Pointfree Spectra of Riesz Spaces

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Abstract

One of the best ways of studying ordered algebraic structures is through their spectra.

The three well-known spectra usually considered are the Brumfiel, Keimel, and the maximal spectra.

The pointfree versions of these spectra were studied by B. Banaschewski for f-rings.

Here, we give the pointfree versions of the Keimel and the maximal spectra for Riesz spaces. Moreover, we briefly mention how one can use the results of this paper to give a pointfree version of the Kakutani duality for Riesz spaces.

Keywords: Frame, Riesz space, uniform and bounded Riesz spaces, Keimel and maximal spectra.

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Introduction

The study of ordered algebraic structures via their spectra is one of the best ways to investigate their behaviour and properties. B. Banaschewski studied the pointfree version of the Brumfiel, Keimel, and the maximal spectra for f-rings.

Here, we define and study the pointfree version of the Keimel and the maximal spectra for Riesz spaces. A short survey of how one can use these results to give the pointfree version of the Kakutani duality for Riesz spaces is also included.

The necessary background on pointfree topology and Riesz spaces is given in Section 1.

In Section 2, we introduce the basic spectrum $\mathcal{L}(E)$ of a Riesz space E and study some of its pointfree properties. Among other things, we show that the frame $\mathcal{L}(E)$ is algebraic, coherently normal, and we give a necessary and sufficient condition under which $\mathcal{L}(E)$ is compact. Moreover, we show that this spectrum is functorial, from the category of Riesz spaces to the category of algebraic frames. And its restriction from the category of bounded Riesz spaces to the category of algebraic compact frames is also functorial.

Pointfree maximal spectrum is usually constructed using the saturation of the basic spectrum. In Section 3, we study this spectrum and investigate its relation with the spectrum of the closed ideals. Moreover, using the spectrum of the closed ideals, we show the functoriality of the maximal spectrum.

As the referee truly suggested, Section 4 contains a survey of the fact that the functoriality of the maximal spectrum and its adjunction to the functor of the real continuous functions provide the pointfree version of the classical Kakutani duality that arises in this context. The details is given in [5].

Regarding the foundations of set theory adopted here, our proofs proceed in Zermelo-Fraenkel set theory (as commonly understood - without the Axiom of Choice) on the basis of the classical logic. So one can treat the present subject in a way that is constructively valid in the sense of Topos Theory.

1 Background

Here, we recall some definitions and results from the literature on frames. For more details see the appropriate references given in the paper.

1.1 A *frame* is a complete lattice L which satisfies the distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$, for all $x \in L$ and $S \subseteq L$. A *frame map* $h : L \to M$ is a lattice morphism preserving arbitrary joins, the unit (top element) e, and the zero (bottom element) 0 of L. The resulting category is denoted by **Frm**.

1.2 As the most familiar examples of frames we have the finite distributive lattices, the complete chains, the complete Boolean algebras, and the lattice $O(X)$ of open sets of a topological space X.

1.3 Let L be a frame. We say that a is rather below b, and write $a \prec b$, if there exists a *separating element* s of L with $a \wedge s = 0$ and $s \vee b = e$.

Notice that $a \prec b$ if and only if $a^* \vee b = e$, where $a^* = \bigvee\{y : y \wedge a = 0\}$ is the pseudocomplement of a.

A frame L is called *regular* if each of its elements is a join of elements rather below it.

1.4 An element a of a frame L is said to be *completely below* b, written $a \prec\!\prec b$, if there exists an interpolating sequence (c_{nk}), $k = 0, 1, \cdots, 2^n$ and $n = 0, 1, \cdots$, between a and b, where $c_{00} = a,\ c_{01} = b,\ c_{nk} = c_{n+1,2k},\ c_{nk} \prec c_{n,k+1}$.

A frame L is called *completely regular* if each $a \in L$ is a join of elements completely below it.

1.5 An element $a \in L$ is called *compact* if $a = \bigvee S$ implies $a = \bigvee T$ for some finite $T \subseteq S$. A frame L is called *compact* whenever its unit e is compact; *algebraic* if every element of L is a join of compact elements; and *coherent* if it is compact, algebraic, and for every compact elements a and b, $a \wedge b$ is compact.

1.6 A frame L is called *normal* if $a \vee b = e$ implies that there exist u and v in L such that $a \vee u = e = b \vee v$ and $u \wedge v = 0$. L is said to be *coherently normal* if it is coherent and for each compact element $c \in L$, the frame $\downarrow c$ is normal.

1.7 A frame L is called *subfit* if $a < b$ implies $a \vee c < e = b \vee c$ for some $c \in L$.

1.8 A frame map $h : M \rightarrow L$ is called

dense if $h(x) = 0$ implies $x = 0$; *codense* if $h(x) = e$ implies $x = e$; and a *quotient map* if it is onto.

1.9 A *nucleus* is a map $n : L \rightarrow L$ on a frame L satisfying:

(N1) $x \leq n(x)$ for all $x \in L$; (N2) $x \leq y$ implies $n(x) \leq n(y)$; (N3) $n^2(x) = n(x)$, for all $x \in L$; (N4) $n(x \wedge y) = n(x) \wedge n(y)$, for all $x, y \in L$.

For any nucleus n on a frame L, the closure system $Fix(n) = \{a \in L : n(a) = a\}$ is a frame such that the map $n : L \rightarrow Fix(n)$ is a quotient map.

For any compact frame L one has the so called *saturation nucleus* s on L which plays an important role in this paper and is defined as follows. Recall that for any $x, a \in L$, x is called a-small if $x \vee y = e$ implies $a \vee y = e$, for all $y \in L$. The saturation nucleus is defined by $s(a) = \bigvee\{x \in L : x \text{ is } a\text{-small} \}$. Note that if L is compact, then $s(a)$ is *a-small* and hence it is the largest a-small element of L.

Now the map $s : L \rightarrow SL = Fix(s)$ is the unique smallest codense quotient of L (Banaschewski-Harting [4]). Further SL is subfit and $s : L \rightarrow SL$ is also the unique codense subfit quotient of L. Finally, SL is compact since s is codense (see [1]).

1.10 An element $p \in L$ is called *prime* if $p < e$ and $a \wedge b \leq p$ imply $a \leq p$ or $b \leq p$. An element $m \in L$ is called *maximal* if $m < e$ and $m \leq x \leq e$ imply $m = x$ or $x = e$. Note that every maximal element is prime.

For each frame L there is a topological space $\sum L$, called the *spectrum* of L, consisting of the prime elements of L, with open sets $\sum_a = \{p \in \sum L : a \not\leq p\}$, $a \in L$. The topological subspace of $\sum L$ consisting of all maximal elements of L is denoted by $Max(L)$.

A frame L is called *spatial* if it is isomorphic to the frame of open sets of some topological space; in fact, it is isomorphic to the frame of open sets of $\sum L$.

Banaschewki has given the following two lemmas:

1.11 Lemma [2] *For any compact frame* L, $Max(L) = Max(SL) = \sum(SL)$.

1.12 Lemma [1] *For any normal coherent frame* L*, the following are equivalent*:
(i) $L = SL$
(ii) L *is subfit.*
(iii) L *is regular.*

Now, we recall what we need about Riesz spaces from [9].

1.13 A vector space E over $\mathbb{Q}$, the field of rational numbers, with a partial order $\leq$ is called a *Riesz space* if $(E, \leq)$ is a lattice and $a \leq b$ implies $(a + c \leq b + c \ \& \ ra \leq rb)$, for all $a, b, c \in E \ \& \ r \in \mathbb{Q}^+$.

The set of all $x \in E$ with $x \geq 0$ is called the *positive cone* of E and is denoted by E^+.

1.14 For a Riesz space $E, a, b \in E$, and $r \in \mathbb{Q}$, define

$$a^+ = a \vee 0, \quad a^- = (-a) \vee 0, \quad |a| = a \vee (-a).$$

Then we have

$$a = a^+ - a^-, \ |a| = a^+ + a^-, \ a^+ \wedge a^- = 0, \ |a + b| \leq |a| + |b|$$

and $|ra| = |r||a|$. Also, for every $r \in \mathbb{Q}^+$ and $a, b \in E$ we have $r(a \wedge b) = ra \wedge rb$, since for $r > 0$ the map $x \mapsto rx$ is an order preserving bijection.

1.15 We have the decomposition property of F. Riesz which says that if $0 \leq x \leq y_1 + y_2 + \cdots + y_n$ and $y_i \geq 0$, for $1 \leq i \leq n$, in a Riesz space E, then there exist $x_i \in E$ such that $0 \leq x_i \leq y_i$ and $x = x_1 + \cdots + x_n$.

1.16 A (vector) subspace I of E is called an *ideal* if $|a| \leq |b|$ and $b \in I$ imply $a \in I$. The set of all ideals of E is denoted by $\mathcal{L}(E)$. Since we do not want to prove any new result in this section, we postpone the proof of the fact that $(\mathcal{L}(E), \subseteq)$ is a frame to Proposition 2.5.

1.17 We write $a \perp b$ if $|a| \wedge |b| = 0$. For $A \subseteq E$, the set $\{x \in E : x \perp a$ for all $a \in A\}$ is denoted by $A^\perp$, which is proved to be a band, and $A^{\perp\perp} = (A^\perp)^\perp$.

Recall that a Riesz space E is called *Archimedean* if $nx \leq a$ for all $n \in \mathbb{N}$ implies $x \leq 0$.

Let E be a (Dedekind) complete Riesz space; that is every bounded subset of E has a supremum and an infimum. For $A \subseteq E$, let $B(A)$ be the band generated by A. Then we have $E = A^\perp \oplus B(A)$, $A^{\perp\perp} = B(A)$. We also have that every complete Riesz space is Archimedean.

2 Basic spectra of a Riesz space

Having in mind the pointfree version of the Keimel spectrum $Spec_K(E) = \Sigma\mathcal{L}(E)$ of a Riesz space E as given in [1] for f-rings, we show that $\mathcal{L}(E)$ is an algebraic frame and give a necessary and sufficient condition under which $\mathcal{L}(E)$ is compact.

The following definition seems to be the right notion in this regard.

2.1 Definition Let E be a Riesz space. We say that E is *bounded* if there exists $u \in E$ such that $u \geq 0$ and $E = [u]$, the ideal generated by u.

To justify the above definition, first we give the following remark.

2.2 Remark For a bounded Riesz space $E = [u]$, the map $\bar{\cdot} \colon \mathbb{Q} \to E$ given by $r \longmapsto \overline{r} = ru$ is linear, one-one, $\wedge$ and $\vee$ preserving, and $\overline{rs} = r\overline{s}$. Also, for any positive $r \in \mathbb{Q}$, $E = [\overline{r}]$.

Recall that a topological space X is pseudocompact if and only if every element of $\mathcal{C}(X)$, the ring of real continuous functions with identity $\mathbf{1}$, is bounded.

Now, noting that $\mathcal{C}(X)$ is naturally a Riesz space, the following proposition justifies the above definition of a bounded Riesz space.

2.3 Proposition *For a topological space X, the Riesz space $\mathcal{C}(X)$ is bounded if and only if X is pseudocompact.*

Proof One direction is trivial, since $\mathcal{C}(X) = [\mathbf{1}]$. To prove the nontrivial direction, let $\mathcal{C}(X)$ be bounded, say $\mathcal{C}(X) = [u]$ for some $u \in \mathcal{C}(X)$. If u is not bounded, taking $f(x) = e^{u(x)}$ it is easy to check that $|f| \not\leq ru$ for all $r \in \mathbb{Q}^+$, which is a contradiction. Thus u, and hence every element of $\mathcal{C}(X)$, is bounded. $\square$

2.4 Remark Recalling that a compact topological space is pseudocompact, we get that for any compact topological space X, $\mathcal{C}(X)$ is a bounded Riesz space. The converse of this statement need not be true. In fact, there are pseudocompact topological spaces which are not compact (see [6], 5I).

Now we prove that $(\mathcal{L}(E), \subseteq)$, consisting of all ideals of a Riesz space E, is in fact an algebraic frame, and it is compact if and only if E is bounded.

2.5 Proposition *Let E be a Riesz space. Then*

(i) *$\mathcal{L}(E)$ is an algebraic frame.*

(ii) *$\mathcal{L}(E)$ is a compact frame if and only if E is a bounded Riesz space.*

Proof It is clear that $\mathcal{L}(E)$ is a complete lattice, since the intersection of ideals of E is an ideal of E.

Also notice that the join of a family $\{J_\gamma : \gamma \in \Gamma\}$ of ideals is given by $\bigvee J_\gamma = \sum J_\gamma$. This is because $\sum J_\gamma$ is an ideal. For, let $a \in E$ and $|a| \leq |a_1 + \cdots + a_n|$, where $a_k \in J_{\gamma_k}$, for $k = 1, \cdots, n$. Hence, $|a| \leq |a_1| + \cdots + |a_n|$, and, by the decomposition property in Riesz spaces, $|a| = x_1 + \cdots + x_n$ for some $0 \leq x_i \leq |a_i|$, and hence $|a| \in \sum J_\gamma$. Similarly, $a^+ \in \sum J_\gamma$, and so $a = 2a^+ - |a| \in \sum J_\gamma$.

To prove the distributivity of $\cap$ over arbitrary $\bigvee$, let $x \in J \cap \sum J_\gamma$. Then $x \in J$ and $|x| = u_1 + \cdots + u_n$, where $u_k \geq 0$, $u_k \in J_{\gamma_k}$, for $k = 1, \cdots, n$. Thus $|x| = |x| \wedge (u_1 + \cdots + u_n) \leq |x| \wedge u_1 + \cdots + |x| \wedge u_n$, where the latter inequality is proved by induction on n. For $n = 2$, we have

$$\begin{aligned} |x| \wedge (u_1 + u_2) &\leq ((2|x|) \wedge (|x| + u_1) \wedge (|x| + u_2) \wedge (u_1 + u_2) \\ &= (|x| + |x| \wedge u_1) \wedge (u_2 + |x| \wedge u_1) \\ &= |x| \wedge u_1 + |x| \wedge u_2. \end{aligned}$$

So $|x| \in \sum J \cap J_\gamma$, that is, $x \in \sum J \cap J_\gamma$. Hence, $\mathcal{L}(E)$ is a frame.

To prove the second part of (i), as well as the statement (ii), it is enough to show that for any ideal J of E, J is a compact element of $\mathcal{L}(E)$ if and only if $J = [a]$ for some $a \in E$.

Suppose that J is compact. So $J = [a_1] + [a_2] + \cdots + [a_n]$, for some $a_i \in J$. It is easy to see that $J = [a]$ for $a = |a_1| \vee \cdots \vee |a_n|$.

Conversely, suppose that $J = [a]$ and $J = \bigvee\{J_\gamma : \gamma \in \Gamma\} = \sum_{\gamma \in \Gamma} J_\gamma$. We can take each $J_\gamma = [a_\gamma]$ for some $a_\gamma \in J_\gamma$. Then, $[a] = \sum [a_\gamma]$ and so $|a| = u_1 + \cdots + u_k$ where $0 \leq u_t \in [a_{i_t}]$. Obviously, $[a] \subseteq [u_1] + \cdots + [u_k] \subseteq [a_{i_1}] + \cdots + [a_{i_k}] \subseteq [a]$, and hence $J = [a_{i_1}] + \cdots + [a_{i_k}]$, so J is compact. $\Box$

2.6 Lemma *In a Riesz space E, for all $a, b \in E^+$, $[a] \cap [b] = [a \wedge b]$.*

Proof The inclusion $[a] \cap [b] \supseteq [a \wedge b]$ is trivial. Let $x \in [a] \cap [b]$. Thus $|x| \leq ra, sb$, for some $r, s \in \mathbb{Q}^+$, hence also $|x| \leq r'a \wedge r'b$ for $r' = r + s$, and then $|x| \leq r'(a \wedge b)$, by 1.14. So $x \in [a \wedge b]$. This proves the lemma. $\Box$

2.7 Proposition *Let E be a Riesz space and P be an element of $\mathcal{L}(E)$. Then, the following are equivalent:*

(i) *P is a prime element of the frame $\mathcal{L}(E)$.*

(ii) *P is a prime ideal; that is $|x| \wedge |y| \in P \Rightarrow x \in P$ or $y \in P$.*

(iii) *$\sum\{a^\perp : a \in E - P\} \subseteq P$.*

(iv) *E/P is totally ordered.*

Proof (i) $\Rightarrow$ (ii): Let $x, y \in E$ and $|x| \wedge |y| \in P$. By Lemma 2.6, $[x] \cap [y] = [|x| \wedge |y|] \subseteq P$, and hence $x \in [x] \subseteq P$ or $y \in [y] \subseteq P$.

(ii) $\Rightarrow$ (i): Let $I \cap J \subseteq P$, $x \in I - P$, and $y \in J - P$. Hence, $|x| \in I - P$ and $|y| \in J - P$. Since P is a prime ideal, $|x| \wedge |y| \in (I \cap J) - P$, which contradicts $I \cap J \subseteq P$.

(ii) $\Rightarrow$ (iii): Let $a \in E - P$. Let $x \in a^{\perp}$. We have $|x| \wedge |a| = 0 \in P$. So $x \in P$, by definition. Then $a^{\perp} \subseteq P$.

(iii) $\Rightarrow$ (iv): Let $x + P, y + P \in E/P$. We have

$$(x - x \wedge y) \wedge (y - x \wedge y) = x \wedge y - x \wedge y = 0.$$

Hence, $x - x \wedge y \in P$ or $y - x \wedge y \in P$, and so $x + P \leq y + P$ or $y + P \leq x + P \in E/P$.

(iv) $\Rightarrow$ (ii): Let $x, y \in E$ and $|x| \wedge |y| \in P$. Assume that $x + P \leq y + P$. We have $x - x \wedge y \in P$, and hence $x \in P$. □

2.8 Remark The spectrum of the frame $\mathcal{L}(E)$ is called the *Keimel spectrum* and is denoted by $Spec_K E$ [1, 7]. Obviously BUT is enough to prove that $\mathcal{L}(E)$ is spatial, for any Riesz space E. Hence by the above proposition, BUT implies that $\bigcap\{P : E/P \text{ is totally ordered }\} = 0$, which means that the homomorphism from E to $\prod\{E/P : E/P \text{ is totally ordered }\}$ is one-one. So, BUT implies that every Riesz space is a subdirect product of totally ordered chains (see [1] for the same remark for f-rings).

2.9 Proposition *For every Riesz space E, $\mathcal{L}(E)$ is coherently normal.*

Proof Lemma 2.6 yields that $\mathcal{L}(E)$ is coherent. Thus we just verify the normality condition. Let $I + J = [a]$ where $I, J \in \mathcal{L}(E)$ and $a \in E^{+}$, which gives the compact element $[a]$ in $\mathcal{L}(E)$. Then $a = b + c$ for some $b \in I$ and $c \in J$.

Let $u = |c| - |b| \wedge |c|$ and $v = |b| - |b| \wedge |c|$. We have $u \in J$ and hence $I + [u] \subseteq [a]$. But

$$a = |a| \leq |b| + |c| = (|b| + |b| \wedge |c|) + u.$$

Thus $a \in I + [u]$. Similarly, $[a] = J + [v]$. □

2.10 Notation The category of all Riesz spaces with Riesz maps (linear-lattice maps) is denoted by $\mathcal{R}\mathbf{sz}$, and the category of all bounded Riesz spaces is denoted by $\mathcal{B}\mathbf{d}\mathcal{R}\mathbf{sz}$.
The categories of all algebraic frames and compact algebraic frames are denoted by $\mathcal{A}\mathbf{Frm}$ and $\mathcal{A}\mathbf{KFrm}$, respectively.

Now we are ready to give the following functors:

2.11 Proposition *The assignment $E \longmapsto \mathcal{L}(E)$ is functorial, from $\mathcal{R}\mathbf{sz}$ to $\mathcal{A}\mathbf{Frm}$. And its restriction to $\mathcal{B}\mathbf{d}\mathcal{R}\mathbf{sz}$ is also functorial, from $\mathcal{B}\mathbf{d}\mathcal{R}\mathbf{sz}$ to $\mathcal{A}\mathbf{KFrm}$.*

Proof For any map $f : E \to D$ in $\mathcal{R}\mathbf{sz}$, the corresponding $\mathcal{L}f : \mathcal{L}(E) \to \mathcal{L}(D)$ maps each ideal J of E to the ideal generated by $f[J]$ in D; that is $\mathcal{L}f(J) =< \bigcup\{[f(a)] : a \in J\} >$. This map obviously preserves arbitrary joins, since it is a left adjoint to the map $I \longmapsto f^{-1}[I]$ from $\mathcal{L}(D)$ into $\mathcal{L}(E)$, and the preservation of finitary meet is easily checked. □

3 Pointfree maximal spectrum

In this section we study $S\mathcal{L}(E)$, the saturation of the frame $\mathcal{L}(E)$, which gives $Spec_M E = \Sigma S\mathcal{L}(E)$, the pointfree version of the maximal spectrum.

First, having in mind the frame of closed ideals of an f-ring given in [1], we give the counterpart of this frame for Riesz spaces, and investigate its relation with the frame $\mathcal{L}(E)$. To do this, recall from [9] that:

If $(x_n)_1^\infty$ is a decreasing sequence in E and $x = \bigwedge\{x_n : n \in \mathbb{N}\}$, then we write $x_n \downarrow x$. Similarly the dual notion $x_n \uparrow x$ is defined.

A sequence $(x_n)_1^\infty$ with $x_n \downarrow 0$ is called a *null sequence.*

A sequence $(y_n)_1^\infty$ is called *order convergent* to y as $n \to \infty$, written as y=o-$lim_{n\to\infty} y_n$, if there exists a null sequence $(x_n)_1^\infty$ with $|y_n - y| \leq x_n$ for all $n \in \mathbb{N}$.

Note that a bounded Riesz space E is Archimedean if and only if $(\overline{1/n})_{n\geq 1}$ is a null sequence.

Applying the definition of ordered limit we get the following lemma:

3.1 Lemma [9] *Assume that* $(x_n)_1^\infty$, $(y_n)_1^\infty$ *are sequences in* E *such that o-*$lim_{n\to\infty} x_n = x$ *and* o-$\lim_{n\to\infty} y_n = y$. *Then,*

(i) o-$\lim_{n\to\infty}(x_n + ay_n) = x + ay$.

(ii) o-$\lim_{x\to\infty}(x_n \vee y_n) = x \vee y$.

(iii) o-$\lim_{n\to\infty}(x_n \wedge y_n) = x \wedge y$.

3.2 Definition A bounded Riesz space E is called *uniform* if for every null sequence (x_n) in E and $r \in \mathbb{Q}^+$ there exists $m \in \mathbb{N}$ such that $0 \leq x_m \leq \overline{r}$.

Let A be a subset of a Riesz space E. Denoting the set of all ordered limits of the sequences in A by $\overline{A}$, we have:

3.3 Lemma *For a uniform Archimedean bounded Riesz space* E, $c_E : \mathcal{L}(E) \to \mathcal{L}(E)$, *given by* $I \longmapsto \overline{I}$, *is a closure operator.*

Proof First notice that for an ideal J of E, $\overline{J}$ is an ideal. This is because first of all $\overline{J}$ is a subspace (Lemma 3.1 (i)). Now assume that $0 < x \leq |a|$ for $a \in \overline{J}$. Let o-$lim_{n\to\infty} a_n = a$, for $a_n \in J$. Then o-$\lim_{n\to\infty} |a_n| = |a|$, hence $x = |a| \wedge x$=o-$\lim_{n\to\infty}(|a_n| \wedge x)$, and so $x \in \overline{J}$, since $|a_n| \wedge x \in J$ for all n. Thus $|x| \leq |a|$ implies that $|x|, x^+ \in \overline{J}$, and hence x belongs to $\overline{J}$. Thus $\overline{J}$ is an ideal.

Second, to prove $\overline{\overline{J}} = \overline{J}$ suppose that $a \in \overline{\overline{J}}$. Then there exist sequences (x_n), $(y_{mn})_{n\geq 1}$, (t_n), and $(w_{mn})_{n\geq 1}$ such that $y_{mn} \in J$ for all $m, n \in \mathbb{N}$, and $t_n \downarrow 0$, $w_{mn} \downarrow 0$, such that for every $m, n \in \mathbb{N}$,

$$|a - x_n| \leq t_n, \quad |x_n - y_{mn}| \leq w_{mn}.$$

Since E is a uniform bounded Riesz space, we can assume that $(t_n)_{n\geq 1} = \overline{(1/n)}_{n\geq 1}$ and $(w_{mn})_{n\geq 1} = \overline{(1/mn)}_{n\geq 1}$. Hence we have

$$|a - y_{nn}| \leq |a - x_n| + |x_n - y_{nn}| \leq \overline{1/n} + \overline{1/nn}.$$

Now, since E is Archimedean, we get that $a=$ o-$\lim_{n\to\infty} y_{nn}$. So $a \in \overline{J}$, which proves the lemma. □

3.4 Lemma *The above closure operator also satisfies the following:*
(i) $\overline{I \cap J} = \overline{I} \cap \overline{J}$.
(ii) *If E has a nonzero null sequence, then $\overline{I}$ is a band of E.*

Proof (i) The inclusion $\overline{I} \cap \overline{J} \supseteq \overline{I \cap J}$ is trivial. Let $x \in \overline{I} \cap \overline{J}$. Thus $x=$ o-$\lim_{n\to\infty} a_n=$ o- $\lim_{n\to\infty} b_n$ for some $(a_n)_{n\geq 1}$ in I, $(b_n)_{n\geq 1}$ in J. Hence, $x^+=$ o-$\lim a_n^+ \wedge b_n^+$, $x^-=$ o-$\lim a_n^- \wedge b_n^-$. But $0 \leq a_n^+ \wedge b_n^+ \leq a_n^+,\ b_n^+,\ 0 \leq a_n^- \wedge b_n^- \leq a_n^-, b_n^-,\ a_n^+, a_n^- \in I$, and $b_n^+, b_n^- \in J$. Thus $a_n^+ \wedge b_n^+, a_n^- \wedge b_n^- \in I \cap J$ and hence $x^+, x^- \in \overline{I \cap J}$. So $x \in \overline{I \cap J}$. Thus $\overline{I} \cap \overline{J} = \overline{I \cap J}$.

(ii) By hypothesis, there exists a nonzero null sequence $(a_n)_{n\in\mathbb{N}}$. Let $S \subseteq \overline{I}$ and $x = \bigvee S$. Thus for each $n \geq 1$, $x - a_n < x$ and hence $x - a_n \leq s_n \leq x$ for some $s_n \in S$. Thus $x=$ o-$\lim_{n\to\infty} s_n$; in fact, $x \in \overline{\overline{I}} = \overline{I}$ and hence $\overline{I}$ is a band. □

We denote the set of all closed ideals of E, that is, ideals I for which $\overline{I} = I$, by $Cl(E)$.

3.5 Theorem *For a uniform Archimedean bounded Riesz space E, $Cl(E)$ is a frame and $c_E : \mathcal{L}(E) \to Cl(E)$ is codense. If in addition E is complete, then $Cl(E)$ is a subfit frame and $c_E : \mathcal{L}(E) \to Cl(E)$ is a subfit quotient.*

Proof By the above lemmas, $c_E : \mathcal{L}(E) \to \mathcal{L}(E)$ is a nucleus, and so $Fix(c_E) = Cl(E)$ is a frame. To prove that c_E is codense, let $\overline{J} = E = [u]$, where $u > 0$, and $J \in \mathcal{L}(E)$. Thus $u=$ o-$\lim_{n\to\infty} u_n$ for some $u_n \in J$. We can assume that $u \geq u_n > 0$. We have $u - u_n \leq x_n$ for some $x_n \downarrow 0$. Since E is a uniform bounded Riesz space, there exists $m \in \mathbb{N}$ such that $x_m \leq \overline{1/2}$. Thus $u - u_m < (1/2)u$, so $0 < u < 2u_m$, and hence $u \in J$, since $u_m \in J$. So $J = E$, which means that c_E is codense.

Moreover, if E is complete, then $Cl(E)$ is subfit. To see this, let $I \subset J$ in $Cl(E)$. We claim that $J^\perp \in Cl(E)$, $E = J + J^\perp$ and $E \neq I + J^\perp$. Let $x \in \overline{J^\perp}$. Then $x=$ o-$\lim_{n\to\infty} x_n$ for some $(x_n)_{n\geq 1}$ in $J^\perp$. We have $|x| \wedge |h|=$ o-$\lim_{n\to\infty}(|x_n| \wedge |h|) =$ o-$\lim_{n\to\infty} 0 = 0$ for all $h \in J$. Thus $x \in J^\perp$ and so $J^\perp$ is a closed ideal.

Since $E = B(J) \oplus J^{\perp}$, J is closed, and E, being Archimedean, has a nonzero null sequence, by Lemma 3.4(ii), $B(J) = \overline{J} = J$. Hence, $E = J \oplus J^{\perp}$.

Now let $x \in J \setminus I$. If $x \in I + J^{\perp}$, then $x = a + b$ for some $a \in I$ and $b \in J^{\perp}$. Thus $b = x - a \in I \cap J^{\perp}$, so $b = 0$ and hence $x \in I$, which is a contradiction to $x \in J \setminus I$. So $x \notin I + J^{\perp}$ and hence $E \neq I + J^{\perp}$. This proves the claim. Note that since c_E is codense, $E \neq \overline{I + J^{\perp}} = I \vee J$ (in $Cl(E)$). It proves that $Cl(E)$ is subfit and $c_E : \mathcal{L}(E) \to Cl(E)$ is a subfit quotient. □

3.6 Corollary *If E is a uniform complete bounded Riesz space, then there is an isomorphism $h : S\mathcal{L}(E) \to Cl(E)$ such that $hs_E = c_E$:*

$$\begin{tikzcd}
\mathcal{L}(E) \arrow[r, "s_E"] \arrow[dr, "c_E"] & S\mathcal{L}(E) \arrow[d, "h"] \\
 & Cl(E)
\end{tikzcd}$$

Proof Apply Theorem 3.5 and 1.9. □

3.7 Proposition *For a bounded Riesz space E, $S\mathcal{L}(E)$ is completely regular, and hence if E is moreover uniform and Archimedean, then $Cl(E)$ is compact completely regular, too.*

Proof We begin by showing that $[(a - \overline{p})^{+}] \prec\prec [a]$ in $\mathcal{L}(E)$, for all $a \in E$ and each rational number p with $0 < p \leq 1$. We will show that $[(a - \overline{p})^{+}] \prec [(a - \overline{q})^{+}]$ for all $0 < q < p$. We have

$$\begin{array}{rcl}
(a - \overline{q})^{+} + (a - \overline{p})^{-} & = & (a - \overline{q}) \vee 0 + (\overline{p} - a) \vee 0 \\
 & = & (a \vee \overline{q}) - \overline{q} + (\overline{p} \vee a) - a \\
 & = & 0 \vee (\overline{q} - a) + (\overline{p} - \overline{q}) \vee (a - \overline{q}) \\
 & \geq & \overline{p} - \overline{q} = \overline{p - q} > 0
\end{array} .$$

Thus $[(a - \overline{q})^{+}] + [(a - \overline{p})^{-}] \supseteq [(a - \overline{q})^{+} + (a - \overline{p})^{-}] \supseteq [\overline{p - q}] = E$.

On the other hand, $(a - \overline{p})^{+} \wedge (a - \overline{p})^{-} = 0$. Hence, by Lemma 2.6, $[(a - \overline{p})^{+}] \cap [(a - \overline{p})^{-}] = 0$. Thus $[(a - \overline{p})^{+}] \prec [(a - \overline{q})^{+}]$.

Now we show the complete regularity of $S\mathcal{L}(E)$. Since for every $J \in S\mathcal{L}(E)$, $J = \bigcup\{[a] : a \in J^{+}\}$ (in $\mathcal{L}(E)$), $J = s(J) = \bigvee\{s[a] : a \in J^{+}\}$ (in $S\mathcal{L}(E)$). So it will be sufficient to show that for each $a \in E^{+}$, $s[a]$ is the join in $S\mathcal{L}(E)$ of the corresponding $s[(a - p)^{+}]$, $0 < p \leq 1$ in $\mathbb{Q}$; that is, $s[a] = s(I)$ for the ideal $I = \bigcup\{[(a - p)^{+}]$: $0 < p \leq 1$ in $\mathbb{Q}\}$. The inclusion $s(I) \subseteq s[a]$ is trivial. So it is enough to show that $[a]$ is I-small.

Let $[a] + J = E$. Thus $\overline{1} \in [a] + J$, and so we have $1 \leq ra + b_1$ for some

$b_1 \in J$ and $r > 0$. We have

$$\begin{array}{ll} & 0 < \frac{1}{r} \leq a + \frac{1}{r}b_1 = a + b \text{ for } b = \frac{1}{r}|b_1| \\ \Rightarrow & [a \vee b] = [a] + [b] \supseteq [a+b] \supseteq [\frac{1}{r}] = E \\ \Rightarrow & 0 < \overline{p} < \overline{q} < a \vee b \text{ for some } p, q \in Q \\ \Rightarrow & \overline{q} - \overline{p} \leq (a - \overline{p}) \vee (b - \overline{p}) \leq (a - \overline{p})^+ \vee b \in I + J \\ \Rightarrow & \overline{q - p} \in I + J = E. \end{array}$$

To show the complete regularity of $Cl(E)$, note that for every $J \in Cl(E)$, $J = \bigcup\{[a] : a \in J^+\}$ (in $Cl(E)$), and hence, $J = \overline{J} = \bigcup\{\overline{[a]} : a \in J^+\}$ (in $Cl(E)$). So, it is sufficient to show that for each $a \in E^+$, $\overline{[a]}$ is the join in $Cl(E)$ of the corresponding $\overline{[(a-p)^+]}$, $0 < p \leq 1$ in $\mathbb{Q}$; that is, $\overline{[a]} = \overline{I}$ for the ideal $I = \bigcup\{[(a-p)^+]$: $0 < p \leq 1$ in $\mathbb{Q}\}$. $\overline{I} \subseteq s[a]$ is trivial. Using the Archimedean condition, $a =$o-$\lim_{n\to\infty} a - \overline{n^{-1}}$, so $a = a^+ =$o-$\lim_{n\to\infty}(a - \overline{n^{-1}})^+$, since $(a - \overline{n^{-1}})^+$ for all $n \in \mathbb{N}$. Hence, $a \in \overline{I}$, and so $\overline{[a]} \subseteq \overline{I}$. And the proof is complete. □

3.8 Remark The frame $Cl(E)$ is normal, because it is the codense image of $\mathcal{L}(E)$ (use 1.2 of [1]). Moreover, since $Cl(E)$ is closed under directed unions, the compact elements of $Cl(E)$ are exactly the $\overline{[a]}, a \in E^+$, and so, using Lemma 3.4(i), we get that $Cl(E)$ is coherent.

Thus, using Lemma 1.12 and Proposition 3.7, we can have Theorem 3.5 and Corollary 3.6 without the condition of completeness. See the following proposition, too.

3.9 Proposition *For a complete uniform Archimedean bounded Riesz space E, $Cl(E)$ is coherently normal.*

Proof By the above remark it remains to prove that $\downarrow \overline{[a]}$ is normal in $Cl(E)$. Let $I, J \in Cl(E)$ with $I \vee J = \overline{[a]}$. Also let $U = I^\perp \cap \overline{[a]}$ and $V = J^\perp \cap \overline{[a]}$. It is clear that U and V are closed ideals. Since $U \cap I = 0$, we have $U \cap J = U \cap (I \vee J) = U \cap \overline{[a]} = U$, hence $U \subseteq J$ and so $U \cap V = 0$.

On the other hand, using completeness of E, we get $\overline{[a]} = \overline{I \oplus U} = I \vee U$, and similarly, $\overline{[a]} = J \vee V$. This proves the proposition. □

In the rest of this section we are going to show that $Cl(E)$ gives a functor from a suitable subcategory of $\mathcal{R}$**sz** to the category **KCRFrm**, of all compact completely regular frames.

3.10 Definition Let A, B be Riesz spaces and $f : A \to B$ be a Riesz map. f is called a *continuous* Riesz map if $f(x_n) \downarrow 0$ whenever $x_n \downarrow 0$ in A.

3.11 Lemma *Let $f : A \to B$ be a Riesz map. Then*

(i) *$f(x_n) \downarrow f(x)$ whenever $x_n \downarrow x$ in A.*

(ii) *if* $a=$ o-$\lim_{n\to\infty} a_n$*, then* $f(a)=$ o-$\lim_{n\to\infty} f(a_n)$.

Proof (i) Let $t_n = x_n - x$. Then $t_n \downarrow 0$, and so $f(t_n) \downarrow 0$. But $f(t_n) = f(x_n) - f(x)$, and so $f(x_n) \downarrow f(x)$.

(ii) Suppose that $|a - a_n| \leq t_n$, where $t_n \downarrow 0$. Hence $|f(a) - f(a_n)| = f(|a - a_n|) \leq f(t_n)$ and $f(t_n) \downarrow 0$, which means $f(a)=$ o-$\lim_{n\to\infty} f(a_n)$. $\square$

3.12 Notation The category of all bounded uniform Archimedean Riesz spaces and Riesz maps is denoted by $\mathcal{B}d\mathcal{UAR}$**sz**.

3.13 Lemma *If* $f : E \to D$ *is a Riesz map,* E *is a uniform Riesz space, and* D *is an Archimedean Riesz space, then* f *is a continuous Riesz map. In particular, every map in the category* $\mathcal{B}\mathbf{d}\mathcal{UAR}sz$ *is continuous.*

Proof Assume that $x_n \downarrow 0$ in E. Let $k \in \mathbb{N}$. Since E is uniform, there exists m_k such that for each $m \geq m_k$, $0 \leq x_m < \overline{k^{-1}}$. So $0 \leq f(x_m) \leq f(\overline{k^{-1}}) = k^{-1}f(u)$ for all $m \geq m_k$, where $E = [u]$. Since D is Archimedean, $\inf\{k^{-1}f(u) : k = 1, 2, \cdots\} = 0$, hence, $\inf\{f(x_m) : m = 1, 2, \cdots\} = 0$, and so $f(x_n) \downarrow 0$.

3.14 Proposition

(i) *If* $f : E \to D$ *is in* $\mathcal{B}\mathbf{d}\mathcal{UAR}$**sz***, then* $\mathcal{L}f(\overline{J}) \subseteq \overline{\mathcal{L}f(J)}$ *for every ideal* J *in* E.

(ii) *The assignment* $E \longmapsto Cl(E)$ *is functorial, from* $\mathcal{B}\mathbf{d}\mathcal{UAR}sz$ *to* **KCRFrm**.

Proof Let $a \in \overline{J}$. Then $a=$ o-$\lim_{n\to\infty} a_n$ for some $a_n \in J$. Since f is a continuous Riesz map, $f(a)=$ o-$\lim_{n\to\infty} f(a_n)$ and we have $f(a_n) \in f[J]$, and hence also $f(a) \in \overline{\mathcal{L}f(J)}$. Thus $f[J] \subseteq \overline{\mathcal{L}f(J)}$ and so $\mathcal{L}f(\overline{J}) \subseteq \overline{\mathcal{L}f(J)}$.

(ii) Using (i) and by general principles concerning frame quotients, there exists a frame map $Cl(f) : Cl(E) \to Cl(D)$ such that the diagram

$$\begin{tikzcd}
\mathcal{L}(E) \arrow[r, "c_E"] \arrow[d, "\mathcal{L}f"'] & Cl(E) \arrow[d, "Cl(f)"] \\
\mathcal{L}(D) \arrow[r, "c_D"] & Cl(D)
\end{tikzcd}$$

commutes, and so $E \longmapsto Cl(E)$ is functorial. $\square$

4 Concluding Remarks

As the referee truly suggested, in this section we present a rather short survey of the developments that flow from the functoriality of the maximal spectrum. For more details, see [5]. Let $\mathcal{C}(L)$ be the ring of pointfree continuous real functions on a frame L given in [3] and studied as a Riesz spaces in [5].

4.1 The Adjunction and its properties

The functor $\mathcal{M} : \mathcal{BdUARsz} \to \mathbf{KCRFrm}$ given by $E \mapsto Cl(E)$ and $f \mapsto Cl(f)$ is a left adjoint to the functor $\mathcal{C} : \mathbf{KCRFrm} \to \mathcal{BdUARsz}$ given by $L \mapsto \mathcal{C}(L)$ and $f \mapsto \mathcal{C}(f)$. The appropriate adjunction maps are

$$\sigma_L : \mathcal{MC}(L) \to L, \quad \tau_E : E \to \mathcal{C}(\mathcal{M}E)$$

such that $\sigma_L(\overline{[a]}) = coz(\alpha)$ for any $\alpha \in \mathcal{C}(L)$ and $\tau_E(a) = \hat{a}$, where $\hat{a} : \mathcal{R} \to \mathcal{M}E$ is given by $\hat{a}(p,q) = \overline{[((a-\overline{p}) \wedge (\overline{q}-a)^{+})]}$, for $p, q \in \mathbb{Q}$. Here $\mathcal{R}$ is the frame of reals, and $\mathcal{R} \simeq O(\mathbb{R})$ in Zermelo-Frankel set theory. Note that the identities

$$coz(\hat{a}) = \overline{[a]}, \quad coz(\hat{\alpha}(p,q)) = \alpha(p,q)$$

give the adjunction identities

$$\sigma_{\mathcal{M}E}\mathcal{M}_{\tau_E} = id_{\mathcal{M}E}, \quad (\mathcal{C}\sigma_L)\tau_{\mathcal{C}L} = id_{\mathcal{C}L}.$$

For every compact completely regular frame L, σ_L is an isomorphism. This is because L *is completely regular if and only if the set* $\{coz(\alpha) : \alpha \in \mathcal{C}(L)\}$ *generates* L.

For any bounded uniform Riesz space E, the monomorphism τ_E is an *embedding*, with $\tau_E(\overline{1}) = 1$, in the sense that $0 < a < \overline{1}$ if and only if $0 < \tau_E(a) < \overline{1}$.

4.2 The Stone-Weierstrass Theorem

Let L *be a compact completely regular frame. Then any subvector lattice* E *of* $\mathcal{C}(L)$ *containing* 1 *such that* $\{coz(\alpha) : \alpha \in E\}$ *generates* L *is uniformly dense in* $\mathcal{C}(L)$.

4.3 The pointfree version of the Kakutani Duality

A bounded uniform Archimedean Riesz space E is called *conditionally complete* if every dense embedding Riesz map from E to another such a Riesz space is an isomorphism. It is proved that for every compact completely regular frame L, $\mathcal{C}(L)$ is conditionally complete. Moreover, by the Stone-Weierstrass Theorem, τ_E is onto if and only if E is conditionally complete. Then one has a pointfree version of the Kakutani duality as follows:

The functors $\mathcal{M}$ *and* $\mathcal{C}$ *induce an equivalence between the category of bounded Archimedean uniform conditionally complete Riesz spaces and the category of compact completely regular frames.*

It is necessary to cue that the classical Kakutani duality is an equivalence between the category of *M-spaces* (that is, Banach lattices with a norm satisfying

$\|x \vee y\| = max\{\|x\|, \|y\|\}$) and the category of compact Hausdorff spaces, where the functor from the second category to the first is given by the ring of continuous real functions. Finally, it is proved that

A bounded Archimedean uniform Riesz space is an M-space if and only if it is conditionally complete.

Notice that, using the Axiom of Choice, one gets the classical Kakutani duality from its pointfree version.

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