Blind Adaptive MIMO Decision Feedback Equalization using Givens Rotations

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Abstract— **An adaptive method is proposed for Multiple-Input-Multiple-Output (MIMO) Decision Feedback Equalization (DFE) in the absence of channel knowledge at the receiver and without requiring a training sequence. The Zero-Forcing (ZF) DFE is shown to consist of an orthogonal matrix (feedforward section) and an upper triangular matrix (feedback section). The feedforward matrix is expressed as the multiplication of Givens rotations terms, where each term is characterized by a rotation angle. An adaptive algorithm is presented for the blind determination of the unknown angles and of the elements of the feedback matrix. The adaptation criteria are decorrelation conditions, which are similar to those previously proposed in [1], [2]. Simulation results are included demonstrating the performance of the method.**

I. Introduction

MIMO communication systems are attracting a lot of attention for both wireless and wireline applications. In wireless, it has been realized that large capacity increases are possible by means of employing multi-element antennas, which fuels the need for a better understanding of MIMO techniques. For wireline systems (such as Digital Subscriber Lines), the demand for yet faster broadband access requires the application of advanced crosstalk mitigation methods on the multiple copper pairs forming the MIMO channel.

Linear and decision feedback equalization for MIMO systems has been examined from several different perspectives. The focus of this paper is on decision feedback equalization, which compared to linear equalization is characterized by milder noise enhancement. Additionally, it has been shown that under certain conditions MIMO DFE is capable of approaching the capacity of the vector channel [3].

The research effort on MIMO DFE's has been intense [4], [5], [3], [6], [7], [8], [9], and several adaptive methods have been suggested. Chen and Roy [10] proposed an adaptive decorrelating detector with decision feedback for CDMA based on a least squares formulation. Rapajic and Vucetic [11] used a stochastic gradient algorithm to adapt the filter coefficients of a CDMA DFE receiver. An adaptive method for a CDMA DFE receiver was presented by Rapajic [12], which can be implemented in a blind fashion given knowledge of signature sequences and timing. Tehrani [13] solved the MIMO DFE adaptation problem by assuming an appropriate state space model.

This paper presents an adaptive method for MIMO DFE equalization that (in contrast with the previously proposed methods) makes no use of training and requires no knowledge about the channel or its structure. The derivation of the ZF DFE reveals that the feedforward section is an orthogonal matrix and the feedback section is an upper triangular matrix. The orthogonal matrix serves to "triangularize" the channel, so that decision feedback can then be applied to remove the "trailing" interference. It is observed that the orthogonal matrix can be decomposed into Givens rotations terms, where each term is parameterized by a single angle. Each such term aims to zero a specific lower-triangular entry of the channel matrix, therefore, the corresponding angle can be estimated adaptively according to a decorrelation criterion. Also, the entries of the feedback matrix are adaptively estimated with an analogous criterion. Similar decorrelating conditions have been employed in [1] (for blind implementation of a DFE to combat ISI) and in [2] (for interference cancellation through linear filtering in CDMA). The provided simulation results demonstrate the convergence properties and the performance of the algorithm.

The outline of the paper is as follows: Section II gives the MIMO channel assumptions, and Section III describes the MIMO DFE with a ZF criterion. Then, Section IV reviews the technique of applying Givens rotations to perform the QR decomposition, and Section V uses this technique to derive an adaptive equalization algorithm. Finally, Section VI includes a number of simulation results, and Section VII concludes this paper.

In the following, $E(\cdot)$ denotes expectation, $< \cdot >$ denotes time average, and $(\cdot)^T$ denotes the transpose operation.

II. Channel Model

A narrowband MIMO system with N receive and N transmit elements is considered. The sampled output of the baseband equivalent system satisfies:

$$\mathbf{y} = H\mathbf{x} + \mathbf{n}' \qquad (1)$$

where $\mathbf{y} = [y_1\, y_2\, \ldots\, y_N]^T$, $\mathbf{x} = [x_1\, x_2\, \ldots\, x_N]^T$ and $\mathbf{n}' = [n_1'\, n_2'\, \ldots\, n_N']^T$ are vectors of the output samples, the transmitted symbols and the noise samples, respectively. The general case of multi-level transmission is considered. The noise is assumed white and gaussian with covariance matrix $R_{\mathbf{n}'\mathbf{n}'} = \frac{N_o}{2} I_N$. Also, H is an $N \times N$ non-singular matrix, where the element h_{ij} equals the complex gain from transmitter element j to receiver element i. All quantities are assumed

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real to facilitate intuitive interpretations, however, extensions to the complex case are conceptually straightforward.

III. MIMO Decision Feedback Equalization

This section re-derives the ZF DFE for the channel described by (1). The motivation for employing the ZF (instead of the MMSE) equalizer will become evident later, where it is shown that the proposed ZF structure has an elegant adaptive implementation. Besides, the inherent problem of noise enhancement for ZF equalizers is not as much pronounced in decision feedback implementations as it is in linear equalizers.

In order to employ ZF decision feedback, an appropriate feedforward matrix must be used to "triangularize" the channel:

$$\mathbf{z} = W^T \mathbf{y} = W^T H \mathbf{x} + \mathbf{n} \tag{2}$$

where W^T is the feedforward matrix and $\mathbf{n} = W^T \mathbf{n}'$. The triangularization condition imposes that $W^T H$ must be upper triangular, which combined with scaling is expressed as:

$$H_{N-k}^T \mathbf{w}_{N-k} = \mathbf{d}_{N-k}\ ,\ k = 0, \dots, N-1 \tag{3}$$

where $H_{N-k} = [\mathbf{h}_1\ \mathbf{h}_2\ \dots\ \mathbf{h}_{N-k}]$ is a matrix composed of the first $N-k$ columns of H, $\mathbf{w}_{N-k}$ is the $(N-k)^{\text{th}}$ column of W, and $\mathbf{d}_i$ is an $i \times 1$ column vector with its i^{th} element equal to 1, and all other elements 0. Since noise enhancement must be minimized, the least-norm solution of $\mathbf{w}_{N-k}$ is obtained:

$$\mathbf{w}_{N-k} = H_{N-k} \left(H_{N-k}^T H_{N-k} \right)^{-1} \mathbf{d}_{N-k} \tag{4}$$

The QR decomposition of the channel gives $H = QR$, where $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \dots\ \mathbf{q}_N]$ is orthogonal and $R = [\mathbf{r}_1\ \mathbf{r}_2\ \dots\ \mathbf{r}_N]$ is upper triangular. Also, $H_{N-k} = QR_{N-k}$, where $R_{n-k} = [\mathbf{r}_1\ \mathbf{r}_2\ \dots\ \mathbf{r}_{N-k}]$ and the upper triangular structure of R implies that $R_{N-k} = \left[I_{(N-k)\times(N-k)}\ O_{k\times(N-k)} \right]^T R'_{N-k}$, where R'_{N-k} (the top $N-k$ rows of R_{N-k}) is invertible. Equation (4) becomes:

$$\begin{aligned} &\mathbf{w}_{N-k} = \\ &Q\,[I\ O]^T R'_{N-k} \left(R'^T_{N-k} R'_{N-k} \right)^{-1} \mathbf{d}_{N-k} = \\ &\mathbf{q}_{N-k} \tfrac{1}{r_{N-k,N-k}} \end{aligned} \tag{5}$$

Therefore, it is found that $W = QD$, where $D = diag\left(r_{11}^{-1}\ r_{22}^{-1}\ \dots\ r_{NN}^{-1} \right)$ and r_{ij} are the elements of R. Consequently:

$$\mathbf{z} = DR\mathbf{x} + \mathbf{n} \tag{6}$$

with $E\left(\mathbf{n}\mathbf{n}^T \right) = D^2 \frac{N_o}{2}$. Thus, a decision feedback structure is obtained as shown in Fig. 1.

It is easy to prove that the above formulation is equivalent to deriving the equalizer from the Cholesky decomposition of $H^* H$ [14]. Typically, decision feedback is combined with appropriate ordering of the dimensions, so that error propagation effects are minimized [6], [7].

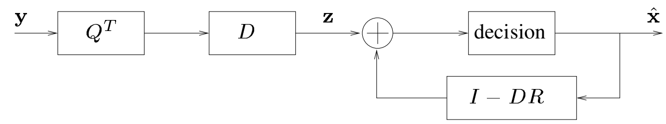

Fig. 1. MIMO Zero-Forcing Decision Feedback Equalizer.

IV. Givens Rotations and QR Decomposition

This section quickly reviews the application of Givens rotations for the derivation of the QR decomposition [15]. A Givens rotation is an orthogonal matrix of the following form:

$$G(i,j,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & \cos(\theta) & \dots & \sin(\theta) & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & -\sin(\theta) & \dots & \cos(\theta) & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \tag{7}$$

where all off-diagonal entries are 0 except for (i,j) and (j,i) that equal $\sin(\theta)$ and $-\sin(\theta)$ respectively, and all diagonal entries are 1 except for (i,i) and (j,j) that are both equal to $\cos(\theta)$.

When a vector is multiplied by G^T, then the plane defined by the (i,j) coordinates is rotated by θ radians in the counter-clockwise direction. If the original vector entries are (x_i, x_j), then the choice $\theta = \tan^{-1}(-x_j/x_i)$ zeros the j^{th} coordinate.

Thus, left multiplication of a matrix H by G^T can be used to zero an element of row j using the element of the same column at row i as a "pivot". The application of this procedure for all lower triangular entries of H can be used to obtain its QR decomposition. An example of the method for a 3×3 case is shown:

$$\begin{aligned} &\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix} \xrightarrow{G_1^T(1,2,\theta_1)} \begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix} \xrightarrow{G_2^T(1,3,\theta_2)} \\ &\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & X & X \end{bmatrix} \xrightarrow{G_3^T(2,3,\theta_3)} \begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{bmatrix} \end{aligned} \tag{8}$$

At most $N(N-1)/2$ Givens rotations are needed to zero all lower triangular entries of H, so that:

$$G_{N(N-1)/2}^T \dots G_2^T G_1^T H = R \tag{9}$$

where G_k are Givens rotations and R is upper triangular. Therefore, Q can be expressed as:

$$Q = G_1 G_2 \dots G_{N(N-1)/2} \tag{10}$$

and thus, the QR decomposition is obtained.

V. Blind MIMO Decision Feedback Equalization

A blind method to compute the Givens rotations is found next. Notice that an alternative definition of the triangularization conditions of (3) is the following set of $N(N-1)/2$ equations:

$$E\left(z_m x_n\right) = 0 \ , \ m = 2, \ldots, N \ , \ n = 1, \ldots, m-1 \quad (11)$$

where z_m are the elements of $\mathbf{z}$. If one defines:

$$\mathbf{z}^{(k)} = G_k^T \ldots G_2^T G_1^T \mathbf{y} \quad (12)$$

then (11) can be replaced by:

$$E\left(z_m^{(k)} x_n\right) = 0 \ , \ m = 2, \ldots, N \ , \ n = 1, \ldots, m-1 \quad (13)$$

where $z_m^{(k)}$ are the elements of $\mathbf{z}^{(k)}$, and it is assumed that the k^{th} rotation zeros element (m, n) of $G_{k-1}^T \ldots G_2^T G_1^T H$. Using (8) as an example, the requirement $E(z_2^{(1)} x_1) = 0$ determines the parameter θ_1 of G_1. Similarly, $E(z_3^{(2)} x_1) = 0$ determines θ_2 of G_2, and $E(z_3^{(3)} x_2) = 0$ determines θ_3 of G_3.

Assuming that the decision outputs of the equalizer are reliable, (13) can be replaced by:

$$E\left(z_m^{(k)} \hat{x}_n\right) = 0 \ , \ m = 2, \ldots, N \ , \ n = 1, \ldots, m-1 \quad (14)$$

Motivated by the idea of a "bootstrap" detector [2], the following adaptive algorithm is proposed for estimating the θ_k's:

$$\bar{\theta}_k \leftarrow \bar{\theta}_k - \mu_k z_m^{(k)} \text{sign}\left(\hat{x}_n\right) \quad (15)$$

where $\bar{\theta}_k$ is the estimate of θ_k, $\mu_k > 0$ is an appropriately chosen step-size, and $k = 1, \ldots, N(N-1)/2$. Clearly for adequately small noise and interference, this method allows convergence to the exact solution of (13). Previous work [16] has shown that the "bootstrap" approach is fairly robust even in environments with relatively large noise. It is later shown that this is also the case here.

Evidently, (15) defines a blind method for the determination of Q in the QR decomposition of H. Note that the use of (13) to determine θ_k assumes that all angles θ_1,…,θ_{k-1} have been accurately estimated. This implies for (15) that for any k, convergence for $\bar{\theta}_k$ requires a satisfactory level of convergence for $\bar{\theta}_{k-1}$. Simulations verify that the convergence time indeed increases with a larger N.

The need to blindly estimate R is handled next. Assuming that Q has achieved convergence, one finds that:

$$\mathbf{z}^{(N(N-1)/2)} = R\mathbf{x} + Q^T \mathbf{n}' \quad (16)$$

In order to perform decision feedback detection, the strictly upper triangular entries of R must be estimated. Let $\tilde{\mathbf{z}}$ (with elements $\tilde{z}_i$) be the vector exactly before the decision device, which ideally must contain no interference. The following set of equations determines $r_{mn} \ , \ n > m$:

$$E\left(\tilde{z}_m x_n\right) = 0 \ , \ m = 1, \ldots, N \ , \ n = m+1, \ldots, N \quad (17)$$

Thus, the following adaptation algorithm is proposed for r_{mn}:

$$\bar{r}_{mn} \leftarrow \bar{r}_{mn} + \lambda_{mn} \tilde{z}_m \text{sign}\left(\hat{x}_n\right) \quad (18)$$

where $\bar{r}_{mn}$ is the estimate of r_{mn}, $\lambda_{mn} > 0$ is an appropriately chosen step size, and $m = 1, \ldots, N \ , \ n = m+1, \ldots, N$.

Even if $r_{mn} \ , \ m = 1, \ldots, N \ , \ n = m+1, \ldots, N$ have been perfectly estimated, $\tilde{\mathbf{z}}$ must be scaled before hard decisions are made. An easy method to estimate the magnitude of the diagonal elements of R (which equal the diagonal elements of D^{-1}) is by averaging over time the magnitude of the elements $\tilde{z}_i$:

$$|\bar{r}_{ii}| = \frac{< |\tilde{z}_i| >}{\sqrt{\mathcal{E}_i}} \quad (19)$$

where $\mathcal{E}_i$ is the average energy of $\mathbf{x}_i$, which must be known to the receiver. A number of time averaging variants can be employed, and the simulations shown later make use of a simple sliding window technique.

Still, the signs of the r_{ii}'s are unknown, and it is clear that it is impossible to resolve them blindly. Thus, after convergence has been achieved, a single training symbol needs to be transmitted to determine the signs of r_{ii}.

Observe that given H, the rotation angles $\theta_i \ , \ i = 1, \ldots, N(N-1)/2$ are not unique. It can be shown that the derived decomposition into $G_1, G_2, \ldots, G_{N(N-1)/2}$ and R is unique up to multiplication of these matrices by diagonal matrices with diagonal elements equal to ± 1. This has no consequence for the equalization method.

Also, it should be noted that the performance of this scheme is significantly affected by the decoding order. However, any reordering operation would have to make use of some channel knowledge that would violate the assumption of a blind technique.

The blind equalization structure is graphically depicted in Fig. 2 for the case of $N = 3$. Observe that one difference with Fig. 1 is that the scaling matrix D has been moved exactly before the decision device.

VI. Simulation Results

The convergence of the algorithm is first demonstrated using a simple example with $N = 2$. The case of $N = 2$ was chosen for illustration, however, simulations indicate that convergence is achieved for higher N, although the required convergence time increases.

The channel matrix is assumed to be:

$$H = \begin{bmatrix} 4 & 1.5 \\ 6 & 4 \end{bmatrix} \quad (20)$$

with noise variace $\sigma^2 = 1$ on all dimensions. The modulation scheme is 4-PAM (also known as 2B1Q), and the minimum distance between constellation points at the transmitter is 2.

The adaptation parameters are chosen to be $\mu_k = 5 \cdot 10^{-4}$ for all k, and $\lambda_{mn} = 10^{-3}$ for all m and n. The time averaging window size is 1000 symbols. Fig. 3 shows the adaptation of

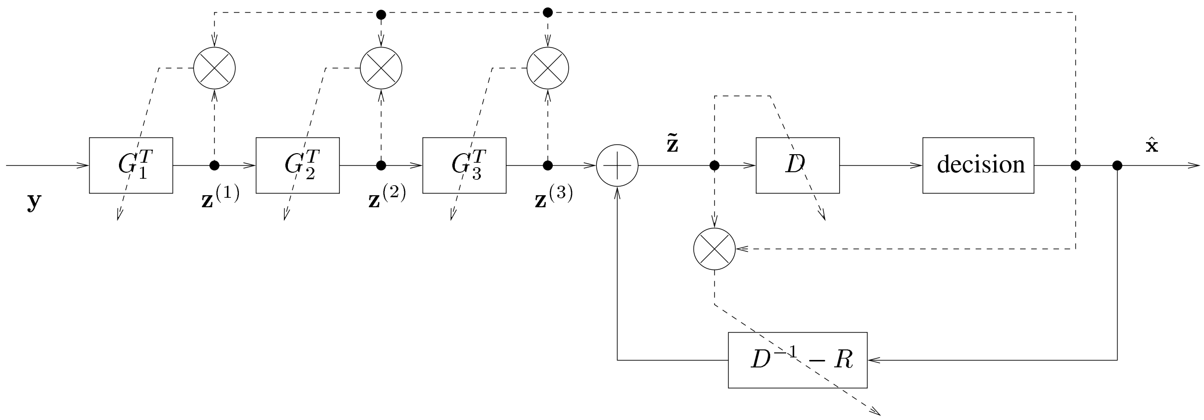

Fig. 2. Blind MIMO Decision Feedback Equalizer (example with $N = 3$).

the rotation angle as well as the exact rotation angle required to make H upper triangular. Fig. 4 shows the magnitudes of the estimates of R as well as the magnitudes of the exact values. In these figures, the estimates are compared against the values obtained by QR decomposition assuming perfect channel knowledge. Finally, Fig. 5 plots the squared errors of the three "streams" (measured at the inputs of the decision devices) and compares them against the theoretical errors of the ZF-DFE. (A stream refers to the symbols transmitted on an input and decoded after equalization at the corresponding output.) It is seen that the measured error of stream 1 converges to the theoretical value, but that of stream 2 remains significantly higher than the theoretical value. This discrepancy is almost entirely due to error propagation effects. Since stream 2 has a significantly large error, detection errors occur frequently, which then lead to a large error for stream 1.

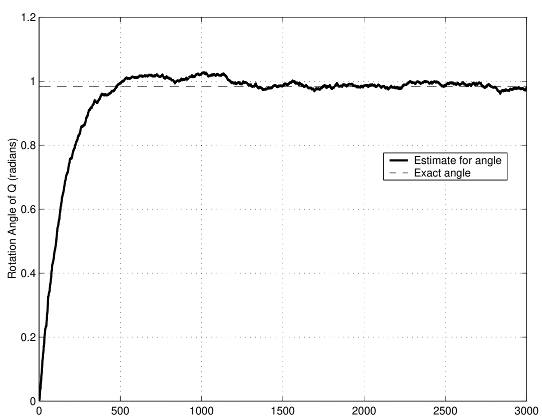

Fig. 3. Adaptation of the rotation angle corresponding to Q.

Next, the channel matrix is assumed to be:

$$H = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 1 & 0.2 \\ 0.5 & 0.2 & 1 \end{bmatrix} \tag{21}$$

and its QR decomposition gives the following upper triangular

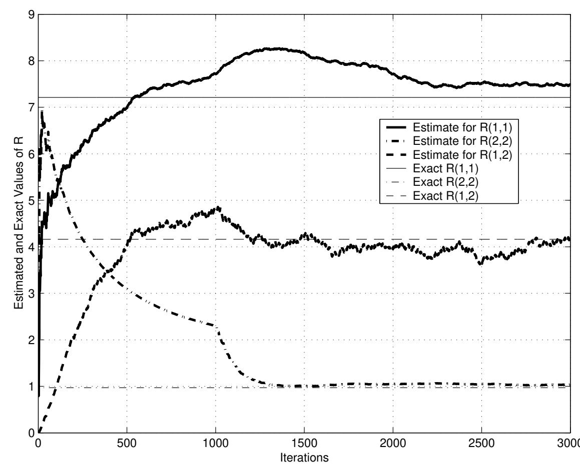

Fig. 4. Adaptation of the upper triangular entries of R.

matrix:

$$R = \begin{bmatrix} -1.187 & -1.095 & -0.910 \\ 0 & -0.672 & 0.068 \\ 0 & 0 & 0.677 \end{bmatrix} \tag{22}$$

Simulation results are now presented studying the probabilities of symbol error in the three streams. Each simulation run involves 10^6 test symbols on which the probabilities of error are measured. For each SNR setting, 10 such runs are performed in order to average over the possible adaptation outcomes. The number of training symbols is 10^3, and the adaptation parameters are chosen as $\mu_k = 8 \cdot 10^{-5}$, and $\lambda_{mn} = 10^{-3}$. All other parameters remain the same as before.

Figure 6 shows the probabilities of error for the three streams, which are compared with the theoretical values obtained from an ideal ZF-DFE where error propagation effects are assumed to have miraculously disappeared. The SNR of a stream is defined as the receiver SNR in the absence of interference. The SNR of streams 1 and 3 is fixed at 19 dB and the SNR of stream 2 varies between 4 and 19 dB in increments of 3.

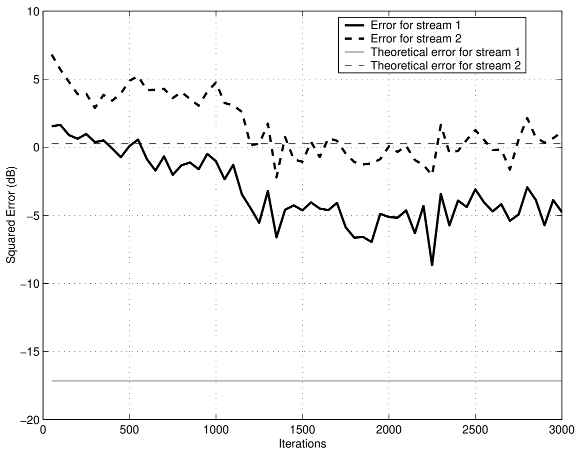

Fig. 5. Squared error before decisions.

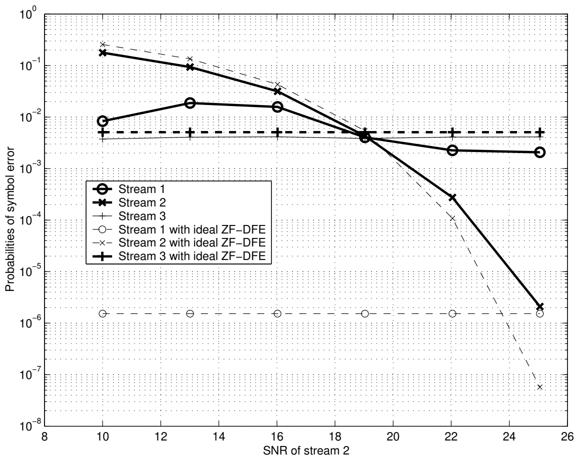

Fig. 6. Probability of error for the three streams, varying the SNR of stream 2.

The first observation is that the probability of error of stream 3 is very close to the theoretical value (and actually lies a bit lower). The assumed parameters result in a large value for the probability of error, which has consequences in the detection of the other two streams. The curve of stream 2 closely follows the theoretical curve, up to the point where the theoretical probability of error of stream 2 becomes significantly smaller than the probability of error of stream 3. From that point on, stream 2 appears to be affected by propagation of errors originating from stream 3. However, the effect is not dominant, and this can be explained by examining (22), where it is evident that the effect of decision feedback on stream 2 is weak. On the other hand, the detection of stream 1 strongly depends on subtracting the interference of streams 2 and 3, therefore error propagation is dominant. At low SNR, stream 1 is degraded due to errors emanating from stream 2, but for large SNR, the degradation is due to errors from stream 3. Thus, stream 1 is incapable of attaining the theoretical values.

Therefore, the divergence from the theoretical values is mainly due to the inherent characteristics of decision feedback. In all cases, the algorithm achieves satisfactory convergence even at relatively low SNR values and with strong interference.

VII. Conclusion

A blind adaptive MIMO DFE technique is proposed based on a ZF formulation. It is shown that the receiver feedforward matrix can be expressed as the multiplication of Givens rotation terms. An adaptive algorithm is defined, where the angles characterizing the rotations are determined using a decorrelation criterion. The feedback coefficients are determined similarly. Simulation results demonstrate the convergence of the algorithm, and compare its performance against an ideal ZF-DFE.

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