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Solution to Problem 167.

For x, y, $z > 0$, find the minimum of $\dfrac{x^3+y^3+z^3}{(y+z)(z+x)(x+y)}$.

Without loss of generality, let's assume $x \ge y \ge z$. Then $x+y \ge x+z \ge y+z$ and $\frac{1}{x} \le \frac{1}{y} \le \frac{1}{z}$. By the principle of rearrangement inequality, we have the following:

$$\begin{aligned}
\frac{x+y}{z}+\frac{x+z}{y}+\frac{y+z}{x} &\ge \frac{x+y}{x}+\frac{x+z}{z}+\frac{y+z}{y}\\
\frac{x+y}{z}+\frac{x+z}{y}+\frac{y+z}{x} &\ge \frac{x+y}{y}+\frac{x+z}{x}+\frac{y+z}{z},
\end{aligned}$$

with equality only if $x=y=z$. Adding the above two inequalities to get

$$\frac{x+y}{z}+\frac{x+z}{y}+\frac{y+z}{x} \ge 6$$

or

$$\begin{aligned}
\frac{x+y}{z}+\frac{x+z}{y}+\frac{y+z}{x}+2 &\ge 8\\
\frac{(x+y)(x+z)(y+z)}{xyz} &\ge 8\\
\frac{xyz}{(x+y)(x+z)(y+z)} &\le \frac{1}{8},
\end{aligned}$$

again, with equality only if $x=y=z$.

We can also use the principle of rearrangement inequality to obtain

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y} \ge \frac{3}{2}.$$

Now for the original problem,

$$\begin{aligned}
\frac{x^3+y^3+z^3}{(y+z)(z+x)(x+y)} &= \frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}-1-\frac{xyz}{(x+y)(x+z)(y+z)}\\
&\ge \frac{3}{2}-1-\frac{1}{8}\\
&= \frac{3}{8}
\end{aligned}$$

with equality only if $x = y = z$.

$\blacksquare$