

## 2.2 Measurements, Estimations and Errors

### 2.2.1 Systems of Standardized Units

- The concept of numbers is abstract. However, in most real situations, numbers have a specific meaning.
- The meaning of a number depends on the units associated with it.
- Examples: 5 apples; 5 hours; 5 feet
- Working with units along with numbers is essential in problem solving.

#### The U.S. Customary System of Measurement (The English System)

- The USCS represents units customarily used in the United States.
- These units include feet, yards, miles, pounds, quarts and so on.
- The USCS originated from ancient systems of measurement used by Middle Eastern civilizations including the Egyptians, Sumerians, Babylonians and Hebrews.

- These systems were further developed in ancient Greece, Rome, and in Europe during the Middle Ages.
- The English standardized many of these ancient measures and brought them to the colonies.
- The USCS is sometimes referred to as the English System.
- The USCS has become very complicated because so many different units have been defined and used.

- Lengths in the USCS are derived from units originally based on body parts.
- Examples:
  - cubit - length of the forearm from the elbow to the tip of the middle finger
  - foot - used to measure distance, equivalent to 12 thumb-widths, called uncia (from which we get the word inch)
  - milia passum - one thousand paces (from which we get the word mile)

- Lengths based on body parts vary from person to person!
- Thus, lengths were standardized based on body parts of royalty (e.g. the foot was based on the King's foot!).
- King Henry I (1100-1135) set the yard as the measure from the tip of his nose to the tip of his thumb on his outstretched arm.
- The official definitions of USCS units are now based on metric standards.
- The official definition of inch is 2.54 centimeters.

## Basic SI (Metric Units)

The metric system was invented to:

- create a coherent and sensible set of standardized units to replace the customary systems in use around the world
- simplify conversions and calculations by organizing relationships among the units by factors of 10

## The Metric System

- Created in the late 1700's in France
- An international conference in 1875 formulated the so-called *Treaty of the Meter*, which was a refinement of the metric system, and which encouraged its international use.
- The modern version of the metric system is known as *Système Internationale d'Unités*, or SI, which was formally established in 1960.
- SI had been adopted by every nation except Burma, Liberia and the U.S.
- Even in the U.S. SI units are legally established and widely used in science, manufacturing and commerce.

## Definitions of basic units

- Precise definitions of SI units are established by the International Bureau of Weights and Measures (based in Sèvres, France). Many definitions have changed over time.
- Current definitions are now based on properties of atoms.
- Before 1967, the second was defined as 1/86,400 of a day. But the precise length of a day changes slowly over time.
- A second is now defined as the amount of time required for 9,192,631,770 vibrations of a particular frequency of light emitted by atoms of the element cesium-133.

- The most precise time-keeping devices are the atomic clocks which work essentially by counting vibrations of light from cesium atoms.
- The meter was originally defined as one ten millionth of the distance from the equator to the north pole. A metal rod was cut to a length of 1 meter and stored in Sèvres, France. This particular bar was the official standard for the meter.
- In 1983, the meter was redefined as the distance travelled by light (in a vacuum) during a time of 1/299,792,458 second.
- A kilogram is presently defined (not based on atomic standards) as the mass of a particular block made of platinum and iridium. This block is being kept in Sèvres, France. A new definition of the kilogram is currently being sought.

## 2.2.2 Estimation

- One way to put large and small numbers in perspective is by comparing them to quantities that are easier to understand.
- Estimation helps in making reasonable comparisons.

- The height of say, 1000 feet, is much easier to visualize if we say that 1000 feet is about the height of a 100-story building (like the Empire State Building)
- Estimation, like problem solving, is more of an art than a science.
- The methods of estimation will help you put large and small numbers into perspective.

## The process of estimation

- \* Decide what you want to estimate, and why.
- \* Devise a strategy for making the estimate.
- \* Carry out the strategy and revise it if necessary.
- \* Look back to check, interpret, and explain your estimate.

## Example: The New York Marathon

Put the total amount of running done by participants in the New York Marathon in perspective.

To do this, we estimate the total distance run by marathon participants (in a single race). This will allow comparison to other known distances, which should provide some perspective.

$$\text{Total distance} = (\text{distance per runner}) \times (\text{number of runners})$$

The marathon distance is 42 kilometers. Newspaper accounts relate that typically, 20,000 runners finish the race each year.

Thus the total distance is

$$42 \frac{\text{km}}{\text{runner}} \times (2 \times 10^4 \text{ runners}) = 8.4 \times 10^5 \text{ km}$$

or about 840,000 kilometers. Comparing this distance to the circumference of the earth (about 40,000 kilometers)

$$\frac{\text{total distance}}{\text{circumference of the earth}} = \frac{8.4 \times 10^5 \text{ km}}{4 \times 10^4 \text{ km}} = 21$$

Thus, run as a relay, the marathon runners could circle the Earth approximately 21 times.

## How long would running such a relay take?

- The runners take an average of 4 hours to finish the marathon. Thus,

$$\text{total time} = \frac{4 \text{ hr}}{\text{runner}} \times (2 \times 10^4 \text{ runners}) = 8 \times 10^4 \text{ hrs}$$

- Converting from hours to years, we get

$$(8 \times 10^4 \text{ hr}) \times \left( \frac{1 \text{ day}}{24 \text{ hr}} \right) \times \left( \frac{1 \text{ yr}}{365 \text{ day}} \right) = 9.1 \text{ years}$$

- That is, the New York Marathon, run as a relay, would take 9.1 years to complete!

## Order of magnitude estimation

- In the example, though we have succeeded in putting the race in perspective, estimates of the number of runners and the average time to complete the race introduce some uncertainty into the results. The actual number could be substantially different. Nevertheless, the estimate may still provide a "feel" for the total distance covered.
- Estimates that involve large uncertainties, but still give a feel for their size, is called order of magnitude.

- An order of magnitude is a power of 10.

- Examples:

100 is an order of magnitude larger than 10;  
10,000 is two orders of magnitude ( $10^2$ , or 100 times) larger than 100;

$10^{23}$  is five orders of magnitude ( $10^5$  or 100,000 times) larger than  $10^{18}$ .

- Example: Make an order of magnitude estimate of the number of atoms in a human body.

↳ In devising an appropriate strategy, we need to recall a few facts from chemistry.

- The human body is made up mostly of water,  $H_2O$
- The atomic weight of hydrogen is 1, and the atomic weight of oxygen is 16. The total atomic weights for the three atoms in a molecule of water is 18. Thus the average atomic weight of three atoms in water is 6. The average atomic weight of the atoms in water, 6, reminds us that 6 grams of water contains  $6 \times 10^{23}$  (Avogadro's number) atoms.
- Thus, 1 gram of water contains  $1 \times 10^{23}$  atoms. As the human body is made up mostly of water, we can reasonably assume that each kilogram of the human body contains about  $10^{26}$  atoms.
- We may safely assume that a person weighs on the order of 10 to 100 kilograms.
- Thus the number of atoms in a human body is on the order of  $10^{27}$  to  $10^{28}$  atoms.

Now, approximately how many atoms are there among all living humans?  
 Answer:  $10^{37}$  to  $10^{38}$  atoms!

## Scaling

- **Scaling** is comparing numbers to other things that you already know, or creating maps and scale models.
- Scaling is another technique for giving meaning to numbers that might otherwise seem incomprehensible

## Three ways of expressing scales:

- ↳ **Verbally.** A scale can be described in words such as "one centimeter on the map represents one kilometer on the ground."
- ↳ **Graphically.** A scale can be shown with a "miniruler" marked to show the represented distances.
- ↳ **As Fractions.** A scale at which 1 centimeter represents 1 kilometer can be written "1 to 100,000" or as a fraction  $1/100,000$ . In this case, 100,000 is the scale factor.

- The scale factor shows directly how distances on the scale compare to actual distances. For example, a scale factor of 100,000 on a map means that actual distance is 100,000 times larger than distances on the map.

$$\text{scale factor} = \frac{\text{actual distance}}{\text{map distance}}$$

- **Exercise:** Suppose that you have a map with a scale expressed as "one inch on the map represents one mile on the ground". Write this scale as a fraction, and state the scale factor.

## Other uses of scales

- Scaling is used by architects and engineers in building scale models.
- Scaling is used to create timelines of historical events on which each centimeter along the timeline represents a certain number of years of history.
- Time-lapse photography allows movies and videos to show events in fast motion, in which one second of video might represent 1 hour of real time.
- On a computer, researchers can create a scale for size or time, or many other physical measurements, which they can change at will. For example, astronomers used a computer to create scale models of two large galaxies and then scaled time so that a few billion years could be represented in just a few minutes. This computer model simulates interactions that take billions of years in nature before our very eyes.

## The Scale of Space and Time

- One of the most ancient of all human questions: How big or how old is the universe?
- **Example:** The planet Jupiter has a diameter of about 140,000 km, and its average distance from the Sun is  $7.8 \times 10^8$  km. What is its diameter and distance from the Sun on a scale of 1 to 10 billion?  
 Answer: scale diameter = 1.4 cm  
 scale distance = 78 cm

- A **light-year** is the distance that light can travel in one year.
- It is used to measure distances among the stars (because miles or kilometers are simply too small to work with easily).
- This unit of distance (it is not a unit of time) is used for the following reasons:
  1. The speed of light is constant. Thus the definition of light-year contains no ambiguity. The speed of light is approximately 300,000 km per second.
  2. The speed of light is the fastest possible speed (as proved by Einstein).
  3. A light-year is an extremely long distance; it is far enough to make it a convenient unit for cosmic scales.
- How far is a light-year?
 

1 light-year = (speed of light) x (1 yr)

$$= \left(3 \times 10^5 \frac{\text{km}}{\text{sec}}\right) \times \left(1 \text{ yr} \times \frac{365 \text{ day}}{1 \text{ yr}} \times \frac{24 \text{ hr}}{1 \text{ day}} \times 60 \frac{\text{min}}{1 \text{ hr}} \times 60 \frac{\text{sec}}{1 \text{ min}}\right) = 9.5 \times 10^{12} \text{ km}$$

or 9.5 trillion kilometers!

- In the region of the Sun, the stars are like grapefruits spaced thousands of kilometers apart.
- **Example:** Imagine trying to travel a thousand light-years (only 1% of the diameter of the Milky Way Galaxy) with present technology, whereby a spacecraft can travel at a maximum of about 100,000 kilometers per hour. How long would the trip take?
 
$$\text{travel time} = \frac{\text{distance}}{\text{speed}} = \frac{10^3 \text{ light-year}}{10^5 \text{ km/hr}} \times \frac{9.5 \times 10^{12} \text{ km}}{1 \text{ light-year}} = 9.5 \times 10^{10} \text{ hr}$$

$$9.5 \times 10^{10} \text{ hr} \times \left(\frac{1 \text{ day}}{24 \text{ hr}}\right) \times \left(\frac{1 \text{ yr}}{365 \text{ day}}\right) = 1.1 \times 10^7 \text{ yr}$$
- Using present technology, to travel just 1% of the distance across the galaxy would take more than 10 million years!

## 2.2.3 Sources of Uncertainty

Measurement and estimation involve uncertainties.

We need to examine uncertainty, how it arises and how it can be handled honestly.

- Most real-life problems involve measurements, and measured numbers are only approximate.
- The numbers encountered in the real world are most always approximate and therefore involve uncertainties.
- Learning to recognize when numbers are approximate, and estimate their level of uncertainty is one of the most important aspects of the concept of number.

- Numbers can be categorized as being exact or approximate.
- Exact numbers generally arise in only two situations:
  1. Numbers from mathematical theory, such as numbers in formulas, are exact.
  2. Numbers obtained in simple counting may be exact if no counting errors are made.

## Sources of Uncertainty

- Uncertainty in Measurement due to:
  1. Different levels of precision of the measuring device (precision error)
  2. The existence of random errors (unavoidable or unpredictable errors that occur when reading any measurement device)
  3. The existence of systematic errors (errors in the process, or system, used in measurement)

### ■ Uncertainty in Estimation due to:

1. Sampling sizes (precision error)
2. Uncontrollable factors in sampling (random errors)
3. Errors in the sampling system (systematic errors)

## 2.2.4 Accuracy: True vs. Measured Values

- Accuracy is the relative uncertainty of a measured or estimated number. (This is a statistical definition.)
- *Example:* Suppose that after accounting for all known sources of uncertainty, you determine your weight to be  $50 \pm 2$  kg. The absolute uncertainty is 2 kg and the relative uncertainty is  $2/50$ , or 4%. Thus, the measurement is accurate to within 4%.

- Accuracy is the closeness of the approximate value to the true value. The closer the approximate value lies to the true value, the greater is the accuracy of the measurement or estimate.
- In reality, you can never know the true value independent of measurements or estimates. Thus, there is no way of knowing the accuracy of measurement. Nevertheless, by taking precautions against both systematic and random errors, you can gain confidence that your approximate value accurately reflects the true value.
- The true value may even change with time. Thus, even if an approximate value accurately reflects a true value at the time of measurement, it might be less accurate some later time.

## 2.2.5 Expressing Uncertainty

- To properly interpret approximate numbers, some idea of the involved uncertainty is required.
- However, the nature of uncertainty is that you can never know whether you have accounted for all possible sources of error. In other words, you will always be uncertain about the degree of uncertainty.
- But it is still possible to identify potential sources of uncertainty.

## How do we express uncertainty?

- Implicitly, by being careful in rounding
- Explicitly, by stating a range of possible values. You can use the plus/minus notation.  
*Example:* Suppose a census finds that the population of a town lies between 22,000 and 25,000. This range can be expressed as  $23,500 \pm 1500$ .
- You can also express your level of confidence in the stated uncertainty range.

- When dealing with order of magnitude estimates, it is best to express the range explicitly.
- *Example:* The age of the universe is *on the order of* 10 billion years, but the range of uncertainty puts the age between 7 and 20 billion years. (You cannot express this uncertainty as a plus/minus range centered on 10 billion years). Instead, you can describe it in terms of a multiplicative factor. Thus we might say that "the age of the universe" is within a factor of 2 of 10 billion years."

## Significant digits

- In real life, uncertainties rarely are stated explicitly. We must make an educated guess concerning the range of uncertainty.
- First, we hope that numbers are rounded to reflect only as much precision as they deserve.
- Any digit in a number that reflects something that was actually measured, or carefully estimated, is called a significant digit.
- Significant digits can be used to determine an implied uncertainty range (if we assume that only significant digits are reported).

*Example:* The implied uncertainty range for a reported measurement of 167.1 cm is from 167.05 to 167.15 cm.

*Example:* The implied uncertainty range for a reported measurement of 12.92 cm is from 12.915 to 12.925 cm.

*Note:* The implied uncertainty range for a whole number is  $\pm 1/2$ .

## When are zeros significant?

- When zeros are involved, the implied range of uncertainty is ambiguous.  
*Example:* In the reported measurement of 210 km, if the 0 was actually measured and therefore significant, the uncertainty range is from 209.5 to 210.5 km. But if the 0 is merely used as a placeholder, the uncertainty range is from 205 to 215 km. Unfortunately, without any further information, we have no way of knowing whether 210 km is precise to the nearest km or to the nearest 10 km.

- We assume that zero is significant only if they are *necessary* in reporting a measurement. Zeros are not significant if they serve simply as placeholders.
- The following are significant digits:

Nonzero digits

Zeros that occur between nonzero digits (as in 4002 or 3.06)


Zeros that follow a nonzero digit and lie to the right of the decimal point (as in 4.20 or 30.00)


- Zeros to the right of the decimal point that serve only to place the location of the decimal point (as in 0.006 or 0.00052) - not significant
- Zeros to the left of the decimal point that follow the last nonzero digit (as in 40,000 or 210) - not significant unless otherwise stated
- *Example:* What is the implied uncertainty range of a population reported as 240,000? What is the implied uncertainty range if it was reported that 240,000 contains 3 significant digits?

Answer: from 235,000 to 245,000  
from 239,500 to 240,500

## Combining Approximate Numbers

- Special care is required when combining approximate numbers. Otherwise you might state an answer with more certainty than it deserves.
- An answer obtained by adding or subtracting approximate numbers should be rounded to the same precision as the least precise number (This rule is based on the assumption that the stated precisions account for all sources of error.)

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- *Example:* If you are told that the distance between Town A and Town B is 37 km and the distance between Town B and Town C is 14.7 km, how far would you drive in going from A to C via B? (Safe answer: 51 to 53 km)
  - An answer obtained by multiplying or dividing approximate numbers should be rounded to the same number of significant digits as the number with the fewest significant digits (This rule is based on the assumption that all sources of error are accounted for in the stated significant digits.)

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- *Example:* Suppose that you measure the side length of a square room to be 3.3 m. What is the area of the room? (Safe answer: 11 sq m)
  - When you multiply or divide order of magnitude estimates, you can find the resulting uncertainty by multiplying the individual uncertainty.
  - *Example:* Suppose that you estimated the volume of a jar to be within a factor of 2 of 5 liters, and that, on average, a one-peso coin occupies 1 cubic cm, within a factor of 2. How many one-peso coins can be contained in the jar? (Answer: within a factor of 4 of 5000, somewhere between 1250 and 20,000)
  - Always keep track of as many significant digits as possible, rounding only when reporting the final answer.