx\, dy \to 0$ as $m, n \to \infty$, then $\int\int_D |f_n(z) - f(z)|\, dx\, dy \to 0$ as $n \to \infty$.

Ans: Suppose that f is analytic in the disk $|z - z_0| \leq a$, with $a > 0$. By Cauchy's formula for the circle $\gamma(t) = z_0 + re^{i\theta}$, one has

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Multiplying this by $r\, dr$, integrating from 0 to a, and dividing by $\int_0^a r\, dr = a^2/2$, we have

$$f(z_0) = \frac{1}{\pi a^2}\int\int_D f(z)\, dA,$$

where $dA = dx\,dy = r\,dr\,d\theta$ is the element of area. This is the mean-value property referred to in the hint.

(a) Using the above, we show that $\{f_n(z)\}$ is locally uniformly bounded in D. If K is any compact subset of D, let $\delta = \text{dist}(K, \partial D)$. Then each point $z \in K$ is contained in the closed disk $\bar{D}(z,\delta) \subset D$ so $f_n(z) = \frac{1}{\pi\delta^2}\int\int_{\bar{D}(z,\delta)} f_n(\zeta)\,dA$. So

$$|f_n(z)| \leq \frac{1}{\pi\delta^2}\int\int_{\bar{D}(z,\delta)} |f_n(\zeta)|\,dA \leq \frac{1}{\pi\delta^2}\int\int_{D} |f_n(\zeta)|\,dA \leq \frac{1}{\pi\delta^2}.$$

Since this bound is independent of n, this proves the local uniform boundedness of the family $\{f_n\}$. By Montel's theorem, a subsequence of $\{f_n\}$ converges normally to an f analytic in D. For convenience, denote this subsequence again by $\{f_n\}$ below.

(b) Let K_k be the standard sequence of compact subsets of D whose union is D, defined by

$$K_k = \{z \in D | \text{dist}(z, \partial D) \leq 1/k \text{ and } |z| \leq k\}.$$

Then $\int\int_{K_k} |f_n(z)|\,dA \leq \int\int_D |f_n(z)|\,dA \leq 1$ since the integrand is non-negative. Since K_k is compact, we have uniform convergence of f_n to f on K_k and hence $\int\int_{K_k} |f(z)|\,dA = \lim_{n\to\infty}\int\int_{K_k} |f_n(z)|\,dA \leq 1$. Now, by a fundamental property of integrals (monotone convergence), we have

$$\int\int_D |f(z)|\,dA = \lim_{k\to\infty}\int\int_{K_k} |f(z)|\,dA \leq 1.$$

(c) If $\int\int_D |f_n(z) - f_m(z)|\,dx\,dy \to 0$, then given $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $\int\int_D |f_n(z) - f_m(z)|\,dx\,dy < \epsilon$, for $m, n \geq N$. By the same argument as in (b), letting $m \to \infty$, we can conclude from this that $\int\int_D |f_n(z) - f(z)|\,dx\,dy \leq \epsilon$ for $n \geq N$. This shows that $\int\int_D |f_n(z) - f(z)|\,dx\,dy \to 0$ as $n \to \infty$.

XI.6–2. Let $\phi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $|f'(z_0)| \leq \phi'(z_0)$, with equality only when $f(z)$ is a constant multiple of $\phi(z)$. *Remark* This shows that $\phi(z)$ is the Ahlfors function of D corresponding to z_0.

Ans: Suppose that $f(z_0) = \alpha$. Let $\phi_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$ and consider $g = \phi_\alpha \circ f \circ \phi^{-1}$. Then ϕ^{-1} maps $\mathbb{D}$ one-one onto D with $\phi^{-1}(0) = z_0$, f maps D into $\bar{\mathbb{D}}$ with $f(z_0) = \alpha$ and ϕ_α maps $\bar{\mathbb{D}}$ one-one onto $\bar{\mathbb{D}}$ with $\phi_\alpha(\alpha) = 0$, so g maps $\mathbb{D}$ into $\bar{\mathbb{D}}$ with $g(0) = 0$. Thus Schwarz' lemma applies to g implying $|g'(0)| \leq 1$. Computing $g'(0)$ by the chain rule, we have

$$g'(0) = \phi_\alpha'(\alpha) f'(z_0)(\phi^{-1})'(0) = \frac{1}{1-|\alpha|^2} f'(z_0) \frac{1}{\phi'(z_0)}.$$

Hence $|f'(z_0)| \leq |g'(0)|(1-|\alpha|^2)\phi'(z_0) \leq \phi'(z0)$, since both $|g'(0)| \leq 1$ and $1-|\alpha|^2 \leq 1$.

For equality to hold, we must have both $|g'(0)| = 1$ and $1-|\alpha|^2 = 0$. The first implies that $g(z) = cz$ for a constant c with $|c| = 1$, and the second that $\alpha = 0$. Thus $f \circ \phi^{-1}(z) = cz$ and so $f(z) = c\phi(z)$ for all $z \in D$.

Remark: Note that the book's hint only applies to the case $\alpha = 0$.

XI.6–3. Let $\phi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $\text{Re}\, f'(z_0) \leq \phi'(z_0)$, with equality only when $f(z) = \phi(z)$.

Ans: Since $\text{Re}\, f'(z_0) \leq |f'(z_0)|$ and $|f'(z_0)| \leq \phi'(z_0)$ by the previous problem, we clearly have $\text{Re}\, f'(z_0) \leq \phi'(z_0)$. If equality holds then by the first inequality, we also must have $|f'(z_0)| = \phi'(z_0)$ and hence by the previous problem we have $f(z) = c\phi(z)$ with $|c| = 1$ a constant. But then we have $\text{Re}\, f'(z_0) = c\phi'(z_0)$ and $\text{Re}\, f'(z_0) = \phi'(z_0)$ so $c = 1$.