Mathematics 440 & 508

Homework #2

V.2-9. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k}$ does not converge uniformly for $|z| < 1$.

Ans: The hint suggests showing first that (*) if the series converges uniformly in $|z| < 1$ then it converges uniformly on $|z| \le 1$. But uniform convergence in $|z| \le 1$ implies convergence at $z = 1$, and this is certainly not true since at $z = 1$ we obtain the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ which is known to diverge. So we are left with proving (*). For this, one needs to use the Cauchy criterion for uniform convergence and the continuity of the partial sums to conclude that if $|\sum_{k=m}^{n} z^n/k| < \epsilon$ for $|z| < 1$ then $|\sum_{k=m}^{n} z^n/k| \le \epsilon$ for $|z| \le 1$. For this approach, see whatever text you used for Math 320 (or equivalent).

To avoid using the Cauchy criterion, you can argue that if the series converges uniformly on $|z| < 1$ then, taking $\epsilon = 1$ in the definition, there is a N such that $|\sum_{k=N+1}^{\infty} z^k/k| < 1$ for all $|z| < 1$ and hence $|-\mathrm{Log}\,(1-\mathrm{z})| \le |\sum_{\mathrm{k}=1}^{\mathrm{N}} \mathrm{z}^{\mathrm{k}}/\mathrm{k}| + |\sum_{\mathrm{k}=\mathrm{N}+1}^{\infty} \mathrm{z}^{\mathrm{k}}/\mathrm{k}| \le \sum_{\mathrm{k}=1}^{\mathrm{N}} 1/\mathrm{k} + 1$ for all $|z| < 1$. That is, $-\mathrm{Log}\,(1-\mathrm{z})$ is uniformly bounded in $|z| < 1$. But this is clearly false since $\lim_{z\to 1} |-\mathrm{Log}\,(1-\mathrm{z})| = \infty$, as z approaches 1 from within the unit disk.

V.2-12. Let $f(z)$ be analytic in a domain D, and suppose $|f(z)| \le M$ for all $z \in D$. Show that for each $\delta > 0$ and $m \ge 1$, $|f^{(m)}(z)| \le m!M/\delta^m$ for all $z \in D$ whose distance from ∂D is at least δ. Use this to show that if $f_k(z)$ is a sequences of analytic functions on D that converges uniformly to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from ∂D.

Ans: Many of you were tempted to use the version of Cauchy's formula on p.114 of the text. But note that this has the hypothesis that ∂D is a finite union of piecewise smooth curves and that f extends smoothly to ∂D, neither of which is assumed here. (See pp.328-329 for some domains with boundaries that would not be a treat to integrate over.)

You really only need to use Cauchy's formula for a circle. Given $z \in D$ at distance at least δ from ∂D, the open disk $D(z, \delta)$ is contained in D and hence the circle $\gamma(t) = z + re^{it}$ lies in D for all $r < \delta$. Certainly $|f(z)| \le M$ on any such circle since this holds for all $z \in D$ and hence we can apply Cauchy's estimate (p.118) to conclude that $|f^{(m)}(z)| \le \dfrac{m!M}{r^m}$ for any $r < \delta$. Letting $r \to \delta-$, we get the desired estimate. (Notice that you can't directly use the circle of radius δ since one or more points of this circle may be on ∂D.)

Now to complete the proof, apply this inequality to $f_k - f$ and we have

$$|f_k^{(m)}(z) - f^{(m)}(z)| \le m!\delta^{-m} \sup_{\zeta \in D} |f_k(\zeta) - f(\zeta)|.$$

The right side of this inequality is a sequence of constants which converges to 0 as $k \to \infty$ by the definition of uniform convergence. Hence the left side converges uniformly to 0 for all z at distance δ from ∂D.

V.6-4. Define the **Bernoulli numbers** B_n by

$$f(z) = \frac{z}{2}\cot\frac{z}{2} = 1 - B_2\frac{z^2}{2!} + B_4\frac{z^4}{4!} - B_6\frac{z^6}{6!} + \dots.$$

Explain why there are no odd terms in this series. What is the radius of convergence of the series? Find the first five Bernoulli numbers.

Ans: I've renumbered the coefficients to agree with the accepted definition of the Bernoulli numbers as those defined by the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}.$$

Notice that $f(z) = g(z)/h(z)$ where $g(z) = \cos(z/2)$ and $h(z) = \sin(z/2)/(z/2)$ if $z \neq 0$ and $h(0) = 1$. Each of $g(z)$ and $h(z)$ is entire. We see this for $h(z)$ by observing that it has an everywhere convergent power series obtained by dividing the power series for $\sin(z/2)$ by $z/2$. The function $h(z)$ has simple zeros at $z = 2\pi n$ for every non-zero integer n. So $f(z)$ is analytic everywhere except at these points (and has simple poles at each of these points).

The Taylor series for $f(z)$ around $z = 0$ converges in the largest disk in which $f(z)$ is analytic. This is $|z| < 2\pi$ because of the poles at $z = \pm 2\pi$. Clearly this is the disk of convergence since $|f(z)| \to \infty$ as z approaches either of these points, so the radius of convergence is 2π.

The function $f(z)$ is an even function, i.e. $f(z) = f(-z)$ for all z in its domain, so by the uniqueness of the Taylor series expansion around $z = 0$, the coefficient of every odd ordered term must be zero. Otherwise the expansions of $f(z)$ and $f(-z)$ would not be identical, contradicting uniqueness.

To find the coefficients, the easiest way is to use the known series for $\sin(z/2)/(z/2)$ and $\cos(z/2)$ and equate terms in the expansion of the product on the left side to the corresponding terms in the right side of

$$(\sum_{n=0}^{\infty} (-1)^n \frac{B_{2n} z^{2n}}{(2n)!})(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^{2n}(2n+1)!}) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^{2n}(2n)!}.$$

This produces the sequence of equations

$$\sum_{k=0}^{n} B_{2k} \binom{2n}{2k} \frac{2^{2k}}{2n - 2k + 1} = 1,$$

for $n = 0, 1, 2, \ldots$, which give a recurrence relation for the B_{2n}. Since there are a lot of terms up to z^{10}, you will be forgiven if you resort to Maple to complete the calculation, once you have explained how it is done.