# Mathematics 440 & 508

## Homework #4

V.8-3. Show that each branch of $\sqrt{z}$ can be continued analytically along any path $\gamma$ in $\mathbb{C}\backslash\{0\}$, and show that the radius of convergence of the power series $f_(z)$ representing the continuation is $|\gamma(t)|$. Show that $\sqrt{z}$ cannot be continued analytically along any path containing 0.

**Ans:** Denote the principal branch of $\sqrt{z}$ by $s(z) = \exp(\frac{1}{2}\mathrm{Log}\, z)$, where $\mathrm{Log}\, z$ is the standard principal branch of $\log z$. Thus $s(z)$ is analytic in the plane minus the nonpositive real axis (the branch cut) but is discontinuous across the cut. The other branch of $\sqrt{z}$ is $-s(z)$ so it is enough to show how to continue $s(z)$ since if $f_t(z)$ is the continuation of $s(z)$ along $\gamma$, $-f_t(z)$ will be the continuation of $-s(z)$. Note that since $f_0(z)^2 = z$ for all $z$ in the domain of $f_0$, the continuation must also satisfy $f_t(z)^2 = z$ for all $z$ in its domain. This is because the unique continuation of the function $z$ to the entire plane is $z$ itself. (See the "permanence of functional equations" principle, i.e. the Theorem on p.157 of Gamelin). Thus for each value of $t$ and each $z$ in the domain of $f_t$, the value $f_t(z) = \epsilon_t(z)s(z)$, where $\epsilon_t(z) \in \{-1, 1\}$. Thus we know what the Taylor expansion of $f_t(z)$ about $\gamma(t)$ is up to sign. So in essence, we just need to find the correct sign to assign to $f_t(\gamma(t))$ for each $t \in [0, 1]$.

So the crucial first step is to find a continuous function $\mu(t)$ such that $\mu(t)^2 = \gamma(t)$ for $t \in [0, 1]$ for which $\mu(0) = s(\gamma(0))$. Note that to do this it is *not* sufficient to take $\mu(t) = s(\gamma(t))$ as many of you did, since this function will have a discontinuity whenever $\gamma(t)$ crosses the branch cut of $s$. It is true that for each value of $t$ one has $\mu(t) = \epsilon(t)s(\gamma(t))$, where $\epsilon(t) \in \{-1, 1\}$ but $\epsilon(t)$ has to be chosen so that the resulting function $\mu(t)$ is continuous. If $\gamma(t)$ crosses the cut at only a finite set of points, then this can be done by introducing a change of sign of $\epsilon(t)$ at each such crossing.

To construct $\mu$ for a general *path* $\gamma$, it seems easiest to first define a continuous value $L(t)$ of $\log\gamma(t)$ by defining $L(0) = \mathrm{Log}\,\gamma(0)$ and then

$$L(t) = L(0) + \int_0^t \frac{\gamma'(s)}{\gamma(s)}\, ds.$$

Then $L(t)$ is a continuous function of $t$ since $\gamma(s)$ is never 0. Also $L'(t) = \gamma'(t)/\gamma(t)$, so $\frac{d}{dt}\big(\gamma(t)\exp(-L(t))\big) = 0$ for all $t$. Thus $\exp(L(t)) = \gamma(t)$ for all $t$ since this holds for $t = 0$ by choice of $L(0)$. Now define $\mu(t) = \exp(\frac{1}{2}L(t)$ so $\mu(t)$ is continuous and satisfies $\mu(t)^2 = \gamma(t)$ for $t \in [0, 1]$. Note that this construction assumes $\gamma$ is piecewise differentiable. The case of a general continuous *curve* takes even more care. (e.g. think about the case of a space-filling curve – but we'd rather not perform continuation along such a curve!)

Now to define the analytic continuation of $s(z)$, we define

$$f_t(z) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \mu(t)^{1-2n}(z - \gamma(t))^n.$$

Then $f_t(z)$ is the Taylor expansion of a branch of $\sqrt{z}$ for which $f_t(\gamma(t)) = \mu(t)$. Clearly the coefficients are continuous functions of $t$ by the way we chose $\mu(t)$. Since $\mu(0) = s(\gamma(0))$, this is the required analytic continuation of $s(z)$. Checking the compatibility conditions $f_s(z) = f_t(z)$ for $s$ near $t$ is easy. We only need to check that $f_s(\gamma(s)) = f_t(\gamma(s))$ and this is obvious since the only other possibility is $f_s(\gamma(s)) = -f_t(\gamma(s))$ and this would make $f_t$ discontinuous at $z = \gamma(s)$. The radius of convergence of the power series is $|\gamma(t)|$ by the ratio test.

To see why we cannot continue either branch analytically along a curve that passes through 0, suppose $\gamma$ passes through zero for the first time at $t = t_0 > 0$. Then for $t < t_0$, $f_t(z)$ is given by the formula above (since the continuation is uniquely determined by the curve). But since the radius of convergence about $\gamma(t)$

is $|\gamma(t)|$, the point $\gamma(t_0) = 0$ is not in the disk of convergence of $f_t(z)$ for any $t < t_0$. Thus continuation is not possible along $\gamma$.

V.8-6. Suppose $f(z) = \sum_n a_n z^n$, where $a_n = 0$ except for $n$ in a sequence $n_k$ that satisfies $n_{k+1}/n_k \geq 1+\delta$ for some $\delta > 0$. Suppose further that the series has radius of convergence $R = 1$. Show that $f(z)$ does not extend analytically to any point of the unit circle. *Remark.* Such a sequence with large gaps between successive nonzero terms is called a **lacunary sequnce.** This result is the **Hadamard gap theorem.** There is a slick proof. If $f(z)$ extends analytically across $z = 1$, consider $g(w) = f(w^m(1+w)/2)$, where $m$ is a large integer. Show that the power series for $g(w)$ has radius of convergence $r > 1$, and that this implies that the power series of $f(z)$ converges for $1 \leq \mathrm{Re}\, z < 1+\epsilon$.

**Ans:** Note that it is enough to consider extension to the point $z = 1$ since we could replace $f(z)$ by $f_\theta(z) = f(ze^{i\theta})$ which has a series of the same form. If we can't extend $f_\theta$ analytically to $z = 1$ then we can't extend $f$ to $z = e^{i\theta}$.

So assume $f$ extends analytically to $z = 1$. The extension is thus analytic a domain containing $z = 1$ in its interior, of the form $D_1 = \{|z| < 1\} \cup \{|z-1| < \epsilon\}$.

Following the suggestion, write $z = w^m(1+w)/2 = h(w)$, say. Note that $h$ maps the closed disk $|w| \leq 1$ into a subset of $\{|z| < 1\} \cup \{1\}$. This follows from the triangle inequality since if $|w| \leq 1$ then $|h(w)| = |(w^m + w^{m+1})/2| \leq 1$, with equality if and only if $w^{m+1}/w^m$ is positive. Thus, for sufficiently small $r > 1$, $h$ maps $|w| < r$ into a subdomain of $D_1$. Hence $g(w) = f(h(w))$ is analytic in $|w| < r$ and so its Maclaurin series has radius of convergence at least $r$. (Note that up to this point we have not used the form of the power series for $f$).

But since $f(z) = \sum_k b_k z^{n_k}$, where $b_k = a_{n_k}$, we may compute the Maclaurin series for $g(w)$ by expanding out the terms in $g(w) = \sum_k b_k w^{mn_k}(1+w)^{n_k}2^{-n_k}$ using the binomial expansion. The powers of $w$ appearing in the $k$th sum lie in the range $mn_k, \ldots, (m+1)n_k$. Note that these ranges do not overlap if $(m+1)n_k < mn_{k+1}$ and this can be insured by choosing $m > 1/\delta$ since then $(1+1/m)n_k < (1+\delta)n_k \leq n_{k+1}$. Thus the Maclaurin series for $g(w)$ is simply the concatenation of these individually expanded terms. We have shown that this must have radius of convergence $r > 1$ and hence must converge for $w$ in the real interval $1 < w < r$. Substituting such a $w$ into the series for $g(w)$ and recombining the expanded terms gives us the power series for $f(z)$ at a point $z$ in the real interval $1 < z < (r^m + r^{m+1})/2$. But this series is supposed to have radius of convergence 1 and hence cannot converge for such $z$, so we have a contradiction.

*Remark:* This rather subtle argument is due to L.J. Mordell. To understand it better, it is instructive to follow it through taking a series without gaps like $f(z) = \sum_{k=0}^{\infty}(-1)^{k-1}z^k = 1/(1+z)$ and $h(w) = (1+w)/2$. Then $f$ does extend analytically to $z = 1$, and $g(w) = f(h(w)) = 2/(3+w)$ is analytic in $|w| < 3$. The Maclaurin series for $g(w)$ has radius of convergence 3 and may be obtained by expanding out the terms $(-1)^{k-1}2^{-k}(1+w)^k$ and summing. However, the fact that this series converges for real $w$ in the range $1 < w < 3$ says nothing about the Maclaurin series for $f(z)$.

V.8-8. Let $f(z)$ be analytic at $z_0$ and let $\gamma(t)$, $0 \leq t \leq 1$ be a path such that $\gamma(0) = z_0$. If $f(z)$ cannot be continued analytically along $\gamma$ show that there is a parameter value $t_1$ such that there is an analytic continuation $f_t(z)$ for $0 \leq t < t_1$, and the radius of convergence of the power series $f_t(z)$ tends to 0 as $t \to t_1$.

**Ans:** (Note that I have taken $a = 0$ and $b = 1$ as is customary). Let $S$ denote the set of $u \leq 1$ such that $f$ can be continued analytically along $\gamma(t)$ for $t \in [0, u]$. Note that $S$ is not empty since $f$ is analytic at $z_0$ and hence we can continue along the initial portion of $\gamma$ that lies within the circle of convergence of the Taylor series of $f$ around $z_0$. Define $t_1 = \sup S$.

We claim that $t_1 \notin S$. If $t_1 = 1$ this follows from the assumption that $f(z)$ cannot be continued along $\gamma$ (to

$\gamma(1)$). If $t_1 < 1$ and $t_1 \in S$ then the expansion of $f_{t_1}(z)$ around $\gamma(t_1)$ has a positive radius of convergence $r(t_1)$ and hence we can continue along the portion of $\gamma(t)$ with $t_1 \leq t < t_1 + \delta$, with $\delta > 0$, for which $|\gamma(t) - \gamma(t_1)| < r(t_1)$. But this contradicts the definition of $t_1$ since all the points in $[t_1, t_1 + \delta)$ lie in S. Thus we have shown that we can continue analytically along $\gamma$ to $\gamma(t)$ for $0 \leq t < t_1$ but not to $\gamma(t_1)$.

Now to show that $\lim_{t \to t_1^-} r(t) = 0$, it is tempting to state that this follows from the continuity of $r(t)$ and the fact that $r(t_1) = 0$. However, the proof of the continuity of $r(t)$ only applies to the half open interval $[0, t_1)$ as is evident from an examination of the proof of the Lemma on p.159. In fact $r(t)$ is not well defined for $t \geq t_1$ since the uniqueness argument defining $f_t(z)$ fails for these $t$.

Instead, we argue that for all $t < t_1$ near $t_1$ the point $\gamma(t_1)$ cannot lie in the disk $|z - \gamma(t)| < r(t)$, or else the series around $\gamma(t)$ would provide a continuation to $\gamma(t_1)$ and we have shown that this doesn't exist. Thus we have $|\gamma(t_1) - \gamma(t)| \geq r(t) > 0$ for $t < t_1$ near $t_1$. Letting $t \to t_1^-$, and using the continuity of $\gamma$, we have $\lim_{t \to t_1^-} |\gamma(t_1) - \gamma(t)| = 0$ and hence by the inequality on $r(t)$ that $\lim_{t \to t_1^-} r(t) = 0$.