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ℓ-modules, Riesz spaces, and the pointfree Kakutani duality

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I dedicate my thesis to

Immanuel Kant

the lofty and majestic philosopher, on the occasion of his 200th death anniversary.
Also, I dedicate to my kindly wife

Farzaneh

and my sympathetic sister

Zohreh

Abstract

There are many classical results proved using the Axiom of Choice. Using the notions of pointfree topology we can state and prove the choice free versions of some of these results. Our aim in this thesis is to investigate the choice free version of the Kakutani duality. To do so, we use the pointfree versions of some classical spectra.

The three well-known spectra usually considered are, Brumfiel, Keimel, and the maximal spectrum. The pointfree versions of these spectra have been studied for f-rings by Banaschewski, and the last two spectra will be studied in this thesis for ℓ-modules and Riesz spaces.

Finally, in the last chapter we study the pointfree version of the representation of the real Riesz maps.

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Contents

Introduction

There are two Theorems that characterize the order algebraic structures real C^*-algebra (See [16] for the definition) and M-space (see Definition 1.3.10) proved by Stone [20, 21] and Kakutani [19], respectively.

The Stone-Gelfand Theorem says that every C^*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X [16]. B. Banaschewski in [2, 6] provides the pointfree version of the Stone-Gelfand Theorem. He uses the complete Archimedean strong bounded f-rings to correspond to the C^*-algebras and proves that they are equivalent (see [6]). The pointfree version of the Stone-Gelfand Theorem says that every complete bounded strong f-ring is lattice isomorphic to $C(L)$ for some compact completely regular frame L. Moreover, if A is an Archimedean strong bounded f-ring then A is lattice isomorphic to $C(\mathcal{M}A)$ where $\mathcal{M}A$ is the frame of closed ℓ-ideals of A (see [2, 6]).

This view of the Stone-Gelfand Theorem by B. Banaschewski, helped us to give the pointfree version of the Kakutani Theorem in this thesis. Our final aim in this thesis is to give the choice free version of the Kakutani Theorem. Our tools are the notions from pointfree topology and the objects to be used are the frames of ℓ-ideals and closed ℓ-ideals objects.

We start the study these notions in chapter 3, and prove our final theorem in chapter 4. To do so, we correspond the complete bounded Archimedean Riesz spaces to the M-spaces and prove that they are equivalence (Proposition 4.5.9). Also, we give the pointfree version of the Stone-Weierstrass Theorem (Theorem 4.4.7) for Riesz spaces (see [18] for the classical version of the theorem).

In chapter 1, we provide the background material necessary to read this thesis.

In chapter 2, we study ℓ-modules, in three sections. In the first section we define f-modules, strong ℓ-modules, and supper f-modules, and study the relations between them. We will see that they all give equational classes. In the second section we study the ℓ-modules of fractions. We prove that the module of fractions inherits the (f) equations but not necessarily the (ℓ) equations from the base module, and give the counterexample in Remark 2.2.4. In the third section we study the topological structures of ℓ-modules and Riesz spaces. There are two topological structures which are studied, the uniform topology for ℓ-modules, and the topology given from the order limit sequence for Riesz spaces.

In chapter 3, we study the pointfree spectra of ℓ-modules, the pointfree Keimel spectrum, the pointfree maximal spectrum, and the spectra of closed ℓ-ideals. In the first section we prove that the compactness of $\mathcal{L}(M)$ is equivalent to the boundedness of the ℓ-module M, and for every f-module M, $\mathcal{L}(M)$ is coherently normal. Also, we prove that $\mathcal{L}(M)$ is isomorphic to $\mathcal{L}(S^{-1}M)$, for almost every multiplicative subset S of M. This is used to prove the complete regularity of pointfree maximal spectrum $S\mathcal{L}(M)$ in the second section. Corresponding to the two above mentioned topological structures we have the two spectra of the closed ℓ-ideals. We study these spectra in the third section.

The final aim of the thesis is studied in Chapter 4. The pointfree Kakutani duality is studied in the final section. In the first section we prove the functoriality of the spectra studied in the previous chapter. The functoriality of the Riesz space of the continuous real functions is studied in the next section, and the adjunction between these functors is given in the third section (Theorem 4.3.1). In section 4 we study the generalized pointfree version of the Stone-Weierstrass Theorem (Theorem 4.4.7), to prove the Kakutani duality.

In chapter 5, we present the pointfree version of a representation theorem in analysis. This theorem is a classical representation theorem for real Riesz maps on $C(X)$, consisting of real-valued continuous functions on a compact Hausdorff space X, which assigns to each real Riesz map $\phi : C(X) \to \mathbb{R}$ with $\phi(1) = 1$ a point $x \in X$ such that $\phi = \hat{x}$, where $\hat{x}$ is given by $\hat{x}(\alpha) = \alpha(x)$ for $\alpha \in C(X)$ (see [16], p. 163). The proof of the classical representation uses the Axiom of Choice and does not explicitely give the point x.

Here we replace the compact Hausdorff space X by a compact completely regular frame M and the map $\hat{x}$ by $\tilde{p}$, where p is a prime element of M. We define $\tilde{p}$ by Dedekind cuts which are used to define the real number $\tilde{p}(\alpha)$ for $\alpha \in C(M)$. Then we prove that each real Riesz map $\phi : C(M) \to \mathbb{R}$ is of the form $\phi(1)\tilde{p}$, where $p = \bigvee coz(ker\phi)$ (Lemma 5.2.2). We then give the exact relation between the real Riesz maps, the prime elements, and the prime ideals which are in $Fix(\eta)$ (Theorem 5.2.4). If M is not completely regular we can find a completely regular frame K_M for which $C(M) \simeq C(K_M)$ and we have a correspondence between the real Riesz maps and $\sum K_M$ (Theorem 5.2.5). Next we study the relation between $\hat{}$ and $\tilde{}$ in Proposition 5.2.6, and see when $\tilde{p}$ and $\tilde{q}$ are equal.

Finally, we study the relationship between the primes elements and the cozero elements. The fact that there is no nonunit cozero element greater than each prime element is a result of this investigation. (Corollary 5.2.7).

Chapter 1

Preliminaries

In this chapter, we provide the background needed to read this thesis.

1.1 Categories

In this section we give a brief review of categorical notions. (see [1, 17]).

1.1.1 Definition A *category* $\mathcal{A}$ consists of,

i) A class $\mathcal{O}$, denoted by $Ob\mathcal{A}$, whose members are called *objects.*

ii) A class $\mathcal{M}$, denoted by $Mor\mathcal{A}$, whose members are called *morphisms* or *arrows.*

iii) Two maps from $\mathcal{M}$ to $\mathcal{O}$, given by the assignments $f \mapsto dom(f)$, called the *domain of* f, and $f \mapsto cod(f)$, called the *codomain of* f. If $A = dom(f)$ and $B = cod(f)$, then we write $f : A \to B$

iv) A map from $P = \{(f, g) : f, g \in \mathcal{M}, dom(f) = cod(g)\}$ into $\mathcal{M}$, given by $(f, g) \mapsto f \circ g$, called the *composite of* f *and* g.

v) For each object A, there exists an arrow $id_A : A \to A$ (or 1_A), called the *identity arrow on* A. This is subject to (1) $(f \circ g) \circ h = f \circ (g \circ h)$, (2) $f \circ 1_A = f$ and $1_A \circ g = g$, whenever the composites are defined.

Further, we postulate that for each pair of objects A, B the class $\{f \in \mathcal{M} : dom(f) = A, cod(f) = B\}$ is a set. We denote it by $Hom_{\mathcal{A}}(A, B)$.

Some of the categories which we will be concerned with are:

Top: the category whose objects are all topological spaces and the arrows are all continuous maps.

Frm: the category whose objects are all frames and the frame maps.

Rsz: the category whose objects are all Riesz space and the arrows are all Riesz map.

1.1.2 Definition Let $\mathcal{A}$ be a category. A category $\mathcal{B}$ is said to be a *subcategory* of $\mathcal{A}$, if $Ob\mathcal{B} \subseteq Ob\mathcal{A}$, and $Mor\mathcal{B} \subseteq Mor\mathcal{A}$. The domain and the codomain and the composition functions of $\mathcal{B}$ are restrictions of the corresponding functions of $\mathcal{A}$, and for each $A \in Ob\mathcal{B}$, the identity arrow on A is in $Mor\mathcal{B}$. A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is said to be a *full subcategory* if for every objects A, B of $\mathcal{B}$, $Hom_{\mathcal{B}}(A, B) = Hom_{\mathcal{A}}(A, B)$.

Let $\mathcal{A}$ be a category. By the *opposite* (or *dual*) *category* of $\mathcal{A}$ we mean a category (denoted by $\mathcal{A}^{op}$) with $Ob\mathcal{A}^{op} = Ob\mathcal{A}$, $Mor\mathcal{A}^{op} = Mor\mathcal{A}$ and by interchanging the role of dom and cod in such a way that for each morphism

$f : A \to B$ in $\mathcal{A}$ we have one and only one morphism $f^{op} : B \to A$ in $\mathcal{A}^{op}$, and when the composite $g \circ f$ is defined in $\mathcal{A}$, then the composite $f^{op} \circ g^{op}$ is defined in $\mathcal{A}^{op}$ with $f^{op} \circ g^{op} = (g \circ f)^{op}$. Notice that $(\mathcal{A}^{op})^{op} = \mathcal{A}$.

1.1.3 Definition Let $\mathcal{A}$ be a category and $f : A \to B$ be a morphism in $\mathcal{A}$.

i) f is called a *monomorphism* (or a *monic*) if for any pair of morphisms $g, h : C \to A$ in A, $f \circ g = f \circ h$ implies $g = h$.

ii) f is called an *epimorphism* (or an *epic*) if for any pair of morphisms $g, h : B \to C$ in A, $g \circ f = h \circ f$ implies $g = h$.

iii) f is called a *retraction* if there exists a morphism $g : B \to A$ such that $f \circ g = 1_B$. In this case we say that B is a *retract* of A.

iv) f is called a *coretraction* (or a *section*) if there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$.

v) f is called an *isomorphism* if it is both a retraction and a section. In this case, we say that A, B are isomorphic and write $A \simeq B$.

1.1.4 Definition Let $\mathcal{A}, \mathcal{B}$ be categories. A *functor* (or more precisely a *covariant functor*) F from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functions $F_0 : Ob\mathcal{A} \to Ob\mathcal{B}$ and $F_1 : Mor\mathcal{A} \to Mor\mathcal{B}$, such that
i) if $f : A \to B$ is in $Mor\mathcal{A}$ then $F_1 f : F_0 A \to F_0 B$ is in $Mor\mathcal{B}$,
ii) $F_1(1_A) = 1_{F_0 A}$, for each $A \in Ob\mathcal{A}$,
iii) $F_1(f \circ g) = (F_1 f) \circ (F_1 g)$, if $f \circ g$ is defined in $\mathcal{A}$.
In this case we write $F : \mathcal{A} \to \mathcal{B}$. Also both F_0 and F_1 are usually denoted by

the same letter F.

In the above definition, if we substitute (iii) with
(iii') $F_1(f \circ g) = (F_1 g) \circ (F_1 f)$, when $f \circ g$ is defined in $\mathcal{A}$, then F is called a *contravariant functor.*

Also, corresponding to any subcategory of a category, the pair of inclusion functions gives the *inclusion functor*.

Notice that we can compose any two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathbf{C}$ by composing the functions F_0, G_0 and F_1, G_1.

1.1.5 Definition Let $F : \mathcal{A} \to \mathcal{B}$ be a (covariant) functor.

i) F is said to be *faithful* if for each pair of morphisms $f, g : A \to B$ in $\mathcal{A}$, $Ff = Fg$ implies $f = g$.

ii) F is said to be *full* if for each pair of objects A, B in $\mathcal{A}$ and each morphism $g : FA \to FB$ in $\mathcal{B}$ there exists a morphism $f : A \to B$ in $\mathcal{A}$ such that $Ff = g$.

iii) F is said to be an *isomorphism* if F is full and faithful and F_0 is a bijection; or equivalently there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that $F \circ G = I_{\mathcal{B}}, G \circ F = I_{\mathcal{A}}$.

iv) F is said to be an *equivalence* if F is full and faithful and for each object B in $\mathcal{B}$ there exists an object A in $\mathcal{A}$ such that $FA \simeq FB$. In this case we say that $\mathcal{A}$, $\mathcal{B}$ are *equivalent*.

1.1.6 Definition Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation* $\tau : F \to G$ is a function $\tau : Ob\mathcal{A} \to Mor\mathcal{B}$ assigning to each A in $\mathcal{A}$ a morphism $\tau_A : FA \to GA$ in $\mathcal{B}$ such that for every morphism $f : A \to B$ in $\mathcal{A}$, $(Gf)(\tau_A) = \tau_B(Ff)$. In this case we denote τ by $(\tau_A)_{A\in\mathcal{A}}$, and call each τ_A a component of τ.

A *natural isomorphism* is a natural transformation whose components are all isomorphisms.

Notice that a functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence if and only if there is a functor $G : \mathcal{B} \to \mathcal{A}$ with natural isomorphisms from $F \circ G$ to $I_{\mathcal{B}}$ and from $G \circ F$ to $I_{\mathcal{A}}$.

1.1.7 Definition Let $G : \mathcal{B} \to \mathcal{A}$ be a functor and $A \in Ob\mathcal{A}$. A *universal arrow* from A to G is a morphism $u : A \to GB$ in $\mathcal{A}$ with B in $\mathcal{B}$ such that for each $D \in Ob\mathcal{B}$ and each morphism $f : A \to GD$ in $\mathcal{A}$ there exists a unique morphism $\overline{f} : B \to D$ in $\mathcal{B}$ with $(G\overline{f}) \circ u = f$.

Dualizing the above definition we get the notion of a couniversal arrow from A to G.

1.1.8 Definition Let $G : \mathcal{B} \to \mathcal{A}$ be a functor. A *left adjoint* to G is a functor $F : \mathcal{A} \to \mathcal{B}$ such that for each $A \in \mathcal{A}, B \in \mathcal{B}$ there is an isomorphism $\alpha_{A,B} : Hom_{\mathcal{B}}(FA, B) \to Hom_{\mathcal{A}}(A, GB)$ which is natural in A and B. In this case we write $F \dashv G$ and say that (F, G, α) is an *adjunction.* Also, G is said to be a *right adjoint* to F.

1.1.9 Theorem Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors. Then, the following are equivalent:

a) F is a left adjoint to G

b) There exists a natural transformation $\eta : I_{\mathcal{A}} \to G \circ F$, called the *unit* or the *front adjunction,* such that for each $A \in Ob\mathcal{A}, \eta_A : A \to (G \circ F)A$ is a universal arrow from A to G.

c) There exists a natural transformation $\varepsilon : F \circ G \to I_{\mathcal{B}}$, called the *counit* or the *back adjunction,* such that for each $B \in Ob\mathcal{B}, \varepsilon_B : (F \circ G)B \to B$ is a couniversal arrow from B to F.

d) There are natural transformations $\eta : I_{\mathcal{A}} \to G \circ F$ and $\varepsilon : F \circ G \to I_{\mathcal{B}}$ such that $\varepsilon_F \circ F\eta = I_F$ and $G\varepsilon \circ \eta_G = I_G$, where I_F, I_G are identity natural transformations on F, G, respectively, and $F\eta$ is the natural transformation $F \to F \circ G \circ F$ whose components are $(F\eta_A)_{A \in Ob\mathcal{A}}$, and $\varepsilon_F : F \circ G \circ F \to F$ is the natural transformation whose components are $(\varepsilon_{FA})_{A \in Ob\mathcal{A}}$, and $G\varepsilon, \eta_G$ are similarly defined.

1.1.10 Proposition *Suppose that $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors such that $F \vdash G$ and F is faithful. Then F preserves epimorphisms.*

1.2 Universal Algebra

Universal algebras are useful to pure as well as applicable mathematics, specially to computer sciences. In this section we bring universal algebra because of Birkhoff Theorem.

1.2.1 Definition Let n_t be a nonnegative integer and A be a nonempty set. An n_t-*ary operation* on A is a function λ_t from A^{n_t} to A, where A^{n_t} is the set of n_t-tuples of elements of A, for $n_t > 0$, and A^0 is a singleton set. If λ_t is an n_t-ary operation on A, then n_t is called the *arity* (or *rank*) of λ_t.

A *language* (or *type*) of algebras is a set τ such that a nonnegative integer n_t and a function symbol λ_t is assigned to each member t of τ.

If τ is a language of algebras, then a *universal algebra* (or simply, an *algebra*) $\mathcal{A}$ of type τ is an ordered pair (A, λ), where A is a nonempty set and $\lambda = (\lambda_t^A : t \in \tau)$ is a family of operations on A, where λ_t is an n_t-ary operation on A.

If there is no confusion we denote λ_t^A by λ_t and use the same notation λ for every algebra $\mathcal{A}$.

1.2.2 Definition Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type τ. A function $f : A \to B$ is called a *homomorphism* if for every $t \in \tau$ and all $a_1, ..., a_{n_t} \in A$,

$$f(\lambda_t(a_1, \cdots, a_{n_t})) = \lambda_t(f(a_1), \cdots, f(a_{n_t}))$$

A subset A of B is called *subalgebra* of B if the inclusion map $\iota : A \hookrightarrow B$ is homomorphism.

1.2.3 Definition Let θ be an equivalence relation on a set A. Then, θ is said to be a *congruence* of an algebra $\mathcal{A}$ if for each $t \in \tau$, $\lambda_t(a_1, \cdots, a_{n_t})\theta\lambda_t(b_1, \cdots, b_{n_t})$, whenever $a_1, \cdots, a_{n_t}$, $b_1, \cdots, b_{n_t} \in A$ satisfy $a_i\theta b_i$, for $i = 1, \cdots, n_t$.

Let θ is an equivalence relation of algebra A. the natural quotient map $\gamma : A \to A/\theta$ is homomorphism if θ is a congruence.

1.2.4 Definition An algebra A is a *subdirect product* of an indexed family $(A_i)_{i\in I}$ of algebras if

(i) $A \leq \prod_{i\in I} A_i$ and

(ii) $\pi_i(A) = A_i$ for each $i \in I$.

An embedding $\alpha : A \to \prod_{i\in I} A_i$ is *subdirect* if $\alpha(A)$ is a subdirect product of the A_i.

1.2.5 Example If $\theta_i \in Con(A)$ for $i \in I$ and $\bigcap_{i\in I} \theta_i = \Delta$, then the natural homomorphism

$$v : A \to \prod_{i\in I} A/\theta_i$$

defined by

$$v(a) = (a/\theta_i)_{i\in I}$$

is a subdirect embedding.

1.2.6 Definition An algebra A is called *subdirectly irreducible* if for every subdirect embedding

$$\alpha : A \to \prod_{i\in I} A_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : A \to A_i$$

is an isomorphism.

1.2.7 Theorem (Birkhoff's Theorem) *Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebra (with are homomorphic image of A).*

1.2.8 Remark The axiom of choice implies the Birkhoff's theorem and it is obviously that this theorem implies the axiom of choice, so it is equivalent with the axiom of choice.

1.3 Ordered algebraic structures

1.3.1 Ordered group A group G together with a partial order $\leq$ is called an *ordered group* if $x \leq y$ implies $axb \leq ayb$ for all $a, b \in G$.

Note that in any ordered group G every translation $x \rightsquigarrow axb, a, b \in G$ is an order automorphism, and its inverse is given by $x \rightsquigarrow a^{-1}xb^{-1}$.

The set $G^+ = \{x \in G : x \geq 0\}$ is called the *positive cone* of G.

1.3.2 Proposition *Let G be a non-trivial ordered group.*
(1) G cannot have universal bounds with respect to the partial ordered set $(G,\leq)$.
(2) If $(G, \leq)$ is an updirect set (in particular a lattice) then G cannot be finite and cannot have any element with finite order.

1.3.3 Definition An *ℓ-group* G is an ordered group which is a lattice with its order.
If G is a lattice we define the following:

$$a^+ = a \vee 0, \quad a^- = (-a) \vee 0, \quad |a| = a \vee (-a)$$

1.3.4 Theorem (Properties of ℓ-groups) *Let G be an ℓ-group.*
(1) Distributive laws: $a(x \vee y)b = axb \vee ayb$, $a(x \wedge y)b = axb \wedge ayb$, *and*

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y), \quad a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$$

(2) $a^+ \wedge a^- = 0$, $|a| \geq 0$, $|a| = a^+a^-$, and $a(a \wedge b)^{-1}b = b \vee a$. In particular $a = a^+(a^-)^{-1}$.

(3) $|(a \vee x)(b \vee y)^{-1}| \leq |a(b)^{-1}|$ *and dually.*

(4) *G is an (additive) abelian group if and only if the triangle inequality holds; that is,* $|a + b| \leq |a| + |b|$.

(5) *If G is an (additive) abelian group then for every* $a, b, c \in G^+$

$$a \wedge (b + c) \leq a \wedge b + a \wedge c$$

If $a \leq b + c$ *then there exist* $0 < b_1 < b$ *and* $0 < c_1 < c$ *such that* $a = b_1 + c_1$.

1.3.5 Definition An ℓ-group G is called *Archimedean* if $a^n \leq b$ for all $n \in \mathbb{N}$ implies $a \leq 0$.

An ℓ-group which is a chain is called an *ordered ℓ-group*. By Birkhoff,s Theorem 1.2.7 in universal algebra, which is a result of the Axiom of Choice, we have:

1.3.6 Theorem *Any abelian ℓ-group is a subdirect product of ordered ℓ-groups. Also, every equation that is true for ordered ℓ-groups is true for all ℓ-groups.*

1.3.7 Definition A Riesz space is a vector space E over $\mathbb{Q}$ with a partial order such that $(E, +)$ is an ℓ-group and for every rational number $r > 0$; $ra \leq rb$ whenever $a \leq b$.

1.3.8 Let E be a Riesz space. For every $r \in \mathbb{Q}$ and $a, b \in E$, $|ra| = |r||a|$, and for every $r \leq 0$ $r(x \wedge y) = rx \wedge ry, r(x \vee y) = rx \vee ry$.

1.3.9 Definition A *Banach lattice* is a real Banach space which is also a

Riesz space in which the order and the norm are related by
$|a| \leq |b|$ implies $\|a\| \leq \|b\|$.

1.3.10 Definition We say that a Banach lattice E is a *(Kakutani) M-Space* if its unit ball has a greatest element 1_E (and hence a least element -1_E).

In this case the norm on E is definable from the partial order, In fact $\|a\| = inf\{q > 0 : |a| \leq q1_E\}$, because

$$|a| \leq q1_E \Rightarrow \|a\| = \|(|a|)\| \leq \|q1_E\| = q\|1_E\| = q$$

On other hand $\|\frac{|a|}{\|a\|}\| = \frac{\|a\|}{\|a\|} = 1$ and then $\frac{|a|}{\|a\|} \leq 1_E$ hence $|a| \leq \|a\|1_E$. It also follows that $\|a \vee b\| = max\{\|a\|, \|b\|\}$ (See [18], Proposition 1.2.13). Note that the last equality is sufficient for a Banach lattice to be an M-space.

Now we recall two theorems by Kakutani whose proofs uses the Axiom of Choice.

1.3.11 Theorem (Kakutani) *For every M-space E there is a compact Hausdorff topological space X such that E is a lattice as well as a norm isomorphic to $C(X)$.*

1.3.12 Kakutani Duality *The contravariant functor* **C** *from the category of compact Hausdorff spaces to the category of M-spaces with lattice-norm preserving linear maps (M-morphism) is a dual equivalence.*

1.3.13 Definition An *ordered ring* is a ring A which is also a poset under a partial order $\leq$ in which $(A, +)$ is a ordered group and for every $a, b \in A^+$, $ab \geq 0$.

The ordered ring A is called an *ℓ-ring* if $(A, +)$ is an ℓ-group. And it is called an *f-ring* if for every $c \in A^+$, $c(a \vee b) = ca \vee cb$.

1.4 Pointfree Topology

Here we recall some definitions and results from the literature on frames. For more details see the appropriate references given in the thesis.

1.4.1 Definition A *frame* is a complete lattice L which satisfies the arbitrary distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$, for all $x \in L$ and arbitrary subsets $S \subseteq L$. A *frame map* $h : L \to M$ is a lattice morphism preserving arbitrary joins, the unit *top element* e, and the zero (bottom element) 0 of L. The resulting category is denoted by **Frm**.

1.4.2 Example As the most familiar examples of frames we have the finite distributive lattices, the complete chains, the complete Boolean algebras, and the lattice $O(X)$ of open subsets of a topological space X.

1.4.3 Definition Let L be a frame. We say that a is *rather below* b, and write $a \prec b$, if there exists a *separating element* s of L with $a \wedge s = 0$ and $s \vee b = e$.

Notice that $a \prec b$ if and only if $a^* \vee b = e$, where $a^* = \bigvee\{y : y \wedge a = 0\}$ is the pseudocomplement of a.

A frame L is called *regular* if each of its elements is a join of elements rather below it.

1.4.4 Definition An element a of a frame M is said to be *completely below* b, written as $a \prec\!\prec b$, if there exists a sequence $(c_q)_{q \in [0,1] \cap \mathbb{Q}}$ where $c_0 = a,\ c_1 = b$, and if $p < q$ then $c_p \prec c_q$. A frame M is called *completely regular* if each $a \in M$ is a join of elements completely below it.

1.4.5 Definition An element $a \in L$ is called *compact* if $a = \bigvee S$ implies $a = \bigvee T$ for some finite $T \subseteq S$. A frame L is called: *compact* whenever its unit e is compact; *algebraic* if every element of L is a join of compact elements; *coherent* if it is compact, algebraic, and for compact elements a and b, $a \wedge b$ is compact.

1.4.6 Definition A frame L is called *normal* if $a \vee b = e$ implies that there exist u and v in L such that $a \vee u = e = b \vee v$ and $u \wedge v = 0$. L is said to be *coherently normal* if it is coherent and for each compact element $c \in L$, $\downarrow c$ is normal.

1.4.7 Definition A frame L is called *subfit* if $a < b$ implies $a \vee c < e = b \vee c$ for some $c \in L$.

1.4.8 Definition A frame map $h : M \to L$ is called *dense* if $h(x) = 0$ implies $x = 0$; *codense* if $h(x) = e$ implies $x = e$; and a *quotient map* if it is onto.

1.4.9 Lemma *Suppose that $f : L \to M$ is a codense quotient, if L is compact (normal) then M is compact (normal).*

1.4.10 Definition A *nucleus* is a map $n : L \to L$ on a frame L satisfying:

(N1) $x \leq n(x)$ for all $x \in L$; (N2) $x \leq y$ implies $n(x) \leq n(y)$; (N3) $n^2(x) = n(x)$ for all $x \in L$; (N4) $n(x \wedge y) = n(x) \wedge n(y)$.

For any nucleus n on a frame L, the closure system $Fix(n) = \{a \in L : n(a) = a\}$ is a frame such that the map $n : L \to Fix(n)$ is a quotient map.

1.4.11 Lemma *Suppose that* $\eta : L \to L$, $\kappa : M \to M$ *are nucleui, and* $f : L \to M$ *is a frame homomorphism such that* $f\eta(x) \leq \kappa f(x)$ *for all* $x \in L$. *Then there exists a frame homomorphism* $\tilde{f} : \eta(L) \to \kappa(M)$ *such that* $\tilde{f} \circ \eta = \kappa \circ f$.

Proof Define $\tilde{f}(\eta(x)) = \kappa(f(x))$. It is enough to show that $\tilde{f}$ is well defined. Let $x, y \in L$ such that $\eta(x) = \eta(y)$. We have

$$\kappa(f(x)) \leq \kappa(f(\eta(x))) = \kappa(f(\eta(y))) \leq \kappa(\kappa(f(y))) = \kappa(f(y))$$

Similarly, it is proved that that $\kappa(f(y)) \leq \kappa(f(x))$. Hence $\tilde{f}(\eta(x)) = \tilde{f}(\eta(y))$, and so $\tilde{f}$ is well defined and the proof is complete. $\square$

1.4.12 Note For any compact frame L one has the so called *saturation nucleus* s on L which plays an important role in this thesis and is defined as follows.

Recall that for any $x, a \in L$, x is called *a-small* if $x \vee y = e$ implies $a \vee y = e$, for all $y \in L$. *The saturation* nucleus is defined by $s(a) = \bigvee\{x \in L | x$ is a-small $\}$. Note that if L is compact then $s(a)$ is a-small and hence it is the largest a-small element of L.

1.4.13 Proposition [2,5] *The map* $s : L \to SL = Fix(s)$ *is the unique smallest codense quotient of* L *(Banaschewski-Harting [8]). Further* SL *is subfit and* $s : L \to SL$ *is also the unique codense subfit quotient of* L. *Finally,* SL *is* *compact since s is codense.*

1.4.14 Definition An element $p \in M$ is called *prime* if $p < e$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m \in M$ is called *maximal* if $m < e$ and $m \leq x \leq e$ implies $m = x$ or $x = e$.

Note that every maximal element is prime.

1.4.15 For each frame L there is a topological space called the *spectrum* of L and is denoted by $\sum L$. $\sum L$ consists of all prime elements of L, and $\sum_a = \{p \in \sum M | a \not\leq p\}$ $(a \in L)$ as its open sets. For a frame map $h : L \rightarrow M$, $\sum h : \sum M \rightarrow \sum L$ takes $p \in \sum M$ to $h_*(p) \in \sum L$, where $h_* : M \rightarrow L$ is the *right adjoint* of h, characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. Note that h_* preserves primes and arbitrary meets, and $h_*(b) = \bigvee\{x \in L : h(x) \leq b\}$.

1.4.16 By notation of 1.4.15, $L \mapsto \sum L$ and $h \mapsto \sum h$ defines a functor from the category of **Frm** to the category of $\mathbf{Top^{op}}$. The subspace $\sum L$ of all maximal element of L is denoted by $Max(L)$.

1.4.17 Lemma [5] *For any compact frame L, $Max(L) = Max(SL) = \sum(SL)$.*

1.4.18 Lemma [2,5] *For any normal coherent frame L, the following are equivalent*:

(i) *$L = SL$.*

(ii) *L is subfit.*

(iii) *L is regular.*

1.4.19 A frame L is called *spatial* if it is isomorphic to the frame of open sets of some topological space; in fact isomorphic to the frame of open sets of $\sum L$, the spectrum of L.

1.4.20 Recall from [4] that the frame $\mathcal{R}$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$

(R3) $(p, q) = \bigvee\{(r, s) | p < r < s < q\}$

(R4) $e = \bigvee\{(p, q) |$ all $p, q\}$.

Note that the pairs (p, q) in $\mathcal{R}$ and the open intervals $(p, q) = \{x \in \mathbb{R} : p < x < q\}$ in the frame $O\mathbb{R}$ of open sets have the same role; in fact there is a frame isomorphism $\lambda : \mathcal{R} \to O\mathbb{R}$ such that $\lambda(p, q) = (p, q)$.

Also, there is a homeomorphism $\tau : \sum \mathcal{R} \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ and equivalently $r \leq \tau(p) \leq s$ if and only if $(-, r) \vee (s, -) \leq p$. Where $(-, r) = \bigvee_{x<r}(x, r)$ and $(s, -) = \bigvee_{s<x}(s, x)$, correspond the intervals $(-\infty, r)$ and $(s, +\infty)$, respectively.

1.4.21 The set $C(M)$ of all frame homomorphisms from $\mathcal{R}$ to M has been studied as an f-ring by B. Banaschewski [2, 4].

Corresponding to every continuous operation $\diamond : \mathbb{R}^2 \to \mathbb{R}$ (in particular $+, ., \wedge, \vee$) we have an operation on $C(M)$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(z,w) : (r,s) \diamond (z,w) \leq (p,q)\}$$

where $(r,s) \diamond (z,w) \leq (p,q)$ means that for each $r < x < s$ and $z < y < w$ we have $p < x \diamond y < q$. It is easy to check that

$$\alpha \diamond \beta(p,-) = \bigvee\{\alpha(r,-) \wedge \beta(s,-) : (r,-) \diamond (s,-) \leq (p,-)\}$$

$$\alpha \diamond \beta(-,p) = \bigvee\{\alpha(-,r) \wedge \beta(-,s) : (-,r) \diamond (-,s) \leq (-,p)\}$$

$C(M)$ is also a Riesz space with the scalar multiplication $r\alpha = (r1).\alpha$, where $r \in \mathbb{Q}$, $\alpha \in C(M)$ and . is the ring multiplication of $C(M)$ (see [4]).

1.4.22 For each $\alpha \in C(M)$ and $A \subseteq C(M)$, let $coz(\alpha) = \alpha(-,0) \vee \alpha(0,-)$, and $coz(A) = \{coz(\alpha) : \alpha \in A\}$. For any $\alpha, \beta \in C(M)$ and $0 \neq r \in \mathbb{R}$ we have:

- $coz(0) = 0$, and $coz(1) = e$,
- $coz(\alpha + \beta) \leq coz(\alpha) \vee coz(\beta)$, and if $\alpha, \beta \geq 0$ equality holds,
- $coz(|\alpha|) = coz(\alpha) = coz(r\alpha)$,
- if $|\alpha| \leq |\beta|$ then $coz(\alpha) \leq coz(\beta)$, and
- if $\alpha, \beta \geq 0$ then $coz(\alpha \wedge \beta) = coz(\alpha) \wedge coz(\beta)$.

1.4.23 M is completely regular if and only if $coz(C(M))$ generates M.

The categories of all algebraic frames and compact algebraic frames, compact Normal regular frames, and compact completely regular frames are denoted, respectively, by $\mathcal{A}$**Frm**, $\mathcal{A}$**KFrm**, **KNRFrm**, and **KCRFrm**.

Chapter 2

ℓ-modules and Riesz spaces

In this chapter we study ℓ-modules, in tree section. In first section we define f-modules, strong ℓ-modules, and supper f-modules, and study relations between them. Note that all of them are equational class with equations in Remark 2.1.6.

In second section we study ℓ-modules of fractions. We prove that f equations are preserved from a module to its module of fractions but ℓ equations are not preserved, in Remark 2.2.4 we have a counterexample.

In third section we study topological structures of ℓ-modules and Riesz spaces. There are two topological structure that are studied. Uniform topology for ℓ-modules, and order limit sequence convergence for Riesz spaces.

2.1 ℓ-modules, f-modules, and Riesz spaces

In this section we study ℓ-modules. All rings are commutative with a unit.

2.1.1 Definition Let A be an ordered ring. An A-module M is called an *ordered module* if it is a an ordered abelian group and for every $a \in A^+$ and $x \in M^+$, $ax \in M^+$; an *ℓ-module* if it is a lattice with its order; an *f-module* if for every $a \in A^+$, $x, y \in M$; $a(x \wedge y) = ax \wedge ay$.

2.1.2 Definition Let A be an ℓ-ring, M is called a *strong* ℓ-module if for every $a, b \in A$ and $x \in M^+$, $(a \wedge b)x = ax \wedge bx$.
M is called a *supper f-module* if the equality $|a||x| = |ax|$ is true for every $a \in A$ and $x \in M$.

2.1.3 Theorem *Assume that A is an ℓ-ring and M is an ℓ-module.*
(1) *For every $a \in A$ and $x \in M$, $|ax| \leq |a||x|$.*
(2) *M is an f-module if and only if for every $a \in A^+$, $a(x \vee y) = ax \vee ay$.*
(3) *M is an f-module if and only if for every $a \in A^+$, $|ax| = a|x|$.*
(4) *M is a strong ℓ-module if and only if for every $x \in M^+$, $|ax| = |a|x$.*
(5) *If M is a super f-module then it is a strong ℓ-module and f-module.*

Proof (1) First assume that $a \in A^+$. We have $a(-x), ax \leq a|x|$ and hence $|ax| = (-ax) \vee ax \leq a|x| = |a||x|$. Now let $a \in A$ be arbitrary. We have

$$|ax| = |(a^+ - a^-)x| \leq |a^+x| + |a^-x| \leq a^+|x| + a^-|x| = (a^+ + a^-)|x| = |a||x|$$

(2) Using the equality $-(x \vee y) = (-x) \wedge (-y)$ we get the result.
(3) Suppose that M is an f-module and $a \in A^+$. By (2),

$$a|x| = a(x \vee (-x)) = ax \vee (-ax) = |ax|$$

Conversely, Assume that the equality holds for every $a \in A^+$. Let $x, y \in M$.

We have

$$2a(x \vee y) = a(|x - y| + x + y) = |ax - ay| + ax + ay = 2(ax) \vee (ay)$$

Since M has no nonzero element with finite order, Proposition 1.3.2(2), $a(x \vee y) = (ax) \vee (ay)$, and so by (2) M is an f-module.

(4) Suppose that M is a strong ℓ-module and $x \in M^+$. By Definition 2.1.2 and the equality $-(a \vee b) = (-a) \wedge (-b)$ we have

$$|a|x = (a \vee (-a))x = ax \vee (-ax) = |ax|$$

Conversely, Assume that the equality is true for every $x \in M^+$. Let $a, b \in M$. We have

$$2(a \vee b)x = (|a - b| + a + b)x = |ax - by| + ax + bx = 2(ax) \vee (bx)$$

Since M has no nonzero element with infinite order Proposition 1.3.2(2), hence $(a \vee b)x = (ax) \vee (bx)$, so, using the equality $-(a \vee b) = (-a) \wedge (-b)$, M is a strong ℓ-module.

(5) Follows from (3), (4), and Definition 1.2.2. □

2.1.4 Definition Suppose that M and N are l-modules over the ordered ring A. $f : M \to N$ is called an o-homomorphism if it is an A-homomorphism which preserves order. And it is called an ℓ-homomorphism if it is an A-homomorphism which preserves $\vee$ and $\wedge$. For Riesz spaces we say *Riesz map* use to ℓ-homomorphism.

2.1.5 Remark If f is a ℓ-homomorphism then

$$f(|x|) = |f(x)|, f(x^+) = f(x)^+, f(x^-) = f(x)^-$$

2.1.6 Remark Suppose that A is an ℓ-ring. Consider all algebras of type $\langle +, \vee, \wedge, -, (\lambda_a)_{a\in A}, 0\rangle$, where $\lambda_a : M \to M$ is the unary operation with $x \rightsquigarrow ax$. We have following equations:

The module equations:

(M_1) $(x+y)+z = x+(y+z)$,

(M_2) $x+y = y+x$,

(M_3) $0+x = x$,

(M_4) $x+(-x) = 0$,

(M_5) $(ab)x = a(bx)$,

(M_6) $a(x+y) = ax+ay$,

(M_7) $(a+b)x = ax+bx$,

(M_8) $1x = x$.

The lattice equations:

(L_1) $(x\vee y)\vee = x\vee(y\vee z)$,

(L_2) $x\vee y = y\vee x$,

(L_3) $x\vee x = x$.

The ℓ-equations:

(ℓ_1) $(x\vee y)+z = (x+z)\vee(y+z)$,

(ℓ_2) $a^+x^+\wedge 0 = 0$,

(ℓ_3) $-(x\vee y) = (-x)\wedge(-y)$.

The f-equations:

(f) $|a||x| = ||a|x|$,

$(\mathrm{s}\ell)$ $|a||x| = |a|x||$,

$(\mathrm{s}f)$ $|a||x| = |a||x|$.

Now, It is easy to check that the class of all ℓ-modules is an equational class with equations (M_1)-(M_8), (L_1)-(L_3), and (ℓ_1)-(ℓ_3). By Theorem 2.1.3, the class of all f-modules is equational by adding the equation (f) given above, the class of all strong ℓ-modules is an equational class by adding the equation (sℓ) given above, and finally, the class of all super f-modules is an equational class by adding the equation (sf) to its equations. We have the following relation between these four equational classes:

2.1.6 Definition Let M be an ℓ-module over an ordered ring A. A submodule I of M is called an ℓ-ideal if for every $x \in M$ and $y \in I$ with $|x| \leq |y|$ we have $x \in I$.

Since an arbitrary intersection of ℓ-ideals is an ℓ-ideal, for every subset $X \in M$ there is the smallest ℓ-ideal containing X which is called the *ℓ-ideal generated by* X and is denoted by $\langle X \rangle$. The following lemma describes the elements of $\langle X \rangle$:

2.1.7 Lemma *Let M be an ℓ-module over an ℓ-ring A, and $X \subseteq M$. Then $x \in \langle X \rangle$ if and only if $|x| \leq |a_1||x_1| + ... + |a_n||x_n|$ for some $n \in \mathbb{N}$, $x_i \in X$ and $a_i \in A$. In particular, $\langle x \rangle = \{y \in M : |y| \leq |a||x|, a \in A\}$.*

Proof Let $J = \{x : |x| \leq |a_1||x_1| + ... + |a_n||x_n|\}$. Then J is a submodule of M. Since for every $x, y \in J$ and $a \in A$ there exist $n, m \in \mathbb{N}$, $a_i, b_j \in A$ and $x_i, y_j \in X$ for $i = 1, ..., n$ and $j = 1, ..., m$,

$$\begin{aligned} |ax + y| \leq |a||x| + |y| &\leq |a|(|a_1||x_1| + ... + |a_n||x_n|) + |b_1||y_1| + ... + |b_m||y_m| \\ &= |a||a_1||x_1| + ... + |a||a_n||x_n| + |a||b_1||y_1| + ... + |a||b_m||y_m| \end{aligned}$$

Since $|a||a_i|, |a||b_j| \geq 0$, $ax + y \in J$. Let $|x| \leq |y|$ such that $y \in J$. Suppose that $|y| \leq |a_1||x_1| + ... + |a_n||x_n|$, hence $|x| \leq |y| \leq |a_1||x_1| + ... + |a_n||x_n|$ and so $x \in J$. Thus J is an ℓ-ideal containing X, and hence $\langle X \rangle \subseteq J$. This proves the nontrivial side of the equality $\langle X \rangle = J$. □

2.1.8 Definition Let M be an ordered module over an ordered ring A. An element $u \geq 0$ is called a *strong unit* if for every $x \in M^+$ there exist $a \in A^+$ such that $0 \leq x \leq au$. M is called *bounded* if it has a strong unit.

2.1.9 Remark Let M be an ℓ-module over an ℓ-ring A. By Lemma 2.1.7, M is bounded if and only if there exist a positive element $u \in M$ such that $M = \langle u \rangle$.

To justify Definition 2.1.8, first we give the following remark.

2.1.10 Remark Let M be a bounded ordered module over an ordered ring A with a fix strong unit u. For every $a \in A$ we denote $au = a$, and in particular $1u = 1$. Define the map $\iota : A \to M$ given by $a \rightsquigarrow a$. We have the following:

(1) $\iota : A \to M$ is an ordered module homomorphism.

(2) If A is an ℓ-ring and M is an ℓ-module then $\iota : A \to M$ is an ℓ-module homomorphism whenever M is a strong f-module.

(3) For a bounded Riesz space E, $\iota : \mathbb{Q} \to E$ is a monomorphism Riesz map. Also, for any positive $r \in \mathbb{Q}$, $E = \langle r \rangle$.

Recall that a topological space X is pseudocompact if and only if every element of $\mathcal{C}(X)$, the ring of real continuous functions with identity $\mathbf{1}$, is bounded.

Now, noting that $\mathcal{C}(X)$ is naturally a Riesz space, the following proposition justifies the Definition 2.1.8 of a bounded Riesz space.

2.1.11 Proposition *For a topological space X, the Riesz space $\mathcal{C}(X)$ is bounded iff X is pseudocompact.*

Proof One direction is trivial, since $\mathcal{C}(X) = [\mathbf{1}]$. To prove the nontrivial direction, let $\mathcal{C}(X)$ be bounded, say $\mathcal{C}(X) = [u]$ for some $u \in \mathcal{C}(X)$. If u is not bounded, taking $f(x) = e^{u(x)}$, it is easy to check that $|f| \not\leq ru$ for all $r \in \mathbb{Q}^+$, which is a contradiction. Thus u, and hence every element of $\mathcal{C}(X)$, is bounded. □

2.1.12 Notation Let A be an ℓ-ring. The category of all ℓ-modules over A with ℓ-homomorphism is denoted by $\ell\mathbf{Mod(A)}$, the category of all f-modules over A is denoted by $f\mathbf{Mod(A)}$. The category of all bounded ℓ-modules and bounded f-modules over A are denoted by $\mathbf{B}\ell\mathbf{Mod(A)}$ and $\mathbf{B}f\mathbf{Mod(A)}$, respectively.

Also, the category of all Riesz spaces with Riesz maps is denoted by **Rsz**, and the category of all bounded Riesz spaces is denoted by **BRsz**.

2.2 ℓ-module of fractions

Let A be an ordered ring and S be a multiplicative subset of A containing 1. Consider the ring $S^{-1}A$ of fractions. If $S \subseteq A^+$, we define an order on $S^{-1}A$

by $a/s \leq b/t$ if and only if there exists $w \in S$ such that $w(ta - sb) \leq 0$, which makes $S^{-1}A$ an ordered ring. If A is an f-ring, then so is $S^{-1}A$ with $a/s \diamond b/t = (ta \diamond sb)/st$ for each $\diamond \in \{+, \wedge, \vee\}$ and $|a/s| = |a|/s$. For $S = \{s \in A : s \geq 1\}$, we denote $S^{-1}A$ by $\tilde{A}$. Note that $\tilde{A}$ is a *strong* ordered ring, in the sense that every $x \geq 1$ is invertible. Moreover, the correspondence $A \mapsto \tilde{A}$ gives a coreflector from the category of f-rings to the category of strong ℓ-rings [2].

2.2.1 Definition Suppose that M is an ordered module over the ordered ring A, and S is a multiplicative subset of A with $S \subseteq A^+$. Define a partial order on $S^{-1}M$ by $x/s \leq y/t$ if there exists $w \in S$ such that $w(tx - sy) \leq 0$.

2.2.2 Proposition *Suppose that A is an ordered ring and $S \subseteq A^+$ is a multiplicative subset with $0 \notin S$.*

(1) *$S^{-1}M$ is an ordered module over the ordered ring $S^{-1}A$.*

(2) *$S^{-1}M$ is also an ordered module over the ordered ring A and $i : M \to S^{-1}M$ which is given by $i(x) = x/1$ is an A-ordered module homomorphism.*

(3) *If M is an f-module then $S^{-1}M$ is an f-module, and i is an A-ℓ-homomorphism.*

For every f-ring A;

(4) *If M is a strong ℓ-module then $S^{-1}M$ is a strong ℓ-module over $S^{-1}A$.*

(5) *If M is a super f-module then $S^{-1}M$ is a super f-module over $S^{-1}A$.*

Proof (1) First we show that the relation defined on $S^{-1}M$ in Definition 2.2.1 is a partial order. It is obviously reflexive. To prove antisymmetry, let $x/s \leq y/t$ and $x/s \geq y/t$. Hence there exist $w, w' \in S$ such that $w(sy-tx) \geq 0$ and $w'(tx - sy) \geq 0$. Thus $ww'(ta - sb) = 0$ and so $x/s = y/t$. To prove transitivity, let $x/s \leq y/t$ and $y/t \leq z/k$. So there exist $w, w' \in S$ such that $w(sy - tx) \geq 0$ and $w'(tz - ky) \geq 0$. Thus

$$ww't(sz - kx) = ww'(stz - sky) + ww'(sky - tkx) \geq 0$$

and so $x/s \leq z/k$. Now to prove that $S^{-1}A$ is an ordered module, let $x/s \leq y/t$ and $z/k \in S^{-1}M$. Hence

$$\begin{aligned} w(tx - sy) \leq 0 \ &\Rightarrow \ wk^2(tx - sy + stz - stz \leq 0 \\ &\Rightarrow \ w(k^2tx + stkz - k^2sy - sktz) \leq 0 \\ &\Rightarrow \ w(tk(kx + sz) - sk(ky + tz)) \leq 0 \\ &\Rightarrow \ (kx + sz)/sk \leq (ky + tz)/tk \\ &\Rightarrow \ x/s + z/k \leq y/t + z/k \end{aligned}$$

Suppose that $a/s \geq 0$ and $x/t \geq 0$ where $a \in A$ and $x \in M$. So there exist $w, w' \in S$ such that

$$\begin{aligned} wx \geq 0, \quad w'a \geq 0 \ &\Rightarrow \ ww'ax \geq 0 \\ &\Rightarrow \ (ax)/(st) \geq 0 \\ &\Rightarrow \ (a/s)(x/t) \geq 0 \end{aligned}$$

Thus $S^{-1}M$ is an ordered set.

(2) For every $a \in A$ and $x/s \in S^{-1}M$, define $a(x/s) = (ax)/s$. If $a \geq 0$ then $a/1 \geq 0$, and hence $(ax)/s = (a/1)(x/s) \geq 0$. Thus $S^{-1}M$ is an ordered module over A. For the second part note that for every $x \leq y$ we have $x/1 \leq y/1$.

(3) First we show that $S^{-1}M$ is an ℓ-module. Let $x/s, y/t \in S^{-1}M$. We claim that

$$x/s \wedge y/t = (tx \wedge sy)/(st), \quad x/s \vee y/t = (tx \vee sy)/(st)$$

First note that $x/s, y/t \leq (tx \vee sy)/(st)$, since

$$t(tx \vee sy) - sty = t^2x \vee tsy - tsy \geq 0$$

$$s(tx \vee sy) - stx = t^2y \vee tsx - tsy \geq 0$$

Now, suppose that $x/s, y/t \leq z/k$, and hence there exist $w_1, w_2 \in S$ such that

$$w_1(sz - kx) \geq 0, \quad w_2(tz - ky) \geq 0$$

So

$$w_1w_2stz \geq w_1w_2tkx, \quad w_1w_2stz \geq w_1w_2sky$$

and hence

$$\begin{aligned} w_1w_2stz &\geq w_1w_2skx \vee w_1w_2sky \\ &= w_1w_2(tx \vee sy) \end{aligned}$$

Thus

$$w_1w_2(stz - k(tx \vee sy)) \geq 0 \Rightarrow z/k \geq (tx \vee sy)/st$$

and so, $x/s \vee y/t = (tx \vee sy)/st$.

To prove $x/s \wedge y/t = (tx \wedge sy)/st$, first note that $x/s, y/t \geq (tx \wedge sy)/(st)$, since

$$t(tx \wedge sy) - sty = t^2t \wedge tsy - tsy \leq 0$$

$$s(tx \wedge sy) - stx = t^2y \wedge tsx - tsx \leq 0$$

Suppose that $x/s, y/t \geq z/k$, and so there exist $w_1, w_2 \in S$ such that

$$w_1(sz - kx) \leq 0, w_2(tz - ky) \leq 0$$

Thus

$$w_1w_2stz \leq w_1w_2tkx, \quad w_1w_2stz \leq w_1w_2sky$$

so

$$\begin{aligned} w_1w_2stz &\leq w_1w_2skx \wedge w_1w_2sky \\ &= w_1w_2(tx \wedge sy) \end{aligned}$$

Hence

$$w_1w_2(stz - k(tx \wedge sy)) \leq 0 \Rightarrow z/k \leq (tx \wedge sy)/st$$

and so, $x/s \wedge y/t = (tx \wedge sy)/st$. This proves the claim and so $S^{-1}M$ is an ℓ-module.

Now we prove the f-module condition for $S^{-1}M$. For this, let $a/k \geq 0$, and so there exists a w such that $wa \geq 0$. So,

$$\begin{aligned} (a/k)(x/s \vee y/t)) &= (wa)/(kw)(x/s \vee y/t)) \\ &= (wa)/(wk)((tx \vee sy)/(st)) \\ &= (wtax \vee wsay)/(wkst) \end{aligned}$$

$$\begin{aligned} &= w(tax \vee say)/(wkst) \\ &= (tax \vee say)/(kst) \\ &= (ktax \vee ksay)/(k^2 st) \\ &= (ax/ks) \vee (ay/kt), \end{aligned}$$

thus

$$a/k(x/s \vee y/t) = (ax)/ks \vee (ay/kt).$$

It means that $S^{-1}M$ is an f-module.

By the formulas for $\vee$ and $\wedge$ in $S^{-1}M$ in the proof of ℓ-module, we have

$$x/1 \vee y/1 = (x \vee y)/1, \quad x/1 \wedge y/1 = (x \wedge y)/1$$

So i preserves $\vee$ and $\wedge$, and it is an A-ℓ-homomorphism.

(4) Since A is an f-ring, for every $a/s \in S^{-1}A$ we have $|a/s| = |a|/s$. Let $x/t \in S^{-1}M, x \geq 0$, and hence $|a/s|(x/t) = |a|/s(x/t) = |a|x/st = |ax|/st$. So, by Theorem 2.1.3(4), $S^{-1}M$ is a strong ℓ-module over $S^{-1}A$.

(5) Since M is a super f-module, it is an f-module, and so $|x/t| = |x|/t$ is true for $x/t \in S^{-1}M$. Let $a/s \in S^{-1}A$. Thus $|a/s||x/t| = (|a|/s)(|x|/t) = |a||x|/st = |as|/st = |(a/s)(x/t)|$. This proves that $S^{-1}M$ is a super f-module over $S^{-1}A$. □

2.2.3 Remark If M is an f-module then by the formulas for $\vee$ and $\wedge$ in $S^{-1}M$ we have the following formulas which will be used to prove Proposition 2.2.3:

$$|x/s| = (x/s) \vee (-x/s) = (x \vee (-x))/s = |x/s|$$

$$(x/s)^+ = (x/s) \vee (0/1) = (x \vee (0))/s = x^+/s$$

$$(x/s)^- = (-x/s) \wedge (0/1) = ((-x) \wedge 0)/s = x^-/s$$

2.2.4 Remark If M is an ℓ-module then $S^{-1}M$ may not be an ℓ-module. For example, $A = \mathbb{Q}[x]$ with $a_0 + ... + a_n x^n \geq 0$ if and only if $a_i \geq 0, 0 \leq i \leq n$, is an ℓ-ring but $S^{-1}A$ is not an ℓ-ring, where $S = \{p \in A : p > 0\}$. To prove this we will show that $((1-x)/1)^+$ does not exist. Assume that $((1-x)/1)^+ = f/g$, where $f > 0$ and $g \in S$. Hence $f \geq (1-x)g$, and so $f \geq ((1-x)g)^+$. Since $f/g \geq ((1-x)g)^+/g$ and $((1-x)g)^+/g \geq (1-x)/1$, we have $f = ((1-x)g)^+$ and so $((1-x)/1)^+ = ((1-x)g)^+/g$. Suppose that $g = r_0 + r_1 x + ... + r_n x^n$, where $r_i \geq 0$, $(0 \leq i \leq n-1)$, and $r_n > 0$. Let $h = r_0 + r_1 x + ... + r_n x^n + q x^{n+1}$, where $0 < q < r_n$. Hence

$$(1-x)h = r_0 + (r_1 - r_0)x + ... + (r_n - r_{n-1})x^n + (q - r_n)x^{n+1} - q x^{n+2}$$

Thus

$$\begin{aligned} ((1-x)h)^+ &= r_0 + (r_1 - r_0)^+ x + ... + (r_n - r_{n-1})^+ x^n + (q - r_n)^+ x^{n+1} \\ &= r_0 + (r_1 - r_0)^+ x + ... + (r_n - r_{n-1})^+ x^n \\ &= ((1-x)g)^+ \end{aligned}$$

So, $0, (1-x)/1 < ((1-x)h)^+/h < ((1-x)g)^+/g$ and it contradicts $((1-x)/1)^+ = ((1-x)g)^+/g$.

2.2.5 Notation For $S = \{s \in A : s \geq 1\}$ denote $S^{-1}A$ by $\tilde{A}$ and $S^{-1}M$ by $\tilde{M}$.

2.2.6 Remark The ordered ring $\mathbb{Z}$ of all integers with its natural order is a bounded f-ring. Every abelian ℓ-group G is a super f-module over $\mathbb{Z}$. On the other hand $\tilde{\mathbb{Z}} = \mathbb{Q}$. So every Riesz space is of the form $\tilde{G}$ and hence it is a super f-module over $\mathbb{Q}$. Conversely, if G is an abelian ℓ-group then $\tilde{G}$ is a Riesz space.

2.3 Topological structures on ℓ-module and Riesz spaces

2.3.1 Definition Let A be a strong bounded ℓ-ring. The neighbourhoods

$$V_n(a) = \{x \in A : |x - a| < 1/n\}$$

for each $a \in A$ and $n \in \mathbb{N}$ determine a uniform topology on A with $\{V_n(a) : a \in A, n \in \mathbb{N}\}$ as its basis. Moreover, A is a topological ring with this topology. In fact, the operations $\diamond : A^2 \to A$ are uniformly continuous, where $\diamond \in \{+, \wedge, \vee\}$, and the multiplication is continuous (see [2]).

2.3.2 Definition For an ℓ-module M over a strong ordered ring A the neighbourhoods

$$V_n(a) = \{x \in M : |x - a| < 1/n\}$$

for each $a \in M$ and $n \in \mathbb{N}$, determine a uniform topology on M with $\{V_n(a) : a \in A, n \in \mathbb{N}\}$ as its basis. Moreover, we have:

2.3.3 Lemma *Suppose that M is an ℓ-module over a strong ordered ring A. Then the operations $\diamond : M^2 \to M$ are uniformly continuous, where $\diamond \in$*

$\{+, \wedge, \vee\}$.

Furthermore, if A is a bounded ℓ-ring then the scalar multiplication $A \times M \to M$ is continuous.

Proof Uniform continuity of $\diamond$ results from the following inequality:

$$|x \diamond y - x' \diamond y'| \leq |x - y| + |x' - y'|$$

For the continuity of scalar multiplication, let $a_0 \in A$ and $x_0 \in M$. We have

$$\begin{aligned} |ax - a_0x_0| &= |(a - a_0)(x - x_0) + a_0(x - x_0) + (a - a_0)x_0| \\ &\leq |(a - a_0)(x - x_0)| + |a_0(x - x_0)| + |(a - a_0)x_0| \\ &\leq |a - a_0||x - x_0| + |a_0||x - x_0| + |a - a_0||x_0| \end{aligned}$$

Let $k \in \mathbb{N}$. Take $m, n \in \mathbb{N}$ to be large enough such that $1/mn + |a_0|/n + |x_0|/m < 1/k$. Now let $|x - x_0| < 1/n$ and $|a - a_0| < 1/m$. Hence

$$\begin{aligned} |ax - a_0x_0| &\leq |a - a_0||x - x_0| + |a_0||x - x_0| + |a - a_0||x_0| \\ &< 1/mn + |a_0|/n + |x_0|/m \\ &< 1/k \end{aligned}$$

This proves the continuity of $A \times M \to M$. $\square$

We study another topology which is defined by *order convergence* of sequences [18]. This topology is studied for Riesz space.

Let E be a Riesz space. If $(x_n)_1^\infty$ is a decreasing sequence in E and $x = \bigwedge\{x_n | n \in \mathbb{N}\}$, then we write $x_n \downarrow x$. Similarly the dual notion $x_n \uparrow x$ is defined. A sequence $(\epsilon_n)_1^\infty$ with $\epsilon_n \downarrow 0$ is called a *null sequence*.

A sequence $(y_n)_1^\infty$ is called *order convergent* to y as $n \to \infty$, written as y=o-$\lim_{n\to\infty} y_n$ if there exists a null sequence $(\epsilon_n)_1^\infty$ with $|y_n - y| \leq \epsilon_n$ for all $n \in \mathbb{N}$.

Note that a bounded Riesz space E is Archimedean if and only if $(1/n)_{n\geq 1}$ is a null sequence.

Applying the definition of ordered limit we get the following lemma:

2.3.4 Lemma [18] *Assume that $(x_n)_1^\infty$, $(y_n)_1^\infty$ are sequences in E such that o-$\lim_{n\to\infty} x_n = x$ and o-$\lim_{n\to\infty} y_n = y$. Then,*

(i) o-$\lim_{n\to\infty}(x_n + ay_n) = x + ay$.

(ii) o-$\lim_{x\to\infty}(x_n \vee y_n) = x \vee y$.

(iii) o-$\lim_{n\to\infty}(x_n \wedge y_n) = x \wedge y$.

2.3.5 Definition A bounded Riesz space E is called *uniform* if for every null sequence (x_n) in E and $r \in \mathbb{Q}^+$ there exists $m \in \mathbb{N}$ such that $0 \leq x_m \leq r$.

2.3.6 Definition Let X be a subset of a Riesz space E. Denoting the set of all ordered limits of the sequences in X by $\overline{X}^o$, X is called *o-limit closed* if $X = \overline{X}^o$. In the case of a uniform Archimedean bounded Riesz space, o-limit closed subsets are the closed subsets of some topological space. In fact:

2.3.7 Lemma *For a uniform Archimedean bounded Riesz space E, $X \longmapsto \overline{X}^o$, is a closure operator.*

Proof It is clear that $X \subseteq \overline{X}^o$ and if $X \subseteq Y$ then $\overline{X}^o \subseteq \overline{Y}^o$. Hence it is enough to show that $\overline{\overline{X}^o}^o = \overline{X}^o$.

Suppose that $a \in \overline{\overline{X}^o}^o$. Then there exist sequences (x_n), $(y_{mn})_{n\geq 1}$, (ϵ_n), and $(\delta_{mn})_{n\geq 1}$ such that $y_{mn} \in X$ for all $m, n \in \mathbb{N}$, and $\epsilon_n \downarrow 0$, $\delta_{mn} \downarrow 0$, such that for every $m, n \in \mathbb{N}$,

$$|a - x_n| \leq \epsilon_n, \quad |x_n - y_{mn}| \leq \delta_{mn}$$

Since E is a uniform bounded Riesz space, we can assume that $(\epsilon_n)_{n\geq 1} = (1/n)_{n\geq 1}$ and $(\delta_{mn})_{n\geq 1} = (1/mn)_{n\geq 1}$. Hence we have

$$|a - y_{nn}| \leq |a - x_n| + |x_n - y_{nn}| \leq 1/n + 1/n^2$$

Now, since E is Archimedean we get that a= o-$\lim_{n\to\infty} y_{nn}$. So $a \in \overline{X}^o$ which proves the lemma. □

But what is the relation between these two topologies? In the case of bounded Archimedean uniform Riesz spaces the two topologies are equal. In fact:

2.3.8 Theorem *Suppose that E is a bounded Archimedean uniform Riesz space and X is a subset of E. Then $a \in \overline{X}$, with the uniform topology, if and only if there exists a sequence (x_n) in X such that o-$\lim x_n = a$.*

Proof Assume that $a \in \overline{X}$. Hence there is a sequence $x_n \in X$ such that $|x_n - a| < 1/n$. But E being Archimedean, $1/n \downarrow 0$, so o-$\lim x_n = a$, by the definition of order convergent.

Conversely, suppose that $o\text{-}\lim x_n = a$. Thus there is a null sequence (ε_n) such that $|x_n - a| \leq \varepsilon_n$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since E is uniform, there exist an index n such that $\varepsilon_n < 1/k$. This proves that $a \in \overline{X}$. $\square$

Now, we study the continuity of maps with respect to these two topologies. First we begin with the uniform topology for ℓ-modules:

2.3.9 Lemma *Suppose that A is a strong ordered ring, and $f : M \to N$ is a map between two ℓ-modules M and N.*

(1) *f is uniformly continuous if and only if for every $k \in \mathbb{N}$ there exists a $n \in \mathbb{N}$ such that*

$$|x - y| < 1/n \Rightarrow |f(x) - f(y)| < 1/k$$

(2) *f is continuous if and only if for every $x_0 \in M$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$|x - x_0| < 1/n \Rightarrow |f(x) - f(x_0)| < 1/k$$

Proof (1) Assume that f is uniformly continuous and let $k \in \mathbb{N}$. Hence $f^{-1}(V_k)$ contains an entourage V_n in uniform topology of M for some integer n. So, $|x - y| < 1/n$ implies $|f(x) - f(y)| < 1/k$.

Conversely, suppose that the condition is true. Let $V_k = \{(x, y) : |x - y| < 1/k\}$ be an entourage, and n be an integer such that $|x - y| < 1/n \Rightarrow |f(x) - f(y)| < 1/k$. Let $(x, y) \in V_n$, and so $|x - y| < 1/n$. Thus, by the condition

$|f(x) - f(y)| < 1/k$, $V_n \subseteq (f \times f)^{-1}(V_k)$. This means that f is uniformly continuous.

(2) It is similar to (1), replacing V_k and V_n by $V_k(x_0)$ and $V_n(x_0)$, respectively. □

2.3.10 Proposition *Suppose that A is a strong ordered ring and M, N are bounded ℓ-modules. Every ℓ-homomorphism $f : M \to N$ is uniformly continuous.*

Proof Let $k \in \mathbb{N}$. we take $n \in \mathbb{N}$ such that $m/n < 1/k$, where $0 \leq f(1) \leq m$. If $|x - y| < 1/n$, then

$$|f(x) - f(y)| = f(|x - y|) \leq f(1/k) = (1/k)f(1) \leq m/n < 1/k$$

So f is uniformly continuous, by Lemma 2.3.9. □

Now, we study the continuity of Riesz maps with respect to another topology which defined by convergence for Riesz spaces.

2.3.11 Theorem *Let E, D be Riesz spaces and $f : E \to D$ be a Riesz map. Then f is continuous if and only if $f(\epsilon_n) \downarrow 0$ whenever $\epsilon_n \downarrow 0$ in E.*

Proof First we note that f is continuous if and only if for every $X \subseteq E$, $f(\overline{X}^o) \subseteq \overline{f(X)}^o$. Let f be continuous and $\epsilon_n \in E$ such that $\epsilon_n \downarrow 0$. Consider $X = \{\epsilon_n : n \in \mathbb{N}\}$. We have $0 \in \overline{X}^o$, so $0 = f(0) \in \overline{X}^o$. Thus there is a subsequence ϵ_{k_n} such that o-$lim f(\epsilon_{k_n}) = 0$. Since x_n is decreasing and f preserves order, $f(\epsilon_n) \downarrow 0$.

Conversely, assume that the condition is true. Let $X \subseteq E$, $x \in \overline{X}^o$, and $x_n \in X$ be a sequence such that o-$lim x_n = x$. Hence there is a sequence $\epsilon_n \in E$ such that $\epsilon_n \downarrow 0$ and $|x_n - x| \leq \epsilon_n$. Since f is a Riesz map, $|f(x_n) - f(x)| = f(|x_n - x|) \leq f(\epsilon_n)$. Let $\delta_n = f(\epsilon_n)$. We have $\delta_n \downarrow 0$. So, $f(x) =$o-$lim f(x_n)$. Thus $f(\overline{X}^o) \subseteq \overline{f(X)}^o$ and the proof is complete. □

2.3.12 Corollary *Let $f : A \to B$ be a continuous Riesz map. Then*

(i) $f(x_n) \downarrow f(x)$ *whenever* $x_n \downarrow x$ *in* A.

(ii) *if* $a=$ o-$\lim_{n\to\infty} a_n$, *then* $f(a)=$ o-$\lim_{n\to\infty} f(a_n)$.

Proof (i) Let $\epsilon_n = x_n - x$. Then $\epsilon_n \downarrow 0$. By Theorem 2.3.11, $f(\epsilon_n) \downarrow 0$. But $f(\epsilon_n) = f(x_n) - f(x)$, and so $f(x_n) \downarrow f(x)$.

(ii) Suppose that $|a - a_n| \leq \epsilon_n$, where $\epsilon_n \downarrow 0$. Hence $|f(a) - f(a_n)| = f(|a - a_n|) \leq f(\epsilon_n)$ and, by Theorem 2.3.11, $f(\epsilon_n) \downarrow 0$ which means that $f(a)=$ o-$\lim_{n\to\infty} f(a_n)$. □

Convergence continuity of Riesz maps is obtained from the construction of the algebraic operations of Riesz space similar to uniform continuity of ℓ-homomorphisms which is provided by boundedness in Proposition 2.3.10. But for Convergence continuity of Riesz maps obtained from the conditions uniform and Archimedean, in fact:

2.3.13 Lemma *If $f : E \to D$ is a Riesz map, E is a bounded uniform Riesz space, and D is an Archimedean Riesz space, then f is a continuous Riesz map.*

Proof Assume that $\epsilon_n \downarrow 0$ in E. Let $k \in \mathbb{N}$. Since E is uniform, there exists m_k such that for each $m \geq m_k$, $0 \leq \epsilon_m < \overline{k^{-1}}$. So $0 \leq f(\epsilon_m) \leq f(\overline{k^{-1}}) = k^{-1}f(u)$ for all $m \geq m_k$, where $E = \langle u \rangle$. Since D is Archimedean, $\inf\{k^{-1}f(u)|k = 1, 2, \cdots\} = 0$. Hence $\inf\{f(\varepsilon_m)|m = 1, 2, \cdots\} = 0$, and so $f(\epsilon_n) \downarrow 0$. □

From Theorem 2.3.10 we have:

2.3.14 Corollary *Suppose that A is a strong ordered ring. For a strong ordered ring A, every map in the category* **BℓMod(A)** *is uniformly continuous.*

And from Theorem 2.3.13 we have:

2.3.15 Corollary *Every map in the category* **BUARsz** *is convergence continuous.*

Chapter 3

Pointfree spectra

In this chapter we study tree pointfree spectra of ℓ-modules, pointfree Keimel spectrum, pointfree maximal spectrum, and spectra of closed ℓ-ideals.

In first section we prove that compactness of $\mathcal{L}(M)$ is equivalent to boundedness of M, and for every f-module $\mathcal{L}(M)$ is coherently normal. Also, we prove that $\mathcal{L}(M)$ is isomorphic to $\mathcal{L}(S^{-1}M)$.

The completely regularity of pointfree maximal spectrum $S\mathcal{L}(M)$ is proved in second section.

Corresponding of two topological structures we have two spectra of closed ℓ-ideal. We study these spectra in third section.

3.1 Pointfree Keimel spectrum

Suppose that A is an ordered ring and M is an ℓ-module over A. The set of all ℓ-ideals of M is denoted by $\mathcal{L}(M)$ (see 1.2.6 for definition of ℓ-ideal).

The spectrum $\sum \mathcal{L}(M)$ of $\mathcal{L}(M)$ is called the *Keimel Spectrum* and is denoted by $spec_K(M)$ (see [2, 16]).

3.1.1 Theorem *Let A be an ordered ring and M be an ℓ-module over A. Then*

(1) *$\mathcal{L}(M)$ is a frame.*

(2) *$J \in \mathcal{L}(M)$ is a compact element if and only if there exists $m \in M$ such that $J = [m]$, where $[m]$ is the l-ideal generated by m.*

Proof (1) It is clear that $\mathcal{L}(M)$ is a complete lattice, since the intersection of ℓ-ideals of M is an ℓ-ideal of M. Also notice that the join of a family $\{J_\gamma | \gamma \in \Gamma\}$ of ℓ-ideals is given by $\bigvee J_\gamma = \sum J_\gamma$. This is because $\sum J_\gamma$ is an ℓ-ideal. To see this, let $a \in M$ and $|a| \leq |a_1 + \cdots + a_n|$, where $a_k \in J_{\gamma_k}$ for $k = 1, \cdots, n$. Hence, $|a| \leq |a_1| + \cdots + |a_n|$, and by Lemma 1.1.4(5), $|a| = x_1 + \cdots + x_n$ for some $0 \leq x_i \leq |a_i|$, and hence $|a| \in \sum J_\gamma$. Similarly $a^+ \in \sum J_\gamma$, and so $a = 2a^+ - |a| \in \sum J_\gamma$.

To prove the distributivity of $\cap$ over arbitrary $\bigvee$, let $x \in J \cap \sum J_\gamma$. Then $x \in J$ and $|x| = u_1 + \cdots + u_n$, where $u_k \geq 0$, $u_k \in J_{\gamma_k}$ for $k = 1, \cdots, n$. Now, by Lemma 1.1.4(5), $|x| = |x| \wedge (u_1 + \cdots + u_n) \leq |x| \wedge u_1 + \cdots + |x| \wedge u_n$, and so $|x| \in \sum J \cap J_\gamma$, that is, $x \in \sum J \cap J_\gamma$. Therefore, $\mathcal{L}(M)$ is a frame.

(2) Suppose that J is compact. So $J = [a_1] + [a_2] + \cdots + [a_n]$ for $a_i \in J$. It is easy to see that $J = [a]$ for $a = |a_1| \vee \cdots \vee |a_n|$.

Conversely, suppose that $J = [a]$ and $J = \bigvee\{J_\gamma | \gamma \in \Gamma\} = \sum_{\gamma \in \Gamma} J_\gamma$. We can take each $J_\gamma = [a_\gamma]$ for some $a_\gamma \in J_\gamma$. Then $[a] = \sum [a_\gamma]$, and so $|a| = u_1 + \cdots + u_k$, where $0 \leq u_t \in [a_{i_t}]$. Obviously, $[a] \subseteq [u_1] + \cdots + [u_k] \subseteq$

$[a_{i_1}] + \cdots + [a_{i_k}] \subseteq [a]$, and hence $J = [a_{i_1}] + \cdots + [a_{i_k}]$. So J is compact. □

3.1.2 Corollary *For every ℓ-module M over an ordered ring A, the following are equivalent:*

(1) *M is bounded;*

(2) *$\mathcal{L}(M)$ is compact.*

3.1.3 Lemma *Let M be an ℓ-module over an ordered ring A. Then M is an f-module if and only if for every $m, n \in M^+$, $[m] \cap [n] = [m \wedge n]$.*

Proof Suppose that the equality holds for all $m, n \in M^+$. So $m \wedge n = 0$ implies $am \wedge an = 0$ for all $a \in A^+$ and $m, n \in M^+$. Let $a \in A^+$ and $m, n \in M^+$. We have

$$(m - m \wedge n) \wedge (n - m \wedge n) = m \wedge n - m \wedge n = 0$$

So

$$\begin{aligned} 0 = a(m - m \wedge n) \wedge a(n - m \wedge n) &= (am - a(m \wedge n)) \wedge (an - a(m \wedge n)) \\ &= am \wedge an - a(m \wedge n) \end{aligned}$$

Hence, $a(m \wedge n) = am \wedge an$. This means that M is an f-module. This proves the nontrivial part of the lemma. □

3.1.4 Theorem *For every f-module over an ordered ring A, $\mathcal{L}(M)$ is coherently normal.*

Proof By Theorem 3.1.1(2) and Lemma 3.1.3, $\mathcal{L}(M)$ is coherent. Thus we just verify the normality condition. Let $I + J = [a]$, where $I, J \in \mathcal{L}(M)$

and $a \in M^+$, which gives the compact element $[a]$ in $\mathcal{L}(M)$. Then $a = b + c$ for some $b \in I$ and $c \in J$.

Let $u = |c| - |b| \wedge |c|$ and $v = |b| - |b| \wedge |c|$. We have $u \in J$, and hence $I + [u] \subseteq [a]$. But

$$a = |a| \leq |b| + |c| = (|b| + |b| \wedge |c|) + u$$

Thus $a \in I + [u]$, and $[a] = I + [u]$. Similarly $[a] = J + [v]$. □

3.1.5 Remark Since every Riesz space and abelian ℓ-group are f-module by Theorems 1.1.6 and 1.1.8, for every Riesz space E, $\mathcal{L}(E)$ is coherently normal. Also for every abelian ℓ-group G, $\mathcal{L}(G)$ is coherently normal. In fact,

3.1.6 Theorem *For every f-module over an ordered ring A and a multiplicative subset S of A such that for every $s \in S$, $s \geq 1$, we have $\mathcal{L}(M) \cong \mathcal{L}(S^{-1}M)$ as frames. In particular, $\mathcal{L}(M) \cong \mathcal{L}(\tilde{M})$.*

Proof Consider the maps $\phi : \mathcal{L}(M) \to \mathcal{L}(S^{-1}M)$ given by $\phi(J) = S^{-1}J = \{a/s | a \in J, s \in S\}$, and $\psi : \mathcal{L}(S^{-1}M) \to \mathcal{L}(M)$ given by $\psi(I) = I \cap M$. First note that ϕ and ψ are both well defined, since $S^{-1}J$ and $I \cap M$ are ℓ-ideals of $S^{-1}M$ and M, respectively. Also ϕ and ψ preserve inclusions. So it is enough to show that ϕ and ψ are inverse of each other, that is,

$$S^{-1}J \cap M = J, \quad S^{-1}(I \cap M) = I$$

It is obvious that $S^{-1}(I \cap M) \subseteq I$ and $J \subseteq S^{-1}J \cap M$. Conversely, let $x \in S^{-1}J \cap M$. Hence, there exist $a \in J$ and $s \in S$ such that $a/s = x$. Since $s \geq 1$, we have $|a/s| = |a|/s \leq |a|$, and so $a/s \in J$, since J is an ℓ-ideal. Thus $x \in J$. Hence, $S^{-1}J \cap M = J$.

Now assume that $a/s \in I$ where $s \in S$ and $a \in M$. So

$$a = sa/s \in I \Rightarrow a \in I \cap M \Rightarrow a/s \in S^{-1}(I \cap M).$$

This yields $S^{-1}(I \cap M) = I$, and the proof is complete. $\square$

The fact that every f-ring A is a subdirect product of chains (of f-rings) is proved using the Axiom of Choice in [10] (Lemma 4, p.404). Here we prove that every f-module over the ordered ring A is a subdirect product of chains (of modules) using BUT, and give the chains explicitley.

3.1.7 Proposition *Let M be an f-module over the ordered ring A, and P be an element of $\mathcal{L}(M)$. Then the following are equivalent:*

(i) *P is a prime element of the frame $\mathcal{L}(M)$.*

(ii) *P is a prime ℓ-ideal; that is, $|x| \wedge |y| \in P \Rightarrow x \in P$ or $y \in P$.*

(iii) *M/P is totally ordered.*

Proof (i) $\Rightarrow$ (ii): Let $x, y \in M$ and $|x| \wedge |y| \in P$. By Lemma 3.1.3, $\langle x\rangle \cap \langle y\rangle = \langle |x| \wedge |y|\rangle \subseteq P$. Hence, $x \in \langle x\rangle \subseteq P$ or $y \in \langle y\rangle \subseteq P$.

(ii) $\Rightarrow$ (i): Let $I \cap J \subseteq P$, $x \in I - P$ and $y \in J - P$. Hence, $|x| \in I - P$ and $|y| \in J - P$. Since P is a prime ℓ-ideal, $|x| \wedge |y| \in I \cap J - P$, which contradicts $I \cap J \subseteq P$.

(ii) $\Rightarrow$ (iii): Let $x + P, y + P \in M/P$. We have

$$(x - x \wedge y) \wedge (y - x \wedge y) = x \wedge y - x \wedge y = 0$$

Hence, $x - x \wedge y \in P$ or $y - x \wedge y \in P$, so $x + P \leq y + P$ or $y + P \leq x + P \in M/P$.

(iii) $\Rightarrow$ (ii): Let $x, y \in M$ and $|x| \wedge |y| \in P$. Assume that $x + P \leq y + P$. We have $x - x \wedge y \in P$, and hence $x \in P$. $\square$

3.1.8 Remark The spectrum of the frame $\mathcal{L}(M)$ is called the *Keimel spectrum* and is denoted by $Spec_K M$ [2, 16]. *BUT* is enough to prove that $\mathcal{L}(M)$ is spatial, for any f-module M. Hence, by the above proposition, *BUT* implies that $\bigcap\{P : M/P \text{ is totally ordered }\} = 0$, which means that the homomorphism from M to $\prod\{M/P : M/P \text{ is totally ordered }\}$ is one-one. So *BUT* implies that every f-module is a subdirect product of totally ordered chains (see [2] for the same remark for f-rings).

3.2 Pointfree maximal spectrum

As for f-rings [2,4,5], we consider $S\mathcal{L}(M)$, the saturation of the frame $\mathcal{L}(M)$ of ℓ-ideals of a bounded f-module M over a bounded ordered ring A is considered as the maximal spectrum.

We show, among other things, that these two spectra are completely regular and, under some mild conditions, naturally isomorphic.

3.2.1 Lemma *Let M be a bounded f-module over a strong ordered ring A. Suppose that $\kappa : \mathcal{L}(M) \to \mathcal{L}(M)$ is a nucleus such that for each $a \in M^+$, $\kappa([a]) = \kappa(J)$, where $J = \bigcup\{[(a-p)^+],\ 0 < p \leq 1\}$. Then $\kappa\mathcal{L}(M) = Fix(\kappa)$ is a completely regular frame. In particular, if $\kappa : \mathcal{L}(M) \to \kappa\mathcal{L}(M)$ is codense then there is a frame isomorphism $h : S\mathcal{L}(M) \to \kappa\mathcal{L}(M)$ such that $\kappa = hs_M$.*

Proof We begin by showing $[(a-p)^+] \prec\prec [a]$ in $\mathcal{L}(M)$ for all $a \in M$ and each rational number p with $0 < p \leq 1$. We will show that $[(a-p)^+] \prec [(a-q)^+]$ for all $0 < q < p$. We have

$$\begin{aligned}(a-q)^+ + (a-p)^- &= (a-q) \vee 0 + (p-a) \vee 0 \\ &= (a \vee q) - q + (p \vee a) - a \\ &= 0 \vee (q-a) + (p-q) \vee (a-q) \\ &\geq p - q = p - q > 0\end{aligned}$$

Thus $[(a-q)^+] + [(a-p)^-] \supseteq [(a-q)^+ + (a-p)^-] \supseteq [p-q] = M$.

On the other hand, $(a-p)^+ \wedge (a-p)^- = 0$. Hence, by Lemma 3.1.3, $[(a-p)^+] \cap [(a-p)^-] = 0$. Thus $[(a-p)^+] \prec [(a-q)^+]$.

Now we show the complete regularity of $\kappa\mathcal{L}(M)$. Since for every $J \in \kappa\mathcal{L}(M)$, $J = \bigcup\{[a] | a \in J^+\}$ (in $\mathcal{L}(M)$), $J = \kappa(J) = \bigvee\{\kappa[a] | a \in J^+\}$ (in $\kappa\mathcal{L}(M)$). So it is sufficient to show that for each $a \in M^+$, $\kappa[a]$ is the join in $\kappa\mathcal{L}(M)$ of the corresponding $\kappa[(a-p)^+]$, $0 < p \leq 1$ in $\mathbb{Q}$; that is to show that $\kappa[a] = \kappa(I)$ for the ℓ-ideal $I = \bigcup\{[(a-p)^+], 0 < p \leq 1 \text{ in } \mathbb{Q}\}$ which is given by the hypothesis. The second part of the lemma follows from Lemma 1.2.12. □

3.2.2 Lemma *If M is a bounded f-module over a strong bounded ordered ring A, then $S\mathcal{L}(M)$ is completely regular.*

Proof Using Lemma 3.2.1, it is sufficient to show that for each $a \in M^+$, $s[a] = s(I)$ for the ℓ-ideal $I = \bigcup\{[(a-p)^+], 0 < p \leq 1 \text{ in } \mathbb{Q}\}$. The inclusion $s(I) \subseteq s[a]$ is trivial. So it is enough to show that $[a]$ is I-small. Let $[a]+J = M$. Thus $1 \in [a] + J$, and so we have $1 \leq ra + b_1$ for some $b_1 \in J$ and $r > 0$. Since

A is bounded, there exists a natural number n such that $r < n$. We have

$$\begin{aligned} & 0 < \tfrac{1}{n} \leq a + \tfrac{1}{n} b_1 = a + b \quad \text{for } b = \tfrac{1}{n}|b_1| \\ \Rightarrow\ & [a \vee b] = [a] + [b] \supseteq [a + b] \supseteq [\tfrac{1}{n}] = M \\ \Rightarrow\ & 0 < p < q < a \vee b \ \text{ for some } p, q \in \mathbb{Q} \\ \Rightarrow\ & q - p \leq (a - p) \vee (b - p) \leq (a - p)^+ \vee b \in I + J \\ \Rightarrow\ & q - p \in I + J = M. \\ \Rightarrow\ & I + J = M. \end{aligned}$$

This completes the proof. □

3.2.3 Theorem *If M is a bounded f-module over a bounded ordered ring A then $S\mathcal{L}(M)$ is completely regular.*

Proof By the previous lemma and Theorem 3.1.6, it is enough to show that $\tilde{A}$ and $\tilde{M}$ are bounded. Let $a/s \in \tilde{A}$ where $a \in A$ and $s \geq 1$. There exists $n \in N$ such that $|a| \leq n$. So

$$|a/s| = |a|/s \leq |a|/1 \leq n/1 = n$$

This proves the boundedness of $\tilde{A}$. Similarly, $\tilde{M}$ is bounded. □

3.2.4 Corollary *For each bounded Riesz space E, $S\mathcal{L}(E)$ is completely regular.*

3.3 Spectra of closed ℓ-ideals

Let M be an ℓ-module over a strong ordered ring A and $\overline{X}$ denote the closure of $X \subseteq M$ with respect to the uniform topology generated by neighbourhoods $V_n(a)$ $n \in \mathbb{N}$ and $a \in M$. We define a map $c_M : \mathcal{L}(M) \to \mathcal{L}(M)$ given by $c_M(I) = \overline{I}$.

3.3.1 Lemma *For any ℓ-module M over a strong bounded ordered ring A, c_M is a nucleus.*

Proof First we show that $\overline{I}$ is an ℓ-ideal. It is clear that $\overline{I}$ is a submodule of M. Assume that $|x| \leq |a|$ and $a \in \overline{I}$. Let $n \in \mathbb{N}$. Take $y \in I$ such that $|y - a| < 1/n$. Hence

$$||x| - |y| \wedge |x|| = ||a| \wedge |x| - |y| \wedge |x|| \leq ||a| - |y|| \leq ||a| - |y|| < 1/n$$

Thus $|x| \in \overline{I}$, since $|y| \wedge |x| \in I$. Similarly, $x^+ \in \overline{I}$, and so $x = 2x^+ - |x|$ is in $\overline{I}$. Hence $\overline{I}$ is an ℓ-ideal.

To prove that c_M is a nucleus, note that $(N1)$ is trivial. For $(N2)$ let $a \in \overline{\overline{I}}$ and $n \in \mathbb{N}$. Hence $|a - x| < 1/2n$ and $|x - y| < 1/2n$, for some $x \in \overline{I}$ and $y \in I$. Thus

$$|a - y| \leq |a - x| + |x - y| \leq 1/2n + 1/2n = 1/n$$

and so $a \in \overline{I}$.

Now let $a \in \overline{I} \cap \overline{J}$ and $n \in \mathbb{N}$. We have $|a - x| < 1/2n, |a - y| < 1/2n$ for some $x \in I$ and $y \in J$. So

$$||a| - |x| \wedge |y|| = ||a| \wedge |a| - |x| \wedge |y|| \leq ||a| - |x|| + ||a| + |y|| < 1/2n + 1/2n = 1/n$$

But $|x| \wedge |y| \in I \cap J$, so $|a| \in \overline{I \cap J}$. This proves $(N3)$, and the proof is complete. □

Let $C\mathcal{L}M$ be the frame $Fix(c_M)$, consisting of all closed ℓ-ideals of M.

3.3.2 Theorem *For any bounded f-module M over a strong bounded ordered ring A, $C\mathcal{L}(M)$ is completely regular.*

Proof By Lemma 3.2.1, it is enough to show that for every $a \in M^+$, $\overline{[a]} = \overline{I}$, where $I = \bigcup\{[(a-p)^+] | 0 < p \leq 1\}$. We have

$$|a - (a - 1/n)^+| = |a - (a - 1/n) \vee 0| \leq 1/n \wedge a \leq 1/n$$

Thus $a \in \overline{I}$, and hence $\overline{[a]} \subseteq \overline{I}$ which is the nontrivial side of $\overline{[a]} = \overline{I}$. □

3.3.3 Theorem *Suppose that M is a bounded ℓ-module over a strong bounded ordered ring A. Then*

(1) *$c_M : \mathcal{L}(M) \to C\mathcal{L}(M)$ is a codense quotient.*

(2) *$C\mathcal{L}(M)$ is compact normal.*

Moreover, if M is an f-module then

(3) *$C\mathcal{L}(M)$ is coherent.*

(4) *$c_M : \mathcal{L} \to C\mathcal{L}(M)$ is a subfit codense quotient.*

(5) *There is an isomorphism $h_M : C\mathcal{L}(M) \to S\mathcal{L}(M)$ such that $h_M c_M = s_M$.*

Proof (1) Let $\overline{I} = M = [1]$, where $I \in \mathcal{L}(M)$. Hence, $1 \in \overline{I}$ and there exists $a \in I$ such that $|1 - a| < 1/2$. So

$$1 - |a| \wedge 1 = |1 - |a| \wedge 1| \leq |1 - a| < 1/2$$

Thus $1 \leq 2(|a| \wedge 1)$. Since $|a| \wedge 1 \in I$, we have $1 \in I$, and hence $I = M$.

(2) follows from (1), Corollary 3.1.2, and Theorem 3.1.4 and 1.4.9.

(3) First we show that $I \in C\mathcal{L}(M)$ is compact if and only if $I = \overline{[a]}$ for some $a \in I$. Assume that $I \in C\mathcal{L}(M)$ is a compact element. Since $I = \bigvee_{a \in I^+} \overline{[a]}$, there exists $n \in \mathbb{N}$ such that

$$I = \overline{\overline{[a_1]} + ... + \overline{[a_n]}} = \overline{[a_1] + ... + [a_n]} = \overline{[a]}$$

where $a = a_1 \vee ... \vee a_n$.

Conversely, suppose that $a \in M$ and $\overline{[a]} = \bigvee D$, where D is an updirected subset of $C\mathcal{L}(M)$. Hence, $\overline{[a]} = \bigcup D$, and so $a \in I$ for some $I \in D$. Thus $\overline{[a]} = \bigcup D$, and so $a \in I$ for some $I \in D$. Thus $\overline{[a]} = I$ for some $I \in D$. This proves that $\overline{[a]}$ is a compact element.

Using

$$\overline{[a]} \cap \overline{[b]} = \overline{[a] \cap [b]} = \overline{[|a| \wedge |b|]}$$

we see that the meet of two compact elements of $C\mathcal{L}(M)$ is compact. This means that $C\mathcal{L}(M)$ is coherent. And this proves (3).

(4) Since $C\mathcal{L}(M)$ is normal coherent and by Theorem 3.3.2 is regular, hence by Lemma 1.2.12, $C\mathcal{L}(M)$ is subfit. So $c_M : \mathcal{L}(M) \to C\mathcal{L}(M)$ is a subfit codense quotient.

(5) follows from (4). □

By the notations of Definition 2.3.6 we have a closure operator on $\mathcal{L}(E)$, and another closed spectrum for Riesz space E. We begin with the following lemma.

3.3.4 Lemma *For a uniform Archimedean bounded Riesz space E, c_E^o : $\mathcal{L}(E) \to \mathcal{L}(E)$, given by $I \longmapsto \overline{I}^o$, is a closure operator.*

Proof First, notice that for an ℓ-ideal J of E, $\overline{J}^o$ is an ideal. This is because first of all $\overline{J}^o$ is a subspace (Lemma 2.3.4(i)). Now assume that $0 < x \leq |a|$ for $a \in \overline{J}^o$. Let o-$lim_{n\to\infty} a_n = a$, for $a_n \in J$. Then o-$\lim_{n\to\infty} |a_n| = |a|$, and hence $x = |a| \wedge x$=o-$\lim_{n\to\infty}(|a_n| \wedge x)$ and so $x \in \overline{J}^o$, since $|a_n| \wedge x \in J$ for all n. So $|x| \leq |a|$ implies that $|x|, x^+ \in \overline{J}^o$, and hence $x \in \overline{J}^o$. Thus $\overline{J}^o$ is an ideal.

Second, to prove $\overline{\overline{J}^o}^o = \overline{J}^o$ suppose that $a \in \overline{\overline{J}^o}^o$. Then there exist sequences (x_n), $(y_{mn})_{n\geq 1}$, (ϵ_n), and $(\delta_{mn})_{n\geq 1}$ such that $y_{mn} \in J$ for all $m, n \in \mathbb{N}$, and $\epsilon_n \downarrow 0$, $\delta_{mn} \downarrow 0$, such that for every $m, n \in \mathbb{N}$,

$$|a - x_n| \leq \epsilon_n, \quad |x_n - y_{mn}| \leq \delta_{mn}$$

Since E is a uniform bounded Riesz space, we can assume that $(\epsilon_n)_{n\geq 1} = (1/n)_{n\geq 1}$ and $(\delta_{mn})_{n\geq 1} = (1/mn)_{n\geq 1}$. Hence, we have

$$|a - y_{nn}| \leq |a - x_n| + |x_n - y_{nn}| \leq 1/n + 1/nn$$

Now, since E is Archimedean, we get that a= o-$\lim_{n\to\infty} y_{nn}$. So $a \in \overline{J}^o$ which proves the lemma. □

3.3.5 Lemma *The above closure operator also satisfies the following:*

(i) $\overline{I \cap J}^o = \overline{I}^o \cap \overline{J}^o$.

(ii) If E has a nonzero null sequence, then $\overline{I}^o$ is a band of E.

Proof (i) The inclusion $\overline{I}^o \cap \overline{J}^o \supseteq \overline{I \cap J}^o$ is trivial. Let $x \in \overline{I}^o \cap \overline{J}^o$. Thus, x= o-$\lim_{n\to\infty} a_n$= o- $\lim_{n\to\infty} b_n$ for some $(a_n)_{n\geq 1}$ in I, $(b_n)_{n\geq 1}$ in J. Thus x^+=

o-lim $a_n^+ \wedge b_n^+$, $x^-=$ o-lim $a_n^- \wedge b_n^-$. But $0 \leq a_n^+ \wedge b_n^+ \leq a_n^+,\ b_n^+,\ 0 \leq a_n^- \wedge b_n^- \leq a_n^-, b_n^-,\ a_n^+, a_n^- \in I$, and $b_n^+, b_n^- \in J$. Thus $a_n^+ \wedge b_n^+, a_n^- \wedge b_n^- \in I \cap J$ and hence $x^+, x^- \in \overline{I \cap J}^o$. So $x \in \overline{I \cap J}^o$. Thus $\overline{I}^o \cap \overline{J}^o = \overline{I \cap J}^o$.

(ii) By hypothesis, there exists a nonzero null sequence $(a_n)_{n\in\mathbb{N}}$. Let $S \subseteq \overline{I}^o$ and $x = \bigvee S$ exists. Thus for each $n \geq 1$, $x - a_n < x$ and hence $x - a_n \leq s_n \leq x$ for some $s_n \in S$. Thus $x=$ o-$\lim_{n\to\infty} s_n$; in fact $x \in \overline{\overline{I}^o}^o = \overline{I}^o$ and hence $\overline{I}^o$ is a band. □

We denote the set of all closed ℓ-ideals of E, that is ideals I for which $\overline{I}^o = I$, by $C^o(E)$.

3.3.6 Remark But what is the relation of two frame $C^o\mathcal{L}(E)$ and $C\mathcal{L}(E)$? We should say that using Theorem 2.3.8, in case of bounded uniform Archimedean Riesz space $C^o\mathcal{L}(E) = C\mathcal{L}(E)$. In other case the frame $C^o\mathcal{L}(E)$ dose not mean, since $c_E^o : \mathcal{L}(E) \to \mathcal{L}(E)$ is not a closure operator.

So, Here we have all facts for $C^o\mathcal{L}(E)$ that are true for $C\mathcal{L}(E)$:

3.3.7 Theorem *For a uniform Archimedean bounded Riesz space E, $C^o\mathcal{L}(E)$ is a frame and $c_E^o : \mathcal{L}(E) \to C^o\mathcal{L}(E)$ is codense. If in addition E is complete, then $C^o\mathcal{L}(E)$ is a subfit frame and $c_E^o : \mathcal{L}(E) \to C^o\mathcal{L}(E)$ is a subfit quotient.*

3.3.8 Corollary *If E is a uniform complete bounded Riesz space then there* *is an isomorphism* $h : S\mathcal{L}(E) \to C^o\mathcal{L}(E)$ *such that* $hs_E = c^o_E$*:*

$$\begin{array}{ccc} \mathcal{L}(E) & \xrightarrow{s_E} & S\mathcal{L}(E) \\ & \overset{c^o_E}{\searrow} & \downarrow h \\ & & C^o\mathcal{L}(E) \end{array}$$

3.3.9 Proposition *For a bounded uniform Archimedean Riesz space* E*,* $C^o\mathcal{L}(E)$ *is compact completely regular.*

3.3.10 Definition A Riesz space is called *Dedekind complete* if $sup(S)$ exists for all $S \subseteq E$.

3.3.11 Proposition *For a Dedekind complete uniform Archimedean bounded Riesz space* E*,* $C^o\mathcal{L}(E)$ *is coherently normal.*

Proof By the above remark it remains to prove that $\downarrow \overline{\langle a\rangle}^o$ is normal in $C^o\mathcal{L}(E)$. Let $I, J \in Cl(E)$ with $I \vee J = \overline{\langle a\rangle}^o$. Also let $U = I^\perp \cap \overline{\langle a\rangle}^o$ and $V = J^\perp \cap \overline{\langle a\rangle}^o$. It is clear that U and V are closed ideals. Since $U \cap I = 0$ we have $U \cap J = U \cap (I \vee J) = U \cap \overline{\langle a\rangle}^o = U$, hence $U \subseteq J$ and so $U \cap V = 0$.

On the other hand, using completeness of E, we get $\overline{\langle a\rangle}^o = \overline{I \oplus U}^o = I \vee U$, and similarly $\overline{\langle a\rangle}^o = J \vee V$. This proves the proposition. $\Box$

Chapter 4

Pointfree Kakutani duality

This chapter is the final aim of the thesis. Pointfree Kakutani duality is studied in final section, before it, we study the functorial property of spectra that are studied in previous chapter. The continuous real functions Riesz space as a functor is studied in next section, and the adjunction property of these functors is studied in section 3 (Theorem 4.3.1). In section 4 we study the generalized pointfree version of Stone-Weierstrass theorem (Theorem 4.4.7).

4.1 The spectra functors

First we see that the pointfree Keimel spectrum is functorial:

4.1.1 Proposition *Let A be an ordered ring. The assignment $M \longmapsto \mathcal{L}(M)$ is functorial from $\ell\mathbf{Mod(A)}$ to $\mathbf{AFrm}$. And its restriction on $\mathbf{B}\ell\mathbf{Mod(A)}$ is also functorial to $\mathbf{AKFrm}$.*

Proof For any map $f : M \to N$ in $\ell\mathbf{Mod(A)}$, the corresponding map $\mathcal{L}f : \mathcal{L}(M) \to \mathcal{L}(N)$ takes each ideal J of M to the ideal generated by $f[J]$ in N; that is, $\mathcal{L}f(J) = \langle \bigcup\{\langle f(a)\rangle | a \in J\}\rangle$. This map obviously preserves arbitrary joins, since it is a left adjoint to the map $I \longmapsto f^{-1}[I]$ from $\mathcal{L}(N)$ into $\mathcal{L}(M)$, and the preservation of finitary meet is easily checked. □

The functor in Proposition 4.1.1 is called the *Keimel functor* and is denoted by $\mathcal{L}$.

Now we verify the functor of uniform closed spectrum for ℓ-modules and convergence closed spectrum for Riesz space.

4.1.2 Theorem *Suppose that A is a strong ordered ring. Then the correspondence $M \mapsto C\mathcal{L}(M)$ defines a functor from* $\mathbf{B\ell Mod(A)}$ *to* **KNRFrm**, *such that for every M, $C\mathcal{L}f \circ c_M = c_M \circ f$. Moreover, The restriction of this functor on* $\mathbf{B\ell Mod(A)}$ *is a functor to* **KCRFrm**.

Proof By Theorem 3.3.3(2) it is well defined on objects. Let $f : M \to N$ be an ℓ-module homomorphism. By Corollary 2.3.14, f is continuous. Hence for every $J \in \mathcal{L}(M)$, $f[\overline{J}] \subseteq \overline{f[J]}$. Thus

$$\mathcal{L}f \circ c_M(J) \subseteq c_N \circ \mathcal{L}f(J)$$

By Lemma 1.2.10, there exists a frame map denoted by $C\mathcal{L}f : C\mathcal{L}(M) \to \mathcal{L}(N)$ such that $C\mathcal{L}f \circ c_M = c_N \circ \mathcal{L}f$. This implies that $C\mathcal{L}$ with this definition on maps is a functor. The second part of the theorem results from 3.3.2. □

The functor in Proposition 4.1.2 is called the *uniform closed functor* and is denoted by $C\mathcal{L}$. We have:

4.1.3 Corollary *Suppose that A is a strong ordered ring. Then* $\mathbf{c} = (c_M)_M$ *is a natural transformation from $\mathcal{L}$ to $C\mathcal{L}$.*

The functor in Proposition 4.1.4 is called the *maximal functor* and is denoted by $S\mathcal{L}$. We have:

4.1.4 Theorem *Suppose that A is a strong ordered ring. Then the correspondence $M \mapsto S\mathcal{L}(M)$ defines a functor from* $\mathbf{B}f\mathbf{Mod(A)}$ *to* **KCRFrm** *such that* $\mathbf{h} = (h_M)_M$ *is a natural isomorphism from $C\mathcal{L}$ to $SC\mathcal{L}$.*

Proof Apply Theorems 3.3.3(5) and 4.1.2. □

From Theorem 4.1.3 we have:

4.1.4 Corollary *Suppose that A is a strong ordered ring. Then* $\mathbf{s} = (s_M)_M$ *is a natural transformation from $\mathcal{L}$ to $S\mathcal{L}$.* Also, the two natural transformation $\mathbf{c}$ and $\mathbf{s}$ are naturally isomorphic.

4.1.5 Remark By Remark 3.3.11, the assignment $\mathbf{C^o}\mathcal{L} : E \longmapsto C^o\mathcal{L}(E)$ is functorial from **BUARsz** to **KCRFrm**, and it is equal to $\mathbf{C}\mathcal{L}$, and $\mathbf{c^o} = (c^o_E)_E$

is a natral transformation.

$$\begin{array}{ccccc} & \mathcal{L}(E) & \xrightarrow{c^o_E} & C^o\mathcal{L}(E) & \\ \mathcal{L}(f) & \downarrow & & \downarrow & C^o\mathcal{L}(f) \\ & \mathcal{L}(D) & \xrightarrow{c_D} & C^o\mathcal{L}(E) & \end{array}$$

We denote the functor $\mathbf{C^o}\mathcal{L} = \mathbf{C}\mathcal{L}$ by $\mathcal{M}$.

4.2 The functor of continuous real functions

In this section we consider $C(L)$ as a Riesz space with the scaler multiplication $r\alpha = r.\alpha$, where $r \in \mathbb{Q}$, $\alpha \in C(L)$ and the operation . is the ring multiplication of $C(L)$. Part (1) of the following lemma completes the proof of $C(L)$ to be a Riesz space, and the rest describe the order of $C(L)$.

4.2.1 Lemma *Let L be a frame, $r, s \in \mathbb{Q}$, and $\alpha \in C(L)$.*

(1) *For each continous operation $\diamondsuit : \mathbb{Q}^2 \to \mathbb{Q}$, we have $r1\diamondsuit s1 = (r\diamondsuit s)1$.*

(2) *$\alpha \geq 0$ if and only if $\alpha(-,0) = 0$, where $(-,0) = \bigvee\{(p,0)|p < 0\}$.*

(3) *$\alpha \leq 0$ if and only if $\alpha(0,-) = 0$, where $(0,-) = \bigvee\{(0,p)|p > 0\}$.*

(4) *If $\alpha \leq \beta$ and $\beta \in C(L)$, then $\alpha(p,-) \leq \beta(p,-)$ for all $p \in \mathbb{Q}$, where $(-,p) = \bigvee_{q<p}(q,p)$ and $(p,-) = \bigvee_{p<q}(p,q)$.*

Proof (1) Let $p, q \in \mathbb{Q}$. We denoted $r1 = r^*$, $s1 = s^*$, and $(r\diamond s)1 = (r\diamond s)^*$ for easiness.

Case 1: Suppose that $0 = (r\diamondsuit s)^*(p,q)$. We will prove that $r^*\diamondsuit s^*(p,q) = 0$. Let $x, y, z, w \in \mathbb{Q}$ such that $(x,y)\diamondsuit(z,w) \leq (p,q)$.

Since $[[p < r\diamond s < q]] = 0$, either $x < r < y$ or $z < s < w$ is false. So $[[x < r < y]] \wedge [[z < s < w]] = 0$, and hence

$$r^*\diamond s^*(p,q) = \bigvee\{[[x < r < y]] \wedge [[z < s < w]] | (x,y)\diamond(z,w) \leq (p,q)\} = 0$$

Case 2: Suppose that $(r\diamond s)^*(p,q) = 1$, so $p < r\diamond s < q$.
Let $V = \{(t_1,t_2) \in \mathbb{Q}^2 | p < t_1\diamond t_2 < q\}$. In fact, V is the inverse image of the open internal (p,q) under the continouse map $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$.
Then V is an open set and $(r,s) \in (x,y) \times (z,w) \subseteq V$, for some $x,y,z,w \in \mathbb{Q}$. It means that $(x,y)\diamond(z,w) \leq (p,q)$ and $[[x < r < y]] = 1 = [[z < s < w]]$. So

$$1 = \bigvee\{[[x < r < y]] \wedge [[z < s < w]] | (x,y)\diamond(z,w) \leq (p,q)\} = r\diamond s(p,q).$$

Hence, $r^*\diamond s^* = (r\diamond s)^*$.

(2) Suppose that $\alpha \geq 0$. Let $p < 0$. We have

$$\begin{aligned}\alpha(p,0) = (0 \vee \alpha)(p,0) &= \bigvee\{[[r < 0 < s]] \wedge (z,w) | (r,s) \vee (z,w) \subseteq (p,0)\}\\ &= \bigvee\{[[r < 0 < s]] \wedge \alpha(z,w) | [[r < 0 < s]] = 0\}\\ &= 0\end{aligned}$$

Thus $\alpha(-,0) = 0$.

Conversely, suppose that $\alpha(-,0) = 0$ and $p,q \in \mathbb{Q}$. It is enough to show that $\alpha \wedge 0(p,q) = [[p < 0 < q]] = 0(p,q)$.

Case 1: Suppose that $[[p < 0 < q]] = 1$. Since $(p,r) \wedge (p,q) \leq (p,q)$, for all $r > q$, we can write

$$\begin{aligned}\alpha \wedge 0(p,q) &= \bigvee\{\alpha(r,s) \wedge [[z < 0 < w]] | (r,s) \wedge (z,w) \leq (p,q)\}\\ &\geq \bigvee\{\alpha(p,r) \wedge [[p < 0 < q]] | r > q\}\\ &= \alpha(\bigvee_{r>q}(p,r))\\ &= \alpha((p,-) \vee (-,0))\\ &= \alpha(e) = 1\end{aligned}$$

(note that $\alpha(-,0) = 0$). Thus $\alpha \wedge 0(p,q) = 1$.

Case 2: Suppose that $[[p < 0 < q]] = 0$.

Let $0 \leq p < q$. So, $(r,s) \wedge (z,w) \leq (p,q)$ implies $[[z < 0 < w]] = 0$. Thus

$$\alpha \wedge 0(p,q) = \bigvee\{\alpha(r,s) \wedge [[z < 0 < w]] | (r,s) \wedge (z,w) \leq (p,q)\} = 0$$

Let $p < q \leq 0$. So, $(r,s) \wedge (z,w) \leq (p,q)$ and $[[z < 0 < w]] = 1$ imply $(r,s) \leq (p,q)$. Hence,

$$\begin{aligned} \alpha \wedge 0(p,q) &= \bigvee\{\alpha(r,s) \wedge [[z < 0 < w]] | (r,s) \wedge (z,w) \leq (p,q)\} \\ &\leq \{\alpha(r,s) \wedge [[z < 0 < w]] | (r,s) \leq (p,q)\} \\ &\leq \bigvee\{\alpha(r,s) | (r,s) \leq (p,q)\} \\ &\leq \alpha(\bigvee\{(r,s) | (r,s) \leq (p,q)\}) \\ &= \alpha(p,q) \leq \alpha(-,0) = 0 \end{aligned}$$

(3) Results from (2).

(4) Let $p \in \mathbb{Q}$. Then

$$\alpha(p,-) = (\alpha \wedge \beta)(p,-) = \bigvee\{\alpha(r,s) \wedge \beta(z,w) | r, z \geq p\} \leq \bigvee_{z \geq p} \beta(z,w) = \beta(p,-)$$

Then the proof is complete. □

Note that since $C(L)$ is Archimedean as an f-ring, it is so as a Riesz space, too.

For any frame homomorphism $f : L \to L'$ define $\mathcal{C}(f) : C(L) \to C(L')$ by $\alpha \longmapsto f \circ \alpha$.

Since $C(f)$ is an f-ring homomorphism [4], it preserves $+, ., \wedge, \vee$. Now, let $r \in \mathbb{Q}$. We have

$C(f)(r\alpha) = C(f)(r).\ C(f)(\alpha) = r.C(f)(\alpha) = rC(f)(\alpha)$,

since $C(f)(r) = r$ (f preserves 0,1). Thus $C(f)$ is also a Riesz map.

Hence, we have the following Corollary:

4.2.2 Corollary $C : \mathbf{Frm} \to \mathbf{Rsz}$, *given as above, is a functor.*

4.2.3 Lemma *For any compact completely regular frame* L,

(i) $C(L)$ *is uniform.*

(ii) *For any frame map* $f : L \to L'$ *in* **KCRfrm**, $C(f)$ *is a continuous Riesz map.*

(iii) *If* L *is a compact, then* $C(L)$ *is bounded.*

Proof (i) Assume that $\alpha_n \in C(L)$ is a sequence with $\alpha_n \downarrow 0$ and hence $r > 0$. Let $0 < p < r$. We have $\alpha_n(-, p) \leq \alpha_{n+1}(-, p)$ and $\alpha_n(p, -) \geq \alpha_{n+1}(p, -)$ for all $n \geq 1$.

We prove that $\alpha_n \leq p$ for some $n \in \mathbb{N}$ and some p with $0 < p < r$. Suppose that $\alpha_n \nleq p$ for all n, p. Thus $\alpha_n(-, p) \neq 1$ and $\alpha_n(p, -) \neq 0$. Let $a_p = \bigvee_{n=1}^{\infty} \alpha_n(-, p) \neq 1$ and $b_p = \bigwedge_{n=1}^{\infty} \alpha_n(p, -)$. We have $a_p \wedge b_p = 0$. We have $b_p \neq 0$ for all $p < r$. Because, let $b_q = 0$ for some $0 < q < r$. For each p with $q < p < r$ we have $a_p \geq \alpha_m(-, p) \vee \alpha_m(q, -) = 1$ So, $a_p = 1$, and hence, by compactness of L, $\alpha_n(-, p) = 1$ for some $n \in \mathbb{N}$, which is a contradiction. Now, since L is completely regular, cozero sets of L generate L, we can find an $\alpha \in C(L)$ such that $0 < coz(\alpha) \leq b_p$. Let $\beta = \frac{1}{n}|\alpha|$, where $n \in \mathbb{N}$ with $np \geq \alpha$ exists by boundedness of $\mathcal{C}(L)$). We have $coz(\beta) \wedge a_p = 0$ and $0 < \beta \leq p$. Thus

$$\begin{aligned}
(\beta-\alpha_n)(0,-) &= \bigvee\{\beta(r,s)\wedge\alpha_n(z,w)|(r,s)-(z,w)\le(0,-)\}\\
&= \bigvee\{\beta(r,s)\wedge\alpha_n(z,w)|z<w\le r<s\}\\
&= \bigvee\{\beta(r,s)\wedge\alpha_n(z,w)|z<w\le r<s \text{ and } w>0 \text{ and } r\le p\}\\
&\quad (\text{since } \alpha_n>0 \text{ ans } 0<\beta\le p)\\
&\le coz\beta\wedge\alpha_n(-,p)\le coz\beta\wedge a_p=0.
\end{aligned}$$

Hence, $0<\beta\le\alpha_n$ for all $n\ge 1$, by Lemma 4.2.1(3). Which contradicts $\alpha_n\downarrow 0$. Hence there exists $0<p<r$ such that $\alpha_n\le p<r$ for some n. This proves that $C(L)$ is uniform.

(ii) Results from (i) and Corollary 2.3.15.

(iii) Let $\alpha\in C(L)$. We have $\bigvee_{n=1}^{\infty}\alpha(-n,n)=1$. Since L is compact, there exists n such that $\alpha(-n,n)=1$. Thus
$0=\alpha(n,-)=(\alpha-n)(0,-)$ and hence $0=\alpha(-,-n)=\alpha+n(-,0)$. So, by Lemma 4.2.1(2),(3), we have $-n\le\alpha\le n$. Thus $\alpha\in\langle 1\rangle$. Therefore, $C(L)=\langle 1\rangle$, and hence $C(L)$ is bounded. □

Now we have the following Corollary:

4.2.4 Corollary $C:KCRFrm\to\mathbf{BUAR}sz$ *is functorial.*

4.2.5 Proposition $C:\mathbf{Frm}\to\mathbf{Rsz}$ *is a faithful functor.*

Proof Suppose that $f,g:L\to M$ are frame maps such that $C(f)=C(g)$ where L is completely regular. Let $a\in L$. Since L is completely regular, $a=\bigvee\{coz(\alpha):coz(\alpha)\le a\}$. We have

$$\begin{aligned} f(a) = \bigvee\{f(coz(\alpha)) : coz(\alpha) \leq a\} &= \bigvee\{f(\alpha((-,0) \vee (0,-)) : coz(\alpha) \leq a\} \\ &= \bigvee\{g(\alpha((-,0) \vee (0,-)) : coz(\alpha) \leq a\} \\ &= g(a) \end{aligned}$$

So **C** is faithful.

4.3 The adjunction between functors

Recall from Theorem 4.1.2 $\mathbf{C}\mathcal{L} = \mathcal{M} : \mathbf{BUARsz} \to \mathbf{KCRFrm}$ given by $E \longmapsto \mathcal{M}E$ is functorial. In the following theorem we will prove that $\mathcal{M}$ is a left adjoint to C.

4.3.1 Proposition *$\mathcal{M}$ is a left adjoint to C.*

Proof Let L be a compact completly regular frame. We define $\varphi_L : \mathcal{L}C(L) \to L$ by $\varphi_L(I) = \bigvee_{x \in I^+} coz(x)$. It is easy to check that φ_L is a frame homorphism. The map φ_L is onto, since L is completely regular. To prove that φ_L is codense, note that $coz(\alpha) = 1$ implies $\langle\alpha\rangle = C(L)$. Since
$coz(|\alpha|) = 1$
$\Rightarrow |\alpha|(0,-) = 1$
$\Rightarrow |\alpha|(q,-) = 1$, for some $q > 0$
$\Rightarrow |\alpha| > q > 0$, for some $q > 0$
$\Rightarrow \langle\alpha\rangle = C(L) = 1_{\mathcal{L}C(L)}$.

By Proposition 3.3.3, $c_{C(L)} : \mathcal{L}C(L) \to \mathcal{M}C(L)$ is the smallest codense quotient of $\mathcal{L}C(L)$. So there exists a unique frame map $\sigma_L : \mathcal{M}C(L) \to L$ such that $\sigma_L \circ C_{\mathcal{C}(L)} = \varphi_L$. In particular, for all $\alpha \in C(L)$, $\sigma_L(\overline{\langle\alpha\rangle}) = coz\alpha$ where $\overline{\langle\alpha\rangle} = C_{C(L)}(\langle\alpha\rangle) = \overline{\langle\alpha\rangle}$.

Now for every $E \in \mathbf{BUARsz}$ we define $\tau_E : E \to C(\mathcal{M}(E))$ by $\tau_E(a) = \hat{a}$, where $\hat{a}(p,q) = \overline{\langle ((a-p) \wedge (q-a))^+ \rangle}$. We will prove that (σ_L, τ_A) is the adjunction.

To begin, we show that τ_A is a continuous Riesz map. We must show that for $\diamond = +, \vee, \wedge$, for every $a, b \in E$ and $r \in \mathbb{Q}$, $\widehat{a \diamond b} = \hat{a} \diamond \hat{b}$, and $\widehat{(ra)} = r\hat{a}$. First note that $(x \wedge y)^+ = x^+ \wedge y^+$ for all $x, y \in E$. Hence, the equations

$$\widehat{(-a)}(p,q) = \overline{\langle (-a-p)^+ \wedge (q+a)^+ \rangle} = \hat{a}(-q,-p) = (-\hat{a})(p,q)$$

and for $r \geq 0$

$$\widehat{(ra)} = \overline{\langle (ra-p)^+ \wedge (q-ra)^+ \rangle} = \overline{\langle (a-r^{-1}p)^+ \wedge (r^{-1}q-a)^+ \rangle} = \hat{a}(r^{-1}p, r^{-1}q) = (r\hat{a})(p,q)$$

prove $\widehat{(ra)} = r\hat{a}$.

Let $p, q, r, s, w, z \in \mathbb{Q}$ such that $(r,s) + (w,z) \leq (p,q)$. We have

$$\begin{aligned} (a+b-p) \geq (a-r+b-w) &= 2((a-r) \wedge (b-w) + (a-r) \wedge (b-w)) \\ &= 2(2((a-r) \wedge (b-w)) + d) \geq 4((a-r) \wedge (b-w)), \end{aligned}$$

where $d = (a-r) \vee (b-w) - (a-r) \wedge (b-w) \geq 0$. Thus $(a-r)^+ \wedge (b-w)^+ \leq \frac{1}{4}(a+b-p)^+$. Similarly, $(s-a)^+ \wedge (z-b)^+ \leq \frac{1}{4}(q-a-b)^+$. Hence,

$$\Big\langle ((a-r) \wedge (s-a))^+ \Big\rangle \cap \Big\langle ((b-w) \wedge (z-b))^+ \Big\rangle \subseteq \Big\langle ((a+b-p) \wedge (q-a-b)^+ \Big\rangle$$

since, $< x > \cap < y > = < x \wedge y >$ for all $x, y \geq 0$. Hence $\hat{a} + \hat{b}(p,q) \leq \widehat{a+b}(p,q)$. So, $\hat{a} + \hat{b}$, $\widehat{a+b}$ are frame homomorphisms from $L(\mathbb{R})$ to $\mathcal{M}E$ such that for every $u \in L(\mathbb{R})$, $\hat{a} + \hat{b}(u) \leq \widehat{a+b}(u)$. So we can prove the equality using the regularity of $L(\mathbb{R})$.

Next, we consider the operation $\wedge$. Let $p, q, r, s, t, u \in \mathbb{Q}$ such that $(r,s) \wedge (t,u) \leq (p,q)$. Thus $p \leq r \wedge t, s \wedge u \leq q$ and hence

$$(a-r) \wedge (b-t) \leq (a-p) \wedge (b-p) = (a \wedge b) - p.$$

On the other hand,

$$(s-a)\wedge(u-b)\leq(q-a)\vee(q-b)=q-(a\wedge b)$$

So $\hat{a}\wedge\hat{b}(p,q)\leq\widehat{a\wedge b}(p,q)$, which shows that $\hat{a}\wedge\hat{b}=\widehat{a\wedge b}$, again by regularity of $L(\mathbb{R})$. Finally, to prove $\tau_E(e)=\hat{1}$, note that for each $p,q\in\mathbb{Q}$, $\hat{1}(p,q)=1$ if and only if $\hat{p}<e<\hat{q}$ or $p<1<q$.

For the join operation, use $a\vee b=-((-a)\wedge(-b))$. Finally, by Lemma 2.3.13, since E is uniform and $\mathcal{C}(\mathcal{M}E)$ is Archimedean, $\tau_E: E\rightarrow\mathcal{C}(\mathcal{M}E)$ is a continuous Riesz map. Now we prove the adjunction identities:

$$\sigma_{\mathcal{M}E}\mathcal{M}_{\tau_E}=id_{\mathcal{M}E},(C\sigma_L)\tau_{CL}=id_{CL}$$

It is enough to show that

$$coz(\hat{a})=<a> \text{ and } coz(((\alpha-p)\wedge(q-\alpha))^+)=\alpha(p,q)$$

for all $a\in E$, $\alpha\in C(L)$, and $p,q\in\mathbb{Q}$, that $p<q\leq 0$, we have $q-a\leq -a$. Thus $0\leq(q-a)^+\leq(-a)^+$, and hence $(q-a)^+\in\langle a\rangle$. So $((a-p)\wedge(q-a))^+\in<a>$. In the case $0\leq p<q$ we have $a-p<a$. Thus $0<(a-p)^+<a^+\in<a>$, and hence $((a-p)\wedge(q-a))^+\in<a>$.
So $coz(\hat{a})=\bigvee_{\substack{0\leq p<q\\ p<q\leq 0}}\hat{a}(p,q)\leq<a>$. But for a rational number q with $q>0$ and $q>2a$, we have $\hat{a}(0,q)=\langle(a\wedge(q-a))^+\rangle=<a^+>$. Thus $a^+\in\hat{a}(0,-)$. Similarly, $a^-\in\hat{a}(-,0)$. So $a=a^+-a^-\in coz\hat{\alpha}$, and hence $coz(\hat{a})=<a>$.

The second identity of adjunction is proved as follows:

$$\begin{aligned}
coz\left((\alpha-p)^{+}\wedge(q-\alpha)^{+}\right) &= \left((\alpha-p)^{+}\wedge(q-\alpha)\right)^{+}(0,-)\\
&= \bigvee\{(\alpha-p)\vee\circ(r,s)\wedge(q-\alpha)\vee\circ(z,w)|r,z\geq 0\}\\
&= \bigvee\{(\alpha-p)(r,s)\wedge(q-\alpha)(z,w)|r,z>0\}\\
&= \bigvee\{\alpha(r+p,s+p)\wedge\alpha(q-w,q-z)|r,z\geq 0\}\\
&= \alpha\left(\bigvee_{\substack{r,z\geq 0\\ s>r\ u>z}}(r+p,s+p)\wedge(q-w,q-z)\right)\\
&= \alpha((p,-)\wedge(-,q))\\
&= \alpha(p,q)
\end{aligned}$$

Thus the proof is complete. □

4.4 Generalized pointfree Stone-Weierstrass Theorem

Now we prove the Stone-Weierstrass Theorem for $C(L)$ as a Riesz space, which we call it as the Generalized Pointfree Stone-Weierstrass Theorem because it is the pointfree version of the classical Stone-Weierstrass Theorem proved in [18] for $C(X)$ for a compact Hausdorff space X, and it is a generalization of the pointfree Stone-Weierstrass Theorem proved for $C(L)$ as an f-ring [4].

We refer [4] for reason of some definitions and results for the Stone-Weierstrass Theorem in $C(L)$ as f-ring:

4.4.1 Definition Let L be a frame. A subset $S \subseteq C(L)$ is called *separating* if L is generated by the set $Coz(S)$.

4.4.2 Remark The classic definition of separating set is the same as the pointfree definition of separating set, because:

4.4.3 Theorem *Let X be a compact Hausdorff space and A be a unitary subalgebra of $C(X)$. If the cozero sets of the elements of A generate $O(X)$, then A is point-separating, and conversely whenever A is also a sublattice of $C(X)$.*

4.4.4 Theorem (Pointfree Stone-Weierstrass Theorem) *For a compact completely regular frame M, any separating unitary $\mathbb{R}$-subalgebra A of $C(L)$ is uniformly dense in $C(L)$.*

Now we need the notion of *restriction* of $\alpha \in C(L)$ to some $w \in L$ corresponding to the topological notion of restricting continuous maps on a space to some open subspace:

4.4.5 Definition $\alpha|_w$ is the homomorphism $\mathcal{R} \to \downarrow w$ such that $\alpha|_w(r,s) = \alpha(r,s) = \alpha(r,s) \wedge w$, that is, the composite of α with the quotient map $L \to \downarrow w$ which $x \rightsquigarrow x \wedge w$.

4.4.6 Lemma *For any $\alpha \in C(L)$ and any natural k, there exist w_m, $m \in \mathbb{Z}$, covering L and $r_m \in \mathbb{Q}$ such that*

$$-1/k \leq (\alpha - r_m)|_{w_m} \leq 1/k$$

for all $m \in \mathbb{Z}$.

4.4.7 Theorem (Generalized pointfree Stone-Weierstrass Theorem)

Any separating subvector lattice E of $C(L)$ containing 1 is uniformly dense in $C(L)$ for a compact completely regular frame L.

Proof Let $\alpha \in C(L)^+$ and k be a natural number. Take the cover $C = \{w_m | m \in \mathbb{Z}\}$ of L and $r_m \in \mathbb{Q}$ as in Lemma 4.4.6 which have

$$-1/k \leq (\alpha - r_m)|_{w_m} \leq 1/k, \quad m \in \mathbb{Z}$$

Let $w \in C$. By complete regularity of L,

$$\begin{aligned} w = \bigvee\{x \in L : x \prec w\} &= \bigvee\{\bigvee\{coz(\alpha) : \alpha \in E, coz(\alpha) \leq x\} : x \prec w\} \\ &= \bigvee\{coz(\alpha) | \alpha \in E, coz(\alpha) \prec w\} \end{aligned}$$

Hence, the set $\{coz(\alpha) | \alpha \in E, coz(\alpha) \prec w$ for some $w \in C\}$ is a cover of L. Thus by compactness of L we can find $\alpha_1, \cdots, \alpha_n$ such that $coz(\alpha_1) \vee \cdots \vee coz(\alpha_n) = e$ and $coz(\alpha_i) \prec w_i$. Hence, there is a rational number $r > 0$ such that $r \leq |\alpha_1| + \cdots + |\alpha_n|$. Let $\beta_i = \frac{|\alpha_i|}{r} \wedge 1/n$. We have $\beta_i \in E$, $coz(\beta_i) = coz(\alpha_i) \prec w_i$, and $1 = \beta_1 + ... + \beta_n$. Now let $\beta = r_1\beta_1 + \cdots + r_n\beta_n$, $\beta \in E$. We claim that $|\alpha - \beta| \leq 1/k$.
For this, take any $w \in C' = \{w_1, coz(\beta_1)^*\} \wedge \cdots \wedge \{w_n, coz(\beta_n)^*\}$ which is a cover, by the proof of Proposition 1 of [3]. we have

$$(\alpha - \beta)|_w = \sum(\alpha\beta_i - r_i\beta_i)|_w = \sum((\alpha - r_i)|_w)(\beta_i|w)$$

Now for each $i, w \leq w_i$ or $w \leq coz(\beta_i)^*$, and therefore $-1/k \leq (\alpha - r_i)|_w \leq 1/k$ or $\beta_i = 0$, the latter because $coz(\beta_i|_w) = coz(\beta_i) \wedge w$. It follows that

$$-1/k\beta_i|_w \leq ((\alpha - r_i)|_w)(\beta_i|_w) \leq 1/k\beta_i|_w$$

So that $-1/k \leq (\alpha - \beta)|_w \leq 1/k$ by summation over i, and since the w form a cover, this implies $-1/k \leq \alpha - \beta \leq 1/k$, as desired. □

4.4.8 Remark Since every closed $\mathbb{R}$-subalgebra of $C(L)$ is a sublattice of $C(L)$([4]), Theorem 4.4.7 implies Theorem 4.4.4.

4.5 The Pointfree version of Kakutani duality

This section is the final aim of the thesis. Here we construct the pointfree version of Kakutani duality. Every bounded uniform Archimedean Riesz space E can be embedded into $C(L)$ for some compact completely regular frame L. If E is *complete* bounded uniform archimedean Riesz space then this embedding is isomorphism, which is the pointfree version of Kakutani duality.

4.5.1 Lemma *Let L be a compact completely regular frame, and E be a bounded uniform Archimedean Riesz space.*

(1) $\sigma_L : \mathcal{M}C(L) \to L$ *is an isomorphism.*

(2) $\tau_E : E \to C(\mathcal{M}E)$ is one-one.

(3) $\tau_{C(L)} : C(L) \to \mathcal{M}C(L)$ *is an isomorphism.*

Proof (1) Using 1.4.23, $\sigma_L : \mathcal{M}C(L) \to L$ is onto. It is trivial that $\sigma_L : \mathcal{M}C(L) \to L$ is dense, and since both L and $\mathcal{M}C(L)$ are compact regular, $\sigma_L : \mathcal{M}C(L) \to L$ is one-one. Thus $\sigma_L : \mathcal{M}C(L) \to L$ is an isomorphism.

(2) Suppose that $\hat{a} = 0$, and hence $\overline{\langle 0 \rangle} = coz(\hat{a}) = \overline{\langle a \rangle}$. Thus $\overline{\langle a \rangle} = \overline{\langle 0 \rangle}$. Since we have $\overline{\langle 0 \rangle} = \langle 0 \rangle$, thus $a = 0$. So τ_E is one-one.

(3) For every $\alpha \in C(L)$ and $p, q \in \mathbb{Q}$,

$$\begin{aligned}(C(\sigma_L)) \circ \tau_{C(L)}(\alpha)(p,q) &= \sigma_L(\overline{\langle(\alpha - p)^+ \wedge (q-\alpha)^+\rangle} \\ &= coz((\alpha - p)^+ \wedge (q-\alpha)^+) = \alpha(p,q),\end{aligned}$$

hence $(C(\sigma_L)) \circ \tau_{C(L)} = id_{C(L)}$, and using (1) $\tau_{\tau_{C(L)}}$ is an isomorphism. □

τ_E may not be onto. Here we prove that τ_E is onto if and only if E is *complete*, in the sence that:

4.5.2 Definition A Riesz map is called an *embedding* if $0 < a < 1$ is equivalent to $0 < f(a) < 1$, and is called *dense* if its image is dense in its codomain. A bounded uniform Archimedean Riesz space E is called *complete* if every dense embedding Riesz map from E to another such Riesz space is an isomorphism Riesz map.

From 1.3.10, we know that every bounded Archimedean Riesz space has a *natural M-norm* which is defined by $\|a\| = inf\{q \in \mathbb{Q} : |a| < q\}$. We have

4.5.3 Theorem *If $f : E \to D$ is a dense embedding then $\mathcal{M}f : \mathcal{M}E \to \mathcal{M}D$ is an isomorphism.*

Proof Since f is an embedding, we can assume that E is a sub Riesz space of D, and f is the inclusion map. Let J be a closed ℓ-ideal of D. Take $I = J \cap E$. We have

$$\mathcal{M}f(I) = \overline{\langle I \rangle_D} = \overline{\langle J \cap E \rangle_D} = J,$$

since E is dense in D. Thus $\mathcal{M}f$ is onto, and so it is an isomorphism. □

4.5.4 Lemma *Riesz map f from a bounded Archimedean Riesz space to another such Riesz space is an embedding if and only if f preserves the natural M-norm, that is, $\|a\| = \|f(a)\|$.*

Proof Suppose that E is a bounded Archimedean Riesz space and $f : E \to D$ is an embedding. Let $a \in E$ and $q \in \mathbb{Q}$. We have

$$|a| < q \Leftrightarrow 0 < |a|/q < 1 \Leftrightarrow 0 < f(|a|/q) = |f(a)|/q < 1 \Leftrightarrow |f(a)| < q.$$

Hence,

$$\|f(a)\| = inf\{q \in \mathbb{Q} : |f(a)| < q\} = inf\{q \in \mathbb{Q} : |a| < q\} = \|a\|$$

This proves the nontrivial part of the Lemma. □

4.5.5 Theorem *For every compact completely regular frame L, $C(L)$ is complete.*

Proof Let $f : C(L) \to E$ be a dense embedding. We have

$$\begin{array}{ccccc} & C(L) & \xrightarrow{f} & E & \\ \tau_{C(L)} & \downarrow & & \downarrow & \tau_E \\ & C(\mathcal{M}C(L)) & \xrightarrow{C(\mathcal{M}f)} & C(\mathcal{M})E & \end{array}$$

But, by Lemma 4.5.1(3), $\tau_{C(L)}$ and, by Theorem 4.5.3, $C(\mathcal{M}f)$ are isomorphisms. Hence, $\tau_E \circ f$ is an isomorphism. Thus there is a Riesz map $g : E \to C(L)$ such that $g \circ f = id_C(L)$, and so f is an isomorphism. □

4.5.6 Theorem *$\tau_E : E \to C(\mathcal{M}E)$ is onto if and only if E is complete.*

Proof We easily have that τ_E is an embedding, $\tau_E(E)$ is a subvector lattice containing $\hat{1} = 1$. And since for each $a \in E$, $coz(\hat{a}) = \overline{\langle a \rangle}$ it is separating. By Stone-Weierstras Theorem, $\tau_E(E)$ is dense in $C(\mathcal{M}E)$. So, by and Theorem 4.5.5, E is complete if and only if τ_E is onto. □

4.5.7 Remark We have an equevalence between the two categories of compact completely regular frames and bounded uniform Archimedean complete Riesz spaces, with Riesz maps. It is a pointfree version of the classic Kakutani duality(see [16,18]).

Note that, by Proposition 4.5.9, the category of all bounded complete uniform Riesz spaces with Riesz maps is isomorphic to the category of uniform M-spaces (Banach space with M-norm(see 1.3.10)), with Riesz maps, and the proof is free from the Axiom of Choice. It means that the pointfree version of Kakutani duality together with the Axiom of Choice imply the classical Kakutani duality.

4.5.8 Lemma *Suppose that E is a bounded Riesz space, and $\hat{E}$ is the norm completion of the normed space E with the M-norm $\|a\| = inf\{q : |a| < q\}$. Then $\hat{E}$ is a bounded Riesz space. Also, $\hat{E}$ is a M-space, and the inclusion map $\iota : E \to \hat{E}$ is a dense embedding Riesz map.*

proof Let $\hat{E}$ be the norm completion of E with its M-norm. The oper-

ations $+, \wedge, \vee$ and the scaler multiplication, each as a unary operations, are uniformly continuous by applying following relations:
$|x \wedge z - y \wedge w| \leq |x - y| + |z - w|$, $|x \vee z - y \vee w| \leq |x - y| + |z - w|$,
$|x + z - (y + w)| \leq |x - y| + |z - w|$, and $|rx - ry| \leq |r||x - y|$ for all $r \in \mathbb{Q}$.

So for $\diamond = \{+, \wedge, \vee\}$ and $f_r : E \longrightarrow E$ given by $f_r(x) = rx$, for all $r \in \mathbb{Q}$. Hence, there exist continuous operations $\hat{\diamond}$ and $\hat{f}_r$ such that the following diagram commute:

$$\begin{array}{ccc} E \times E & \xrightarrow{\diamond} & E \\ \iota \times \iota \downarrow & & \downarrow \iota \\ \hat{E} \times \hat{E} & \xrightarrow{\hat{\diamond}} & \hat{E} \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{f_r} & E \\ \iota \downarrow & & \downarrow \iota \\ \hat{E} & \xrightarrow{\hat{f}_r} & \hat{E} \end{array}$$

Where ι is the inclusion map. Suppose that $w = w'$ is an equation that holds in E, where w, w' are two terms. We prove that it is true in $\hat{E}$. Since $\hat{+}$, $\hat{\wedge}$, $\hat{\vee}$ and $\hat{f}_r$ are uniformly continuous. Note that $w_{\hat{E}} : \hat{E}^n \to \hat{E}$ and $w'_{\hat{E}} : \hat{E}^m \to \hat{E}$ are uniformly continuous which can proved by induction. Let $a_1, ..., a_n \in \hat{E}$ and $b_1, ..., b_m \in \hat{E}$. Now we can use from topology of convergence sequence by Theorem 2.3.8 to find sequences $(a_{ki})_{i \in \mathbb{N}}$ and $(b_{ti})_{i \in \mathbb{N}}$; $1 \leq k \leq n$ and $1 \leq t \leq m$. Such that $a_k = \lim_{i \to \infty} a_{ki}$ and $b_t = \lim_{i \to \infty} b_{ti}$. Hence

$$\begin{aligned} w_{\hat{E}}(a_1, \cdots, a_n) &= w_{\hat{E}}(\lim_{i \to \infty} a_{1i}, \cdots, \lim_{i \to \infty} a_{ni}) \\ &= \lim_{i \to \infty} w_{\hat{E}}(a_{1i}, \cdots, a_{ni}) \\ &= \lim_{i \to \infty} w'_{\hat{E}}(b_{1i}, \cdots, a_{mi}) \\ &= w'_{\hat{E}}(\lim_{i \to \infty} b_{1i}, \cdots, \lim_{i \to \infty} b_{mi}) \\ &= w'_{\hat{E}}(b_1, \cdots, b_m) \end{aligned}$$

Thus $w = w'$ holds in $\hat{E}$. Since E is a Riesz space $\hat{E}$ is a Riesz space too. Now we prove that

$$\|a\| \leq q \Leftrightarrow |a| \leq$$

for every $a \in \hat{E}$ and $q \in \mathbb{Q}$.

First we note that if $0 \leq a \leq b$ then $\|a\| \leq \|b\|$. Assume that $a = \lim_{n\to\infty} a_n$, $b = \lim_{n\to\infty} b_n$, and let $c_n = a_n^+$ and $d_n = a_n^+ \vee b_n^+$. We have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n \vee 0 = (\lim_{n\to\infty} a_n) = a \vee 0 = a$$

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} a_n^+ \vee b_n^+ = (\lim_{n\to\infty} a_n^+) \vee (\lim_{n\to\infty} b_n^+) = a \vee b = b$$

Also, $0 \leq c_n \leq d_n$, hence $\|c_n\| \leq \|d_n\|$ and we have $\|a\| = \lim_{n\to\infty} \|c_n\| \leq \lim_{n\to\infty} \|d_n\| = \|b\|$. And $\||a|\| = \|a\|$. Let $a = \lim_{n\to\infty} a_n$, we have $|a| = \lim_{n\to\infty} |a_n|$, hence $\||a|\| = \lim_{n\to\infty} \||a_n|\| = \lim_{n\to\infty} \|a_n\| = \|a\|$.

Now, suppose that $|a| \leq q$, thus $\|a\| = \||a|\| \leq \|q\| = q$. Conversely, suppose that $\|a\| \leq q$, and $a = \lim_{n\to\infty} \||a_n|\| \leq q$, hence there exists N, such that for every $n > N$; $\|a_n\| < q$, or $|a_n| < q$. Thus

$$|a| = \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |a_n| \wedge q = (\lim_{n\to\infty} |a_n|) \wedge q = |a| \wedge q$$

It means that $|a| < q$. For boundedness of $\hat{E}$, we show that $\hat{E} = [1]$, let $a \in \hat{E}$, and $\|a\| < n$, hence by previous argument $|a| < n1$. So, $\hat{E}$ is a M-space. For the second part, note that for every $a \in E$; $\|a\|$ in E is equal to $\|a\|$ in $\hat{E}$, thus $\|a\| = \|\ell(a)\|$, hence by Lemma 4.5.4 we have ℓ is an embedding. Obviously that it is ℓ is monomorphism and dense. So, it is a dense embedding. And the proof is complete.□

4.5.9 Proposition *A bounded Archimedean uniform Riesz space is an M-space if and only if it is complete.*

proof Suppose that E is complete. By previous lemma $\hat{E}$ is a M-space, and $\iota : E \to \hat{E}$ is a dense embedding, hence it is an isomorphim, and so $E = \hat{E}$.

Convesely, suppose that E is an M-space. Let $f : E \longrightarrow D$ be a dense embedding map. By the above remark, f is an M-norm preserving from E to D. Suppose that $d \in D$. Since $f(E)$ is dense, $d = \lim f(e_n)$ is a Cauchy sequence. So e_n is a Cauchy sequence, since f preserves the norm. Hence,

$$f(x) = f(\lim x_n) = \lim f(x_n) = d$$

So f is an isomorphism. This proves the Lemma.□

Chapter 5

Pointfree prime representation of real Riesz map

There is a classical representation theorem for real Riesz maps on $C(X)$ consisting of real-valued continuous functions on a compact Hausdorff space X which assigns to each real Riesz map $\phi : C(X) \to \mathbb{R}$ with $\phi(1) = 1$ a point $x \in X$ such that $\phi = \hat{x}$, where $\hat{x}$ is given by $\hat{x}(\alpha) = \alpha(x)$ for $\alpha \in C(X)$ (see [16], p. 163). The proof uses the Axiom of Choice and does not explicitely give the point x.

In this chapter we present the pointfree version of this representation. We replace the compact Hausdorff space X by a compact completely regular frame M and the map $\hat{x}$ by $\tilde{p}$, where p is a prime element of M. We define $\tilde{p}$ by Dedekind cuts that are used to define the real number $\tilde{p}(\alpha)$ for $\alpha \in C(M)$. Then we prove that each real Riesz map $\phi : C(M) \to \mathbb{R}$ is of the form $\phi(1)\tilde{p}$, where $p = \bigvee coz(ker\phi)$ (Lemma 5.2.2).

We then give the exact relation between real Riesz maps, prime elements, and prime ideals which are in $Fix(\eta)$ (Theorem 5.2.4). If M is not completely regular we can find a completely regular frame K_M which $C(M) \simeq C(K_M)$ and we have a correspondence between real Riesz maps and $\sum K_M$ (Theorem 5.2.5). Next we study the relation between $\hat{}$ and $\tilde{}$ in Proposition 5.2.6, and see when $\tilde{p}$ and $\tilde{q}$ are equal. Finally, we study the relationship between the primes elements and the cozero elements. The fact that there is no nonunit cozero element greater than each prime element is a result of this investigation. (Corollary 5.2.7).

5.1 Pointfree version of $\hat{x}$

Let M be a frame, $a \in M$, and $\alpha \in C(M)$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$ are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively. For $a \neq e$ it is obvious that for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$, $r \leq s$. In fact, we have:

5.1.1 Lemma *For a frame M, a prime element $p \in M$, and $\alpha \in C(M)$, $L = L(p, \alpha), U = U(p, \alpha)$ is a Dedekind cut, denoted by $\widetilde{p}(\alpha)$.*

Proof Since p is prime, using $\alpha(-, r) \wedge \alpha(r, -) = 0$ we get $L \cup U = \mathbb{Q}$. Since $\bigvee_L \alpha(-, r) \leq p$, $L \neq \mathbb{Q}$, and similarly, $U \neq \mathbb{Q}$. Obviously, L is a downset and U is an upset. This proves the lemma. $\square$

5.1.2 Corollary *For each prime element p there exists a unique map $\widetilde{p} : C(M) \longrightarrow \mathbb{R}$ such that for each $\alpha \in C(M)$, $r \in L(p, \alpha)$ and $s \in U(p, \alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.*

If $M = O(X)$ for a topological space X, then for every $x \in X$, $\hat{x} : C(X) \longrightarrow \mathbb{R}$ given by $\hat{x}(\alpha) = \alpha(x)$ factors through $\widetilde{\overline{\{x\}}'}$; note that in $O(X)$ the prime elements are exactly elements of the form $\overline{\{x\}}'$. In fact, $\hat{x} = \widetilde{\overline{\{x\}}'} \circ \phi$, where $\phi : C(X) \longrightarrow C(OX)$ is the isomorphism given by $\phi(\alpha)(p,q) = \alpha^{-1}(p,q)$. Hence, $\widetilde{\overline{\{x\}}'}$ is equal to $\hat{x}$ up to isomorphism. By the following proposition, $\tilde{p}$ is an f-ring homomorphism and a bounded Riesz map.

5.1.3 Proposition *If p is a prime element of a frame M, then $\widetilde{p} : C(M) \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. In particular, it is a bounded Riesz map.*

Proof Let $\diamond : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous map. We prove that $\widetilde{p}(\alpha\diamond\beta) = \widetilde{p}(\alpha) \diamond \widetilde{p}(\beta)$ for each α, β in $C(M)$. Let $r \in L(p, \alpha\diamond\beta)$ and $s \in U(p, \alpha\diamond\beta)$. We have $\alpha\diamond\beta(-,r) \leq p$ and hence, by the definition of the operations in $C(M)$,

$$\bigvee\{\alpha(-,z) \wedge \beta(-,w) : (-,z) \diamond (-,w) \leq (-,r)\} \leq p$$

Let $z, w \in \mathbb{Q}$ be such that $(-,z) \diamond (-,w) \leq (-,r)$. So $\alpha(-,z) \wedge \beta(-,w) \leq p$ and hence, since p is prime, $\alpha(-,z) \leq p$ or $\beta(-,w) \leq p$. Thus $z \leq \widetilde{p}(\alpha)$ or $w \leq \widetilde{p}(\beta)$. So

$$(\widetilde{p}(\alpha), \widetilde{p}(\beta)) \notin (-\infty, z) \times (-\infty, w)$$

This means that

$$(\widetilde{p}(\alpha), \widetilde{p}(\beta)) \notin \bigcup\{(-\infty, z) \times (-\infty, w) : (-,z) \diamond (-,w) \leq (-,r)\}.$$

Since $\diamond$ is continuous, $\widetilde{p}(\alpha) \diamond \widetilde{p}(\beta) \notin (-\infty, r)$, or $r \leq \widetilde{p}(\alpha) \diamond \widetilde{p}(\beta)$. Similarly, $\widetilde{p}(\alpha) \diamond \widetilde{p}(\beta) \leq s$. So, by Corollary 5.1.2, $\widetilde{p}(\alpha) \diamond \widetilde{p}(\beta)$ must be equal to $\widetilde{p}(\alpha\diamond\beta)$.

Now, since $+, ., \vee, \wedge$ are continuous operations, $\widetilde{p} : C(M) \longrightarrow \mathbb{R}$ preserves $+, ., \vee, \wedge$ and the scaler multiplication. It remains to prove that $\widetilde{p}(1) = 1$. For this, let $r \in L(p, 1)$ and $s \in U(p, 1)$. We have $1(-, r) \leq p$, hence $1(-, r) = 0$ and so $r \leq 1$. Similarly, $1 \leq s$. Thus $\widetilde{p}(1) = 1$. □

Now we want to prove that every real Riesz map on $C(M)$ is of the form $\pi\widetilde{p}$ for some prime element $p \in M$ and a real number $\pi \geq 0$. We begin by the following lemma; recall that $\mathcal{L}C(M)$ is the frame of ℓ-ideals of the Riesz space $C(M)$.

5.2 Prime representation of real Riesz maps of $C(M)$

5.2.1 Lemma *Suppose that M is a frame. Consider $e : \mathcal{L}C(M) \rightarrow M$ given by $e(I) = \bigvee Coz(I)$ and $\ell : M \rightarrow \mathcal{L}C(M)$ given by*

$$\ell(x) = \{\alpha \in C(M) : coz(\alpha) \leq x\}.$$

(1) *e is a left adjoint to ℓ,*

(2) *$\eta = \ell \circ e : \mathcal{L}C(M) \rightarrow \mathcal{L}C(M)$ is a nucleus and $I \in Fix(\eta)$ if and only if $I = \{\alpha : coz(\alpha) \leq x\}$ for some $x \in M$,*

(3) *If M is completely regular then $e \circ \ell = id_M$.*

(4) *If M is compact then η is codense.*

Proof (1) Let $I \in \mathcal{L}C(M)$ and $x \in M$. Suppose that $e(I) \leq x$ and $\alpha \in I$. We have $coz(\alpha) \leq e(I) \leq x$, and hence $\alpha \in \ell(x)$. So $I \subseteq \ell(x)$.

Conversely, suppose that $I \subseteq \ell(x)$. We have

$$e(I) = \bigvee Coz(I) \leq \bigvee Coz(\ell(x)) \leq x$$

So e is a left adjoint to ℓ.

(2) The extension property and idempotency follow from (1). Now let $I, J \in \mathcal{L}C(M)$, $\alpha \in \eta(I) \cap \eta(J)$, and hence

$$coz(\alpha) \leq (\bigvee Coz(I)) \wedge (\bigvee Coz(J)) = \bigvee Coz(I \cap J)$$

Thus $\alpha \in \eta(I \cap J)$, and so $\eta(I) \cap \eta(J) \subseteq \eta(I \cap J)$ which is the nontrivial side of $\eta(I) \cap \eta(J) = \eta(I \cap J)$. Also let $\eta(I) = I$. Then $\{\alpha : coz(\alpha) \leq \bigvee Coz(I)\} = \ell e(I) = \eta(I) = I$.

Conversely, let $I = \{\alpha : coz(\alpha) \leq x\}$ and $\alpha \in \eta(I)$. Then $coz(\alpha) \leq e(I) = \bigvee Coz(I) \leq x$, so $\alpha \in I$. Thus $\eta(I) = I$.

(3) Assume that M is completely regular. Thus $e\ell(x) = \bigvee\{\alpha \in C(M) : coz(\alpha) \leq x\} = x$

(4) Let M be compact and $\eta(I) = C(M)$. Then $1 \in \eta(I)$, so $\bigvee coz(I) = e$. Thus there exist $\alpha_1, \cdots, \alpha_k \in I$ such that $coz(\alpha_1) \vee \cdots \vee coz(\alpha_k) = e$. Let

$\alpha = |\alpha_1| + \cdots + |\alpha_k| \in I$. We have $\alpha(0, -) = coz(\alpha) = e$ and by compactness of M, $\alpha(r, -) = e$ for some rational number $r > 0$. Hence, $0 < r \leq \alpha$, and so $r \in I$. Thus $I = C(M)$. Therefore, η is codense. □

5.2.2 Lemma *Let M be a frame.*

(1) *For each nonzero bounded Riesz map $\phi : C(M) \to \mathbb{R}$ and a prime element p with $e(ker\phi) \leq p$ we have $\phi = \tilde{p}$.*

(2) *If $e(ker\phi) < e$ then $ker\phi \in Fix(\eta)$, and if M is completely regular then $e(ker\phi)$ is a prime element of M.*

Proof (1) Let $\alpha \in ker\phi$. Then $coz(\alpha) \leq e(ker\phi) \leq p$, so for each $r, s \in \mathbb{Q}$ with $r < 0 < s$ we have

$$\alpha(-, r) \vee \alpha(s, -) \leq coz(\alpha) \leq p$$

Hence, by the definition of $\widetilde{p}$, $\widetilde{p}(\alpha) = 0$, so $\alpha \in ker\widetilde{p}$ which proves that $ker\phi \subseteq ker\widetilde{p}$. Thus $\widetilde{p}$ factors through ϕ and so there exists a linear map $d : \mathbb{R} \to \mathbb{R}$ such that $d \circ \phi = \widetilde{p}$. Since $d(1) = d(\phi(1)) = \widetilde{p}(1) = 1$ and the identity is the only linear map preserving 1 form $\mathbb{R}$ to $\mathbb{R}$, we have $\phi = \widetilde{p}$.

(2) First we show that $ker\widetilde{p} = \ell(p)$ for all prime elements $p \in M$. Let $\alpha \in C(M)$. Then $\alpha \in ker\widetilde{p}$ if and only if for every rational numbers r, s with $r < 0 < s$, $\alpha(-, r) \vee \alpha(s, -) \leq p$ if and only if $coz(\alpha) \leq p$, and hence, if and only if $\alpha \in \ell(p)$. Now assume $e(ker\phi) < e$. By the Prime Ideal Theorem, there is a prime element $p \in M$ such that $e(ker\phi) \leq p$. Hence using (1) we get

$ker\phi = ker\widetilde{p} = \ell(p)$. So $\eta(ker\phi) = ker\phi$. Finally, let $x \wedge y \leq e(ker\phi)$. Then

$$\ell(x) \wedge \ell(y) = \ell(x \wedge y) \leq \ell e(ker\phi) = \eta(ker\phi) = ker\phi$$

Since $ker\phi$ is a prime ℓ-ideal, $\ell(x) \leq ker\phi$ or $\ell(y) \leq ker\phi$. Since M is completely regular, by Lemma 5.2.1(3), $x = e\ell(x) \leq e(ker\phi)$ or $y = e\ell(y) \leq e(ker\phi)$. Thus $e(ker\phi)$ is a prime element. □

5.2.3 Lemma *Let M be a compact completely regular frame, $\phi : C(M) \to \mathbb{R}$ a real Riesz map, p a prime element of M, and U be a prime ℓ-ideal of $C(M)$ such that $U \in Fix(\eta)$. Then $\phi = \phi(1)\widetilde{e(ker\phi)}$, $e(ker\tilde{p}) = p$, and $U = ker\widetilde{e(U)}$.*

Proof If $\phi(1) = 0$ then $\phi = 0$. Assume that $\phi(1) \neq 0$. Let $\phi' = \frac{\phi}{\phi(1)}$ which is a nonzero bounded Riesz map with $ker\phi' = ker\phi$. If $e(ker\phi) = e$ then $\eta(ker\phi) = \ell e(ker\phi) = \ell(e) = C(M)$. Now, since M is compact, η is codense, by Lemma 5.2.1(4), and so $ker\phi = C(M)$ which is a contradiction to ϕ being nonzero. By Lemma 5.2.2(2), $e(ker\phi)$ is a prime element and so, by Lemma 5.2.2(1), $\frac{\phi}{\phi(1)} = \phi' = \widetilde{ker\phi}$. Thus $\phi = \phi(1)e\widetilde{(ker\phi)}$.

By the proof of Lemma 5.2.2(2) with $ker\widetilde{p} = \ell(p)$ we get $e(ker\tilde{p}) = e\ell(p) = p$.

Finally, let $\alpha \in C(M)$. Then $\alpha \in U$ if and only if $\alpha(-, r) \vee \alpha(s, -) \leq coz(\alpha) \leq e(U)$ for all $r < 0 < s$ and, equivalently, $\widetilde{e(U)}(\alpha) = 0$. So it is enough to show that $e(U)$ is a prime element. Suppose that $x \wedge y \leq e(U)$. Hence, $\ell(x) \wedge \ell(y) = \ell(x \wedge y) \leq \ell e(U) = \eta(U) = U$, and thus $\ell(x) \leq U$ or $\ell(y) \leq U$. So, by complete regularity of M, $x = e\ell(x) \leq e(U)$ or $y = e\ell(x) \leq e(U)$, which yields that $e(U)$ is a prime element. □

Let $R(C(M),\mathbb{R})$ denote the set of all nonzero real Riesz maps on $C(M)$ and $\mathcal{U}$ be the set of all prime ideals U of $Fix(M)$. Then we have the following maps:

$F : R(C(M),\mathbb{R}) \to \mathbb{R}^+ \times \sum M$ given by $F(\phi) = (\phi(1), e(ker\phi))$,

$G : \mathbb{R}^+ \times \sum M \to R(C(M),\mathbb{R})$ given by $G(r,p) = r\tilde{p}$,

$H : R(C(M),\mathbb{R}) \to \mathbb{R}^+ \times \mathcal{U}$ given by $H(\phi) = (\phi(1), ker\phi)$, and

$K : \mathbb{R}^+ \times \mathcal{U} \to R(C(M),\mathbb{R})$ given by $K(r,U) = \widetilde{re(U)}$.

5.2.4 Theorem *For a compact completely regular frame M, F is inverse to G, and H is inverse to K. Also the restrictions of e and ℓ to $\mathcal{U}$ and $\sum M$, respectively, are inverse to each other.*

Proof Let $\phi \in R(C(M),\mathbb{R})$, $p \in \sum M$, $U \in \mathcal{U}$, and $r \in \mathbb{R}^+$. By Lemma 5.2.2(2), $e(ker\phi)$ is prime, and $\phi(1) > 0$. We have

$$G \circ F(\phi) = G(\phi(1), e(ker\phi)) = \phi(1)e\widetilde{(ker\phi)} = \phi$$

$$F \circ G(r,p) = F(r\tilde{p}) = (r\tilde{p}(1), e(ker(r\tilde{p}))) = (r, e\ell(p)) = (r,p)$$

$$K \circ H(\phi) = K(\phi(1), ker\phi) = \phi(1)e\widetilde{(ker\phi)} = \phi$$

$$H \circ K(r,U) = H(\widetilde{re(U)}) = (\widetilde{re(U)}(1), ker(\widetilde{re(U)})) = (r,u)$$

Finally, by Lemma 5.2.1(3), we have $e \circ \ell = id_M$ and $\ell \circ e(U) = \eta(U) = U$. Hence, the result. □

5.2.5 Proposition *Let K_M be the subframe of M generated by $Coz(M)$. Then*

(1) $\omega = c(i) : C(K_M) \to C(M)$ *is an f-ring isomorphism, where $i : K_M \hookrightarrow M$ is the inclusion map.*

(2) K_M *is completely regular. Moreover, if M is compact then so is K_M.*

(3) If M is compact then $F : R(C(M), \mathbb{R}) \rightleftarrows \mathbb{R}^+ \times \sum K_M : G$ *are inverse each other, where* $F(\phi) = (\phi(1), e(ker(\phi \circ \omega)))$ and $G(r, p) = r\widetilde{p} \circ \omega^{-1}$.

Proof (1) Trivially ω is an f-ring monomorphism. To show that ω is onto, let $\alpha : \mathcal{R} \to M$ be in $C(M)$. For every $r, s \in Q$, $\alpha(r, s) = coz((\alpha - r)^+ \wedge (s - \alpha)^+) \in Coz(M) \subseteq K_M$, and hence $Im(\alpha) \subseteq K_M$. Thus $\omega(\alpha) = i \circ \overline{\alpha} = \alpha$, where $\overline{\alpha}$ is the image restriction of α.

(2) For each $\alpha : \mathcal{R} \to M$, $coz(\alpha) = coz(\overline{\alpha})$, and for each $\beta : \mathcal{R} \to K_M$, $coz(\beta) = coz(\omega(\beta))$. So $Coz(K_M) = Coz(M)$. Hence, K_M is generated by $Coz(K_M)$. Thus K_M is completely regular. Moreover, if M is compact then every subframe of M is also compact.

(3) By (2), K_M is compact completely regular, and by Lemma 5.2.2(2), $e(ker(\phi \circ \omega))$ is a prime element of K_M, and hence F is well defined. Now let $(r, p) \in \mathbb{R}^+ \times \sum K_M$ and $\phi \in R(C(M), \mathbb{R})$. Then

$$F \circ G(r, p) = F(r\widetilde{p} \circ \omega^{-1})$$

$$= (r\tilde{p}(1), e(ker(r\tilde{p} \circ \omega^{-1} \circ \omega))) = (r, e(ker(r\tilde{p}))) = (r, e\ell(p)) = (r, p)$$

$$G \circ F(\phi) = G(\phi(1), e(\widetilde{ker(\phi \circ \omega)})) = \phi(1)e(\widetilde{ker(\phi \circ \omega)}) \circ \omega^{-1} = \phi \circ \omega \circ \omega^{-1} = \phi$$

Hence the result. □

5.2.6 Proposition *For any frame M, there is an ℓ-ring homomorphism $\psi : C(M) \to C(\sum M)$ given by $\psi(\alpha) = \tau \circ \sum \alpha$ with the property that for every prime elements p, q with $p \leq q$, $\hat{p} \circ \psi = \tilde{q}$.*

Proof Since for every $\alpha \in C(M)$, $\psi(\alpha)$ is continuous, ψ is well defined. Let $\diamond \in \{+, \vee, \wedge, .\}$. We show that $\psi(\alpha \diamond \beta) = \psi(\alpha) \diamond \psi(\beta)$.

First note that for every $\alpha \in C(M)$ and $p \in \sum M$, $\psi(\alpha)$ is a real number such that for every $r, s \in Q$,

$$r \leq \psi(\alpha)(p) \leq s \Leftrightarrow \alpha((-, r) \vee (s, -)) \leq p$$

Let $\psi(\alpha)(p) = a$ and $\psi(\beta)(p) = b$. We have

$$r \leq a \leq s \Leftrightarrow \alpha((-, r) \vee (s, -)) \leq p$$

$$r \leq b \leq s \Leftrightarrow \beta((-, r) \vee (s, -)) \leq p$$

We show that

$$r \leq a \diamond b \leq s \Leftrightarrow (\alpha \diamond \beta)((-, r) \vee (s, -)) \leq p$$

Suppose that $r \leq a \diamond b \leq s$. Let $x, y \in \mathbb{Q}$ such that $(-, x) \diamond (-, y) \leq (-, r)$. Thus

$$\begin{aligned} (-\infty, x) \times (-\infty, y) \subseteq \diamond^{-1}(-\infty, r) &\Rightarrow (a, b) \notin (-\infty, x) \times (-\infty, y) \\ &\Rightarrow a \nleq x \ \ or \ \ b \nleq y \\ &\Rightarrow \alpha(-, x) \wedge \beta(-, y) \leq p \end{aligned}$$

Hence

$$\alpha \diamond \beta(-, r) = \{\alpha(-, x) \wedge \beta(-, y) : (-, x) \diamond (-, y) \leq (-, r)\} \leq p$$

Similarly, $\alpha \diamond \beta(s, -) \leq p$.

Conversely, let $\alpha \diamond \beta((-, r) \vee (s, -)) \leq p$. It is enough to show that $(a, b) \notin \diamond^{-1}(-\infty, r), \diamond^{-1}(s, -)$. Suppose that $(-\infty, x) \times (-\infty, y) \subseteq \diamond^{-1}(-\infty, r)$, hence $(-, x) \diamond (-, y) \leq (-, r)$. Thus $\alpha(-, x) \wedge \beta(-, y) \leq p$, so $\alpha(-, x) \leq p$ or $\beta(-, y) \leq p$. Hence $x \leq a$ or $y \leq b$. Therefore, $(a, b) \notin (-\infty, x) \times (-\infty, y)$. So

$$(a, b) \notin \bigcup\{(-\infty, x) \times (-\infty, y) : (-, x) \diamond (-, y) \leq (-, r)\} = \diamond^{-1}(-\infty, r)$$

Similarly, $(a, b) \notin \diamond^{-1}(s, +\infty)$.

For the second part assume that p, q are prime elements with $p \leq q$ and let $\alpha \in C(M)$. We will prove that $\hat{p} \circ \psi(\alpha) = \tilde{q}$ or $\psi(\alpha)(p) = \tilde{q}$. Let $r \leq \psi(\alpha)(p) \leq s$, and hence $\alpha((-, r) \vee (s, -)) \leq p \leq q$. Thus $r \leq \tilde{q}(\alpha) \leq s$, and so $\psi(\alpha)(p) = \tilde{q}(\alpha)$. □

5.2.7 Corollary *Let p, q be prime elements of the frame M. Then*

(1) $\tilde{p} = \tilde{q} \Leftrightarrow Coz(M) \cap \downarrow p = Coz(M) \cap \downarrow q$,

(2) *If $p \leq q$ then $(p, q) \cap Coz(M) = \emptyset$ is the empty set, where $(p, q) = \{x \in M : p < x < q\}$,*

(3) $Coz(M) \cap (p, e) = \emptyset$.

Proof (1) We have $\tilde{p} = \tilde{q}$ if and only if for every $\alpha \in C(M)$ and $r, s \in \mathbb{Q}$, $\alpha(r, s) \leq p \Leftrightarrow \alpha(r, s) \leq q$. Since every element of $Coz(M)$ is of the form $\alpha(r, s)$ for some $r, s \in \mathbb{Q}$ we get the result.

(2) Assume that $p \leq q$. By Proposition 5.2.6, $\tilde{p} = \hat{p}\psi = \tilde{q}$, and by (1) we have $Coz(M) \cap \downarrow p = Coz(M) \cap \downarrow q$. Hence, $(p, q) \cap Coz(M) = \emptyset$.

(3) Let $x \in (p, e)$. Hence, by the Prime Ideal Theorem, there exists a prime element $q \geq x$. We have $x \in (p, q)$ and by (2) $(p, q) \cap Coz(M) = \emptyset$, hence $x \notin Coz(M)$. Thus $Coz(M) \cap (p, e) = \emptyset$. $\square$

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