Mathematics 440 & 508

Homework #5

VI.1-6. Fix an annulus $D = \{a < |z| < b\}$ and let $f(z)$ be a continuous functions on its boundary ∂D. Show that $f(z)$ can be approximated uniformly on ∂D by polynomials in z and $1/z$ if and only if $f(z)$ has a continuous extension to the closed annulus $D \cup \partial D$ that is analytic on D.

Ans: There are two directions to prove. It is tempting to try to use the Laurent expansion but this needn't converge on ∂D even when f is continuous on $\bar{D} = D \cup \partial D$.

$(\Rightarrow)$

First assume that f can be approximated uniformly on ∂D by polynomials in z and $1/z$. Thus there is a sequence of Laurent polynomials $P_n(z) = \sum_k a_{n,k} z^k$ for which $\max_{z \in \partial D} |P_n(z) - f(z)| \to 0$ as $n \to \infty$. (Note that we cannot assume that $a_{n,k}$ is independent of n). Thus P_n is a uniform Cauchy sequence in the sense that $\max_{z \in \partial D} |P_n(z) - P_m(z)| \to 0$ for all $m > n$, as $n \to \infty$. By the maximum modulus principle $\max_{z \in \bar{D}} |P_n(z) - P_m(z)| \to 0$ as $n \to \infty$, so that P_n is a Cauchy sequence in the sense of uniform convergence on $\bar{D}$ and hence converges uniformly on $\bar{D}$. Denote this limit by $f(z)$, thus extending the definition of f from ∂D to $\bar{D}$. Uniform convergence on $\bar{D}$ is stronger than normal convergence in D and hence f is analytic in D. Also f is continuous on $\bar{D}$ since it is the uniform limit of the $P_n(z)$ which are continuous on $\bar{D}$.

$(\Leftarrow)$

Now assume that f is analytic in D and continuous in $\bar{D}$. Then by the Laurent decomposition, we can write $f = g + h$, where g is analytic in $|z| < b$ and h is analytic in $|z| > a$. Since f and h are continuous on $|z| = b$, we see that g is continuous in $|z| \le b$ and similarly that h is continuous in $|z| \ge a$. So it suffices to show that g can be uniformly approximated by polynomials $P_n(z)$ in $|z| \le b$. It then follows by changing z to $1/z$ that h can be uniformly approximated by polynomials $Q_n(z)$ in $|z| \ge a$ completing the proof.

As noted above, the problem is that the Maclaurin series for g needn't converge on $|z| = b$ (this is basically a fact from the theory of Fourier series). We could now use Fejér's famous theorem from Fourier series that the Cesàro averages of the partial sums of the Fourier series of a continous function converge uniformly to the function. So the Cesàro averages of the partial sums of the Maclaurin series for g converge uniformly to g on $|z| = b$ and hence on $|z| \le b$ by the maximum modulus principle, showing that g is the uniform limit in $|z| \le b$ of this sequence of polynomials.

However, we can avoid this by arguing as follows: since g is continuous on $|z| \le b$, it is uniformly continuous and hence it is the uniform limit on $|z| \le b$ of the functions $g_r(z) = g(rz)$ as $r \to 1-$. But each of the g_r is the uniform limit in $|z| \le b$ of the partial sums of its Maclaurin series, since this is just the series $\sum_k b_k r^k z^k$, where $\sum b_k z^k$ is the Maclaurin series for g, and this converges uniformly in $|z| \le rb$ for any $r < 1$. By the triangle inequality, g is thus the uniform limit in $|z| \le b$ of a sequence of polynomials of the form $\sum_{k=0}^n b_k r_n^k z^k$ for a suitable sequence $r_n \uparrow 1$ as $n \to \infty$.

VI.2–11. Suppose $f(z) = \sum_{n=0}^\infty a_n z^n$ is analytic for $|z| < R$, and suppose that $f(z)$ extends to be meromorphic for $|z| < R + \epsilon$, with only one pole z_0 on the circle $|z| = R$. Show that $a_n/a_{n+1} \to z_0$ as $n \to \infty$.

Ans: Although the wording of the question would allow other poles in $|z| < R + \epsilon$, we may assume that ϵ is chosen so small that there is exactly one pole in $|z| < R + \epsilon$, i.e. the one at $z = z_0$. Let $h(z) = \sum_{k=1}^m c_k (z - z_0)^{-k}$ with $c_m \ne 0$ denote the principal part of $f(z)$ at z_0 so that $g(z) = f(z) - h(z)$ is analytic in $|z| < R + \epsilon$. Thus if $g(z) = \sum_n b_n z^n$ for $|z| < R + \epsilon$, we have $b_n = O(R_1^{-n})$, for any R_1 with $R < R_1 < R + \epsilon$.

By the binomial expansion, $(z - z_0)^{-k} = (-z_0)^{-k}(1 - z/z_0)^{-k} = (-z_0)^{-k} \sum_{n=0}^\infty (-1)^n \binom{-k}{n} z_0^{-n} z^n$, where $\binom{-k}{n} = (-k)(-k-1)\dots(-k-n+1)/n!$. Just use the formula for the Maclaurin series of $(z - z_0)^{-k}$. Each of these series converges for $|z| < |z_0| = R$. Thus $a_n = b_n + \sum_{k=1}^{m} (-1)^{k+n} c_k \binom{-k}{n} z_0^{-n-k} = b_n + p(n) z_0^{-n}$, where $p(n)$ is a polynomial in n of degree $m-1$, since it is easy to see that $(-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1}$ is a polynomial of degree $k-1$ in n and we have assumed $c_m \neq 0$.

Thus $a_n/a_{n+1} = z_0(b_n z_0^n + p(n))/(b_{n+1} z_0^{n+1} + p(n+1)) \to z_0$, using $b_n z_0^n \to 0$ and $p(n+1)/p(n) \to 1$ as $n \to \infty$.

VI.2–12. show that if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity for $g(z) = e^{f(z)}$.

Ans: (1) The simplest answer uses the two Theorems on p.175 which characterize poles and essential singularities in terms of the behaviour of $f(z)$ in a neighbourhood of z_0. Note, by the way, that this problem and the next problem VI.3–3 are equivalent, using the Laurent decomposition and the change of variable $w = 1/(z - z_0)$. That is, write $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is analytic at $z = z_0$ and $f_1(z) = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k}$ is analytic except at $z = z_0$. So $f_1(z) = h(1/(z - z_0))$ where $h(w)$ is an entire function. Since $e^{f_0(z)}$ is analytic at $z = z_0$, the type of singularity of $e^{f(z)}$ at z_0 is the same as that of $e^{f_1(z)}$ at $z = z_0$ which is the same as that of $e^{h(w)}$ at $w = \infty$. See the next problem for the completion of the solution.

Remark: This argument shows that Liouville's theorem on bounded entire functions (p.118) is equivalent to Riemann's theorem on removable singularities (p.172). If you examine the proofs you will see that they use Cauchy's estimate for the coefficients of a Taylor or Laurent expansion in exactly the same way.

(2) One can give a different argument based directly on the Laurent series as follows: It is clear that z_0 can't be a removable singularity of $g(z)$ so it is either a pole or an essential singularity. Suppose that it is a pole so that $g(z) = \sum_{n=-m}^{\infty} b_n(z - z_0)^n$ near z_0. Then $g'(z) = \sum_{n=-m}^{\infty} n b_n (z - z_0)^{n-1}$ and hence $f'(z) = g'(z)/g(z)$ has a simple pole of residue $-m$ at $z = z_0$ (see p.225 for the calculation). But if $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, then $f'(z)$ contains no term in $(z - z_0)^{-1}$ (this is obvious from the series, but also obvious from the fundamental theorem of calculus since the integral of $f'(z)$ around any closed path is 0, hence its residue at z_0 is 0). Thus we have a contradiction, so z_0 cannot be a pole of $g(z)$ and hence must be an essential singularity.

Remark: Some of you tried to give an argument based on computing the Laurent expansion of $e^{f(z)}$ from that of $f(z)$, but most didn't give a convincing explanation of why this series had to have $a_n \neq 0$ for infinitely many $n < 0$. The expression one obtains for a_n by expanding out the exponential series contains a large number of terms and it is not at all clear that these don't cancel to give $a_n = 0$ for all but finitely many $n < 0$.

VI.3–3. Show that if $f(z)$ is a nonconstant entire function then $g(z) = e^{f(z)}$ has an essential singularity at $z = \infty$.

Ans: Since $f(z)$ is entire and nonconstant, so is $g(z)$, so it is clear that it doesn't have a removable singularity at ∞. If $g(z)$ has a pole at ∞ then it is a non-constant polynomial, and hence has a non-trivial zero in $\mathbb{C}$ by the fundamental theorem of Algebra. But we can see that $1/g(z) = e^{-f(z)}$ is entire which is impossible if g vanishes anywhere. So ∞ is not pole of $g(z)$ and hence $g(z)$ has an essential singularity at infinity.

Remark The hint in the book suggests a direct application of Liouville's theorem (p.118), i.e. repeat the proof given for the fundamental theorem of Algebra on p.118.