Channel Diagonalization through Orthogonal Space-Time Coding

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Abstract— **Space-time block codes from Orthogonal Designs were proposed in [1] for communication over fading channels with multiple transmit and receive antennas. These codes require no knowledge of the channel at the transmitter, and linear processing at the receiver achieves maximum-likelihood decoding. This paper presents an alternative approach to code design, where the code criterion is such that the transmitter and receiver operations essentially "diagonalize" any arbitrary matrix channel. Such diagonalization greatly simplifies decoding. The transmitter is assumed to have no knowledge of the channel. The desired properties of such a code are derived. It is shown that with some additional constraints, the resulting class of codes is identical to the codes introduced in [1]. Hence, an alternative interpretation of space-time block codes from orthogonal designs is given. The alternative code construction criteria that are derived may facilitate the design of more general classes of codes.**

I. Introduction

Multiple antenna systems promise much higher spectral efficiencies and greater reliability for communication over fading channels relative to single antenna systems ([2], [3]). Space-time vector coding was proposed in [4], and was shown to be capable of achieving channel capacity. These codes require perfect channel knowledge at the transmitter. However, this assumption is sometimes questionable in rapidly changing mobile environments. Under the constraint of no channel knowledge at the transmitter, Tarokh et al. provided in [5] code construction criteria for space-time trellis-based codes that maximize diversity gain and coding gain over fading channels using multiple transmit antennas. With increased demand for high data rate wireless communication, and strong commercial interest in multiple antenna technology, there has been intensive research for space-time codes with simplified decoding structures. In [6], Alamouti introduced a code for two transmit antennas that provides maximum two-antenna transmit diversity, and requires only linear processing at the receiver. A simple variant of Alamouti's scheme has already been included in the WCDMA Third Generation Wireless Standard [7]. In [1], Tarokh et al. introduced space-time block codes from orthogonal designs and showed that Alamouti's scheme is a specific example of this general coding scheme. The resulting transmit diversity techniques require no channel state information at the transmitter, achieve maximum-likelihood decoding through linear processing at the receiver, and exhibit maximum diversity. Further work has been motivated by the need to design space-time codes for channels that are known neither to the transmitter nor to the receiver. In [8], Marzetta and Hochwald introduced unitary space-time signals. Even more general designs were recently described in [9].

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This paper continues the search for space-time codes with simplified decoding structures. It considers a class of codes that require no channel knowledge at the transmitter but can simultaneously diagonalize any arbitrary channel matrix by appropriate linear processing at both the transmitter and receiver. Achieving this diagonalization leads directly to a very simple decoding strategy, where symbol-by-symbol detection is the optimal decoding algorithm. Code construction criteria are presented and the properties of such "diagonalizing" codes are investigated. It is found that under certain conditions, the problem of diagonalization is equivalent to the problem of designing space-time block codes from orthogonal designs [1].

The paper is organized as follows: Section II presents the system model. Section III derives the optimal receiver design with the assumption that diagonalization of the channel is achievable. Section IV examines the code construction criteria and finally section V proves that the specified class of codes is equivalent to space-time codes using Orthogonal Designs.

The following notation is used throughout the paper: Vectors are lower-case bold characters, and matrices are upper-case bold characters. The symbols $(\cdot)'$ and $(\cdot)^*$ denote the transpose and the conjugate-tranpose operations correspondingly. $\Re[\mathbf{F}]$ is the real part of matrix $\mathbf{F}$, and $<\mathbf{x},\mathbf{y}>=\mathbf{x}^*\mathbf{y}$ is the inner product of the column vectors $\mathbf{x}$ and $\mathbf{y}$.

II. System Model

Assume that there are n transmit antennas and m receive antennas, and that the $m\times n$ complex channel matrix $\mathbf{H}$ is static for the duration of the code sequence. The received vector at time t is expressed as:

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t + \mathbf{n}_t \tag{1}$$

where $\mathbf{x}_t$ is the vector of transmitted symbols at time t and $\mathbf{n}_t$ is the vector of white noise samples at time t. Given a code length of T, the vectors transmitted for the code duration are compactly represented as $\mathbf{x} = [\mathbf{x}_t \ \mathbf{x}_{t+1} \ \ldots \ \mathbf{x}_{t+T-1}]'$. This results in the relationship:

$$\mathbf{y} = \tilde{\mathbf{H}}\mathbf{x} + \mathbf{n} \tag{2}$$

where $\mathbf{y} = [\mathbf{y}_t\, \mathbf{y}_{t+1}\, \ldots\, \mathbf{y}_{t+T-1}]'$, $\mathbf{n} = [\mathbf{n}_t\, \mathbf{n}_{t+1}\, \ldots\, \mathbf{n}_{t+T-1}]'$ and $\tilde{\mathbf{H}}$ is the following block diagonal channel matrix:

$$\tilde{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & O & \ldots & O \\ O & \mathbf{H} & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & \mathbf{H} \end{bmatrix} \tag{3}$$

Space-time coding is performed by a linear operation on the "uncoded" information symbols. This can be represented by an $nT \times 2k$ matrix $\mathbf{A}$ as shown below:

$$\mathbf{x} = \mathbf{A}\mathbf{v} \tag{4}$$

where $\mathbf{v}$ is the $2k$-dimensional vector of real, uncoded symbols. The code rate is defined as $\frac{1}{2}\frac{2k}{T}$, where the factor of $\frac{1}{2}$ accounts for the fact that $\mathbf{v}$ is basically a real vector representation of a complex k-dimensional input symbol. Depicting a complex k-dimensional input vector as a real $2k$-dimensional input vector provides the extra flexibility to represent complex conjugate operations through the linear transformation performed by the complex matrix $\mathbf{A}$.

III. Optimal Receiver Design

At this point, the assumption is made that given $\mathbf{A} \in \mathcal{C}^{nT\times 2k}$, for any matrix $\tilde{\mathbf{H}} \in \mathcal{C}^{mT\times nT}$, and for any number of receive antennas m, the following requirement holds:

$$\Re[(\tilde{\mathbf{H}}\mathbf{A})^*(\tilde{\mathbf{H}}\mathbf{A})] = \mathbf{\Lambda} \tag{5}$$

where $\mathbf{\Lambda}$ is a diagonal matrix. It is easily deduced that the elements of $\mathbf{\Lambda}$ are non-negative.

With channel knowledge available at the receiver, and since the noise is white and gaussian, an optimal receiver determines the real-vector $\hat{\mathbf{v}}$ that achieves minimization of the following metric:

$$||\mathbf{y} - \tilde{\mathbf{H}}\mathbf{A}\hat{\mathbf{v}}||^2 = ||\mathbf{y}||^2 + ||\tilde{\mathbf{H}}\mathbf{A}\hat{\mathbf{v}}||^2 - 2\Re[\hat{\mathbf{v}}^*\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{y}] \tag{6}$$

From (5), it follows that the off-diagonal elements of $(\tilde{\mathbf{H}}\mathbf{A})^*(\tilde{\mathbf{H}}\mathbf{A})$ are imaginary, and since $\hat{\mathbf{v}}$ is a real-vector, $||\tilde{\mathbf{H}}\mathbf{A}\hat{\mathbf{v}}||^2 = \hat{\mathbf{v}}'\mathbf{\Lambda}\hat{\mathbf{v}}$. The minimization can be expressed as:

$$||\mathbf{y} - \tilde{\mathbf{H}}\mathbf{A}\hat{\mathbf{v}}||^2 = ||\mathbf{y}||^2 + \hat{\mathbf{v}}'\mathbf{\Lambda}\hat{\mathbf{v}} - 2\hat{\mathbf{v}}'\Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{y}] \tag{7}$$

Since the term $||\mathbf{y}||^2$ is irrelevant for minimization, it is equivalent to instead minimize the following quantity:

$$\begin{aligned} ||\mathbf{\Lambda}^{-1/2}\Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{y}] - \mathbf{\Lambda}^{1/2}\hat{\mathbf{v}}||^2 &= ||\tilde{\mathbf{y}} - \mathbf{\Lambda}^{1/2}\hat{\mathbf{v}}||^2 && (8) \\ &= \sum_{i=1}^{2k} |\tilde{y}_i - \sqrt{\lambda_i}\hat{v}_i|^2 && (9) \end{aligned}$$

where $\tilde{\mathbf{y}} \stackrel{\text{def}}{=} \mathbf{\Lambda}^{-1/2}\Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{y}]$, $\tilde{y}_i$ are the elements of $\tilde{\mathbf{y}}$, $\hat{v}_i$ are the elements of $\hat{\mathbf{v}}$, and λ_i are the diagonal elements of $\mathbf{\Lambda}$. Also, the following observation can be made:

$$\begin{aligned} \Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{y}] &= \Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\tilde{\mathbf{H}}\mathbf{A}]\mathbf{v} + \Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{n}] && (10) \\ &= \mathbf{\Lambda}\mathbf{v} + \tilde{\mathbf{n}} && (11) \end{aligned}$$

where the covariance matrix of $\tilde{\mathbf{n}}$, $E(\tilde{\mathbf{n}}\tilde{\mathbf{n}}')$, is diagonal (see appendix for proof). Hence, it is sufficient to decode each $\hat{v}_i$ symbol independently, by minimizing the distance from $\tilde{y}_i$.

It has now been shown that, if (5) holds, then the optimal receiver operation is matched filtering, extracting the real part, and performing symbol-by-symbol detection (SBS). The complete system is shown in figure 1.

IV. Code Structure

This section determines sufficient and necessary conditions for the matrix $\mathbf{A}$, such that $\Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\tilde{\mathbf{H}}\mathbf{A}]$ is diagonal. An interpretation of $\mathbf{A}$ is that it is the matrix that simultaneously diagonalizes all matrices of the form $\tilde{\mathbf{H}}^*\tilde{\mathbf{H}}$, where $\tilde{\mathbf{H}}$ is block diagonal with identical blocks.

Let $\mathbf{A} = [\mathbf{A}_t\ \mathbf{A}_{t+1}\ \ldots\ \mathbf{A}_{t+T-1}]'$, where $\mathbf{A}_t$ represents the linear operation performed at time t, so that:

$$\mathbf{x}_t = \mathbf{A}_t\mathbf{v} \tag{12}$$

Then the following relationships hold true:

$$\begin{aligned} \mathbf{Z} &\stackrel{\text{def}}{=} \mathbf{A}^*\tilde{\mathbf{H}}^*\tilde{\mathbf{H}}\mathbf{A} && (13) \\ &= \sum_{i=0}^{T-1}(\mathbf{A}^*_{t+i}\mathbf{H}^*\mathbf{H}\mathbf{A}_{t+i}) \\ &= \sum_{i=0}^{T-1}(\mathbf{A}^*_{t+i}\mathbf{R}\mathbf{A}_{t+i}) && (14) \end{aligned}$$

where $\mathbf{R} \stackrel{\text{def}}{=} \mathbf{H}^*\mathbf{H}$ is of size $n \times n$. With these definitions, it follows that:

$$Z_{ij} = \sum_{l=0}^{T-1}\sum_{f=0}^{n-1}\sum_{g=0}^{n-1} r_{fg}a^{*(l)}_{fi}a^{(l)}_{gj} \tag{15}$$

where

$$\begin{aligned} Z_{ij} &= (i,j)^{\text{th}} \text{ component of } \mathbf{Z} && (16) \\ r_{fg} &= (f,g)^{\text{th}} \text{ component of } \mathbf{R} && (17) \\ a^{(l)}_{ij} &= (i,j)^{\text{th}} \text{ component of } \mathbf{A}_{t+l} && (18) \end{aligned}$$

Thus,

$$\begin{aligned} Z_{ij} &= \sum_{f=0}^{n-1}\sum_{g=0}^{n-1} r_{fg}\sum_{l=0}^{T-1} a^{*(l)}_{fi}a^{(l)}_{gj} && (19) \\ &= \sum_{f=0}^{n-1}\sum_{g<f} r_{fg}\sum_{l=0}^{T-1} a^{*(l)}_{fi}a^{(l)}_{gj} \\ &+ \sum_{f=0}^{n-1} r_{ff}\sum_{l=0}^{T-1} a^{*(l)}_{fi}a^{(l)}_{fj} \\ &+ \sum_{f=0}^{n-1}\sum_{g>f} r_{fg}\sum_{l=0}^{T-1} a^{*(l)}_{fi}a^{(l)}_{gj} && (20) \end{aligned}$$

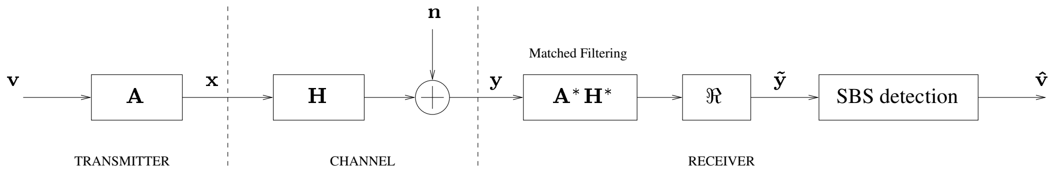

Fig. 1. Space-time code implementation.

Using the relationship:

$$\sum_{f=0}^{n-1}\sum_{g>f} r_{fg}\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{gj}^{(l)} = \sum_{f=0}^{n-1}\sum_{g<f} r_{gf}\sum_{l=0}^{T-1} a_{gi}^{*(l)} a_{fj}^{(l)} \quad (21)$$

and the fact that $\mathbf{R}$ is hermitian, i.e. $r_{gf} = r_{fg}^*$, (20) can be rewritten as:

$$\begin{aligned} Z_{ij} &= \sum_{f=0}^{n-1}\sum_{g<f}(r_{fg}\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{gj}^{(l)} + r_{fg}^*\sum_{l=0}^{T-1} a_{gi}^{*(l)} a_{fj}^{(l)}) \\ &+ \sum_{f=0}^{n-1} r_{ff}\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{fj}^{(l)} \end{aligned} \quad (22)$$

The code construction criterion is that $\Re[\mathbf{Z}]$ is diagonal. Given that the transmit matrix $\mathbf{A}$ must be selected independently of the channel, the only way to satisfy the constraints for the off-diagonal elements in (22) for all possible channel matrices is to design $\mathbf{A}$, such that:

$$\Re\left[\sum_{f=0}^{n-1}\sum_{g<f}(r_{fg}\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{gj}^{(l)} + r_{fg}^*\sum_{l=0}^{T-1} a_{gi}^{*(l)} a_{fj}^{(l)})\right] = 0 \quad (23)$$

and

$$\Re\left[\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{fj}^{(l)}\right] = 0 \quad (24)$$

for all $i \neq j$. Since for a complex number q, $\Re[q - q^*] = 0$, the code construction criterion becomes:

$$\sum_{l=0}^{T-1} a_{fi}^{*(l)} a_{gj}^{(l)} = -\sum_{l=0}^{T-1} a_{gi}^{(l)} a_{fj}^{*(l)}, \forall (f,i) \neq (g,j) \quad (25)$$

This is both a sufficient and necessary condition for a linear space-time transmit filter that diagonalizes all channel matrices. Note that (25) implies that (24) holds. If $\mathbf{A}$ satisfies (25), then the diagonal elements of $\Re[\mathbf{Z}]$ are:

$$\Re[Z_{ii}] = \sum_{f=0}^{n-1} r_{ff}\sum_{l=0}^{T-1} |a_{fi}^{(l)}|^2 \quad (26)$$

V. Equivalence to Orthogonal Designs

In order to interpret the code properties defined by (25), the matrix $\mathbf{\Omega}$ is introduced, where its element Ω_{fl} is defined as the signal transmitted from the f^{th} antenna at the l^{th} time. Using (12), one finds:

$$\Omega_{fl} = \sum_{i=0}^{2k-1} a_{fi}^{(l)} v_i \quad (27)$$

or more compactly:

$$\mathbf{\Omega} = \sum_{i=0}^{2k-1} v_i \mathbf{B}^{(i)} \quad (28)$$

where $B_{fl}^{(i)} = a_{fi}^{(l)}$ are the elements of $\mathbf{B}^{(i)}$. The matrix $\mathbf{B}^{(i)}$ essentially defines how the uncoded symbol v_i is transmitted over the multiple antennas at different times.

Let $\mathbf{b}_f^{(i)}$ be the f^{th} row of $\mathbf{B}^{(i)}$. Then, the code construction criteria presented in (25) can be rewritten as:

$$< \mathbf{b}_g^{(j)}, \mathbf{b}_f^{(i)} > = - < \mathbf{b}_g^{(i)}, \mathbf{b}_f^{(j)} >, \forall (f,i) \neq (g,j) \quad (29)$$

These relationship can be expressed more succintly as:

$$\begin{aligned} \mathbf{B}^{(j)}\mathbf{B}^{*(i)} &= -\mathbf{B}^{(i)}\mathbf{B}^{*(j)}\ ,\ i \neq j \quad (30) \\ \mathbf{B}^{(i)}\mathbf{B}^{*(i)} &= \mathbf{D}_i \quad (31) \end{aligned}$$

where $\mathbf{D}_i$ is a diagonal matrix. Hence, if $\mathbf{A}$ is chosen according to (25), then:

$$\mathbf{\Omega}^*\mathbf{\Omega} = \sum_{i=0}^{2k-1} v_i^2 \mathbf{D}_i \quad (32)$$

In order to show equivalence to the generalized linear orthogonal designs presented in [1], two additional constraints are necessary:

$$\begin{aligned} \sum_{l=0}^{T-1} |a_{fi}^{(l)}|^2 &> 0, \forall f \quad (33) \\ |a_{fi}^{(l)}| &= |a_{f,i+k}^{(l)}| \quad (34) \end{aligned}$$

Imposing the additional constraint of (33) ensures that $\Re[Z_{ii}]$ has terms corresponding to contributions from all the diagonal elements of $\mathbf{R}$. This in turn implies that $\Re[Z_{ii}]$ has contributions from all the elements of the channel matrix $\mathbf{H}$. Hence, with this additional constraint, the resulting space-time codes provide *maximum diversity*. The constraint of (34) assumes that the i^{th} complex transmitted symbol is represented as $v_i + jv_{i+k}$. Then, the constraint essentially means that the scaling of the real part should have the same magnitude as the scaling of the imaginary part of the uncoded symbol. Therefore, the code properties become:

$$\mathbf{B}^{(j)}\mathbf{B}^{*(i)} = -\mathbf{B}^{(i)}\mathbf{B}^{*(j)}, \; i \neq j \tag{35}$$

$$\mathbf{B}^{(i)}\mathbf{B}^{*(i)} = \mathbf{D}_i \tag{36}$$

$$\mathbf{B}^{(i+k)}\mathbf{B}^{*(i+k)} = \mathbf{D}_i \tag{37}$$

$$\mathbf{\Omega}^*\mathbf{\Omega} = \sum_{i=0}^{k-1}(v_i^2 + v_{i+k}^2)\mathbf{D}_i \tag{38}$$

where $\mathbf{D}_i$ is a full-rank diagonal matrix.

The code construction criterion of (38) is identical to the definition for *generalized linear complex orthogonal designs* presented in [1].

Existence Conditions

With the additional constraints of (33) and (34), the existence conditions for the diagonalizing space-time codes introduced in this paper are the same as those for space-time block codes from orthogonal designs. However, it is possible to design codes with reduced diversity order by allowing $\sum_{l=0}^{T-1} |a_{fi}^{(l)}|^2 = 0$ for some f. It is expected that reducing the diversity order increases the permissible rates for these codes. The existence conditions for codes without the constraints of (33) and (34) is still an open problem.

VI. Conclusion

We introduced space-time block codes that effectively achieve diagonalization of any arbitrary matrix channel. The optimal receiver for these codes is simply a matrix matched filter, followed by extraction of the signal's real part. Under certain constraints, these codes were shown to be equivalent to space-time block codes using orthogonal designs. The diagonalizing property of these codes provides an intuitive interpretation of the codes proposed in [1]. Moreover, the alternative design approach adopted in this paper may facilitate the discovery of more general classes of space-time codes.

Appendix

Lemma 1: Suppose that the matrix $\mathbf{T}$ is such that $\Re[\mathbf{T}^*\mathbf{T}]$ is diagonal, and also assume that $\mathbf{n}$ is white with uncorrelated real and imaginary parts. Then, $E(\Re[\mathbf{Tn}]\Re[\mathbf{n}^*\mathbf{T}^*])$ is diagonal.

Proof: Let $\mathbf{n} = \mathbf{n}_R + j\mathbf{n}_I$, and $\mathbf{T} = \mathbf{T}_R + j\mathbf{T}_I$. Then:

$$\begin{aligned} E(\Re[\mathbf{Tn}]\Re[\mathbf{n}^*\mathbf{T}^*]) &= \mathbf{T}_R E(\mathbf{n}_R\mathbf{n}_R')\mathbf{T}_R' - \mathbf{T}_R E(\mathbf{n}_R\mathbf{n}_I')\mathbf{T}_I' \\ &- \mathbf{T}_I E(\mathbf{n}_I\mathbf{n}_R')\mathbf{T}_R' + \mathbf{T}_I E(\mathbf{n}_I\mathbf{n}_I')\mathbf{T}_I' \end{aligned}$$

Since the noise is white with uncorrelated real and imaginary parts, $E(\mathbf{n}_R\mathbf{n}_R') = E(\mathbf{n}_I\mathbf{n}_I') = \sigma^2\mathbf{I}$, and $E(\mathbf{n}_R\mathbf{n}_I') = E(\mathbf{n}_I\mathbf{n}_R') = \mathbf{0}$. Hence,

$$\mathbf{E}(\Re[\mathbf{Tn}]\Re[\mathbf{n}^*\mathbf{T}^*]) = \sigma^{\mathbf{2}}(\mathbf{T_R T_R'} + \mathbf{T_I T_I'}) \tag{39}$$

Now, $\Re[\mathbf{T}^*\mathbf{T}] = \mathbf{T}_R\mathbf{T}_R' + \mathbf{T}_I\mathbf{T}_I'$. Since, $\Re[\mathbf{T}^*\mathbf{T}]$ is a diagonal matrix, it follows that $E(\Re[\mathbf{Tn}]\Re[\mathbf{n}^*\mathbf{T}^*])$ is also diagonal, and $\tilde{\mathbf{n}} = \Re[\mathbf{A}^*\tilde{\mathbf{H}}^*\mathbf{n}]$ is a white-noise process.

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