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Exit Time

An asset, currently priced S, follows a geometric Brownian motion. The probability density function of the expected time to touch a barrier H is given by f(t):

$$M := \mu - \frac{\sigma^2}{2}$$

$$f(t) := \frac{\ln\left(\frac{H}{S}\right) \cdot t^{\frac{-3}{2}}}{\sigma \cdot \sqrt{2\pi}} \cdot e^{\frac{-1}{2} \cdot \left(\frac{\ln\left(\frac{H}{S}\right) - M \cdot t}{\sigma \cdot \sqrt{t}} \right)^2} \qquad [1]$$

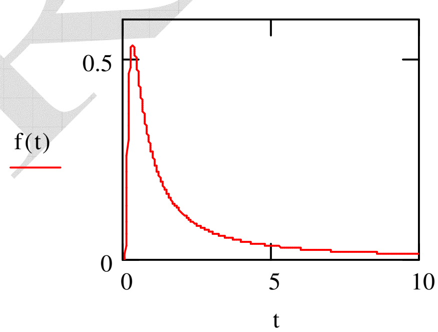
<u>Example:</u> Find the expected time to touch a barrier H=1.1 starting from S=1 given monthly drift 0.01 and monthly standard deviation 0.1.

$\mu := 0.01 \quad H := 1.1 \quad S := 1$

$\sigma := 0.1$

$$\int_0^{\infty} t \cdot f(t)\, dt = 19.062$$

So, the answer is about 19 months. (Mathcad was used to perform the calculations).



Graph: The pdf of Exit Time

<u>Exit Time for the Kelly bettor</u>

The expected time it takes to cross above an up wealth barrier (b) or bellow a down wealth barrier (a) for someone following a fractional (k) Kelly betting strategy is given by:

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$$E(T) = \frac{1}{r} \cdot \ln\left(\frac{b^q}{a^{q-1}}\right) \qquad [2]$$

where $r = \left(k - \frac{k^2}{2}\right) \cdot \left(\frac{\mu}{\sigma}\right)^2$ and $q = \frac{1 - a^{1-\frac{2}{k}}}{b^{1-\frac{2}{k}} - a^{1-\frac{2}{k}}}$

If we are only interested for the expected time to touch the up barrier then:

$$E(T) = \frac{1}{r} \cdot \ln(b) \qquad [3]$$

We can use formula [3] for solving the previous example:

$Kelly = \left(\frac{\mu}{\sigma^2}\right) = 1$, so by owning the asset S it would be equivalent of having invested k=1 Kelly of our wealth in this asset. If for example we had $\left(\frac{\mu}{\sigma^2}\right) = 2$ then it would be equivalent of having invested k=0.5 Kelly of our wealth in this asset.

We use [3] with b=1.1 and k=1 and we have E(T) = 19.062 months as we expected.

Note: We have to be careful and always normalize b as the starting point of S must always be 1.

<u>Example:</u> S is currently trading at 2.5. Find the expected time of either touching 3.5 (up barrier) or 2 (down barrier). The drift is 0.02 per month and volatility is 0.1 per month.

$Kelly = \left(\frac{\mu}{\sigma^2}\right) = \left(\frac{0.02}{0.1^2}\right) = 2$, so k=1/2=0.5. Also we have a = 2/2.5 =0.8 and b = 3.5/2.5 =1.4.

$$q = \frac{1 - 0.8^{1-\frac{2}{0.5}}}{1.4^{1-\frac{2}{0.5}} - 0.8^{1-\frac{2}{0.5}}} = \frac{1 - 0.8^{-3}}{1.4^{-3} - 0.8^{-3}} = 0.6 \quad , \quad r = \left(0.5 - \frac{0.5^2}{2}\right) \cdot \left(\frac{0.02}{0.1}\right)^2 = 0.015$$

From [2] we have:

$$E(T) = \frac{1}{0.015} \cdot \ln\left(\frac{1.4^{0.6}}{0.8^{0.6-1}}\right) = 7.508 \text{ months.}$$