Mathematics 440 & 508

Homework #8

VIII.4–4. Let $f(z)$ be an analytic function on the open unit disk $\mathbb{D} = \{|z| < 1\}$. Suppose there is an annulus $U = \{r < |z| < 1\}$ such that the restriction of $f(z)$ to U is one-to-one. Show that $f(z)$ is one-to-one on $\mathbb{D}$.

Ans: Let $\gamma(t) = \rho e^{it}$, with $r < \rho < 1$. Since the range Γ of γ lies in U, f is one-one on Γ so the curve $f \circ \gamma$ is a simple closed curve in $f(D)$. By the Jordan curve theorem, for any point w inside $f \circ \gamma$, the winding number $W(f \circ \gamma, w) = \pm 1$. But we know that this value must be non-negative, hence $+1$, since f is analytic inside Γ. Since $W(f \circ \gamma, w)$ counts the number of solutions of $f(z) = w$ inside Γ, the number of such solutions is exactly 1 for each w inside $f \circ \gamma$ so f is one-one inside γ. Since $\rho < 1$ is arbitrary, f is one-one on $\mathbb{D}$.

VIII.4–6. Let $f(z)$ be a meromorphic function on the complex plane, and suppose that there is an integer m such that $f^{-1}(w)$ has at most m points for all $w \in \mathbb{C}$. Show that $f(z)$ is a rational function.

Ans: By reducing m if necessary, we may assume that there is a point w_0 for which $\# f^{-1}(w_0) = m$ (counting multiplicity of course). Choose R so that $\{z : f(z) = w_0\} \subset \{|z| < R\}$ and so that no pole of f lies on $|z| = R$ (this is clearly possible since the number of poles is countable). Let N_∞ denote the number of poles of $f(z)$ in $\{|z| < R\}$. Then, if γ is the positively oriented circle $|z| = R$, we have $W(f \circ \gamma, w_0) = m - N_\infty$ by the argument principle. For w in the connected component G of $\mathbb{C} \backslash \mathrm{range}(f \circ \gamma)$ containing w_0, we have $W(f \circ \gamma, w) = m - N_\infty$ and hence $f(z) = w$ has m solutions inside γ. Since this is the maximal number of solutions, all the solutions of $f(z) = w$ lie inside γ. Suppose that G contains $|w - w_0| < \delta$, then all the solutions of $f(z) = w$ for $|w - w_0| < \delta$ lie inside γ, so for z outside γ, we must have $|f(z) - w_0| \geq \delta$.

Thus the meromorphic function $g(z) = 1/(f(z) - w_0)$ satisfies $|g(z)| \leq 1/\delta$ for $|z| \geq R$ and hence has a removable singularity at ∞ by Riemann's theorem on bounded singularities. So $f(z) = w_0 + 1/g(z)$ has a pole or a removable singularity at ∞ and hence is meromorphic in the extended plane $\mathbb{C}^*$. Finally, by the Theorem on p.179, $f(z)$ is a rational function.

IX.2–5. Show that any conformal self-map of the upper half-plane has the form $f(z) = (az + b)/(cz + d)$, where a, b, c, d are real numbers satisfying $ad - bc = 1$. When do two such coefficient choices for a, b, c, d determine the same conformal self-map of the upper half-plane?

Ans: Let f be a conformal self-map of the upper half-plane $\mathbb{H}$. The LFT $g(z) = \dfrac{z - i}{z + i}$ maps $\mathbb{H}$ onto $\mathbb{D}$ and so $g \circ f \circ g^{-1}$ is a conformal self map of the unit disk $\mathbb{D}$. The theorem on p.263 tells us that $g \circ f \circ g^{-1}$ is a LFT, hence so is f. That is, there exist $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ and $f(z) = (az + b)/(cz + d)$. Since a nonzero scalar multiple of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will determine the same LFT we may assume without loss of generality that $ad - bc = 1$. Finally, since the LFT f maps $\mathbb{H}$ onto $\mathbb{H}$ it must map $\mathbb{R} \cup \{\infty\}$ onto $\mathbb{R} \cup \{\infty\}$ and so we may choose distinct $z_1, z_2, z_3 \in \mathbb{R}$ such that $f(z_1), f(z_2), f(z_3) \in \mathbb{R}$. Since f is determined by the cross ratios via $[z, z_1, z_2, z_3] = [w, f(z_1), f(z_2), f(z_3)]$ it follows that we may choose $a, b, c, d \in \mathbb{R}$.

IX.2–7. Show that every conformal self-map of the complex plane has the form $f(z) = az + b$, where $a \neq 0$. *Hint.* The isolated singularity of $f(z)$ at ∞ must be a simple pole.

Ans: Since $f(z)$ is assumed to be one-one, $f^{-1}(w)$ is a single point for each $w \in \mathbb{C}$ so we can apply problem VIII.4–6 above with $m = 1$ to conclude that f is a rational function. Since $f(z)$ is analytic everywhere (i.e. entire) and rational, it is a polynomial. By the fundamental theorem of calculus $\# f^{-1}(w)$ is the degree of f and hence the degree of f is 1 so $f(z) = az + b$. Clearly $a \neq 0$ is needed since a constant function is not one-one, and is sufficient since $f^{-1}(w) = (w - b)/a$. (I don't seem to have used the hint explicitly!)

IX.2–8. Show that every conformal self-map of the Riemann sphere $\mathbb{C}^*$ is given by a fractional linear transformation.

Ans: Let f be such a map. If $f(\infty) = \infty$ then f is a self-map of $\mathbb{C}$ and hence is of the form $az + b$ by the previous problem, so a fractional linear transformation.

Otherwise, if $f(\infty) = c \neq \infty$, let $g(z) = 1/(z - c)$ and $h(z) = g(f(z))$. Then f and g are self-maps of $\mathbb{C}$ and hence so is h. But $h(\infty) = g(f(\infty)) = g(c) = \infty$ so by the previous paragraph, $h(z) = az + b$. So $f = g^{-1} \circ h$ is the composition of two fractional linear transformations and hence is fractional linear.

Extra Problem: Show that if $z = g(w)$ is the solution of $w = z(1 + z^2)$ that takes on the value 0 at $w = 0$, then

$$\tan^{-1} g(w) = w - \frac{4}{1!}\frac{w^3}{3} + \frac{6 \cdot 7}{2!}\frac{w^5}{5} - \frac{8 \cdot 9 \cdot 10}{3!}\frac{w^7}{7} + \ldots \quad \text{for} \quad |w| < 2\sqrt{3}/9.$$

Hint: First consider the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{f(z) - w}\,dz,$$

where $f(z) = z(1 + z^2)$, $h(z)$ is the principal branch of $\tan^{-1} z$, and γ is a small circle around 0.

Ans: The function $h(z)$ is defined to be the function analytic in the plane slit along rays along the imaginary axis from $\pm i$ to ∞ and with $h(0) = 0$ and $h'(z) = 1/(1 + z^2)$ in this domain. Hence h is analytic in $|z| < 1$.

Note that $z = 0$ is a simple zero of $f(z)$ and the next smallest zeros of $f(z)$ are $z = \pm i$, so if the radius of γ is chosen smaller than 1, there is exactly one solution of $f(z) = 0$ inside γ and hence exactly one solution of $f(z) = w$ for each w in the connected component of the complement of the range of $f \circ \gamma$ that contains 0. As usual, call this solution $z = g(w)$. Then, as we showed in class, the integral given in the hint is the residue at the simple pole of the integrand at $z = g(w)$. By the formula for the residue at a simple pole, this equals

$$\lim_{z \to g(w)} \frac{(z - g(w))h'(z)}{f(z) - w} = \frac{h'(g(w))}{f'(g(w))} = h'(g(w))g'(w) = \frac{d}{dw}h(g(w)).$$

If we expand the integral in a series in w, we need only integrate it term by term to obtain $h(g(w))$. The integrand is $\dfrac{1}{(1 + z^2)(z(1 + z^2) - w)}$. Provided $|z(1 + z^2)| > |w|$, we may expand this in the geometric series

$$\frac{1}{(1 + z^2)(z(1 + z^2) - w)} = \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}(1 + z^2)^{n+2}}.$$

Thus if $|z| = r$, we need $|w| < r(1 - r^2) = \min_{|z|=r} |z(1 - z^2)|$. Choosing $0 < r < 1$ to maximize this upper bound we take $r = 3^{-1/2}$ so $|w| < 2\sqrt{3}/9$. The series converges uniformly for any $|w| \leq \rho < 2\sqrt{3}/9$ and so can be integrated term-by-term to obtain

$$\frac{d}{dw}h(g(w)) = \sum_{n=0}^{\infty} w^n \mathrm{Res}_{z=0} \frac{(1 + z^2)^{-n-2}}{z^{n+1}}.$$

Either by Cauchy's formula for the derivative or by Taylor's formula, the residue appearing here is the coefficient of z^n in the Maclaurin expansion of

$$(1 + z^2)^{-n-2} = \sum_{k=0}^{\infty} \binom{-n-2}{k} z^{2k}.$$

This is 0 if n is odd and is $\dbinom{-2k-2}{k}$ if $n = 2k$ is even. After integrating term-by-term we obtain the series

$$h(g(w)) = \sum_{k=0}^{\infty} \binom{-2k-2}{k} \frac{w^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} (-1)^k \binom{3k+1}{k} \frac{w^{2k+1}}{2k+1}.$$

Check that the first few terms are as given in the statement of the problem. We have already determined that this formula is valid if $|w| < 2\sqrt{3}/9$. In fact, it is easy to check that this is the full disk of convergence using the ratio test but this is not really necessary.