Mathematics 440 & 508

Homework #9

XI.2–3. Let D be a domain in the complex plane whose complement in $\mathbb{C}^*\backslash D$ consists of a finite number of closed connected sets, not all of which are points. Show that D can be mapped conformally onto a bounded domain whose boundary consists of a finite number of points and simple closed analytic curves.

Ans: Note that we are assuming the Riemann Mapping Theorem that any simply connected domain of $\mathbb{C}$ with the exception of $\mathbb{C}$ itself can be conformally mapped onto the open unit disk U. It is important to realize that this conformal mapping does not necessarily extend to any points of the boundary of the domain or to its complement. We use the characterization (v) in the Theorem on p. 254 which says that a domain D is simply connected if and only if its complement in $\mathbb{C}^*$ is connected.

Let $F_1, \ldots, F_n$ be the closed connected components of $\mathbb{C}^*\backslash D$ with $F_1, \ldots, F_k$ not being points and $F_{k+1}, \ldots, F_n$ being points. Let $D_1 = \mathbb{C}^*\backslash F_1$. Then D_1 is open since F_1 is closed. Also $D_1 = D \cup F_2 \cup \ldots \cup F_n$ is connected since this is a disjoint union of connected sets and each of the F_k shares a boundary point with D by definition of the F_k. Hence D_1 is a domain and moreover a simply connected domain since F_1 is connected. Thus there is a conformal mapping f_1 of D_1 onto U. Let $U_1 = f_1(D)$. Then U_1 is a subdomain of U whose complement is the union of $|z| \geq 1$, $f_1(F_2)$, $\ldots f_1(F_n)$ and whose boundary thus consists of the unit circle $T = \{z : |z| = 1\}$, which is a simple closed analytic curve, and the sets $f_1(\partial F_2), \ldots, f_1(\partial F_n)$ of which we only know that $f_1(\partial F_{k+1}), \ldots, f_1(\partial F_n)$ are points. (Note that it is not necessarily true that $T = f_1(\partial F_1)$ since f_1 may not even be defined on ∂F_1, but T is nevertheless a component of the boundary of $U_1 = f_1(D)$).

If F_2 is not a point, consider $D_2 = \mathbb{C}^*\backslash f_1(F_2)$ which is an unbounded simply connected domain containing $f_1(D)$ and containing the unit circle T. Let f_2 be a conformal mapping of D_2 onto U_2 and let $U_2 = f_2(U_1) = f_2 \circ f_1(D)$. U_2 is a domain whose complement in $\mathbb{C}^*$ is the union of $|z| \geq 1$, $f_2(|z| \geq 1)$, $f_2 \circ f_1(F_3), \ldots, f_2 \circ f_1(F_n)$. The boundary of U_2 is the union of the boundaries of these sets, namely T, $f_2(T)$, which are simple closed analytic curves, and $f_2 \circ f_1(\partial F_3), \ldots, f_2(\partial F_n)$. (The same remark applies as at step 1, i.e. T is not necessarily equal to $f_2(\partial F_2)$ since f_2 need not even be defined on ∂F_2, but $T \subset U_2$ so it is true that f_2 is defined and analytic on T and hence that $f_2(T)$ is a simple closed analytic curve.)

If F_3 is not a point, then consider $D_3 = \mathbb{C}^*\backslash f_2 \circ f_1(F_3)$ and let f_3 be a conformal mapping of D_3 onto U, etc. After k steps we arrive at a conformal mapping $f_k \circ \ldots \circ f_1$ of D onto a subdomain U_k of U whose boundary consists of $T, f_k(T), f_k \circ f_{k-1}(T), \ldots, f_k \circ \ldots \circ f_2(T)$, which are simple closed analytic curves, together with $f_k \circ \ldots \circ f_1(F_j)$ for $j = k+1, \ldots, n$ which are points. This is what we wanted to achieve.