Challenge Questions

The last 2 or 3 questions on the Euclid contest, the challenge questions, are used to separate the very best mathematicians from the rest of the contestants. As such the questions are more difficult and less predictable, but there are still some patterns that can be observed. Students who are aware of these patterns may find these questions more accessible.

The topics involved and mathematical tools needed to solve these challenge questions are usually not very different than those explored in the previous workshops. However exactly which ideas to use in a specific question may be far less obvious and several ideas will generally be required to solve these problems. Many of these problems will involve geometry or will involve the properties of integers (number theory). Before looking at these more difficult problems, the student should have looked at the previous Euclid workshop, “Euclidean Geometry” and the first FGH workshop, ”Solve it with Integers”.

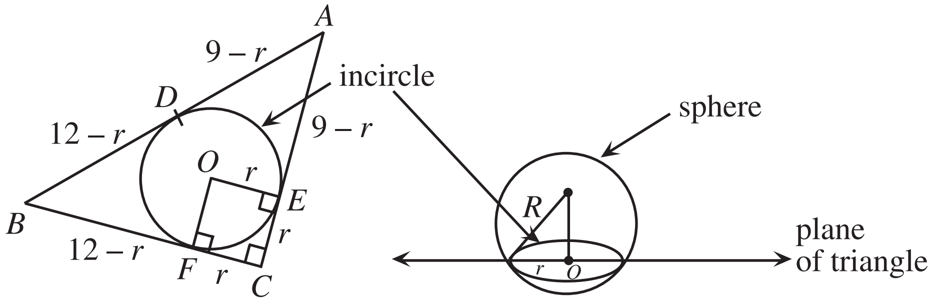
The other generalization that may be made about these challenge problems is that they are problems requiring careful proof or very detailed argument. The examples below will attempt to show how the student should present such proofs.

Example #1

Three thin metal rods of lengths 9, 12 and 15 are welded together to form a right-angled triangle, which is held in a horizontal position. A solid sphere of radius 5 rests in the triangle so that it is tangent to each of the three sides. Assuming that the thickness of the rods can be neglected, how high above the plane of the triangle is the top of the sphere?

Solution:

Although this problem is not intrinsically difficult, the 3 dimensional nature of the problem causes difficulty. Many students find it difficult to visualize that the circle that is the intersection of the sphere and the plane of the triangle is also the in-circle of the triangle.



Therefore, if we can calculate the radius r of the in-circle of the triangle, we should be able to use the theorem of Pythagoras to calculate the distance of the plane of the triangle from the centre of the sphere.

If the triangle is labeled as in the diagram, the square OECF has 4 sides of length r. OECF is a square because it has 4 right angles (recall that a tangent is perpendicular to the radius at the point of contact) and has 2 adjacent sides that are radii. Thus the equality of tangents from an external point, such as AD and AE, gives the remaining labeled parts. Since the hypotenuse is 15, we can arrive at the equation, (9-r)+(12-r)=15. Thus r=3. But the sphere has radius 5 and so the vertical distance between the centre and the triangle is 4 (using the Pythagorean theorem for the right triangle in the second diagram) and so the top of the sphere is 9 units above the triangle.

Example 2 (Question #10,2002)

A triangle is called *Heronian* if each of its side lengths is an integer and its area is also an integer. A triangle is called *Pythagorean* if it is right-angled and each of its side lengths is an integer.

(a) Show that every Pythagorean triangle is Heronian.
(b) Show that every odd integer greater than 1 is a side length of some Pythagorean triangle.
(2) Find a Heronian triangle which has all side lengths different, and no side length divisible by 3, 5, 7 or 11.

<u>Solution:</u>

This is a very difficult problem that is presented here to demonstrate the knowledge and techniques required for its solution. In the solution below we will show a proof by contradiction and make substantial use of the general formula for Pythagorean triples.

Part (a)

We represent the sides of the triangle by the integers a, b, and c where $a^2+b^2=c^2$. We need to show that the area is integral. But the formula for the area of such a right triangle is $\frac{1}{2}ab$. This area will be an integer as long as either a or b is even. This observation leads us to consider the well known series of Pythagorean triples of integers such as 3,4,5 or 5,12,13 or 8,15,17 etc. It seems that a and b cannot both be odd but we need to prove this result. To prove this, we assume the opposite, that a and b are both odd, and show that this assumption leads to a contradiction.

If a and b are odd then c must be even since the sum of the squares of 2 odd numbers is even. So we let a=2l+1, b=2m+1 and c=2k where k , l , m are integers. Then $a^2+b^2=c^2$ gives $4l^2+4l+1+4m^2+4m+1=4k^2$, But this is impossible since the right side is divisible by 4 while the left side leaves a remainder of 2 when divided by 4. Therefore the left side is not equal to the right side. This is our contradiction! So either a or b is even and the area is integral.

Part (b): We are required to show that all odd integers are the sides of some Pythagorean triangle. In this regard we observe that $(2k^2+2k+1)^2 = \left(2k^2+2k\right)^2 + (2k+1)^2$. (Prove this algebraic identity!)

It follows that there must be right triangles with sides 2k+1, $2k^2+2k$ *and* $2k^2+2k+1$. Since 2k+1 takes on all odd values we are done!

But where did this 'magic' formula come from?

The general formula, a=2uv, b=u^2-v^2 and c= u^2+v^2 where u and v are any pair of positive integers which are relatively prime and of different parity, gives all primitive Pythagorean triples.(We state his very useful formula without proof here) If we put b=2k+1=$u^2-v^2=(u+v)(u-v)$ then we can choose u+v=2k+1 and u-v=1 giving u=k+1 and v=k and generating the formula given above.

Part (c): We observe that pasting together 2 Pythagorean triangles with a common side generates a Heronian triangle. For example the Heronian triangle with sides 13,14, and 15 and area 84 comes from pasting a 9,12,15 triangle to a 5,12,13 triangle. Our strategy then will be to join the triangle with sides 2uv, u^2-v^2, u^2+v^2 to the triangle with sides 2mn, m^2-n^2, m^2+n^2. This requires uv=mn and our new triangle has sides m^2+n^2, u^2+v^2,and $m^2-n^2+u^2-v^2$. Now if let uv=3x5x7x11=mn we can assign these 4 prime factors to u and v differently than to m and n. This ensures that the first 2 sides are different and are not divisible by any of these 4 factors since *exactly* one the 2 terms in m^2+n^2(or in u^2+v^2) will be divisible by that factor . Therefore neither of these 2 sides is divisible by 3,5,7,or 11. Thus the only thing left to do is find values of u,v,m,n such that $m^2-n^2+u^2-v^2$is not divisible by any of 3,5,7, or 11. This accomplished by choosing m=35 n=33, u=385 and v=3(finding this required extensive trial and error with the various ways of factoring uv=3x5x7x11=mn) giving a Heronian triangle with sides 2314,148234,and 148352 and area 171346560. An alternate solution and smaller triangle can be found in the solution in past contests on the CMC website

Example 3:

In triangle ABC, $AB=8$, and $<CAB=60°$. Sides BC and AC have integer lengths *a* and *b*, respectively. Find all possible values of *a* and *b*.

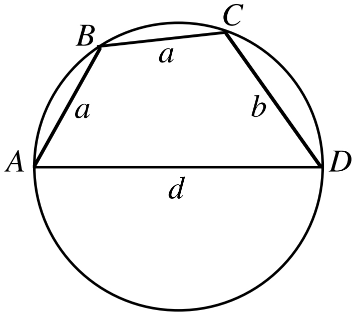
Solution : The solution to this problem will involve only the cosine law and some algebra The cosine law yields $64 = a^2+b^2-2ab\cos(60^\circ) \Rightarrow a^2+b^2-ab=64$. Now multiplying this equation by 4 and completing the square on the left side we have $4a^2-4ab+b^2+3b^2=256$ or $(2a-b)^2+3b^2=256$. Since both terms are squares and thus positive, there are only a limited number of solutions possible. In fact $3b^2<256$ which gives b<$\sqrt{\frac{256}{3}} \approx 9.2$. Thus b=1,2,…..9. Trial and error, gives b=8 and 2a-b=8 ie a=b=8 as the only integer values that solve the equation.

Example 4:

$ABCD$ is a cyclic quadrilateral, as shown, with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$ and $CD = b$. If a, b and d are integers $a \neq b$,

(a) prove that d cannot be a prime number.

(b) Determine the *minimum* value of d.



Solution:

While there are some elegant ideas that would shorten the solution of this problem such as rearranging the sides of the quadrilateral or using Ptolemy's theorem, we simply plow ahead using more elementary ideas. What the solution needs is an algebraic relationship involving only a, b and d. This relationship will then allow the use of various properties of integers to solve the 2 parts of the problem.

What do we know? First that since d is a diameter, d is larger than a or b , and we know that <ABD and <ACD are right angles. Also <BAD=180-<BCD. So we draw in the line BD and let the length of BD be x. Thus we know:

(1) $a^2 + x^2 = d^2$

(2) $x^2 = a^2 + d^2 - 2ad\cos < BAC$

(3) $x^2 = a^2 + b^2 - 2ab\cos < BDC$

Our task is now to eliminate the cosine terms and then eliminate x from these equations. We observe that cos(<BAC)=-cos(<BDC) since these angles are supplementary (the quadrilateral is cyclic). Therefore we multiply the second equation by b and the third equation by d and add the 2 equations. This gives $(d+b)x^2 = (b+d)a^2 + bd^2 + b^2d$. Noting that b+d is also a factor of the last 2 terms and is not zero(band d are positive) we divide out b+d to get $x^2 = a^2 + bd$. Using equation (1) with this new equation we arrive at $d(d-b) = 2a^2$. But d>a and so all prime factors of $2a^2$ are less than d. Since all of these variables are integers d cannot be a prime factor of the right side, and so d cannot be be prime.

$b \pm \sqrt{b^2 + 8a^2}$

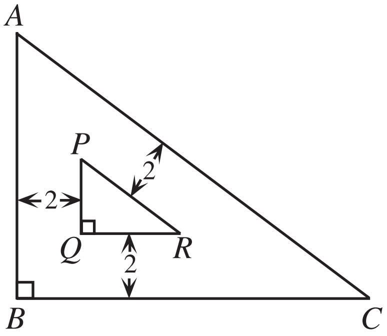
Part (b)

We now find the minimum value of d. Solving $d(d-b) = 2a^2$as a quadratic we get $d = \dfrac{b \pm \sqrt{b^2 + 8a^2}}{2}$. Since $b < \sqrt{b^2 + 8a^2}$ the only positive root is $d = \dfrac{b + \sqrt{b^2 + 8a^2}}{2}$. Notice that if d is to be an integer we require $b^2 + 8a^2 = m^2$, where m is an integer. But d is minimal when a is as small as possible since increasing a also increases the square root term. So if a=1 then $b^2 + 8 = m^2$and then $+8 = m^2 - b^2 = (m+b)(m-b)$. This has a solution only when we factor the 8 into 4x2 which gives m=3 , b=1 and d=2. But we cannot have a=b=1 by the condition of the problem. Thus we need a=2 and so $32 = m^2 - b^2 = (m+b)(m-b)$. Now we use the 2 factorings

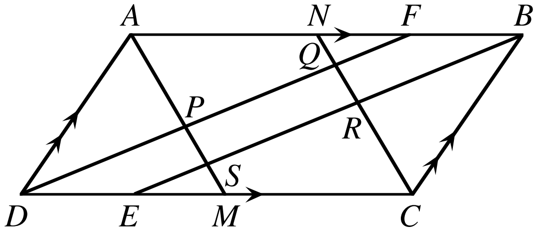
16x2 or 8x4 . After some calculation, these factorings lead to a=b=2 which is invalid, or to a=2 b=7 and d=8, our solution to the problem.

Exercises

1. Triangle ABC is right-angled at B and has side lengths which are integers. A second triangle, PQR, is located inside ΔABC as shown, such that its sides are parallel to the sides of ΔABC and the distance between parallel lines is 2. Determine the side lengths of all possible triangles ABC, such that the area of ΔABC is 9 times that of ΔPQR.

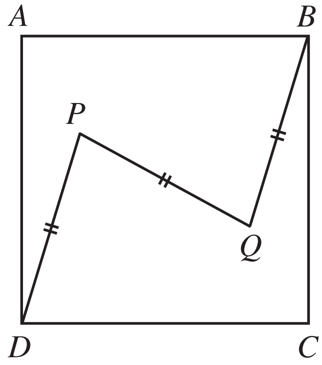


3. In parallelogram $ABCD$, $AB = a$ and $BC = b$, where $a > b$. The points of intersection of the angle bisectors are the vertices of quadrilateral $PQRS$.
   (a) Prove that $PQRS$ is a rectangle.
   (b) Prove that $PR = a - b$.



4. A permutation of the integers 1, 2, ..., n is a listing of these integers in some order. For example, $(3, 1, 2)$ and $(2, 1, 3)$ are two different permutations of the integers 1, 2, 3. A permutation $(a_1, a_2, ..., a_n)$ of the integers 1, 2, ..., n is said to be "fantastic" if $a_1 + a_2 + ... + a_k$ is divisible by k, for ***each*** k from 1 to n. For example, $(3, 1, 2)$ is a fantastic permutation of 1, 2, 3 because 3 is divisible by 1, $3+1$ is divisible by 2, and $3+1+2$ is divisible by 3. However, $(2, 1, 3)$ is not fantastic because $2+1$ is not divisible by 2.
   (a) Show that no fantastic permutation exists for $n = 2000$.
   (b) Does a fantastic permutation exist for $n = 2001$? Explain.

5. Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.



6. A 'millennium' series is any series of consecutive integers with a sum of 2000. Let m represent the first term of a 'millennium' series.
   (a) Determine the minimum value of m.
   (b) Determine the smallest possible positive value of m.

7. The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.
   (a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).

(b) Determine q in terms of a and b.