

An Extension of Seshadri's Identities for Brownian Motion

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Abstract

In this note we extend and clarify some identities in law for Brownian motion proved by V. Seshadri [8] using a new identity in law obtained by H. Matsumoto and M. Yor [6].

Key words and phrases: Brownian motion, Brownian functional, Asian option.
MSC2000: 60H05, 60J65.

1 Introduction

Let $B = (B_t)_{t \geq 0}$ be a one-dimensional standard Brownian motion with $B_0 = 0$. For a real constant ν and $t \geq 0$, set

$$A_t^{(\nu)} = \int_0^t \exp(2(B_s + \nu s)) ds \quad \text{and} \quad A_t = A_t^{(0)}.$$

Let \mathbf{e} be a standard exponential random variable independent from B and let L_t denote the local time of B at 0.

Recently, Matsumoto and Yor [6] proved the following result concerning the joint law of (A_t, B_t) .

Theorem 1.1 (Matsumoto-Yor) *For fixed $t > 0$, the following identity in law holds:*

$$(\mathbf{e} e^{-B_t} A_t, B_t) \stackrel{\text{law}}{=} (\cosh(|B_t| + L_t) - \cosh(B_t), B_t). \quad (1.1)$$

Our aim in this note is to show that this result helps us to find a nontrivial extension of some identities in law (see Theorem 2.1 below) first proved by V. Seshadri. Motivated by the aim to study the joint law of (A_t, B_t) Matsumoto and Yor [6] focused on the left-hand side of the identity (1.1). Here on the contrary we shall be mainly concerned with the right-hand side of this identity.

We also note that Donati-Martin et al. ([1], [2]) used the identity (1.1) in their computations to rederive the expression for the moments of $A_t^{(\nu)}$ earlier obtained by Dufresne [3].

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2 Main Result

We begin by recalling Seshadri's identities in law [8], following closely the presentation given by M. Yor [10].

Theorem 2.1 (Seshadri) *For $t \geq 0$ given and fixed, the following identities in law hold:*

$$(|B_t|L_t, L_t - |B_t|) \stackrel{\text{law}}{=} (t \mathbf{e}/2, B_t) \quad (2.1)$$

$$\left(\frac{2|B_t| + L_t}{2} \cdot \frac{L_t}{2}, |B_t| \right) \stackrel{\text{law}}{=} \left(\frac{2L_t + |B_t|}{2} \cdot \frac{|B_t|}{2}, L_t \right) \stackrel{\text{law}}{=} (t \mathbf{e}/2, |B_t|). \quad (2.2)$$

Remarks:

1) Seshadri's result (2.1) asserts that for a fixed $t > 0$, the two variables $|B_t|L_t$ and $L_t - |B_t|$ are mutually independent, and $|B_t|L_t$ is exponentially distributed with parameter $\lambda = 2/t$. A similar explanation goes for (2.2).

2) Note that $|B_t|$ and L_t play a symmetric role in (2.2).

To understand better the title of this note, we *reformulate* Matsumoto-Yor's result as follows

$$\left(\frac{1}{c} \sinh(\sqrt{c} \cdot \frac{2|B_t| + L_t}{2}) \cdot \sinh(\sqrt{c} \cdot \frac{L_t}{2}), |B_t| \right) \stackrel{\text{law}}{=} \left(\mathbf{e}/2 e^{-\sqrt{c}B_t} \int_0^t e^{2\sqrt{c}B_s} ds, |B_t| \right)$$

by means of the scaling property of B and simple hyperbolic identities, where $c > 0$. Letting now c tend to zero, the result 2.2 of Seshadri follows.

Similarly we have:

Theorem 2.2 *For $t \geq 0$ given and fixed, the following identity in law holds for all $c > 0$:*

$$\left(\frac{1}{c} \sinh(\sqrt{c}|B_t|) \sinh(\sqrt{c}L_t), L_t - |B_t| \right) \stackrel{\text{law}}{=} \left(\mathbf{e}/2 e^{-\sqrt{c}B_t} \int_0^t e^{2\sqrt{c}B_s} ds, B_t \right) \quad (2.3)$$

Proof: A scaling argument shows that only the case $c = 1$ need to be considered. As a well-known consequence of Itô-Tanaka's formula we have

$$|B_t| = \beta_t + L_t \quad \text{where} \quad \beta_t = \int_0^t \text{sgn}(B_t) dB_t$$

in terms of which (2.1) may be rewritten as

$$(|B_t| (|B_t| - \beta_t), -\beta_t) = (|B_t|L_t, L_t - |B_t|) \stackrel{\text{law}}{=} (t \mathbf{e}/2, L_t - |B_t|).$$

Recalling the well-known result concerning the joint law of (B_t, β_t) (see e.g. [9])

$$P(B_t \in dx; \beta_t \in dy) = \frac{2|x| - y}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{(2|x| - y)^2}{2t} \right\} \mathbf{1}_{\{y \leq |x|\}} dx dy$$

and consequently

$$(B_t \in dx; L_t \in du) = \frac{|x| + u}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(|x| + u)^2}{2t}\right\} \mathbf{1}_{\{u \geq 0\}} dx du$$

it follows that

$$P(B_t \in dx | \beta_t = y) = \frac{2|x| - y}{t} \exp\left\{-\frac{(2|x| - y)^2}{2t} + \frac{y^2}{2t}\right\} \mathbf{1}_{\{|x| \geq y\}} dx$$

for all $y \in \mathbf{R}$. Thus for every bounded Borel function f we have for all y by substituting $u = 2|x| - y$ and using another well-known hyperbolic identity that

$$\begin{aligned} E[f(\sinh(|B_t|) \cdot \sinh(L_t)) | |B_t| - L_t = y] &= E[f(\sinh(|B_t|) \cdot \sinh(|B_t| - y)) | \beta_t = y] \\ &= \int_{|y|}^{\infty} \frac{u}{t} \exp\left\{\frac{y^2}{2t}\right\} \exp\left\{-\frac{u^2}{2t}\right\} f((\cosh(u) - \cosh(y))/2) du \\ &= \int_0^{\infty} \frac{v + |y|}{t} \exp\left\{\frac{y^2}{2t}\right\} \exp\left\{-\frac{(v + |y|)^2}{2t}\right\} f((\cosh(v + |y|) - \cosh(|y|))/2) dv \\ &= E[f((\cosh(|B_t| + L_t) - \cosh(|B_t|))/2) | B_t = y] \end{aligned}$$

which by (1.1) equals

$$E[f(e/2 e^{-B_t} A_t) | B_t = y] = E\left[f\left(e/2 e^{-B_t} \int_0^t e^{2B_s} ds\right) | B_t = y\right].$$

Altogether we have proved that for every bounded Borel function f the following identity

$$E[f(\sinh(|B_t|) \cdot \sinh(L_t)) | |B_t| - L_t = y] = E\left[f\left(e/2 \cdot e^{-B_t} \int_0^t e^{2B_s} ds\right) | B_t = y\right]$$

is true for all $y \in \mathbf{R}$ from which the result follows observing that B_t and $|B_t| - L_t$ are identically distributed.

3 Moments of $A_t^{(\nu)}$

In this section we compute moments of certain exponential Brownian functionals connected to the evaluation of Asian options. The techniques used are very simple compared to former proofs (see e.g. [4], [11]) of the same results and furthermore they can be applied in more general situations.

We shall compute all moments of the random variable $\int_0^t \exp((B_s + \nu s)) ds$ i.e. we shall determine the numbers

$$E\left[\left(\int_0^t \exp((B_s + \nu s)) ds\right)^n\right]$$

for all $n \geq 1$ with $\nu \in \mathbf{R}$ and $t > 0$ given and fixed.

The computation will be based on the following well-known simple fact:

Lemma 3.1 *If (M_t) is a non-negative right-continuous martingale and (C_t) a continuous increasing process such that $C_0 \equiv 0$, then*

$$E\left[\int_0^t M_s dC_s\right] = E[M_t C_t]$$

for all $t \geq 0$.

Since the arguments apply not only to the Brownian motion we will assume that we are given a probability space (Ω, \mathcal{F}, P) and a right-continuous process $X = (X_s)_{0 \leq s \leq T}$ defined on (Ω, \mathcal{F}, P) that starts at 0 and has stationary independent increments (shortly called a Lévy process).

Here we assume that the Lévy exponent ψ of X defined by

$$E[\exp(aX_t)] = \exp(t\psi(a))$$

for $t \in [0, T]$ and $a \in \mathbf{R}$ is finite. In the case when X is a standard Brownian motion we have $\psi(a) = a^2/2$.

Straightforward calculations show that

$$(M_s) := (\exp(X_s - s\psi(1)))_{0 \leq s \leq T}$$

is a non-negative right-continuous martingale starting at 1 and that $(X_s)_{0 \leq s \leq T}$ is a Lévy process on $[0, T]$ under \tilde{P} , where \tilde{P} denotes the probability measure on (Ω, \mathcal{F}) defined by

$$d\tilde{P} := M_T dP.$$

The corresponding Lévy exponent $\tilde{\psi}$ is easily seen to be given by

$$\tilde{\psi}(a) = \psi(a+1) - \psi(1) \quad (\text{for } a \in \mathbf{R}).$$

Theorem 3.1 *Let $(X_s)_{0 \leq s \leq T}$ be a Lévy process on $[0, T]$ with exponent ψ . Define for $n \geq 1$, $t \in [0, T]$ and $v \in \mathbf{R}$*

$$C_n(t, v, \psi) = E\left[\left(\int_0^t \exp(X_s + vs) ds\right)^n\right]$$

Then for $n \geq 2$ we have the following recursive relation:

$$C_n(t, v, \psi) = n \int_0^t C_{n-1}(s, v, \tilde{\psi}) \exp(\psi(1)s + vs) ds \quad (3.1)$$

for all $t \in [0, T]$ and all $v \in \mathbf{R}$.

Proof: Using Lemma 3.1 and the integration by parts formula we obtain for $n \geq 2$:

$$\begin{aligned} C_n(t, v, \psi) &= E\left[\left(\int_0^t \exp(X_s + vs) ds\right)^n\right] \\ &= n \cdot E\left[\int_0^t \left(\int_0^s \exp(X_u + vu) du\right)^{n-1} \exp(X_s + vs) ds\right] \\ &= n \cdot E\left[\int_0^t M_s \left(\int_0^s \exp(X_u + vu) du\right)^{n-1} \exp(\psi(1)s + vs) ds\right] \\ &= n \cdot E\left[M_t \int_0^t \left(\int_0^s \exp(X_u + vu) du\right)^{n-1} \exp(\psi(1)s + vs) ds\right] \\ &= n \int_0^t \tilde{E}\left[\left(\int_0^s \exp(X_u + vu) du\right)^{n-1}\right] \exp(\psi(1)s + vs) ds \end{aligned}$$

i.e.

$$C_n(t, v, \psi) = n \int_0^t C_{n-1}(t, v, \tilde{\psi}) \exp(\psi(1)s + vs) ds.$$

□

Using induction in (3.1) the recursive formula for $(C_n(t, v, \psi))_{n \geq 1}$ can be found, and in the Brownian case we obtain the following closed expression.

Corollary 3.1 *For all $n \geq 1$ and $t \geq 0$ we have:*

$$C_n(t, v, a^2/2) = E\left[\left(\int_0^t \exp(B_s + vs) ds\right)^n\right] = n! \sum_{j=0}^n \frac{1}{\prod_{i=0, i \neq j}^n (a_j^v - a_i^v)} \exp(ta_j^v)$$

where for each $0 \leq i \leq n$

$$a_i^v = \psi(i) + iv = \frac{i^2}{2} + iv.$$

Remark:

A negative answer to the long time unsolved question of whether or not the law of $A_t^{(\nu)}$ is determined by its moments has recently been given by A. Nikeghbali [7].

Acknowledgement.

The authors would like to thank G. Peskir for interesting discussions.

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