

## Mathematics 440 & 508

### Homework #7

VIII.2-9. Let  $f(z)$  be a continuously differentiable function on a domain  $D$ . Suppose that for all complex constants  $a$  and  $b$ , the increase in the argument of  $f(z) + az + b$  around any small circle in  $D$  on which  $f(z) + az + b \neq 0$  is nonnegative. Show that  $f(z)$  is analytic.

**Ans:** Of course if you assume that  $f(z)$  is differentiable in the complex sense at all points of  $D$  then it is analytic in  $D$  and so the problem is a little too easy! So what the assumptions must mean is that the partial derivatives  $f_x$  and  $f_y$  exist and are continuous in  $D$ . Then the Taylor formula for functions of two real variables says that if  $h = k + im$ , then  $f(z_0 + h) = f(z_0) + f_x(z_0)k + f_y(z_0)m + o(|h|)$ , as  $h \rightarrow 0$ . This is more nicely written in the complex form  $f(z_0 + h) = f(z_0) + f_z(z_0)h + f_{\bar{z}}(z_0)\bar{h} + o(|h|)$  (more or less as on p.126 except there he assumes the error is  $O(|h|^2)$  which would require the existence of second partials). Then  $f_{\bar{z}}(z_0) = 0$  is the complex form of the Cauchy-Riemann equations and equivalent to the existence of  $f'(z_0)$  (and then that  $f'(z_0) = f_z(z_0)$ ). Suppose that  $z_0$  is a point in  $D$  where  $f_{\bar{z}}(z_0) = c \neq 0$ . then choose  $a$  and  $b$  so that  $f(z_0) + f_z(z_0)(z - z_0) + az + b = 0$  and then we have  $f(z) = c(z - z_0) + o(|z - z_0|)$ . If  $z = z_0 + re^{it}$  for small  $r$  then  $f(z) = cre^{-it} + o(r)$  so the change in argument of  $f$  around a small enough circle is  $-2\pi$ , contradicting the assumption that it be nonnegative. So we have  $f_{\bar{z}}(z_0) = 0$  for all  $z_0 \in D$  and hence  $f$  is analytic in  $D$ .