Explicit calculation of some polynomials introduced by W. Gautschi

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1 Introduction

The polynomials in question were introduced by W. Gautschi when he analyzed the numerical condition of the method of modified moments: Let σ be a given positive measure with infinite support $S(\sigma)$. Then there uniquely exists a family of monic polynomials π_j with

$$\int \pi_l(x)\pi_j(x)\, d\sigma(x) = 0 \quad \text{for} \quad l < j \quad \text{and} \quad \int \pi_j^2(x)\, d\sigma(x) > 0, \quad j = 0, 1, \ldots$$

They satisfy a three-term recurrence relation

$$\pi_{j+1}(x) = (x - \alpha_j)\pi_j(x) - \beta_j \pi_{j-1}(x) \quad \text{for} \quad j = 0, 1, \ldots,$$

if we set $\pi_{-1}(x) \equiv 0$. It is well known, that the generation of the orthogonal polynomials (or, equivalently, of the coefficients α_j and β_j) from ordinary moments

$$\mu_k = \int x^k\, d\sigma(x)$$

is severely ill-conditioned, see the classical paper of Walter Gautschi [4]. More promising are modified moments

$$m_k = \int q_k(x)\, d\sigma(x), \tag{1}$$

with some properly chosen set of polynomials q_k (e.g. Chebyshev polynomials or other polynomials orthogonal with respect to some measure s). It would be desirable to calculate or estimate the condition of the map K_n from $m = [m_0, \ldots, m_{2n-1}]^T$ to $\rho = [\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}]^T$. In the papers [5] and [6], W. Gautschi introduced the vector

$$\gamma = [\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n]^T$$

of nodes and weights of the Gauss-Christoffel quadrature rule

$$\int q(x)\, d\sigma(x) = \sum_{k=1}^{n} \sigma_k\, q(\tau_k)$$

for any polynomial q of degree $\leq 2n - 1$. He denoted by G_n the map

$$G_n : \mathbf{R}^{2n} \to \mathbf{R}^{2n} \quad m \to \gamma$$

and by H_n the (well-conditioned) map

$$H_n : \mathbf{R}^{2n} \to \mathbf{R}^{2n} \quad \gamma \to \rho.$$

Then one has the decomposition

$$K_n = H_n \circ G_n,$$

and this equation reduces our task to the investigation of the map G_n. More precisely, Gautschi considered the map defined by

$$\tilde{G}_n : \mathbf{R}^{2n} \to \mathbf{R}^{2n} \quad \tilde{m} \to \gamma,$$

where

$$\tilde{m}_k = d_k^{-\frac{1}{2}}\, m_k \quad \text{with} \quad d_k = \int q_k^2(x)\, ds(x).$$

The map $\tilde{G}_n$ differs from G_n only by a trivial diagonal map, and the sensitivity of the nonlinear map $\tilde{G}_n$ can be measured by the norm of the Fréchet derivative $\tilde{G}_n'$, which is a linear map (the Jacobian of $\tilde{G}_n$) from $\mathbf{R}^{2n}$ into $\mathbf{R}^{2n}$. We consider here only the Frobenius norm

$$\|A\|_F = (\operatorname{Tr} A^T A)^{\frac{1}{2}} = \left(\sum_{j,k=0}^{2n-1} a_{jk}^2 \right)^{\frac{1}{2}}.$$

In the paper [6], Gautschi proved the equation

$$\|\tilde{G}_n'\|_F = \left\{ \int g_n(x)\, ds(x) \right\}^{\frac{1}{2}}$$

with some polynomial g_n of degree $4n-2$ determined by γ, which will be defined in the following section (this is essentially Equation (3.2) of [6], though the result is implicit already in the proof of Theorem 3.1 of the article [5]).

The present paper is devoted to the explicit calculation of the polynomials g_n in some special cases. For a direct approach to the investigation of the map K_n, see the article [2] of the author. The map H_n is investigated in the subsequent papers of W. Gautschi [5] – [8] and in the article [3] of the author.

2 Polynomial interpolation

The polynomials g_n are defined as follows:

$$g_n(x) = \sum_{j=1}^{n} h_j^2(x) + \sum_{j=1}^{n} \sigma_j^{-2} k_j^2(x), \tag{2}$$

where σ_j are the weights of the Gauss-Christoffel quadrature rule and h_j and k_j are the fundamental polynomials of Hermite interpolation in the nodes of the quadrature rule. We define them after recalling Lagrange interpolation. Let $\{\tau_i\}_{i=1,\ldots,n}$ be a set of distinct numbers, and define

$$l_i(x) = \frac{\omega_n(x)}{\omega_n'(\tau_i)(x-\tau_i)} \quad \text{for} \quad i = 1, \ldots, n, \tag{3}$$

where $\omega_n(x) = (x-\tau_1)\cdots(x-\tau_n)$. The Lagrange interpolation formula

$$q(x) = \sum_{i=1}^{n} q(\tau_i)\, l_i(x) \quad \text{for} \quad q \in \mathcal{P}_{n-1} \tag{4}$$

is well known. The following result is well known, too.

Lemma 1 *Let*

$$h_i(x) = \left[1 - \frac{\omega_n''(\tau_i)}{\omega_n'(\tau_i)}(x-\tau_i)\right] l_i^2(x) \quad \textit{and} \quad k_i(x) = (x-\tau_i)\, l_i^2(x). \tag{5}$$

Then we have the equation

$$q(x) = \sum_{i=1}^{n} q(\tau_i)\, h_i(x) + \sum_{i=1}^{n} q'(\tau_i)\, k_i(x) \quad \textit{for} \quad q \in \mathcal{P}_{2n-1}. \tag{6}$$

In our case, the nodes τ_i are the zeros of some orthogonal polynomial p_n. We remark here that in Equations (3) and (5), we can use the (generally not monic) polynomial p_n instead of ω_n, since the leading coefficient will simply cancel out.

In the following, we will need an elementary interpolation identity of higher order.

Lemma 2 *Let* $\{\tau_i\}_{i=1,\ldots,n}$ *be any set of distinct nodes, and let* $b_i \in \mathcal{P}_{4n-1}$ *be polynomials with the property*

$$b_i(\tau_j) = \delta_{ij}, \quad b_i'(\tau_j) = b_i''(\tau_j) = b_i'''(\tau_j) = 0 \quad \textit{for} \quad i,j = 1,\ldots,n.$$

Then the identity

$$\sum_{i=1}^{n} b_i(x) \equiv 1$$

is satisfied.

Proof: Under our assumptions the polynomial

$$r(x) = \sum_{i=1}^{n} b_i(x) - 1 \in \mathcal{P}_{4n-1}$$

vanishs at all nodes τ_i together with all derivatives up to the third, and thus it is divisible by the polynomial $(x-\tau_1)^4 \cdots (x-\tau_n)^4$ having degree $4n$. Consequently, the polynomial r must be identically zero. $\Box$

In fact, the polynomials b_i are well defined by the above properties and can be written down explicitly in terms of the polynomials l_i.

Lemma 3 *Let* l_i *be the fundamental polynomials of Lagrange interpolation for the nodes* $\{\tau_i\}_{i=1,\ldots,n}$ *and let the first few coefficients of the Taylor expansion of* $l_i(x)$ *around* $x = \tau_i$ *be*

$$l_i(x) = 1 + c_1(\tau_i)(x-\tau_i) + c_2(\tau_i)(x-\tau_i)^2 + c_3(\tau_i)(x-\tau_i)^3 + \cdots.$$

Then we have

$$b_i(x) = [1 + d_1(\tau_i)(x-\tau_i) + d_2(\tau_i)(x-\tau_i)^2 + d_3(\tau_i)(x-\tau_i)^3]\, l_i^4(x),$$

where the coefficients d_i *can be obtained from*

$$\begin{aligned} d_1 &= -4c_1, \\ d_2 &= 10c_1^2 - 4c_2, \\ d_3 &= -20c_1^3 + 20c_1c_2 - 4c_3. \end{aligned}$$

Proof: If $1 + d_1(\tau_i)(x-\tau_i) + d_2(\tau_i)(x-\tau_i)^2 + d_3(\tau_i)(x-\tau_i)^3$ coincides with the first few terms of the Taylor series of $l_i(x)^{-4}$, then the $b_i(x)$ as defined in Lemma 3 satisfies $b_i(x) = 1 + O((x-\tau_i)^4)$. Since this b_i clearly has degree $\le 4n-1$ and vanishes at each $\tau_j \neq \tau_i$, it follows that it is identical with the b_i defined in Lemma 2. The Taylor series can be obtained easily: From

$$\sum_{j=0}^{\infty} d_j (x-\tau_i)^j = \left(\sum_{j=0}^{\infty} c_j (x-\tau_i)^j \right)^{-4}$$

by differentiating and multiplying by $\sum_{j=0}^{\infty} c_j(x-\tau_i)^j$ we have the equation

$$\sum_{j=1}^{\infty} j\, d_j (x-\tau_i)^{j-1} \sum_{j=0}^{\infty} c_j (x-\tau_i)^j = -4 \sum_{j=1}^{\infty} j\, c_j (x-\tau_i)^{j-1} \sum_{j=0}^{\infty} d_j (x-\tau_i)^j .$$

A comparison of the coefficients gives the linear system

$$\begin{aligned} d_1 &= -4c_1, \\ c_1 d_1 + 2d_2 &= -4(c_1 d_1 + 2c_2), \\ c_2 d_1 + 2c_1 d_2 + 3d_3 &= -4(c_1 d_2 + 2c_2 d_1 + 3c_3), \end{aligned}$$

and solving for d_j we arrive at our proposition. $\square$

3 Explicit calculation of $\sum\limits_{i=1}^{n} \sigma_i^{-2} k_i^2(x)$ for Jacobi polynomials

Of course, we will be able to compute explicit formulas only in some special cases. The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal with respect to the measure defined by

$$d\sigma(x) = (1-x)^{\alpha}(1+x)^{\beta}\, dx,$$

their leading coefficient (in their traditional standardization as in [1, 22.2.1]) is

$$K_n = \frac{1}{2^n} \binom{2n+\alpha+\beta}{n}$$

and the constant in their orthogonality relation is

$$H_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \, \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}.$$

We will write simply p_n instead of $P_n^{(\alpha,\beta)}$. The three-term recurrence for p_n has the explicit form

$$\begin{aligned} &2n(n+\alpha+\beta)(2n+\alpha+\beta-2)p_n(x) = [(2n+\alpha+\beta-1)(\alpha^2-\beta^2)+ \\ &\quad +(2n+\alpha+\beta-2)(2n+\alpha+\beta-1)(2n+\alpha+\beta)x]p_{n-1}(x) - \\ &\quad -2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)p_{n-2}(x). \end{aligned} \tag{7}$$

For these special polynomials we have some more useful relations. Especially, they satisfy the differential equation

$$(1-x^2)p_n''(x) + [\beta - \alpha - (\alpha+\beta+2)x]p_n'(x) + n(n+\alpha+\beta+1)p_n(x) = 0 \tag{8}$$

and their derivative can be expressed as

$$\begin{aligned} (2n+\alpha+\beta)(1-x^2)p_n'(x) &= n[\alpha - \beta - (2n+\alpha+\beta)x]p_n(x) + \\ &\quad + 2(n+\alpha)(n+\beta)p_{n-1}(x). \end{aligned} \tag{9}$$

Now we can prove the following

Theorem 1 *Let τ_i be the zeros of the Jacobi polynomial p_n. Then the polynomials k_i defined in Equations (5) satisfy the identity*

$$\sum_{i=1}^{n} \sigma_i^{-2} k_i^2(x) = C_n^2 p_n^2(x) \left[p_{n-1}^2(x) - B_n p_{n-2}(x) p_n(x)\right]$$

with the constants

$$C_n = \frac{K_{n-1}}{K_n H_{n-1}} \quad \textit{and} \quad B_n = \frac{(2n+\alpha+\beta)(n+\alpha-1)(n+\beta-1)}{(2n+\alpha+\beta-1)(n+\alpha)(n+\beta)}.$$

Proof: For any set of orthogonal polynomials the weights of Gauss-Christoffel quadrature formulas can be written as

$$\sigma_i^{-1} = C_n p_n'(\tau_i) p_{n-1}(\tau_i).$$

From the definition (5) of the polynomials k_i and $l_i(x) = \dfrac{p_n(x)}{(x-\tau_i)p_n'(\tau_i)}$ we obtain easily

$$\sum_{i=1}^{n} \sigma_i^{-2} k_i^2(x) = C_n^2 p_n^2(x) \sum_{i=1}^{n} p_{n-1}^2(\tau_i)\, l_i^2(x).$$

The sum on the right-hand side can be evaluated from the interpolation identity (6). First we observe, that from the differential equation the explicit formula

$$h_i(x) = \left[1 - \frac{(\alpha+\beta+2)\,\tau_i + \alpha - \beta}{1-\tau_i^2}\,(x-\tau_i)\right] l_i^2(x)$$

can be derived. Thus Equation (6) can be written as

$$\sum_{i=1}^{n} q(\tau_i)\, l_i^2(x) = q(x) - \sum_{i=1}^{n} \left[q'(\tau_i) - \frac{(\alpha+\beta+2)\,\tau_i + \alpha - \beta}{1-\tau_i^2} q(\tau_i)\right] (x-\tau_i)\, l_i^2(x) \qquad (10)$$

for any polynomial $q \in \mathcal{P}_{2n-1}$. Now, we set $q(x) = p_{n-1}^2(x)$. Since we have

$$(x-\tau_i)\, l_i(x) = \frac{1}{p_n'(\tau_i)}\, p_n(x)$$

and from Equation (9)

$$p_n'(\tau_i) = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(1-\tau_i^2)}\, p_{n-1}(\tau_i),$$

we obtain

$$\left[q'(\tau_i) - \frac{(\alpha+\beta+2)\,\tau_i + \alpha - \beta}{1-\tau_i^2} q(\tau_i)\right] \frac{1}{p_n'(\tau_i)} =$$
$$\frac{2n+\alpha+\beta}{(n+\alpha)(n+\beta)} \left\{(1-\tau_i^2)\, p_{n-1}'(\tau_i) - \frac{1}{2}[(\alpha+\beta+2)\,\tau_i + \alpha - \beta]\, p_{n-1}(\tau_i)\right\}.$$

If we express $(1-\tau_i^2)\, p_{n-1}'(\tau_i)$ using (9) and eliminate the terms $\tau_i p_{n-1}(\tau_i)$ using (7), after an easy (but tedious) calculation we arrive at

$$(1-\tau_i^2)\, p_{n-1}'(\tau_i) - \frac{1}{2}[(\alpha+\beta+2)\,\tau_i + \alpha - \beta]\, p_{n-1}(\tau_i) = \frac{(n+\alpha-1)(n+\beta-1)}{2n+\alpha+\beta-1}\, p_{n-2}(\tau_i).$$

This implies

$$\sum_{i=1}^{n}\left[q'(\tau_i)-\frac{(\alpha+\beta+2)\,\tau_i+\alpha-\beta}{1-\tau_i^2}q(\tau_i)\right](x-\tau_i)\,l_i^2(x)=B_n p_n(x)\sum_{i=1}^{n}p_{n-2}(\tau_i)\,l_i(x),$$

and the sum on the right clearly is equal to $p_{n-2}(x)$ due to the Lagrange interpolation formula (4). $\square$

Since we are able to obtain further results in the special case of Chebyshev polynomials, where $\alpha,\beta=\pm\frac{1}{2}$, and the traditional standardization of Chebyshev polynomials is different from that of Jacobi polynomials, we reformulate our result in this case. While Chebyshev polynomials of first and second kinds are well known, the nomenclature for the mixed cases ($\alpha=-\beta$) was introduced by Gautschi, e.g. in [9], and adopted by other authors (in the papers [10] and [11] of Mason and Elliott we can find some properties of Chebyshev polynomials of third and fourth kinds).

First we recall the connection between these special Jacobi polynomials and Chebyshev polynomials (see Equations 22.5.30 – 22.5.32 in [1]):

$$T_n(x)=\frac{n!\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}\,P^{(-\frac{1}{2},-\frac{1}{2})}(x), \tag{11}$$

$$U_n(x)=\frac{(n+1)!\sqrt{\pi}}{2\Gamma\left(n+\frac{3}{2}\right)}\,P^{(\frac{1}{2},\frac{1}{2})}(x) \tag{12}$$

and

$$U_{2n}(x)=\frac{n!\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}\,P^{(\frac{1}{2},-\frac{1}{2})}(2x^2-1).$$

We will rewrite the last equation in a more convenient way. The trigonometric representation of Chebyshev polynomials implies

$$U_{2n}(x)=U_n(2x^2-1)+U_{n-1}(2x^2-1),$$

and this equation gives immediately

$$U_n(x)+U_{n-1}(x)=\frac{n!\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}\,P^{(\frac{1}{2},-\frac{1}{2})}(x).$$

Since we have $P^{(-\frac{1}{2},\frac{1}{2})}(x)=(-1)^n P^{(\frac{1}{2},-\frac{1}{2})}(-x)$, we obtain easily the equation

$$U_n(x)-U_{n-1}(x)=\frac{n!\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}\,P^{(-\frac{1}{2},\frac{1}{2})}(x). \tag{13}$$

That's why we can call the polynomials $V_n(x)=U_n(x)-U_{n-1}(x)$ *Chebyshev polynomials of third kind* and the polynomials $W_n(x)=U_n(x)+U_{n-1}(x)$ *Chebyshev polynomials of fourth kind*. Their trigonometric representation is

$$V_n(\cos\theta)=\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}\quad\text{and}\quad W_n(\cos\theta)=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \tag{14}$$

see Equations (1.3) and (1.4) in the paper [10]. In view of the obvious property $W_n(x)=(-1)^n V_n(-x)$, there will be no need to consider the polynomials of fourth kind in the following.

Our Theorem 1 can be specialized for Chebyshev polynomials.

Corollary 1 *For Chebyshev polynomials of first kind, we have*

$$\sum_{i=1}^{n} \sigma_i^{-2} k_i^2(x) = \frac{1}{\pi^2}\left[1 - x^2 + \frac{3}{2n} T_{n-2}(x) T_n(x)\right] T_n^2(x),$$

whereas for Chebyshev polynomials of second kind,

$$\sum_{i=1}^{n} \sigma_i^{-2} k_i^2(x) = \frac{1}{\pi^2}\left[1 + \frac{3}{2(n+1)} U_{n-2}(x) U_n(x)\right] U_n^2(x),$$

and for Chebyshev polynomials of third kind,

$$\sum_{i=1}^{n} \sigma_i^{-2} k_i^2(x) = \frac{1}{4\pi^2}\left[2(1-x) + \frac{3}{2n+1} V_{n-2}(x) V_n(x)\right] V_n^2(x).$$

Proof: By simple substitution for the left-hand sum we have

$$\frac{1}{\pi^2}\left[T_{n-1}^2(x) - \frac{n-\frac{3}{2}}{n} T_{n-2}(x) T_n(x)\right] T_n^2(x)$$

in the first case,

$$\frac{1}{\pi^2}\left[U_{n-1}^2(x) - \frac{n-\frac{1}{2}}{n+1} U_{n-2}(x) U_n(x)\right] U_n^2(x)$$

in the second case, and

$$\frac{1}{4\pi^2}\left[V_{n-1}^2(x) - \frac{n-1}{n+\frac{1}{2}} V_{n-2}(x) V_n(x)\right] V_n^2(x)$$

in the third case. Now our proposition follows at once from the elementary identities

$$T_{n-1}^2(x) - T_{n-2}(x) T_n(x) = 1 - x^2,$$

$$U_{n-1}^2(x) - U_{n-2}(x) U_n(x) = 1$$

and

$$V_{n-1}^2(x) - V_{n-2}(x) V_n(x) = 2(1-x).\square$$

4 An estimate of $\sum_{i=1}^{n} h_i^2(x)$ for $\alpha, \beta \in (-1, 0]$

Interestingly, the sum $\sum_{i=1}^{n} h_i^2(x)$ in this case is uniformly bounded.

Theorem 2 *Let τ_i be the zeros of $P_n^{(\alpha,\beta)}$ and $\alpha, \beta \in (-1, 0]$. Then the polynomials h_i defined in (5) satisfy the inequality*

$$\sum_{i=1}^{n} h_i^2(x) \le 1 \quad \textit{for} \quad x \in [-1, 1].$$

Proof: All we have to prove is the nonnegativity of $h_i(x)$ in $[-1,1]$: From (6) with $q \equiv 1$ we obtain

$$\sum_{i=1}^{n} h_i(x) \equiv 1,$$

consequently $0 \le h_i(x) \le 1$ and

$$\sum_{i=1}^{n} h_i^2(x) \le \sum_{i=1}^{n} h_i(x) \equiv 1.$$

Since

$$h_i(x) = \left[1 - \frac{(\alpha+\beta+2)\,\tau_i + \alpha - \beta}{1-\tau_i^2}\,(x-\tau_i)\right] l_i^2(x)$$

and all $\tau_i \in (-1,1)$, it is sufficient to show

$$1 - \frac{(\alpha+\beta+2)\,\tau + \alpha - \beta}{1-\tau^2}\,(x-\tau) \ge 0 \quad \text{for} \quad x \in [-1,1], \tau \in (-1,1).$$

The left-hand side is linear in x, so we have to show the inequality only for $x=-1$ and $x=1$. For $x=-1$ we have

$$1 - \frac{(\alpha+\beta+2)\,\tau + \alpha - \beta}{1-\tau^2}\,(-1-\tau) = \frac{(\alpha+\beta+1)\,\tau + 1 + \alpha - \beta}{1-\tau}.$$

Now the numerator is again a linear function in τ, and it takes the values $-2\beta \ge 0$ for $\tau=-1$ and $2+2\alpha > 0$ for $\tau = 1$, hence it is ≥ 0 for all $\tau \in (-1,1)$. The case $x=1$ can be handled in the same way: We have

$$1 - \frac{(\alpha+\beta+2)\,\tau + \alpha - \beta}{1-\tau^2}\,(1-\tau) = \frac{1 - \alpha + \beta - (\alpha+\beta+1)\,\tau}{1+\tau},$$

and the numerator is $2+2\beta > 0$ for $\tau=-1$ and $-2\alpha \ge 0$ for $\tau = 1$. □

Naturally, this bound is sharp — the sum is equal to 1 for $x = \tau_i$.

5 Explicit calculation of $\sum_{i=1}^{n} h_i^2(x)$ for Chebyshev polynomials

The explicit calculation of this sum is possible for Chebyshev polynomials of first, second and third (or mixed) kinds. Our starting point is the interpolation identity from Lemma 2. The coefficients c_j defined in Lemma 3 can be calculated from the differential equation (8) for any values of α and β. But we will see that for the subsequent simplifications we need explicit (and simple) formulas for $p_n'(\tau_i)$. This is possible only for Chebyshev polynomials.

Lemma 4 *For any zero τ of T_n we have*

$$(1-\tau^2)\,T_n'(\tau) = \frac{n}{U_{n-1}(\tau)} \quad \textit{and} \quad T_n'^2(\tau) = \frac{n^2}{1-\tau^2},$$

whereas for any zero τ of U_n,

$$(1-\tau^2)\,U_n'(\tau) = \frac{n+1}{U_{n-1}(\tau)} \quad \textit{and} \quad U_n'^2(\tau) = \left(\frac{n+1}{1-\tau^2}\right)^2,$$

and for any zero τ of the third-kind Chebyshev polynomial $V_n(x)$,

$$(1-\tau^2)\,V_n'(\tau) = \frac{n+\frac{1}{2}}{U_{n-1}(\tau)} \quad \textit{and} \quad V_n'^2(\tau) = \frac{(2n+1)^2}{2(1-\tau^2)(1+\tau)}.$$

Proof: This follows immediately from the well-known trigonometric representation of Chebyshev polynomials. Differentiating $T_n(\cos\theta) = \cos n\theta$ we obtain

$$\sin\theta\, T'(\cos\theta) = n\, \sin n\theta. \tag{15}$$

If $\cos\theta$ is a zero of T_n, then we have $\cos n\theta = 0$ and consequently $\sin^2 n\theta = 1$. This implies

$$(1-\cos^2\theta)\, T'(\cos\theta) = \sin^2\theta\, T'(\cos\theta) = n\, \sin\theta \sin n\theta = \frac{n\, \sin\theta}{\sin n\theta} = \frac{n}{U_{n-1}(\cos\theta)}.$$

From the above equation (15), by taking squares, we obtain

$$(1-\cos^2\theta)\, T'^2(\cos\theta) = \sin^2\theta\, T'^2(\cos\theta) = n^2 \sin^2 n\theta = n^2, \quad \text{if} \quad \cos n\theta = 0.$$

The derivation of the other equations is very similar. If $\cos\theta$ is a zero of U_n, then $\sin(n+1)\theta = 0$ and consequently $\cos^2(n+1)\theta = 1$. From $\sin\theta\, U_n(\cos\theta) = \sin(n+1)\theta$ by differentiating we have the identity

$$-\sin^2\theta\, U_n'(\cos\theta) + \cos\theta\, U_n(\cos\theta) = (n+1)\, \cos(n+1)\theta, \tag{16}$$

and by squaring both sides (using $U_n(\cos\theta) = 0$ and $\cos^2(n+1)\theta = 1$) we obtain

$$(1-\cos^2\theta)^2\, U_n'^2(\cos\theta) = (n+1)^2.$$

From the equation (16) we conclude

$$\begin{aligned} -\sin^2\theta\, U_n'(\cos\theta) &= (n+1)\, \cos(n+1)\theta = \frac{n+1}{\cos(n+1)\theta} \\ &= \frac{(n+1)\, \sin\theta}{\sin\theta\cos(n+1)\theta - \cos\theta\sin(n+1)\theta} = \frac{(n+1)\, \sin\theta}{-\sin n\theta} = -\frac{n+1}{U_{n-1}(\cos\theta)}, \end{aligned}$$

where we used again $\cos^2(n+1)\theta = 1$ and $\sin(n+1)\theta = 0$. Observing $\sin^2\theta = 1-\cos^2\theta$ we arrive at our proposition.

The proof of the last equations relies on the trigonometric representation (14) and the fact that $\cos\left(n+\frac{1}{2}\right)\theta = 0$ implies $\sin^2\left(n+\frac{1}{2}\right)\theta = 1$ and

$$\sin\left(n+\frac{1}{2}\right)\theta = \frac{1}{\sin\left(n+\frac{1}{2}\right)\theta}$$

(we omit details here). $\square$

Now we are able to formulate the main result of this section.

Theorem 3 *In the case of Chebyshev polynomials of first kind ($\alpha = \beta = -\frac{1}{2}$) the following formula holds:*

$$\sum_{i=1}^{n} h_i^2(x) = 1 - \left[\frac{2}{3} - \frac{1}{6n^2} - \frac{1}{2n^2}\cdot\frac{1-\dfrac{1}{2n}\, x\, U_{2n-1}(x)}{1-x^2}\right] T_n^2(x).$$

For Chebyshev polynomials of second kind ($\alpha = \beta = \frac{1}{2}$) we have

$$\begin{aligned} \sum_{i=1}^{n} h_i^2(x) \quad &= \quad 1 - \frac{1}{(n+1)^2}\left[\frac{7}{2}\, x^2 + \frac{2}{3}\,(n^2+n-3)(1-x^2)\right] U_n^2(x) \\ &\quad + \frac{1}{(n+1)^3}\left[\frac{9}{4}\, U_{n-2}(x) - \frac{5}{4}\, U_{n-4}(x)\right] U_n^3(x). \end{aligned}$$

Finally, for Chebyshev polynomials of third kind ($\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$) we find

$$\begin{aligned}\sum_{i=1}^{n} h_i^2(x) &= 1 - \frac{1}{(2n+1)^2}\left[-4x + \frac{4}{3}(n^2+n)(x+1) - s_n(x)\right] V_n^2(x) \\ &\quad + \frac{2}{(2n+1)^3}\,[U_{n-2}(x) + U_{n-3}(x)]\, V_n^3(x),\end{aligned}$$

where

$$s_n(x) = \frac{1 + \dfrac{1}{2n+1}\,[U_{n-2}(x) - 3\,U_{n-1}(x)]\, V_n(x)}{1-x},$$

and

$$V_n(x) = U_n(x) - U_{n-1}(x).$$

Proof: As we can see easily, we have $h_i(x) = \left[1 + \frac{1}{2}\, d_1(\tau_i)(x-\tau_i)\right] l_i^2(x)$ and thus, using Lemma 3,

$$\begin{aligned} h_i^2(x) &= b_i(x) + \left[\left(\frac{1}{4}\, d_1^2(\tau_i) - d_2(\tau_i)\right)(x-\tau_i)^2 - d_3(\tau_i)\,(x-\tau_i)^3\right] l_i^4(x) \\ &= b_i(x) + p_n^2(x)\,\left[\left(\frac{1}{4}\, d_1^2(\tau_i) - d_2(\tau_i)\right) - d_3(\tau_i)\,(x-\tau_i)\right] \frac{1}{p_n'^2(\tau_i)}\, l_i^2(x)\end{aligned}$$

for any set of orthogonal polynomials. Now we use our explicit formulas for $p_n'^2(\tau_i)$ in the three special cases we consider.

Chebyshev polynomials of first kind: Computing the $c_j(\tau_i)$ from the differential equation (8), $d_j(\tau_i)$ from Lemma 3 and observing $h_i(x) = [1 - \tau_i\,(x-\tau_i)/(1-\tau_i^2)]\; l_i^2(x)$ we obtain after an easy (but tedious) calculation

$$h_i^2(x) = b_i(x) + T_n^2(x)\,\left[\frac{1}{2n^2}\cdot\frac{1}{1-\tau_i^2}\, l_i^2(x) - \left(\frac{2}{3} - \frac{1}{6n^2}\right) h_i(x)\right].$$

From the above interpolation identities we know that

$$\sum_{i=1}^{n} h_i(x) = \sum_{i=1}^{n} b_i(x) \equiv 1,$$

but we have to evaluate the polynomial

$$r_n(x) = \sum_{i=1}^{n} \frac{1}{1-\tau_i^2}\, l_i^2(x).$$

The trick here is to calculate $(1-x^2)\, r_n(x)$, since we have

$$\frac{1-x^2}{1-\tau_i^2}\, l_i^2(x) = \frac{1-\tau_i x}{1-\tau_i^2}\, l_i^2(x) - x\,\frac{x-\tau_i}{1-\tau_i^2}\, l_i^2(x) = h_i(x) - \frac{x\,T_n(x)}{n}\, U_{n-1}(\tau_i)\, l_i(x)$$

(where we used $(x-\tau_i)\, l_i(x) = T_n(x)/T_n'(\tau_i)$ and Lemma 4), and this gives immediately

$$(1-x^2)\, r_n(x) = 1 - \frac{x\,T_n(x)}{n}\, U_{n-1}(x) = 1 - \frac{1}{2n}\, x\, U_{2n-1}(x).$$

Substituting this into the above equation we obtain our first proposition.

Chebyshev polynomials of second kind: As above we compute the representation

$$\begin{aligned} h_i^2(x) &= b_i(x) + \frac{1}{(n+1)^2}\, U_n^2(x) \left\{ \left[-\frac{7}{2}\,\tau_i^2 - \frac{2}{3}\,(n^2+2n-3)(1-\tau_i^2) \right] \right. \\ &\quad \left. + \left[\frac{10\,\tau_i^3}{1-\tau_i^2} + \left(\frac{10}{3}\,n^2 + \frac{20}{3}\,n - \frac{15}{2} \right) \tau_i \right] (x-\tau_i) \right\} l_i^2(x). \end{aligned}$$

The sum

$$\sum_{i=1}^{n} \left[-\frac{7}{2}\,\tau_i^2 - \frac{2}{3}\,(n^2+2n-3)(1-\tau_i^2) \right] l_i^2(x)$$

can be evaluated by Equation (10)

$$\left(\text{with} \quad \alpha = \beta = \frac{1}{2} \quad \text{and} \quad q(x) = -\frac{7}{2}\,x^2 - \frac{2}{3}\,(n^2+2n-3)(1-x^2)\right),$$

and this gives

$$\sum_{i=1}^{n} h_i^2(x) = 1 + \frac{1}{(n+1)^2}\, U_n^2(x) \left\{ q(x) + \sum_{i=1}^{n} \frac{\frac{19}{2}\,\tau_i - 10\tau_i^3}{1-\tau_i^2}\,(x-\tau_i)\, l_i^2(x) \right\}.$$

Now we can use $(x-\tau_i)\, l_i(x) = U_n(x)/U_n'(\tau_i)$ and the formula for $U_n'(\tau_i)$ from Lemma 4 to obtain

$$\sum_{i=1}^{n} \frac{\frac{19}{2}\,\tau_i - 10\tau_i^3}{1-\tau_i^2}\,(x-\tau_i)\, l_i^2(x) = \frac{1}{n+1}\, U_n(x) \sum_{i=1}^{n} \left(\frac{19}{2}\,\tau_i - 10\tau_i^3 \right) U_{n-1}(\tau_i)\, l_i(x).$$

Repeatedly applying the recurrence relation for the polynomials U_j and observing $U_n(\tau_i) = 0$, we can calculate

$$\left(\frac{19}{2}\,\tau_i - 10\tau_i^3 \right) U_{n-1}(\tau_i) = \frac{9}{4}\, U_{n-2}(\tau_i) - \frac{5}{4}\, U_{n-4}(\tau_i),$$

and by the Lagrange interpolation formula we have

$$\sum_{i=1}^{n} \left[\frac{9}{4}\, U_{n-2}(\tau_i) - \frac{5}{4}\, U_{n-4}(\tau_i) \right] l_i(x) = \frac{9}{4}\, U_{n-2}(x) - \frac{5}{4}\, U_{n-4}(x).$$

Chebyshev polynomials of third kind: Our starting point is again a representation

$$h_i^2(x) = b_i(x) - \frac{V_n^2(x)}{(2n+1)^2}\, [C_0(\tau_i) + C_1(\tau_i)(x-\tau_i)]\, l_i^2(x),$$

where

$$C_0(\tau) = -4\tau + \frac{4}{3}(n^2+n)(\tau+1) + \frac{1}{\tau-1}$$

and

$$C_1(\tau) = \frac{2}{3}\, \frac{6(n^2+n-2)\tau^2 + 2(n^2+n+3)\tau - 4n^2 - 4n + 3}{\tau^2-1}$$

are rational functions of τ. The sum

$$\sum_{i=1}^{n} q(\tau_i)\, l_i^2(x)$$

with $q(\tau) = -4\tau + \frac{4}{3}(n^2+n)(\tau+1)$ can be calculated from formula (10) again. The sum on the right-hand side of (10) combines with $\sum_{i=1}^n C_1(\tau_i)(x-\tau_i)\, l_i^2(x)$ to give

$$\sum_{i=1}^n \frac{-4\tau_i^2+2}{1-\tau_i^2}(x-\tau_i)\, l_i^2(x) = \frac{V_n(x)}{n+\frac{1}{2}}\sum_{i=1}^n(-4\tau_i^2+2)\, U_{n-1}(\tau_i)\, l_i(x) =$$
$$\frac{V_n(x)}{n+\frac{1}{2}}\sum_{i=1}^n[-U_{n-3}(\tau_i)-U_{n-2}(\tau_i)]\, l_i(x) = \frac{V_n(x)}{n+\frac{1}{2}}[-U_{n-3}(x)-U_{n-2}(x)]$$

(where we used Lemma 4, the recurrence relation for U_j and the fact that $U_n(\tau_i) = U_{n-1}(\tau_i)$). However, we have to calculate the sum

$$s_n(x) = \sum_{i=1}^n \frac{1}{1-\tau_i}\, l_i^2(x).$$

We use the above trick to calculate $(1-x)\, s_n(x)$, since we have the identity

$$\frac{1-x}{1-\tau_i}\, l_i^2(x) = h_i(x) + \frac{\tau_i - 2}{1-\tau_i^2}(x-\tau_i)\, l_i^2(x).$$

With the same procedure as before we easily obtain the proposition.

This finally proves our theorem. We remark here that the simple but tedious algebraic manipulations were performed partially with MAPLE. □

Our last result will be the proof of an inequality conjectured by W. Gautschi already in his paper [6].

Corollary 2 *In the case* $d\sigma(x) = dx/\sqrt{1-x^2}$, *the inequality* $g_n(x) \leq 1$ *holds for all* $x \in [-1,1]$ *and all* $n \geq 2$.

Proof: As an immediate consequence of Theorem 3 and Corollary 1, we obtain the equation

$$g_n(x) = 1 - \left[\frac{2}{3} - \frac{1}{6n^2} - \frac{1}{2n^2}\, r_n(x) - \frac{1}{\pi^2}\left\{1 - x^2 + \frac{3}{2n}\, T_{n-2}(x) T_n(x)\right\}\right] T_n^2(x), \qquad (17)$$

where as in the proof of Theorem 3, we use the abbreviation

$$r_n(x) = \sum_{i=1}^n \frac{1}{1-\tau_i^2}\, l_i^2(x).$$

In order to estimate $r_n(x)$, we use another way to evaluate this sum. From the Equation (3) and Lemma 4 we have

$$\frac{1}{1-\tau_i^2}\, l_i^2(x) = \frac{1}{n^2}\,\frac{T_n^2(x)}{(x-\tau_i)^2} = \frac{1}{n^2}\left[\frac{T_n(x)-T_n(\tau_i)}{x-\tau_i}\right]^2.$$

Now the right-most expression is a polynomial in τ_i of degree $2n-2$, and thus by virtue of the quadrature formula

$$r_n(x) = \frac{1}{\pi n}\sum_{i=1}^n \sigma_i \left[\frac{T_n(x)-T_n(\tau_i)}{x-\tau_i}\right]^2 = \frac{1}{\pi n}\int\left[\frac{T_n(x)-T_n(y)}{x-y}\right]^2 d\sigma(y).$$

Using the identity

$$\frac{T_n(x)-T_n(y)}{x-y} = U_{n-1}(x) + 2\sum_{k=1}^{n-1} U_{k-1}(x) T_{n-k}(y)$$

(which can be derived easily from the generating functions of Chebyshev polynomials) and the orthogonality of T_{n-k} with respect to σ, we obtain

$$r_n(x) = \frac{1}{n}\left[U_{n-1}^2(x) + 2\sum_{k=1}^{n-1} U_{k-1}^2(x)\right].$$

This equation implies the inequality

$$r_n(x) \le \frac{1}{n}\left(n^2 + 2\sum_{k=1}^{n-1} k^2\right) = \frac{2}{3}\,n^2 + \frac{1}{3}.$$

Observing $T_{n-2}(x)T_n(x) = x^2 - 1 + T_{n-1}^2(x) \le x^2$ and thus

$$1 - x^2 + \frac{3}{2n}\,T_{n-2}(x)T_n(x) \le 1 - x^2 + \frac{3}{2n}\,x^2 \le 1,$$

we see that the expression in square brackets in Equation (17) is not less than

$$\frac{1}{3} - \frac{1}{3n^2} - \frac{1}{\pi^2} \ge \frac{1}{4} - \frac{1}{\pi^2} > 0$$

for $x \in [-1,1]$ and $n \ge 2$. $\square$

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