The Story of Mollweide and Some Trigonometric Identities

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1 Introduction

There are many mathematicians who contributed to and helped shape mathematics into its present form because of the techniques they developed. As new technologies revolutionize our world, however, the role of some techniques has diminished. Subsequently, these mathematicians fade into the background on the stage of mathematics. Karl Brandan Mollweide, his equations, and their importance in trigonometry are such an example.

The role of trigonometry has changed throughout the development of mathematics. For hundreds of years, it was a useful tool for astronomers and land surveyors. It is not a coincidence that the founders of trigonometry, Aristarchus and Hipparchus, along with other great contributors to the field, such as Mollweide, were astronomers.

Beginning with Regiomontanus in 1464 though, trigonometry evolved as a field of study distinct from astronomy, with its emphasis initially on computation and the solutions of triangles and then on analysis. Besides deriving trigonometric identities, a significant portion of typical trigonometry textbooks in the past was devoted to the solutions of triangles. In fact, even in 1784, in his *Harmonia Trigonometrica; or, a Short Treatise of Trigonometry*, Henry Owen still defined trigonometry as "the art of measuring or resolving triangles." By the solution (or resolution) of a triangle, we mean that by given some parts (sides and/or angles) of a triangle, the remaining are to be determined using various relationships among the angles and the sides of a triangle. Many trigonometric identities serve well for this purpose, for example, the law of sines and the law of tangents. Furthermore, after solving a triangle, mathematicians in the past routinely checked their calculations. A textbook from 1928 quoted, "It is *absolutely* necessary that the computer should *know* that his results are correct. For this reason *all* work must be checked" [28].

The equations shown below are for unclear reasons attributed to Karl Brandan Mollweide and are known by his name. They are useful for checking the solution of a triangle because each equation involves all three angles and three sides of a triangle:

$$\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} = \frac{a-b}{c} \tag{1}$$
$$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} = \frac{a+b}{c}. \tag{2}$$

Here and throughout this article, I will follow the convention that in $\triangle ABC$, $A = \measuredangle BAC$, $B = \measuredangle ABC$ and $C = \measuredangle ACB$. Also $a = BC$, $b = AC$ and $c = AB$.

With the advent of calculators and computers, however, as Professor Underwood Dudley commented, "we don't check our solutions of triangles much these days" [51]. Hence, most of today's trigonometry textbooks do not even mention the Mollweide equations and do so only in the context of derivation, not in checking the solutions of triangles.

The Mollweide equations are also useful because they allow for easy application of the logarithms, which transform products into sums. In fact, soon after their discovery, the logarithms became an integral part of trigonometry because of their great use in computations.

I first came across the Mollweide equations in a textbook published in 1938 [22]. While trying to prove the Mollweide equations geometrically, I noticed a common motif that ties together some of the trigonometric identities, including the law of tangents [52]. I rediscovered a figure (figure 5) and developed a figure (figure 8) that prove quite a few identities. My curiosity propelled me to delve into the history of Mollweide the mathematician and his equations themselves: their history, proof, and their relationship to the law of tangents and other trigonometric identities, all of which will be the focus of this article.

2 Karl Brandan Mollweide

Mollweide is not considered a major figure in mathematics. Although he published articles and books in astronomy, physics, mathematics, and was the editor of both *Euklid's Elemente* and a mathematics dictionary, these accomplishments were not sufficient to earn him a place in most of the English history books in mathematics. Almost all of the biographical information in this article is from a newspaper article that appeared about a month after his death and the *Allgemeine Deutsche Biographie* [55, 18].

Karl Brandan Mollweide was born in Wolfenbüttel in Brunswick, Germany on February 3, 1774. As a boy, he attended the public school in Wolfenbüttel, but did not show any interest in mathematics. For unknown reasons, his talent in mathematics emerged suddenly after the age of twelve. He began studying calculus on his own from books he found at home, including a calculus text by Johann Hemeling. Mollweide soon advanced to algebra. At age fourteen, he predicted the occurrence of an eclipse based on his calculations. He showed such exceptional knowledge that his teacher, Christian Leiste, commented on his surprising answers, "Has he memorized all the logarithmic tables in his head?" By this time, he had decided to study mathematics. He headed to the University of Helmstedt, where he studied under Johann Friedrich Pfaff. Pfaff, who made significant contributions to the theory of differential equations, was also the advisor of Karl Friedrich Gauss from 1798 to 1799 when Gauss was working on his famous doctoral dissertation on the fundamental theorem of algebra.

After three years of study at the University of Helmstedt, Mollweide became a teacher there. Unfortunately, he gave up his position only a year later because of debilitating hypochondriac symptoms which remained throughout his life. Two years of rest at home, however, improved his condition sufficiently for him to assume the position of professor of mathematics and astronomy at the University of Halle. At Halle he sought to improve upon the weaknesses of the Mercator projection.

Gerhard Krämer, better known to us by his Latinized name Gerardus Mercator, developed a map-making technique known by his name in 1569 as a navigation tool. (Note that nowadays the Mercator projection is taught using logarithms, which were not discovered by Napier until 1614.) The Mercator projection is unique because it preserves angles, and is therefore a conformal map. Spirals of constant compass heading on the globe called rhumb lines (or loxodrome) form straight lines on this map. It was a remarkable achievement for the sailors at the time because it allowed for more accurate navigating. Distance and area, however, are exaggerated in high latitudes on this projection.

With this weakness in mind, Mollweide introduced a homolographic (equal-area) projection in 1805, which is now known as the Mollweide projection.

Mollweide's method involves only trigonometric functions. Unlike the Mercator projection, the Mollweide projection preserves area in relation to each other. As we know, a projection cannot be both conformal and homolographical. For this contribution, Mollweide is better known today in the field of cartography than in mathematics.

After spending eleven years at Halle, in 1811, he accepted the position of professor of astronomy at the University of Leipzig. Here he exerted great influence on his student August Ferdinand Möbius, who later studied astronomy from Gauss at Göttingen and mathematics from Pfaff at Halle. (Pfaff had accepted the appointment as the chair of mathematics at Halle in 1810 when Helmstedt was closed.) Learning from all these masters, Möbius achieved his own immortality in mathematics, as evidenced by the strip, function, inversion formula and transformation all in his name.

Unfortunately, the war between Germany and France in 1813 had a negative effect on Mollweide's research. The war forced him to devote time on geography, and funding for his observatory was limited; even though he headed the observatory at Leipzig, he never practiced or observed astronomy. In 1814, for the above reasons and for the love of mathematics, Mollweide became a professor of mathematics at Leipzig. Möbius succeeded Mollweide's position in astronomy in 1816. While at Leipzig, Mollweide declined a position with better offers from the University of Dorpat (part of Russia at the time) because of his patriotism. He remained a professor at Leipzig until his death on March 10, 1825, from a long bout of dry cough that was eventually complicated by fever.

Only a few anecdotes are known about Mollweide's personal life. He married late, at the age of 40, to the widow of Meissner, a former astronomer at the observatory. But they did not have any children. Accordingly, he always appeared older than his age because of his gray hair.

As a teacher, he tried with his full heart to promote the study of science and mathematics. Anyone who expressed interest in these topics received his support. As a result, although his students were often initially put off by his hypochondriasis, he was loved by those who knew him well; deep down he was truly kind and always wanted only the best for science and mathematics. Mollweide was admired as a lecturer because of his ability to present dry topics in an interesting manner by drawing connections to other topics. He was also known for his penmanship; his ability to draw a "perfect" circle freehand amazed his students.

As a scientist and mathematician, Mollweide fought against the philosophy of indefinite deductions and mystical interpretations. For example, he organized the scattered rules of magic squares into a book, *De Quadratis Magicis Commentatio*, and published in 1816. This book was the first on magic squares devoid of any mystical nature. Furthermore, he strongly admired Euclid and defended him against criticisms of his methods and proofs. But although he admired the geometrical methods of the ancient mathematicians, he acknowledged that modern sciences required the use of modern analytical methods.

Mollweide demonstrated various talents as a mathematician. He was feared as a proofreader for his ability to easily detect and harshly criticize the smallest flaw in papers. Although he did not discover any completely new mathematical methods, he was admired for thoroughly investigating and extending known methods. Among his mathematical contributions, Mollweide was the first to use the modern congruence symbol "≅" in the 1824 edition of Lorenz's German translation of *Euklid's Elemente*. He also took over the work on the mathematical dictionary, *Mathematisches Wörterbuch*, from Georg Simon Klügel, but only published one volume in 1823 prior to his death. Although the Gauss analogies in spherical trigonometry were published [by Gauss] in 1809, they were actually proceeded in 1808 by Mollweide, and in 1807 by Jean Baptiste Joseph Delambre [13].

Although Mollweide is not considered a major figure in the history of mathematics, he did make significant contributions to the field, and even more so to the field of cartography with his map projection method. Our focus now, though, is on the equations that bear his name.

3 The History of Mollweide's Equations

In mathematics, Mollweide is immortalized by the aforementioned trigonometric equations, although he was not the first to discover these equations. Throughout the eighteenth century, a number of mathematicians, from Newton to Cagnoli, have derived the Mollweide equations using various methods.

Let us begin with equation (2), which is also known as Newton's formula. Sir Isaac Newton gives a different form of the equation (see figure 1) in his *Arithmetica Universalis* in 1707 [33]. Problem VI in *Arithmetica Universalis* asks to determine the sides of $\triangle ABC$ if the base AB, the sum of the sides $AC + BC$ and the vertical angle C are given. After solving the problem, Newton commented:

> Si anguli ad basin quærerentur, conclusio foret concinnior; utpote ducatur EC datum angulum bisecans & basi occurrens in E; & erit $AB.AC + BC(:: AE.AC) ::$ sin. ang. ACE. sin. ang. AEC. Et ab angulo AEC ejusque complemento BEC si subducatur dimidium anguli C relinquentur anguli ABC & BAC.

In translation:

> If the angles at the base are looked for, the conclusion is neater. Draw EC bisecting the given angle and meeting the base at E. Then
> $$\frac{AB}{AC+BC} = \frac{AE}{AC} = \frac{\sin(\measuredangle ACE)}{\sin(\measuredangle AEC)}.$$
> If from the angle AEC, and also from its complement BEC you subtract half the angle C, there will be left the angles ABC and BAC.

The notation in the original text, $p.q :: x.y$, (equivalent to our $p : q = x : y$) originates from the inventor of the slide rule, William Oughtred, who used it for geometrical ratio.

Even though Newton does not clearly state equation (2), the last sentence in his remark indicates that he may have been aware of the relationship stated in equation (2), as we shall see later.

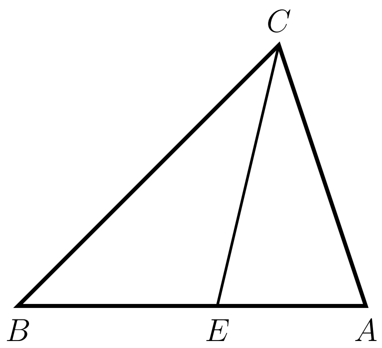

Figure 1

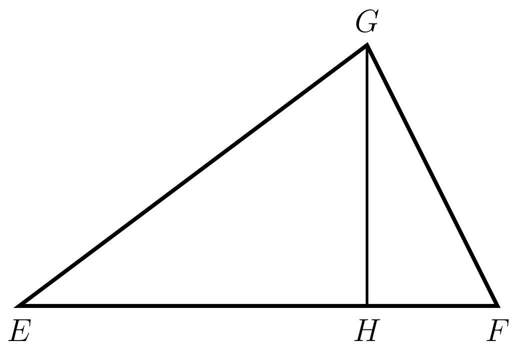

Figure 2

The next major development in the Mollweide equations occurred in 1743. Figure 2 is reproduced from page 5 of *A Miscellany of Mathematical Problems. In Three Volumes* (1743) by Anthony Thacker (?–1744). He refers to Newton's Problem V in *Arithmetica Universalis.* In figure 2, GH is the height of $\triangle EFG$. Thacker then assigns $B = EF$, $S = EG + GF$, $D = EG - GF$, $s = \sin\frac{1}{2}(E+F)$, $q = \cos\frac{1}{2}(E+F)$, $x = \sin\frac{1}{2}(E-F)$ and $y = \cos\frac{1}{2}(E-F)$. Using algebraic computations, he arrives at $qS = By$ and $xB = sD$. Both equations appear on page 6 of the same book. Then on page 13, he summarizes:

> Theorem IX.
>
> As the Base of any Triangle : the Sum of its other two Sides :: the Cosine of half the Sum of the Angles at the Base : the Cosine of half their Difference.
>
> Theorem X.
>
> As the Base of any Triangle : the Difference of the other two Sides :: the Sine of half the Sum of the Angles at the Base : the Sine of half their Difference.

Using the above two theorems, Thacker goes on to solve two triangles given in Seth Ward's *Mathematiks*. In particular, he also states that Newton's problem VI, mentioned previously, may be solved with his theorem IX. While Newton is implicit in stating equation (2), Thacker is explicit, on page 15 of his book (note the correction of the misprint, ~~Co~~sine):

> $AB : AC + BC ::$ ~~Co~~sine of half the Angle C : Cosine of half the Difference betwixt the Angles A and B.

Three years after Thacker, in 1746, the German mathematician Friedrich Wilhelm von Oppel (1720–1769) includes these two statements on page 18 of his *Analysis Triangulorum*:

> §84 Basis trianguli est ad differentiam crurum ut sinus semisummæ angulorum ad basin sitorum ad sinum semidifferentiæ eorundem angulorum.
>
> §85 Basis trianguli est ad summam crurum ut cosinus semisummæ angulorum ad basin ad cosinum semidifferentiæ eorundem.

In more familiar notations, §84 and §85 are, respectively:

$$\frac{c}{a-b} = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \tag{3}$$

$$\frac{c}{a+b} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}. \tag{4}$$

Oppel begins with the same concept as figure 7 in the construction of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$. He follows with laborious algebraic calculations of the sides and applies the law of tangents in order to reach the above two corollaries. Although Oppel's statements are equivalent to Thacker's, his proof is quite different.

Interestingly, all of the authors who studied the history of the Mollweide equations attributed equations (3) and (4) to Oppel [2, 4, 38, 43, 48]. They completely ignored Thacker's explicit statement of equation (2), which was later attributed to Newton. These authors most probably overlooked Thacker because his book title is not specific for trigonometry.

Two years after Oppel, in 1748, Thomas Simpson (1710–1761) gives the present forms of equations (1) and (2) in his *Trigonometry, Plane and Spherical with the Construction and Application of Logarithms.* Simpson, who is best remembered for the numerical methods of integration known as Simpson's rule (actually discovered by Newton), does not reach these equations algebraically like Thacker, as evidenced by his lack of mention of them in his *A Treatise of Algebra* (1745). Instead, Simpson arrives at them geometrically:

> PROP. VII. As the base of any plane triangle ABC is to the sum of the two sides, so is the sine of half the vertical angle to the co-sine of half the difference of the angles at the base.

Simpson proves this proposition using figure 3 (see below). In $\triangle ABC$ with $BC < AC$, extend AC to D so that $CD = BC$. Connect BD. Draw $CE \parallel AB$ and $CF \perp BD$. It follows that $\measuredangle BCF = \frac{1}{2}(A+B)$ and $\measuredangle ECF = \frac{1}{2}(B-A)$.

Various mathematicians, among them, Henry Gellibrand in 1635, Seth Ward in 1654, William Oughtred in 1657, John Caswell in 1685 [15, 45, 34, 6, 20, 49] used figures similar to figure 3 to prove the law of tangents. Even as early as 1595, Pitiscus essentially uses the same construction for the angles $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ to prove the law of tangents in his *Trigonometria.* Yet no one could extract equation (2) out of this figure.

Simpson continues:

> PROP. VIII. As the base of any plane triangle ABC is to the difference of the two sides, so is the co-sine of half the vertical angle to the sine of half the difference of the angles at the base.

For this proposition, he uses a figure similar to figure 4. In $\triangle ABC$, $BD = AB$. Then $CD = a - c$ and $\measuredangle DAC = \frac{1}{2}(A - C)$.

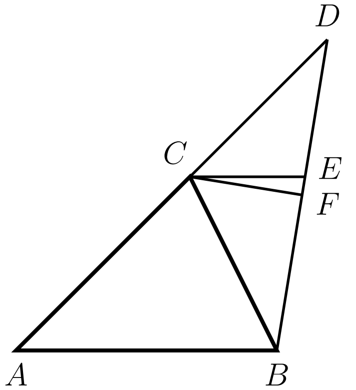

Figure 3

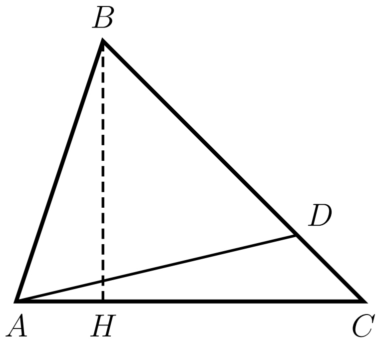

Figure 4

The next year, William Emerson (1701–1782) gives two corollaries to the law of tangents that appear on pages 95 and 96 of *The Elements of Trigonometry* (1749). Figure 4 is a simplification of Emerson's figure. The height BH is added to illustrate his corollaries.

Cor. 1. As the Base CA :
Sum of the Sides, $CB + BA$ ::
So diff. Sides CD :
Diff. Segments of the Base made by a Perpendicular ::
So Cos. $\frac{1}{2}$ Sum op. Angles, or S. $\frac{1}{2}$ the vertical Angle :
Cos. $\frac{1}{2}$ diff. op. Angles, or Cos. $\frac{1}{2}$ diff. vertical Angle, made by a Perp.

Cor. 2. As the Base CA :
Difference of the Sides CD ::
So Sum of the Sides, $CB + BA$:
Diff. Segments by a Perpendicular :
So Sine half Sum op. Angles, $\dfrac{A+C}{2}$:
Sine of half their Difference, $\dfrac{A-C}{2}$.

Emerson uses the shorthand S. as sine. Corollary 1 translates to:

$$\frac{CA}{CB+BA} = \frac{CD}{CH-HA} = \frac{\cos\frac{1}{2}(A+C)}{\cos\frac{1}{2}(A-C)} = \frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}(\measuredangle CBH - \measuredangle HBA)}.$$

And Corollary 2 is:

$$\frac{CA}{CD} = \frac{CB + BA}{CH - HA} = \frac{\sin\frac{1}{2}(A + C)}{\sin\frac{1}{2}(A - C)}.$$

Across the English Channel in France, Antoine René Mauduit (1731–1815) states the Mollweide equations in a scholium to the Napier analogies on pages 83 and 84 in his *Principes d'Astronomie Sphérique ou Traité complet de Trigonométrie Sphérique* (1765):

> 1°. Le sinus du demi-angle vertical est au cosinus de la demi-somme ou de la demi-différence des angles; comme la base ou le côté opposé à'cet angle est à la difference ou à la somme des deux autres côté; c'est-à-dire que $\sin\frac{1}{2}A : \cos\left(\frac{C \pm B}{2}\right) :: BC : AB \mp AC$.
>
> 2°. Le cosinus du demi-angle vertical est au sinus de la demi-somme, ou de la demi-différence des angles opposé; comme la base est à la somme, ou à la difference des côté opposés: c'est-à-dire que $\cos\frac{1}{2}A : \sin\left(\frac{C - B}{2}\right) :: BC : AB \pm AC$.

There is a misprint in the previous statement, $\sin\left(\frac{C - B}{2}\right)$ should be $\sin\left(\frac{C \pm B}{2}\right)$. Mauduit overlooks the fact that $\cos\frac{1}{2}(C + B) = \sin\frac{1}{2}A$ and $\cos\frac{1}{2}A = \sin\frac{1}{2}(C+B)$ since $\frac{1}{2}A$ and $\frac{1}{2}(B+C)$ are complementary. Therefore, half of his above identities do not hold, although his mistakes are corrected in Crakelt's 1768 English translation. Mauduit does mention Newton and Simpson in the preface of his book, which indicates that he was aware of their work.

The Mollweide equations appear again in the English literature before Mollweide's rediscovery. Basil Nikitin (1737–1809) and Prochor Souvoroff give these two equations in the English translation of their [Russian] *Elements of Plane and Spherical Trigonometry* in 1786. The preface to this edition indicates that they were well aware of Simpson's and Emerson's work. They describe these two equations, with a geometric proof, as a corollary to the law of tangents on page 21. They also note that these equations may be used to solve a triangle if two sides and the included angle are given.

Finally, in 1786, the Italian astronomer Antonio Cagnoli (1743–1816) gives an analytic derivation of these two equations by using the law of sines and the sum-to-product formula [5]. He also applies these two identities to solve a triangle when angle A, its opposite side BC and the sum or difference of the other two sides, $(AB + AC)$ or $(AB - AC)$, are given.

We finally arrive at Mollweide's publication of the equations that bear his name. In 1808, he derives these two equations using the law of sines in *Monatliche Correspondenz* [32]. He does mention Cagnoli's trigonometry book at the beginning of his article, which points to further speculation as to why these equations now bear his name. Interestingly, in his paper, Mollweide does not use his equations in the methods for which they would become famous. Although he points out the advantage of these equations in the easy application of logarithms, he does not include any applications. And so we are still left to wonder (1) how the equations came to bear Mollweide's name and (2) how they became applied as a check to the solutions of triangles.

Unfortunately, the evidence for both is not clear. Regarding our first point, after 1808, equations (1) and (2) appear in various German articles and textbooks [14, 16, 19, 21, 36, 37, 46]. Why and when these two equations were named after Mollweide is difficult to understand. According to Tropfke, the wide availability of Mollweide's article may be one reason [43]. Alternatively, the flourish of the German mathematicians in the nineteenth century may also have contributed. For example, quite a few famous German mathematicians–among them, Möbius and Gauss–were familiar with Mollweide and referred to him in their publications. Siegmund Günther gave credit to Mollweide by naming a method in the approximation of $\sqrt{3}$ "*Die Methode von Mollweide*" [17], even though Mollweide's method was previously discovered by other mathematicians. Even the *Allgemeine Deutsche Biographie* states that Mollweide discovered trigonometric formulas involving the sides and angles [18]. In 1897, Hammer credited Mollweide as the discoverer of these equations [19], but Braunmühl and Tropfke corrected this mistake and gave a detailed account on the history of the Mollweide equations in 1901 and 1923, respectively [4, 43]. Neither of them mentioned Thacker, Emerson, Nikitin or Souvoroff, however.

Although many mathematicians credited Mollweide for the equations, on the other hand, more often than not the German textbooks I found did not even mention Mollweide, a fact that raises even more speculation as to why the equations bear his name. In fact, the multitude of texts at the time renders it difficult to trace how these equations were attributed to Mollweide. In the English language, proofs to the Mollweide equations (usually derived from the law of sines) appear in various textbooks [9, 27, 42, 50], but as in the German texts, these authors neither quoted the source of these equations nor used them in checking the solutions of triangles. American authors appear to have inherited the mistake from the Germans. Finally, although equations (1) and (2) were first given by British mathematicians, one by Thacker, then both by Simpson, later British authors simply failed to acknowledge them. None of them gave an eponym to these equations.

The application of the Mollweide equations as a check to the solutions of triangles occurred much later than Mollweide himself, although the origin of this use remains unknown. Again, the plethora of textbooks published renders it difficult to see who first applied these equations in this manner. Most authors in the middle or late nineteenth century were content with giving or deriving the identities without applying the equations. Some authors, however, did apply the Mollweide equations in solving triangles. Among them, Lardner, Serret and Petersen, illustrated their application when two sides and the included angle (a, b and C) are given [27, 39, 35]. Lardner and Serret also applied them when a side, its opposite angle and the sum or difference of the other two sides (C, c and $a \pm b$) are given [27, 39]. Chauvenet (for whom the Chauvenet Prize is named) used them when two angles and a side are given [7]. Furthermore, others noticed that the Mollweide equations could be used to derive Heron's formula [7, 41].

Wentworth was the first author I could identify who labelled the Mollweide equations the "check formula" in his examination question in 1889 [47]. Wilczynski gave a more explicit statement [48]:

> [The Mollweide equations] are particularly convenient for the purpose of checking the accuracy of the numerical solution of a triangle. For each of these equations contains all of the six parts of the triangle, so that an error in any one of these parts would be likely to make itself felt by a lack of agreement between the two members of one of these equations.

He also points out the mistake in crediting the equations to Mollweide.

And so, for unknown reasons, both equations (1) and (2) are mistakenly attributed to Mollweide, although they became used in checking the solutions of triangles because of their unique properties. This recognition leads us to the proof of the equations and their relation to the law of tangents.

4 Proof of the Equations

Surprisingly, these rather complicated looking equations have fairly simple geometric proofs. This insight comes from the construction of the angles $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$ and the segments $(a-b)$ and $(a+b)$ [10, 22, 29, 51].

Figure 5, which I constructed to prove the Mollweide equations, appears as early as 1701 in Samuel Heynes' *A Treatise of Trigonometry, Plane and Spherical, Theoretical and Practical* as a proof to the law of tangents (p.21 and figure IIII on p. 27) and also used by Dickson in 1922 [11]. $\triangle ABC$ is the triangle of interest with $AC < BC$. The construction for obtuse triangle is the same, just move point B along the semicircle. Extend CA to D and AC to F so that $CD = BC = CF = a$. Connect BD and BF.

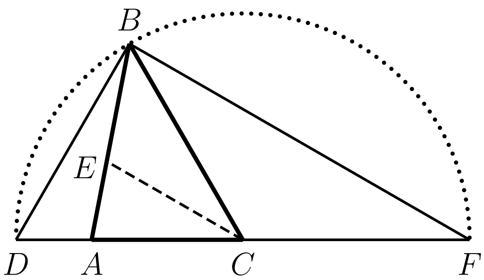

Figure 5

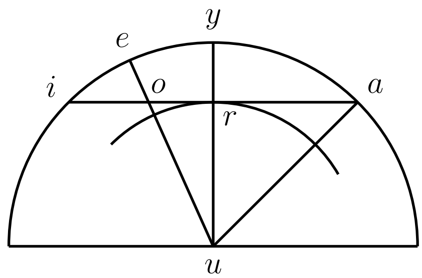

Figure 6

The beauty of this figure is that it contains all the required elements to prove equations (1) through (4). $AF = AC + CF = a + b$. $AD = CD - AC = a - b$. In the isosceles $\triangle BCD$, $\measuredangle D = \measuredangle CBD$. $\measuredangle D + \measuredangle CBD = 180^\circ - C = A + B$. Therefore, $\measuredangle D = \frac{1}{2}(A+B)$. $\measuredangle D + \measuredangle ABD = A$ implies that $\measuredangle ABD = \frac{1}{2}(A-B)$. The isosceles $\triangle BCF$ implies that $\measuredangle F = \frac{1}{2}C$. Finally, notice that $\angle DBF$ is a right angle.

In $\triangle ABD$, using the law of sines, proving equation (3) only takes one step [11, 53]. Since $\angle D$ and $\angle F$ are complementary to each other, $\sin \measuredangle D = \cos \measuredangle F$. Substituting $\measuredangle D = \frac{1}{2}(A+B)$ and $\measuredangle F = \frac{1}{2}C$, $\sin \frac{1}{2}(A+B) = \cos \frac{1}{2}C$, equation (1) follows.

In $\triangle ABF$, again, using the law of sines, we have $\sin \measuredangle ABF / \sin \measuredangle F = (a+b)/c$. Since $\angle ABD$ and $\angle ABF$ are complementary, $\sin \measuredangle ABF = \cos \measuredangle ABD = \cos \frac{1}{2}(A-B)$. Along with $\measuredangle F = \frac{1}{2}C$, a simple substitution gives equation (2). Equation (4) can be obtained by noting that $\angle D$ and $\angle F$ are complementary, or $\sin \frac{1}{2}C = \cos \frac{1}{2}(A+B)$.

To see how Newton's version of the equation relates to equation (2), draw the angle bisector of $\angle C$ so it meets side AB at point E. We conclude $CE \| FB$ since $\angle ACE \cong \angle F$. Since $\triangle ACE \sim \triangle AFB$, we have $AB/(AC+BC) = AE/AC = \sin\measuredangle ACE/\sin\measuredangle AEC = \sin\measuredangle F/\sin\measuredangle ABF = AB/(AC+CF) = c/(a+b)$. The rest is already proven in the previous paragraphs.

As mentioned previously, figure 5 was first used to prove the law of tangents. It is not a coincidence that the Mollweide equations are closely related to the law of tangents.

5 The Law of Tangents

5.1 Derivation of the Law of Tangents from the Mollweide Equations

The law of tangents may be easily derived from the Mollweide equations. Dividing equations (1) and (2) gives a slight variation of the law of tangents, which reflects the symmetry of the Mollweide equations:

$$\frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C} = \frac{a-b}{a+b}.$$

A simple substitution of $\cot\frac{1}{2}C = \tan\frac{1}{2}(A+B)$ gives the law of tangents.

Multiplication gives yet another identity (not related to the law of tangents though):

$$\frac{\sin(A-B)}{\sin C} = \frac{a^2-b^2}{c^2}.$$

William Emerson gives a variant of the above as the third corollary to the law of tangents in the same book mentioned previously:

$$\frac{\sin(A+C)}{\sin(A-C)} = \frac{AC^2}{BC^2-BA^2}.$$

The law of tangents also follows directly from Thacker's and Oppel's equations; dividing equation (3) by equation (4) gives the law of tangents:

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{a+b}{a-b}.$$

5.2 History of the Law of Tangents

Although closely related, the law of tangents has a much older history than the Mollweide equations. François Viète (1540–1603), who led trigonometry to assume a more important role in analysis, was the first to give the modern version of the law of tangents in his *Variorum de Rebus Mathematicis Responsorum Liber VIII* in 1593. His version appears on page 402 of his collected work, *Opera Mathematica* [44]:

> Vt adgregatum crurum ad differentiam eorundem, ita prosinus dimidiæ summæ angulorum ad basin ad prosinum dimidiæ differentiæ.

Viète uses *prosinus* and *transsinuosa* as the trigonometric functions *tangent* and *secant*, respectively. Prior to Viète, though, the Danish mathematician Thomas Fincke (1561–1656), (also known as Finke, Finck, or Fink) had observed that the lines tangent to a circle are related to the sines. Hence, in 1583, he introduced the terms *tangent* and *secant* in trigonometry on pages 73 and 76 respectively in his *Geometriae Rotundi*. But Viète did not agree with Fincke out of fear that confusion might arise from using the same terms in geometry and what would later become the field known as trigonometry. Later authors, however, agreed with Fincke rather than Viète, as we know from our present use of the terms *tangent* and *secant*. Maginus in 1592 (*De Planis Triangulis*, Venice), Blundevile in 1594 (*Exercises, Containing Eight Treatises*, London) and Pitiscus in 1595 (*Trigonometria*, Heidelberg), all adopted Fincke's nomenclature, although without referencing him. With Pitiscus' *Trigonometria*, we also witness the birth of the word *trigonometry*.

Fincke not only preceded Viète in the naming of the tangent and secant functions, but also in the law of tangents. Fincke–who as a professor of mathematics, rhetoric, and medicine was quite the renaissance man [38]–gives the law of tangents in a more complicated but equivalent form on page 292 of his *Geometriae Rotundi*:

> ut semissis summæ crurum ad differentiam summæ semissis alteriusque cruris, sic tangens semissis anguli crurum exterioris ad tangentem anguli quo minor interiorum semisse dicti reliqui minor est, aut major, major.

In our notations (refer to figure 5), the above statement is:

$$\frac{\frac{1}{2}(a+b)}{\frac{1}{2}(a+b)-b} = \frac{\tan\frac{1}{2}\measuredangle BCF}{\tan(\frac{1}{2}\measuredangle BCF - B)}.$$

Indeed, since $\measuredangle BCF = A + B$, the law of tangents follows after a direct substitution. This expression is a more direct translation of Fincke's work, although Tropfke, Smith and Zeller give the following version [43, 40, 54]:

$$\frac{\frac{1}{2}(a+b)}{\frac{1}{2}(a+b)-b} = \frac{\tan\frac{1}{2}(180^\circ - C)}{\tan[\frac{1}{2}(180^\circ - C) - B]}.$$

While today we relate the law of tangents to a triangle, the very first version was not restricted to a triangle. The prelude to the law of tangents appears in Proposition 8 on page 281 of Fincke's *Geometriae Rotundi*. It states that if the sum or the difference of two arcs (or angles) and the ratio of their sines are given, each arc (or angle) can be determined. The essence of the solution is the equation,

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{a+b}{a-b}. \tag{5}$$

At the beginning of Proposition 8, Fincke refers to Regiomontanus (1436–1476), who may have known this identity in 1464 [40]. Fincke thought highly of Regiomontanus, as he states right after the law of tangents in his *Geometriae Rotundi*:

> Cujus certe libri à studiosis avidè legi debent: & cum fructu legi possunt.

In translation,

> His books ought to be read eagerly by students; they are able to be read with profit.

Since Regiomontanus is such an important figure in trigonometry, I will digress here to mention his contributions.

Regiomontanus was born Johann Müller near Königsberg, Franconia (now in Bayern, Germany; not the more famous Königsberg of East Prussia). Following the tradition in which a person is known by the Latinized name of his birthplace, Regiomontanus is the direct translation of Königsberg, or the king's mountain. Regiomontanus, who lived during the transition period from the Dark Ages to the Renaissance, was probably the most influential mathematician in Europe during the fifteenth century. He was the first European to treat trigonometry as an independent subject rather than as a tool for astronomy. His *De Triangulis Omnimodis* (completed in 1464 but not published until 1533) was the first textbook to present trigonometry in the manner we know today. Prior to *De Triangulis Omnimodis*, Regiomontanus completed his teacher Georg Peurbach's half-finished translation of Ptolemy's *Almagest*; their combined efforts resulted in the book *Epitome in Ptolemæi Almagestum*, which, when published in 1462, replaced the medieval version that had been translated from Arabic. Regiomontanus' contributions were pivotal in reviving trigonometry in Europe.

Interestingly, Fincke's Proposition 8 is a slight variation of Theorem XXI in book IV of Regiomontanus' *De Triangulis Omnimodis*, which in turn has its root in Book XI of Ptolemy's *Almagest*. Theorem XXI states that if any known arc less than a semicircumference is divided into two arcs, whose ratio of the sines is given, then each of the arcs can be determined. Regiomontanus, however, does not use the concept of tangents in his *De Triangulis Omnimodis* and thus never really elucidates equation (5). Whether or not he knew the concept of tangent while writing *De Triangulis Omnimodis* is questionable, although he does use the tangent function in a later book in 1467, the *Tabulae Directionum*.

The figure that accompanies Proposition 8, which is quite similar to that of Regiomontanus, is reproduced in figure 6. Although Fincke does not show the proof himself, his figure may be used to prove the law of tangents. In 1900, Braunmühl filled in the details using a similar figure [3]. Briefly, arc ai is divided by the radius ue into ae and ei, and the subtending chord ai into ao and oi. Radius $uy \perp ai$. Let radius $ur = 1$, $ae = A$, $ei = B$, $ao = \alpha$ and $oi = \beta$. Then $\sin A / \sin B = \alpha/\beta$. Furthermore, $ay = yi = \frac{1}{2}(A + B)$ and $ye = \frac{1}{2}(A - B)$. Similarly, $ar = \frac{1}{2}(\alpha + \beta)$ and $ro = \frac{1}{2}(\alpha - \beta)$. Since radius $ur = 1$, $\tan \frac{1}{2}(A + B) = ar = \frac{1}{2}(\alpha + \beta)$ and $\tan \frac{1}{2}(A - B) = ro = \frac{1}{2}(\alpha - \beta)$. Dividing the last two equations gives the law of tangents.

Although Viète gave us the modern version of the law of tangents, Fincke, whose enlightenment rests on Theorem XXI of Regiomontanus, stated the law of tangents for the first time and also demonstrated its application by solving a triangle when two sides and the included angle are given.

5.3 An Old and a New Proof to the Law of Tangents

There are many geometric proofs to the law of tangents [2, 8, 12, 29, 31]. The following proof (figure 7) is a personal communication from Professor Kevin Kolbeck after seeing my proof [52]. Coincidentally, a similar figure appeared in Theorem V of Simpson's *Trigonometry, Plane and Spherical* as a proof to the law of tangents.

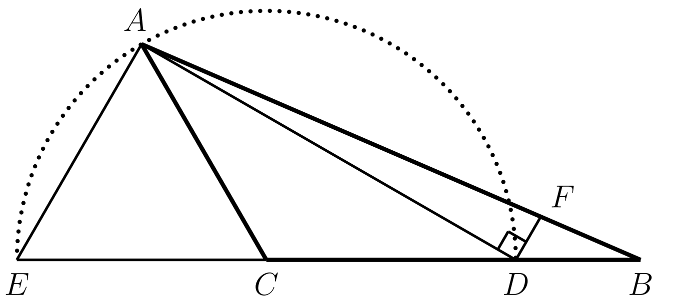

Figure 7

$\triangle ABC$ is the triangle of interest with side $AC < BC$. Extend BC to E and pick point D on BC such that $CE = AC = CD = b$. Connect AD, AE and draw $DF \perp AD$. From this construction, again, we see that $\measuredangle ADC = \frac{1}{2}(A+B)$ and $\measuredangle DAB = \frac{1}{2}(A-B)$. $DB = a-b$ and $EB = a+b$. Note that $\triangle BDF \sim \triangle BEA$.

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan\measuredangle DAF}{\tan\measuredangle ADE} = \frac{DF/AD}{AE/AD} = \frac{DF}{AE} = \frac{DB}{EB} = \frac{a-b}{a+b}$$

Figure 8, a modification of figure 7, is another proof to equation (5). $CD = AC = CE = b$. The height is CH. Draw DG and EJ such that $DG \parallel CH \parallel EJ$. Draw DM and CN so that $DM \perp CH$ and $CN \perp EJ$. Extend BA to meet EJ at J. Again note that $DA \perp EA$.

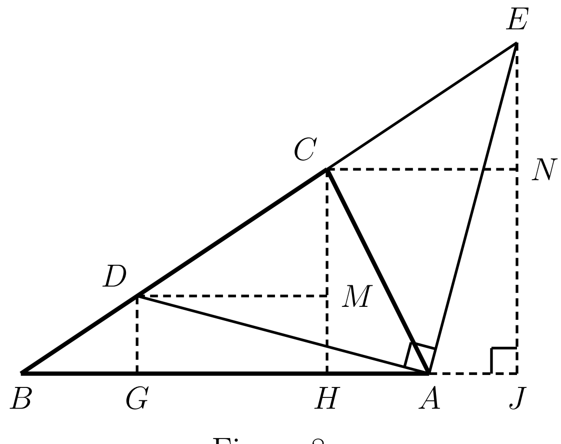

Figure 8

From previous constructions, we know that $BD = a-b$ and $BE = a+b$. In $\triangle AHC$, $CH = b\sin A$. In $\triangle DMC$, $CM = b\sin B$. In $\triangle CNE$, $EN = b\sin B$. Therefore, $DG = MH = b\sin A - b\sin B$ and $EJ = b\sin A + b\sin B$.

Using $\triangle BDG \sim \triangle BEJ$, again we have:

$$\begin{aligned}\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} &= \frac{\tan\measuredangle EDA}{\tan\measuredangle DAG} = \frac{EA/DA}{DG/AG}\\ &= \frac{EJ/AG}{DG/AG} = \frac{EJ}{DG} = \frac{\sin A+\sin B}{\sin A-\sin B}\\ &= \frac{BE}{BD} = \frac{a+b}{a-b}.\end{aligned}$$

From Regiomontanus to Fincke, to Viète, and to Mollweide, the underlying theme among the law of tangents, the Mollweide equations and the various equations mentioned previously is the intricate relationship among the sides of a triangle and the various trigonometric functions of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$. Indeed, the construction of figure 8 leads us to the proofs of more trigonometric identities.

6 Some Identities Involving $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$

Our readers may readily recognize that equation (5) is an immediate consequence of the law of sines and the sum-to-product formula, which again, involve $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$.

$$\sin A+\sin B = 2\cos\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B) \tag{6}$$

$$\sin A-\sin B = 2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B). \tag{7}$$

Interestingly, equation (6) was proven by Viète through a geometric construction [1]. He also developed the formula for $\sin A+\cos B$ and $\cos A+\cos B$. These identities may have enlightened Napier in the development of the logarithms, which also transform products into sums [30].

Yokio Kobayashi and Sidney Kung each gives two elegant geometric proofs to both the above identities and the remaining two [23, 24, 25, 26]:

$$\cos A+\cos B = 2\cos\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B) \tag{8}$$

$$\cos B-\cos A = 2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B). \tag{9}$$

Like Kung's proofs, figure 8 has the advantage of having both angles $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$ in the same figure. $\measuredangle CDA = \measuredangle CAD = \frac{1}{2}(A+B)$

and $\measuredangle DAB = \measuredangle AEJ = \frac{1}{2}(A - B)$. Notice that $\angle GDA$ and $\angle DAB$ are complementary and that $\triangle DGA \sim \triangle AJE$.

Let us label $CD = AC = CE = 1$. Then, in $\triangle ADE$,

$$\begin{aligned} DA &= 2\cos\frac{1}{2}(A+B) \\ AE &= 2\sin\frac{1}{2}(A+B). \end{aligned}$$

In $\triangle ADG$,

$$\begin{aligned} DG &= DA \cdot \sin\measuredangle DAB &= 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B) \\ GA &= DA \cdot \cos\measuredangle DAB &= 2\cos\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B). \end{aligned}$$

Similarly, in $\triangle AJE$,

$$\begin{aligned} AJ &= AE \cdot \sin\measuredangle AEJ &= 2\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+B) \\ EJ &= AE \cdot \cos\measuredangle AEJ &= 2\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+B). \end{aligned}$$

As noted before, $CH = NJ = \sin A$, $AH = \cos A$, $EN = CM = \sin B$ and $CN = DM = GH = HJ = \cos B$. Then,

$$\begin{aligned} DG &= CH - CM &= \sin A - \sin B \\ GA &= GH + AH &= \cos A + \cos B \\ AJ &= CN - AH &= \cos B - \cos A \\ EJ &= NJ + EN &= \sin A + \sin B. \end{aligned}$$

From the above labelling, formula (6), (7), (8) and (9) follow by equating the different representations of EJ, DG, GA and AJ respectively.

There is also enough information in figure 8 to prove the following identities, which are derived by none other than the grand master Leonhard Euler in 1748 in his *Introductio in Analysin Infinitorum*.

$$\tan\frac{1}{2}(A+B) = \frac{\sin A + \sin B}{\cos A + \cos B} \tag{10}$$

$$\tan\frac{1}{2}(A-B) = \frac{\sin A - \sin B}{\cos A + \cos B} \tag{11}$$

$$\cot\frac{1}{2}(A+B) = \frac{\sin A - \sin B}{\cos B - \cos A} \tag{12}$$

$$\cot\frac{1}{2}(A-B) = \frac{\sin A + \sin B}{\cos B - \cos A} \tag{13}$$

$$\tag{14}$$

Using the fact that $\triangle DGA \sim \triangle AJE$, equation (10) follows from $\tan\frac{1}{2}(A+B) = EA/DA = EJ/GA$. Equation (12) can be obtained from $\cot\frac{1}{2}(A+$

$B) = DA/EA = DG/AJ$. Equations (11) and (13) are self evident in $\triangle DGA$ and $\triangle AJE$.

In $\triangle DGA$, $\tan\frac{1}{2}(A-B) = DG/AG$. We also know $\cot\frac{1}{2}(A+B) = DG/AJ$. Then $\cot\frac{1}{2}(A+B)/\tan\frac{1}{2}(A-B) = AG/AJ$, or

$$\frac{\cot\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\cos A + \cos B}{\cos B - \cos A}.$$

We can extract a few more identities from figure 8. Since $EN + CM = EJ - DG$, we have:

$$\sin B = \cos\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B).$$

$CN + DM = GA + AJ$ implies

$$\cos B = \cos\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B).$$

Similar identities for $\sin A$ and $\cos A$ can also be derived by adding equations (6) and (7) or subtracting equation (9) from (8) respectively.

In $\triangle ADG$, using Pythagorean's theorem, $AD^2 = DG^2 + AG^2$, or

$$4\cos^2\frac{1}{2}(A+B) = (\sin A - \sin B)^2 + (\cos A + \cos B)^2$$

Similarly, in $\triangle AJE$, the following can be obtained:

$$4\sin^2\frac{1}{2}(A+B) = (\sin A + \sin B)^2 + (\cos B - \cos A)^2.$$

Heading the same direction, square both sides of identities (7) and (9) and then add them to get:

$$4\sin^2\frac{1}{2}(A-B) = (\sin A - \sin B)^2 + (\cos B - \cos A)^2.$$

Viète applied this identity in the computation of his trigonometric tables.

Using identities (6) and (8) together gives the last of this type of relationship, namely,

$$4\cos^2\frac{1}{2}(A-B) = (\sin A + \sin B)^2 + (\cos A + \cos B)^2.$$

7 Conclusion

The equations named after a mathematician better known for his accomplishment in cartography–Karl Brandan Mollweide–are fascinating because of their symmetry, their ability to check the solutions of triangles and their relationship to the law of tangents and other identities involving $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$. For reasons still unclear to me, Mollweide became famous for a pair of equations that were first discovered by Newton and fully developed by Simpson. The Mollweide equations eventually found a unique place in trigonometry as a computational tool. As new technologies emerge, Mollweide's equations are losing their glory. Nonetheless, we should always remember all those mathematicians who paved the road ahead of us because their achievements demonstrate the ingenuity of the human mind.

For those who are willing to go a little further, I will leave them to stretch their imagination to look for the following identities in figure 8 using $2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B) = \sin(A+B)$ and $2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A-B) = \sin(A-B)$.

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} = \frac{\sin(A+B)}{\sin A - \sin B} = \frac{2\sin^2\frac{1}{2}(A+B)}{\cos B - \cos A} \tag{15}$$

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = \frac{\sin(A+B)}{\sin A + \sin B} = \frac{2\cos^2\frac{1}{2}(A+B)}{\cos A + \cos B} \tag{16}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \tag{17}$$

(*Hint:* Identity (17) can be obtained by noting the area of the trapezoid $DEJG$ is the sum of the areas of $\triangle DAE$, $\triangle DGA$ and $\triangle AJE$.)

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