The fundamental group of the real moduli spaces $\overline{M}_{0,n}$. A preliminary report on the topological aspects of some real algebraic varieties

Gefry Barad
Department of Mathematics, Johns Hopkins University
Baltimore MD 21218, gefry@math.jhu.edu, http://www.math.jhu.edu/~gefry

December 8, 2002

Abstract

We prove that the fundamental group of the space of the real points of $\overline{M}_{0,n}(\mathbf{C})$ is non-abelian (Theorem 14). The technique has a topological nature. Also, we prove that these spaces are *non-orientable* manifolds (Theorem 16). To our knowledge, the results are new.In a forthcoming manuscript, we investigate the associated combinatorial structures. Many of them were already studied in Computational Biology.

1 Introduction

Moduli of curves were first studied in the realm of Algebraic Geometry . The geometry and the topological aspects of the real points of these spaces were studied by Davis et al. The real points form a space of non-positive curvature. A first attempt to extract the basic combinatorics was made by Devadoss and Yoshida. Yoshida connects these spaces with the theory of hypergeometric functions.

1.1 Algebraic Geometry

We would like to summarize the Theorems of Knudsen, Kapranov, Manin, Kontsevich and Keel. We hope that the reader interested or specialized in Algebraic Geometry will have a glimpse of the topic, which is one of the fundamental models of moduli problems.

Start with the $(n-3)$-dimensional projective space P^{n-3} and choose n-1 points $q_1, ..., q_{n-1}$ in linear general position. These are unique up to a projective transformation. We blow up as follows:

1: blow up the points $q_1, ..., q_{n-2}$, then the lines passing through pairs of these points, followed by the planes passing through triples of these points, etc.;

2.blow up the point q_{n-1}, then the lines spanned by pairs of points including q_{n-1} but not q_{n-2}, then the planes spanned by triples including q_{n-1} but not q_{n-2}, etc.;

3.blow up the line $\{q_{n-1},\ q_{n-2}\ \}$ then the planes spanned by $\{q_i,\ q_{n-1},\ q_{n-2}\ \}$, and all subspaces spanned by them in the increasing order of dimension.

4., 5.,...there are n-3 steps of the type above.

The result of this iterated blow up is the space denoted: $\overline{M}_{o,n}(\mathbf{C})$.

1.1.1 Moduli of pointed curves

An *n-pointed curve* is a connected curve Σ with n smooth distinct *labelled points* $\mathbf{p} = (p_1, \cdots, p_n) \in \Sigma^n$, satisfying the following conditions:

1. Σ has only ordinary double points.
2. The arithmetic genus of Σ is equal to 0.

An n-pointed curve is *stable*, if has a finite automorphism group (i.e. on each irreducible component of Σ, the number of singular points plus the number of marked points on it is at least three). Two such curves , $(\Sigma'; \mathbf{p}')$ are *isomorphic* if there exists a biholomorphic equivalance $\Sigma \to \Sigma'$ mapping p_i to p'_i.

Denote by the set of isomorphism classes of n-pointed stable curves. The space is a compactification of the moduli space of n distinct point configuration on $\mathbb{C}^1$ modulo the action of Möbius group $PSL_2(\mathbb{C})$. An automorphism of $\mathbb{C}^1$ is determined by its action on three distinct points. Thus the open stratum of the orbit space forms a non-singular variety which is isomorphic to product $(\mathbb{C}^1 \backslash \{0, 1, \infty\})^{n-3} \backslash \triangle$ of $n-3$ factors where the diagonal $\triangle = \bigcup_{k<l}\{|\text{for two coordinates } p_k = p_l\}$.

1.1.2 Combinatorial type of pointed curves

Each curve in corresponds to a tree of projective lines meeting transversally. Therefore it is convenient to think of isomorphism classes of pointed curves in terms of their (dual) trees.

The (dual) tree τ of a n-pointed curve is an n-tree with

- *vertices* $V_\tau =$ the set of irreducible components of Σ,
- *edges* $E_\tau =$ the set of double points of Σ,
- *tail* $T_\tau =$ the set of marked points,
- *labelling* $T_\tau =$ the ordering of the marked points.

See Figure 1.

Figure 1: A stable 5-tree τ.

For any n-tree τ, there is a subvariety $D_\tau \subset$ parametrizing the curves with the dual tree given by τ. D_τ is isomorphic to $\prod_{v \in V_\tau} M_{|v|}$ where $|v|$ is the valency of the vertex v. We denote the compactification of the set D_τ by $\overline{D}_\tau$. Its codimension equals to the cardinality $|E_\tau|$ of the set of edges E_τ. Specifically, the open stratum of corresponds to the one-vertex n-tree. The codimension one strata of correspond to one-edge n-trees.

1.2 The real points of $\overline{M}_{0,n}(\mathbf{C})$ after Manin and Devadoss

3. Real points of $\overline{M}_{0,n}$. Consider a stable n–labelled curve over $\mathbf{C}$. Endowing it with the complex conjugate structure we will produce another similar curve. Thus we have a conjugation involution acting upon $\overline{M}_{0,n}(\mathbf{C})$. Fixed points of this involution form the space of real points $\overline{M}_{0,n}(\mathbf{R})$. A stable curve living over a real point is itself real in the following sense: it is endowed with a conjugation involution fixing all labelled points and all singular points. Thus, every irreducible component of a real stable labelled curve is a sphere $\mathbf{P}^1(\mathbf{C})$ endowed with a real equator $\mathbf{P}^1(\mathbf{R})$ carrying all labelled points and eventually all intersection points with other components.

Taken all together, such points are called *special ones.* In the dual graph τ, an irreducible component becomes a vertex, and special points become edges at this vertex. Thus, a real structure of the respective curve determines an additional structure on τ: *an unoriented cyclic order on the set of edges at each vertex.*

We will call a choice of such orders *a locally planar structure* on τ.

Manin was the first one who was trying to summarize the already proven results of the space of the real points:

Theorem 1. Manin [12]. *(i)Let S be a set of $n+3$ elements. The set $\overline{M}_{0,S}(\mathbf{R})$ is a connected closed real manifold. Connected components of intersections of $\overline{M}_{0,S}(\mathbf{R})$ with complex boundary strata form a cell decomposition. Cells of it are in one–to–one correspondence with stable locally planar S–labeled trees. The relation "a cell is a codimension one component of the boundary of another cell" corresponds to the relation "a locally planar tree produces another locally planar tree by contracting an internal edge."*

(ii) Fix an unoriented cyclic order on S and consider the respective open cell. Any choice of three consecutive labels with respect to this order allows one to introduce real coordinates which identify the open cell with the simplex Δ_n^0, where $|S| = n+3$. This identification was denoted Φ in (5).

(iii) The closure of each open cell has the structure of a Stasheff polytope. In particular, its boundary strata of codimension 1 are indexed by those stable 2–partitions of S which are compatible with the respective cyclic order: they correspond to breaking the real equator into two connected arcs.

1.2.1 Devadoss' study of the space of the real points. Braid Arrangements and blow-ups.

From Devadoss' study, the reader can admire the visual and the intuitive aspects of the geometry.

Let $V^n \subset \mathbb{R}^{n-1}$ be the hyperplane defined by $\Sigma x_i = 0$. For $1 \leq i < j \leq n-1$, let $H_{ij}^n \subset V^n$ be the hyperplane defined by $x_i = x_j$. The *braid arrangement* is the collection of subspaces of V^n generated by all possible intersections of the H_{ij}^n.

If $\mathcal{H}^n$ denotes the collection of subspaces $\{H^n_{ij}\}$, then $\mathcal{H}^n$ cuts V^n into $(n-1)!$ simplicial cones. Let $\mathbb{S}(V^n)$ be the sphere in V^n and let $\mathbb{P}(V^n)$ be the projective sphere in V^n (that is, $\mathbb{RP}^{n-3}$). Let $\mathcal{B}^n$ to be the intersection of $\mathcal{H}^n$ with $\mathbb{P}(V^n)$; the arrangement $\mathcal{B}^n$ cuts $\mathbb{P}(V^n)$ into $\frac{1}{2}(n-1)!$ open $n-3$ simplices.

Definition 2. *Let $\mathfrak{b}^k$ be a codim k* irreducible *cell of $\mathbb{P}(V^n)$ if $\binom{k+1}{2}$ hyperplanes of $\mathcal{H}^n$ intersect there.*

Looking at the arrangement $\mathcal{B}^n$ on $\mathbb{P}(V^n)$, there turn out to be $n-1$ irreducible points $\{\mathfrak{b}^{n-3}\}$ in *general position.* These points are the vertices of an $n-3$ simplex with an additional point at the center. Between every two $\mathfrak{b}^{n-3}$ points of $\mathcal{B}^n$, there exists a $\mathfrak{b}^{n-4}$ line, resulting in $\binom{n-1}{n-3}$ such irreducible lines. In general, k irreducible points of $\mathcal{B}^n$ span a $k-1$ dimensional irreducible cell; restating this, we get

Blowing up along each point $\mathfrak{b}^{n-3}$ in $\mathcal{B}^n$ uses the following procedure: A small spherical neighborhood is drawn around $\mathfrak{b}^{n-3}$ and the inside of the sphere is removed, resulting in $\mathfrak{sb}^{n-3}$. Observe that this sphere (which we denote as $\mathcal{S}$) is engraved with great arcs coming from $\mathcal{B}^n$. Projectifying, $\mathfrak{sb}^{n-3}$ becomes $\mathfrak{pb}^{n-3}$, and $\mathcal{S}$ becomes the projective sphere $\mathbb{P}\mathcal{S}$. The engraved arcs on $\mathbb{P}\mathcal{S}$ are $\mathcal{B}^{n-1}$, and $\mathbb{P}\mathcal{S}$ can be thought of as $\mathbb{P}(V^{n-1})$. The blown up of $\mathfrak{b}^{n-3}$ cell becomes $\overline{\mathcal{M}}_0^{n-1}(\mathbb{R})$.

The iterated "minimal or wonderful " blow-up of the n-projective space along the irreducible cells, in increasing order of dimension creates the space of the real points $\overline{M}_{0,n}(R)$.

The minimal blow-up of the braid arrangement in the projective space has been shown by Kapranov to coincide with some other spaces of classical importance. [6] [7].

Theorem 3 (Kapranov). . *For all $n \geq 1$, the following varieties are isomorphic.*
(1) The minimal blow-up of the braid arrangement in RP^n.
(2)The Chow quotient $G(2, n+2)//(R^*)^{n+3}$.
(3)The Chow quotient $(RP^1)^{n+3}//PGL(2, R)$.
(4)The real points of the Grothendieck-Knudsen moduli space for stable (n+3)-pointed curves of genus 0.

2 A cellular decomposition of $\overline{M}_{0,n}(R)$, after Kapranov and Devadoss

Let K_{n-1} be the n-3-convex polytope whose partial order set of its faces is isomorphic with the partial order set of an n-gon with several non-intersecting diagonals. The partial orders are given by inclusions. K_{n-1} is called the associahedron. There is such a convex polytope [Ziegler-Lectures on Polytope p.310]. The codimension k faces of the associahedron are indexed by n-gons with k non-intersecting diagonals.

DIAG is the following set of $n(n-3)/2$ involutions from S_n. For every pair (k, l), $0 < k < l < n$, where (k,l) is different from (1, n-1), we define d:=d(k, l)(x)=l+k-x , if x is between k and l. Otherwise, x is a fixed point. Let P be a fixed n-gon, with edges labeled 1,2,3...n. For every diagonal of P we can associate an element of DIAG in the following way: any diagonal determines a partition of 1,2...n. Take the one which doesn't contain n: it is between 2 numbers, k and l. Then the associated d will be d(k,l), and we say that d(k,l) is supported by the diagonal of the n-gon P. **Throughout the paper, the word "diagonal" means a diagonal of the n-gon, or the involution carried by the diagonal.**

Take $n!$ copies of K_{n-1}. For every permutation of S_n, label the edges of the n-gon with $\sigma(1), \sigma(2).....\sigma(k), ...\sigma(n)$. So the codimension k faces are labeled by decorated n-gons with k non-intersecting diagonals. Now we build our space $\overline{M}_{0,n}(R)$. Two codimension k faces of different K_{n-1}'s are identified (glued) if the permutations σ_1 and σ_2 which color the edges of the n-gons satisfy the following condition "flip" or gluing condition: there are $d_1, d_2, ..., d_k$ elements of DIAG, supported by the diagonals of the second face, such that $\sigma_1 = \sigma_2 \circ d_1 \circ d_2 \circ ... \circ d_k$. ($\circ$ means composition of functions).

The top dimensional faces (without diagonals) are identified by the action of D_n, the dihedral group. So we can begin with $(n-1)!/2$ copies of K_{n-1}, indexed over S_n/D_n. Two codimension k faces are identified if their classes modulo the dihedral group D_n contain two permutations which satisfy the flip condition from the previous paragraph.

3 Differential Geometry and Topology

In this section we will prove that $\overline{M}_{0,n}(R)$ are non-orientable manifolds, with non-abelian fundamental group. The proofs use two theorems which are not met in the main papers dedicated to these spaces. The first one appears in 3-manifold theory. The second one has applications in the theory of Selberg-type integrals. Yoshida was the first one who described, in a way convenient for our research, the efect of a real blow-up on a simplex

Theorem 4. *(Poenaru, pag 35)Smith's Theorem: The fundamental group of an aspherical CW-complex of finite dimension is torsion free.*

Theorem 5. *(Yoshida'2002, section 2) (see also Devadoss and Davis): In the real* $(x_1, \ldots, x_n)$*-space, we consider the hyperplanes*

$$x_i = 0, 1, \ x_j \ (j \neq i), \quad i = 1, \ldots, n.$$

They cut the space into $(n+2)!/2!$ *chambers (simplices) defined by*

$$a_1 < \cdots < a_{n+2}, \quad \{a_1, \ldots, a_{n+2}\} = \{0, x_1, \ldots, x_n, 1\},$$

among which are $n!$ *bounded chambers defined by*

$$0 < x_{\sigma(1)} < \cdots < x_{\sigma(n)} < 1, \quad \sigma \in S_n.$$

Now we blow-up along the non-normally crossing loci of the union of these hyperplanes. Each simplex is transformed into an n-polygon called Terada-n. (A Terada-2 is a pentagon, and a Terada-3 appeared in the previous subsection.) We encode the Terada-n coming from the simplex $0 < x_{\sigma(1)} < \cdots < x_{\sigma(n)} < 1$ *by the word*

$$0\sigma(1)\cdots\sigma(n)(n+1)(n+2),$$

which will be called an $(n+3)$*-juzu; note that point* 1 *is coded by* $n+1$*.*

Indirectly, this last theorem gives us the opportunity to label the simplices from the Devadoss' description of the minimal blowup of the braid arrangement.

The following theorems will be used:

Theorem 6. *(Gromov, [5], p.119) A nonpositively curved geodesic space is aspherical (i.e it is an Eilenberg-MacLane spaces* $K(\pi,1)$ *)*

Theorem 7. *(Davis et al., Corollary 3.5.4, and page 3)* $\overline{M}_{0,n}(R)$ *is a compact connected ,nonpositively curved manifold*

A consequence of these two theorems is given by:

Theorem 8. *(Davis)* $\overline{M}_{0,n}(R)$ *are Eilenberg-MacLane spaces* $K(\pi,1)$*, [3].*

Observation 9. *A particular case of the simplicial approximation theorem is: given a function g from* S^1 *to X, there is a simplicial function which is homotopic with this function g. There are only a finite number of simplicial functions between two finite simplicial complexes. A consequence of this paragraph is the following:*

Theorem 10. *The fundamental group of* $\overline{M}_{0,n}(R)$ *is finitely generated.*

We have the necessary theorems to prove our new result:

Theorem 11. *The fundamental group* π *of* $\overline{M}_{0,n}(R)$ *is non-abelian.*

Proof: We will prove by contradiction. Let us say that π is abelian. According to Theorem 12, it is finitely generated. Theorem 7 says that π is torsion free. According to the fundamental theorem of structure of the finitely generated abelian groups, π is a product of copies of Z.

The following equalities can be found in (Spanier, page 424).
S^1 = K(Z,1)
For any two abelian groups A and B, we have the following homotopy equivalent Eilenberg-Maclane spaces: $K(A\text{x}B,1) = K(A,1)$ x $K(B,1)$
Theorem 11, and the uniqueness up to homotopy of the Eilenberg-MacLane spaces imply that the real moduli spaces are products of circles S^1 = K(Z,1). The dimension is a topological invariant of homology manifolds (Munkres, page 201). So, we have n-3 copies of S^1.

According to Theorem 16, the $(n-3)^{th}$ homology group of the real moduli space is zero. The $(n-3)^{th}$ homology group of the cartesian product of n-3 copies of S^1 is isomorphic with $\mathbf{Z}$. So, the real moduli spaces cannot be homotopically products of n-3 circles $S^1 =$ K(Z,1). We arrived at a contradiction. So, π is not abelian. (Q.E.D.)

Theorem 12. *The* $(n-3)^{th}$ *homology group of the real moduli space is zero. More exactly:* $\overline{M}_{0,n}(R)$ *are non-orientable compact smooth manifolds of dimension n-3.*

Proof: We will prove by contradiction. We suppose that our spaces are orientable.

Let us recall an equivalent definition of the orientability of a finite simplicial complex X which is n-homology manifold: X is orientable if we can assign an orientation for every top dimensional simplex, such that two n-simplices which have a common (n-1)-simplex will induce opposite orientations of this common face. The moduli spaces are tilled by contractible associahedra. An orientation of the entire space means a choice of one of the two orientations, such that two codim 1 adjacent associahedra will induce opposite orientation of the common face.

According to Devadoss and Yoshida approach (see section 1.2.1 and [20]) , the effect of the real blow-up on the spherical simplices will be a truncated simplex , called the associahedron (or Terada-n, or Stasheff's polytope). A blow-up along codimension 1 faces does not change anything. So, if two simplices are codim 1-adjacent on the sphere, then they will be adjacent in the moduli space , too. The converse is not true.

An orientation of the real moduli spaces will give an orientation of the projective space of the same dimension. Just load a spherical simplex with the orientation of the corresponding associahedron from the moduli space. For even n-3, this fact is not possible : the projective spaces of even dimensions are non-orientable. So we proved our theorem in this case.

For odd n-3, the projective spaces are orientable, so we cannot apply the reasoning from the last paragraph. Instead, we can apply Keel's inductive construction: The construction of the moduli space is inductive, whose fibers are generically S^1 which is orientable. So, an orientable moduli space will give another one which is orientable. But we already proved that an even dimensional moduli space is non-orientable.

A rigorous proof is given by the description of these spaces as blow-ups of braid arrangements:

On the sphere, blowing up around each vertex of the simplicial cone uses the following procedure: a small disk is removed, resulting a manifold with boundary S^{n-4}. The braid arrangement B_n will intersect S^{n-4} as a braid arrangement of lower dimension B_{n-1}. Blowing up along the lines or planes, etc. of B_n corresponds to blowing up along points, lines, cells of lower dimensions in B_{n-1}. Much more, an orientation of $\overline{M}_{0,n}(R)$ will give an orientation of the conical simplices which have a common point on the sphere. The codimension one faces will receive an orientation,too. All these orientations of the simplicial cones which passes through a point are compatible with the blow-up procedure. An orientation of $\overline{M}_{0,n}(R)$ will give an orientation of $\overline{M}_{0,n-1}(R)$. And we know that the last one is non-orientable (1).

The second proof of the non-orientability of the moduli spaces

The proof is based on the following theorem (Spanier, page 206): if K is a finite n-dimensional orientable pseudomanifold, then $H_{n-1}(K)$ has no torsion.

We will assume that the moduli space is orientable. We will prove that the first homology group has Z_2-torsion. By Poincare duality, $H_{n-1}(K)$ has torsion. We will find an orientable manifold which does not satisfy Spanier's lemma.

So, the main point is to prove that the first homology group has Z_2-torsion. We will use Davis' description of the fundamental group:

The fundamental group π is described by Davis et al. as a kernel of an epimorphism f. Let A be the following group, given by generators and relations:

-involutory generators α_T, for each proper sub-interval T of [1, n-2]. The relations are: if the distance between the intervals X and Y is at least 2, then α_X and α_Y commute. If X $\subset$ Y, then $\alpha_Y \alpha_X \alpha_Y = \alpha_Z$. Z denotes the image of X under the order-reversing involution of Y.

-T is a set of $n(n-3)/2$ involutions from S_n, called reversals. For every pair (k, l), $0 < k < l < n$, where (k,l) is different from (1, n-1), we define d:=d(k, l)(x)=l+k-x , if x is between k and l. Otherwise, x is a fixed point.

The epimorphism f sends the generators of A into the corresponding reversals, which satisfy the same relations. The kernel of f is isomorphic with the fundamental group of $\overline{M}_{0,n}(R)$.

We are doing a simple diagram chasing:

Note: in Devadoss' research, an error was the folowing: he described reversals as transpositions. The consequences of this error are the following: -the imposibility to prove , using the diagram above, the fact that the first homology contains torsion (any transposition will go to an odd permutation, it is not in the kernel) - a wrong estimates of the diameter of the graph which described the incidence relations among associahedra. -This discovery, of the reversals inside the real moduli space, lead us to Computational Biology where these permutations were first studied in the context of Gollan Conjecture. Nothing was published about the circular version of Pevzner and Bafna results.

References

[1] Arbarello; Cornalba *Combinatorial and algebro-geometric cohomology classes on the moduli space of curves.* J.Alg.Geom. 5 (1996), no.4

[2] V. Bafna and P. Pevzner *Genome Rearrangements and sorting by reversals.* SIAM Journal of Computing Vol. 25, No 2 pp 272-289, April 1996.

[3] M.Davis, R.Scott, T.Januszkiewicz *Fundamental groups of blow-ups.* Advances in Mathematics, page 38, Remark 6.2.4 2002.

[4] S. Devadoss *Tessellations of moduli spaces and the mosaic operad..* Homotopy invariant algebraic structures 1998, pp.91-114 Contemp.Math. 239.

[5] M.Gromov *Hyperbolic Groups.* Essays in Group Theory (Gersten, ed.), Springer, Berlin 1988, 75-264

[6] M.Kapranov *Chow quotients of Grassmannians, I..* Adv.in Soviet Math 16(1993), 29-111.

[7] M.Kapranov, I.M. Gelfand, A.Zelevinsky *Discriminants, resultants, and multidimensional determinants.* Birkhuser Boston, Inc., Boston, MA, 1994.

[8] S.Keel *Intersection theory of moduli space of stable n-pointed curves of genus zero.* Trans. Amer. Math. Soc. **330** (1992), no. 2, 545-574.

[9] F. Knudsen *Projectivity of the moduli space of stable curves II.* Math. Scand. **52** (1983), 1225-1265.

[10] M.Kontsevich, Yu. Manin Quantum cohomology of a product. In *Inventiones mathematicae*, 124, 313-339 (1996)

[11] D. Krammer *Braid Groups are linear.* Annals of Mathematics 155 (2002), no. 1, 131-156.

[12] Yu.Manin, A.B. Goncharov *Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$.* math.AG/0204102

[13] Mumford, D.; Fogarty, J.; Kirwan, F. *Geometric invariant theory. Third edition.* Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer-Verlag, Berlin, 1994.

[14] Munkres *Elements of Algebraic Topology* Addison-Wesley'1984

[15] P.A. Pevzner *Computational Molecular Biology.* A Bradford Book. MIT Press, Cambridge, MA, 2000. xviii+314 pp

[16] V. Poenaru *Discrete Groups.* Lecture Notes in Mathematics, Vol. 421, Springer-Verlag, Berlin-New York, 1974

[17] E.Spanier *Algebraic topology.* Springer-Verlag, New York-Berlin, 1981. xvi+528 pp

[18] G. Watterson, W. Ewens, Hall, A. Morgan *The Chromosome Inversion Problem.* Journal of Theoretical Biology , 1982 , 99. p.1-7.

[19] M. Yoshida *The democratic compactification of configuration spaces of point sets on the real projective line.* Kyushu J. of Math. 50 (1996) 493-512

[20] M.Yoshida, K.Mimachi *Intersection numbers of twisted cycles and the correlation functions of the conformal field theory II .* math.AG/0208097

[21] G.Ziegler *Lectures on Polytopes*, page 310. Springer'1995