

Mathematics 440 & 508

Homework #3

V.7-9. Show that if the analytic function $f(z)$ has a zero of order N at z_0 then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z_0) \neq 0$.

Ans: Write $f(z) = (z - z_0)^N h(z)$ as in (7.1), where $h(z)$ is analytic in a neighbourhood of z_0 and $h(z_0) \neq 0$. Let $\log_0 w$ be a branch of $\log w$ with a cut along the ray L from 0 through $-h(z_0)$. By continuity, there is an $r > 0$ so that h is analytic in $U = \{z : |z - z_0| < r\}$ and satisfies $|h(z) - h(z_0)| < |h(z_0)|$ there. Thus $h(U)$ does not intersect the ray L since in fact it lies in a half-plane bisected by $-L$. Thus $\log_0 w$ is analytic in $h(U)$ so $\log_0 h(z)$ is analytic in U . Now define $g(z) = \exp((\log_0 h(z))/N)$ for $z \in U$. Then g is the composite of analytic functions and hence is analytic in U and satisfies $g(z)^N = h(z)$ there.

Remark: This follows the hint in the text, but many of you failed to show how to define the logarithm of $h(z)$. Note that the result is a local one, as hinted by the word “near” in the statement of the problem.

V.7-10. Show that if $f(z)$ is a continuous function on a domain D such that $f(z)^N$ is analytic on D for some integer N , then $f(z)$ is analytic on D .

Ans:

We first show that f is analytic for all values of z with $f(z) \neq 0$. Just write

$$\frac{f(z+h) - f(z)}{h} = \frac{f(z+h)^N - f(z)^N}{h} \cdot \frac{1}{f(z+h)^{N-1} + f(z+h)^{N-2}f(z) + \dots + f(z)^{N-1}}.$$

Let $h \rightarrow 0$ and the first term converges to $\frac{d}{dz}f(z)^N$, while the denominator of the second term converges to $Nf(z)^{N-1} \neq 0$, since there are N terms in the sum, and f is continuous at z . This shows directly that $f'(z)$ exists if $f(z) \neq 0$.

Now we show that f is analytic at values for which $f(z) = 0$. If f^N is identically zero then clearly so is f and hence it is certainly analytic in D . Otherwise, the zeros of f^N in D are isolated, and since these are the same as the zeros of f , the same is true for f . By assumption f is continuous in D and we have shown that it is analytic except possibly at the zeros of f , a set of isolated points in D . In our proof of Goursat's theorem, we observed that in this case $\int_{\partial T} f(z) dz = 0$ around the boundary of a triangle $T \subset D$. Thus we have $\int_{\partial R} f(z) dz = 0$ around the boundary of a rectangle $R \subset D$ by writing ∂R as the sum of two integrals around triangles. Combining this with the assumption that f is continuous in D , we see that Morera's theorem implies that f is analytic in D .

Remark: To give a proof along the lines hinted in the text, you must show that locally $f(z) = (f(z)^N)^{1/N}$ for a suitably chosen branch of the N th root. This requires a discussion similar to that in problem 9, and you must give a convincing argument that this is possible using the assumption that f is continuous. For example, you could choose a branch of the N th root so that the function $g(z) = (f(z)^N)^{1/N}$ is continuous in a neighbourhood of a point z_0 and then observe that $f(z)/g(z)$ is a continuous function on a connected set which takes on values in the discrete set of N th roots of 1, hence is constant. This remark applies both to the case $f(z_0) \neq 0$ and $f(z_0) = 0$.

V.7-13. Let $f_n(z)$ be a sequence of analytic function on a domain D such that $f_n(D) \subset D$, and suppose that $f_n(z)$ converges normally to $f(z)$ on D . Show that either $f(D) \subset D$, or else $f(D)$ consists of a single point in ∂D . (You may assume that D is *regular*, i.e. that D is the interior of its closure).

Ans: We first observe that $f(D) \subset \bar{D} = D \cup \partial D$, since for each $z \in D$, $f(z)$ is a limit of points $f_n(z) \in D$. (This just uses pointwise convergence).

Next, since each f_n is analytic in D and $f_n(z)$ converges normally to $f(z)$, we conclude that f is analytic in D . (This was stated and proved in class. It is proved but not stated in the text on p.137). By the open mapping property (problem #11), since D is open and f is analytic, $f(D)$ is open.

Thus $f(D)$ is an open subset of \bar{D} , i.e. $f(D) \subset \text{int}(\bar{D})$. By our assumption, $\text{int}\bar{D} = D$ and hence we have shown that $f(D) \subset D$.

Remarks: 1. Many of you tried to prove this without using the suggested regularity assumption or else by not stating how it is used. To see how your particular version of the proof fails, try applying it to the example $D = \{0 < |z| < 1\}$ for which $\bar{D} = \{|z| \leq 1\}$ and $\text{int}(\bar{D}) = D_1 = \{|z| < 1\}$. Most of the incomplete proofs would fail if $f(D) = D_1$ which is an open subset of $\text{int}(\bar{D})$ but is not contained in D .

In fact the problem *is* correct without the regularity assumption. But the proof of this is based on a more subtle result than the open mapping theorem, namely a theorem of Hurwitz about the zeros of limits of sequences of analytic functions (p.231 of the text).

2. I had claimed that the question should omit the words “in ∂D ” at the end of the last sentence, since in the case of a constant limit $f_n(z) \rightarrow w_0$, we can have $w_0 \in D$ as the example $f_n(z) = z^n$ and $D = \{|z| < 1\}$ shows. However, as many of you pointed out, in case $w_0 \in D$, it is certainly true that $f(D) = \{w_0\} \subset D$. So the only remaining case is indeed the case of a constant limit $w_0 \in \partial D$ as the problem states.