A lower bound for the rotation graph

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Abstract

We prove that the diameter of the rotation graph is at least of order n/2. We studied this graph for the following reasons: 1. the graph forms the 0 and 1- dimensional faces of the associahedron, a convex polytope which appears in recent math. Thurston , in his article "Shapes of polyhedra...'1998 reviewed some tools from his initial article. 2. the initial and only proof used numerical results, and there is not a clear connection between the conditions from lemma 5 and the result of lemma 6 in Thurston.

3. The proof of the diameter of the rotation graph is the only geometric proof of a combinatorics problem, who did not receive other approach

1 The Rotation Graph and the Associahedron.

There is a 1-to-1 correspondence between binary trees and triangulations of the n-gon, using n-3 non-intersection diagonals.

A diagonal flip is an operation that transforms one triangulation of a polygon into another: a diagonal inside the polygon is removed, creating a face with four sides. The opposite diagonal of this quadrilateral is inserted, restoring the diagram of a new triangulation.

Let $TG(n)$ be a graph with one node for each triangulation of an n-gon and an edge between two nodes if they are related by a diagonal flip.

Theorem 1 (Thurston, Sleator and Tarjan). *The diameter of $TG(n)$ is 2n-10, for n big enough.*

It is possible to draw the graph $TG(n)$ on the boundary of a n-3 dimensional cube, creating a special polytope called the Stasheff's associahedron K_{n-1} [**?**]. Yoshida called it Terada-n [**?**].

K_{n-1} is the n-3 dimensional convex polytope whose partial order set of its faces is isomorphic with the partial order set of an n-gon with several non-intersecting diagonals. The partial orders are given by inclusions. There is such a convex polytope [**?**].The codimension k faces of the associahedron are indexed by n-gons with k non-intersecting diagonals.

Following Thurston, Sleator and Tarjan, every triangulation of an n-gon gives a planar binary rooted tree.

We label the leaves of a tree from left to right by 0,1,... To every tree, we can associate a sequence of numbers: $(x_1, y_1, ...x_k, y_k, ...x_n, y_n)$, obtained in the following way: We label the internal vertices by 1,2...The k^{th} vertex is the one which falls in between the leaves k-1 and k. We denote by x_k and y_k the number of leaves on the left side, resp. right side of the internal vertex k.

Let w(t) be the weight of a tree t, defined as
$w(t) = \sum_k |x_k - y_k|$ /(n+1)

Lemma 1 If two trees are joined by an edge in the rotation graph, then their sequences of numbers $(x_1, y_1, ...x_k, y_k, ...x_n, y_n)$ and $(z_1, t_1, ...z_k, t_k, ...z_n, t_n)$ satisfy the following property: there are 2 indices k and l such that $(z_k, t_k) = (x_k \pm x_l, y_k)$ and $(z_l, t_l) = (x_k, y_k \mp y_l)$.

Proof It is easy to describe the flip operation for the triangulation of the n-gon. It is more complicated to describe the flip , or rotation operation for trees. The picture will help. Also, the picture shows the content of the lemma above.

Details of the picture:
$(z_k, t_k) = (x_k - x_l, y_k)$ and $(z_l, t_l) = (x_k, y_k + y_l)$.

In the second tree, obtained by a rotation operation, the edge m dissapears, half of the edge v dissapears and a new vertex called B appears. The vertex A dissapears. The edges in the wave style are added.

Lemma 2 The difference between the weights of two adjacent trees is at most one.

Proof. This lemma is a consequence of Lemma 1.

Theorem 2. *The diameter of $TG(n)$ is at least n/2 - c, for n big enough.*

Proof: If we find two trees such that the difference between their weights is of order n/2, then, according to lemma 2, the diameter will be at least of order n/2 i.e there are at least n/2 edges between these 2 trees. The picture shows the required trees:

The sequence of the first tree is (1,1,1,2,1,3,1,4,1,5...), so its weight is $n(n-1)/2(n+1)$. The second tree is "the most balanced tree". If n is a power of 2, it weight is zero. Otherwise, its weight shows the difference between some parts of the tree, which is of order Clog(n)/n. As a consequence, the difference between their weight is of order n/2.

1.1 Basic Ideas

In the real moduli spaces, we found "reversals", some special involutions. They led us to the Gollan Conjecture. Its proof inspired us in the proof of the lower bound of the diameter above.

1.Pevzner and Bafna idea is to assign a weight function for every permutation, b(σ) - c(σ), such that the change in this quantity is at most 1, if we make a reversal. In computer sciences, there is a technique to prove upper bounds on the performance of an algorithm using potential functions.

2. we built a combinatorial potential, attached to the rotation graph above.

References

[1] V. Bafna and P. Pevzner *Genome Rearrangements and sorting by reversals.* SIAM Journal of Computing Vol. 25, No 2 pp 272-289, April 1996.

[2] Sleator, Daniel D.; Tarjan, Robert E.; Thurston, William P. *Rotation distance, triangulations, and hyperbolic geometry.* . J. Amer. Math. Soc 1 (1988), no. 3, 647–681