On the Relation between V-BLAST and the GDFE

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Abstract— **This paper shows that the receiver processing of the Vertical Bell Labs Layered Space-Time (V-BLAST) architecture may be viewed as a Generalized Decision Feedback Equalizer (GDFE) as applied to a Multiple-Input-Multiple-Output (MIMO) channel. It is proven that, depending on the criterion chosen to reduce the interference, the receiver operations of the V-BLAST system are fundamentally equivalent to either a Zero-Forcing (ZF) or a Minimum-Mean-Square-Error (MMSE) GDFE.**

I. Introduction

The GDFE [1] is a generalization of the well-known decision feedback idea to the case of vector transmission, and is capable of approaching the communication upper bounds imposed by information theory. The V-BLAST architecture [2], [3] has been recently proposed for realizing very high spectral efficiencies over wireless channels with rich scattering. This paper proves that the V-BLAST receiver processing can be interpreted as a special case of the GDFE. Although this result is generally believed to be true, the authors have not encountered any prior proof in the literature. Adopting this alternative view has certain benefits: It gives further insight regarding the functions of V-BLAST, provides a link between V-BLAST and the vast accumulated knowledge about DFE's, and easily produces a decomposition into parallel sub-channels that simplifies SNR and capacity calculations. In the following, ^{T} and * denote the transpose and conjugate-transpose operations.

II. GDFE

The sampled output of a narrowband MIMO system with an equal number of receive and transmit elements satisfies:

$$\mathbf{y} = H\mathbf{x} + \mathbf{n}' \tag{1}$$

where $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{n}'$ are vectors of the output samples, the transmitted symbols and the noise samples, respectively. Noise is assumed white, gaussian with variance $N_o/2$. Also, H is an $N \times N$ non-singular matrix with the element h_{ij} denoting the complex gain from transmitter element j to receiver element i.

The GDFE equalization structure is shown in Figure 1. Although the GDFE in general includes transmit filtering, it is here assumed that $A = I$, so that $\mathbf{u} = \mathbf{x}$ and $\hat{\mathbf{u}} = \hat{\mathbf{x}}$. From (1), the *forward canonical model* is obtained:

$$\mathbf{z} = R_f\mathbf{x} + \mathbf{n} \tag{2}$$

where $\mathbf{z} = H^*\mathbf{y}$, $R_f = H^*H$, $\mathbf{n} = H^*\mathbf{n}'$ and the noise covariance is $R_{\mathbf{nn}} = \frac{N_o}{2}R_f$. The *backward canonical model* is obtained by deriving the MMSE estimate of $\mathbf{x}$ given $\mathbf{z}$:

$$\mathbf{x} = R_b\mathbf{z} + \mathbf{e} \tag{3}$$

where $R_b = R_{\mathbf{uz}}R_{\mathbf{zz}}^{-1}$, and $\mathbf{e}$ is the MMSE error. It is shown in [1] that $R_b^{-1} = R_f + \frac{N_o}{2}R_{\mathbf{xx}}^{-1}$ and $R_{\mathbf{ee}} = R_b$.

A. ZF-GDFE

The Cholesky factorization $R_f = G^*S_oG$ is first derived, where S_o is diagonal with positive elements, and G is upper triangular and monic (having ones along the diagonal). Processing $\mathbf{z}$ by the lower triangular matrix $S_o^{-1}G^{-*}$ yields:

$$S_o^{-1}G^{-*}\mathbf{z} = G\mathbf{x} + S_o^{-1}G^{-*}\mathbf{n} = G\mathbf{x} + \tilde{\mathbf{e}} \tag{4}$$

where $R_{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} = \frac{N_o}{2}S_o^{-1}$ is diagonal. Since G is upper triangular and $\tilde{\mathbf{e}}$ has uncorrelated components, $\mathbf{x}$ can be recovered by back-substitution combined with symbol-by-symbol detection. Thus, a decision feedback structure is derived with the feedforward filter being $S_o^{-1}G^{-*}$ and the feedback filter being $I - G$ as shown in Figure 1.

Further simplification is possible by using the QR decomposition $H = QR$, where Q is unitary and R is upper triangular. Then, $R_f = R^*R$, and $S_o^{1/2} = diag\,(r_{11}, r_{22}, \ldots, r_{NN})$, where r_{ij} are the elements of R. Also, $G = S_o^{-1/2}R$, and the elements of G are $g_{ij} = \frac{r_{ij}}{r_{ii}}$. The combined matched-feedforward filter becomes:

$$S_o^{-1}G^{-*}H^* = S_o^{-1}S_o^{1/2}R^{-*}R^*Q^* = S_o^{-1/2}Q^* \tag{5}$$

which can be interpreted as a rotation/reflection operation performed by Q^*, followed by a scaling operation performed by $S_o^{-1/2}$. The equalization procedure is described as follows:

ZF-GDFE Algorithm

$\tilde{\mathbf{z}} := S_o^{-1/2}Q^*\mathbf{y}$

`for k = 0 to N-1`

$\quad \hat{x}_{N-k} := decision\left(\tilde{z}_{N-k} - \sum_{i=1}^{k} g_{N-k,N-k+i}\hat{x}_{N-k+i}\right)$

`end`

where $\hat{x}_i$, $\tilde{z}_i$ are the elements of $\hat{\mathbf{x}}$ and $\tilde{\mathbf{z}}$ correspondingly.

B. MMSE-GDFE

The Cholesky factorization $R_b^{-1} = G^*S_oG$ is derived, where S_o and G have the same properties as before. All derivations and results hold exactly as for the ZF case, except for the fact that the structure cannot be simplified using the QR decomposition. Then, the algorithm becomes:

MMSE-GDFE Algorithm

$\tilde{\mathbf{z}} := S_o^{-1}G^{-*}H^*\mathbf{y}$

`for k = 0 to N-1`

$\quad \hat{x}_{N-k} := decision\left(\tilde{z}_{N-k} - \sum_{i=1}^{k} g_{N-k,N-k+i}\hat{x}_{N-k+i}\right)$

`end`

The MMSE-GDFE always achieves a larger geometric SNR than the ZF-GDFE [1].

This work was supported by gifts from Samsung, Intel, TI, France Telecom, Telcordia, SK Telecom, and CommTech. The authors are with STARLab, Dept. of Electrical Engineering, Stanford University.

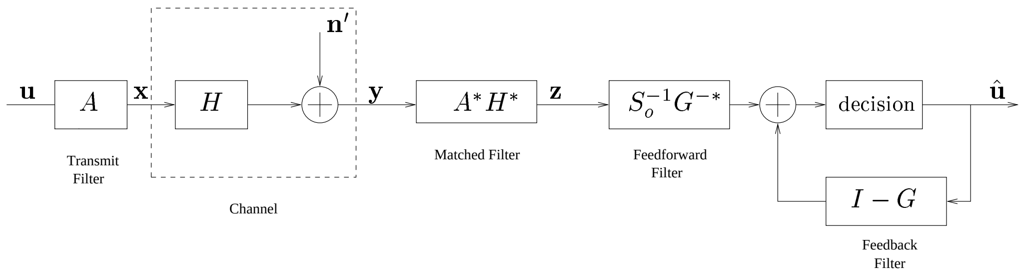

Fig. 1. GDFE equalization structure.

III. V-BLAST Receiver Processing

With V-BLAST [2], [3], the symbols are detected sequentially. Prior to the detection of a symbol, interference from previously detected symbols is subtracted, and the received vector elements are linearly weighted to null (or reduce) interference from the yet undetected symbols.

Assume that the channel is again described by (1), and $H = [\mathbf{h}_1\, \mathbf{h}_2\, \ldots\, \mathbf{h}_N]$. The detection algorithm is described as follows:

V-BLAST Algorithm

$\mathbf{v}_1 := \mathbf{y}$
`for k = 0 to N-1`
  $\hat{x}_{N-k} := decision\left(\mathbf{w}^*_{N-k}\mathbf{v}_{k+1}\right)$
  $\mathbf{v}_{k+2} = \mathbf{v}_{k+1} - \mathbf{h}_{N-k}\hat{x}_{N-k}$
`end`

(Reordering operations can be taken into account by rearranging the rows and columns of H.) The vector $\mathbf{w}_{N-k}$, performing the linear weighting of the elements of $\mathbf{v}_{k+1}$, is chosen to satisfy either a ZF or an MMSE criterion.

A. ZF V-BLAST

With the ZF criterion, interference from the yet undetected symbols is nulled. This is compactly expressed by:

$$H^*_{N-k}\mathbf{w}_{N-k} = \mathbf{c}_{N-k}\ ,\ k = 0, \ldots, N-1 \tag{6}$$

where $H_{N-k} = [\mathbf{h}_1\, \mathbf{h}_2\, \ldots\, \mathbf{h}_{N-k}]$ (full-rank for any k), and $\mathbf{c}_i$ is an $i \times 1$ column vector with its i^{th} element equal to 1, and all other elements 0. The least-norm (and thus noise enhancement minimizing) solution of (6) is:

$$\mathbf{w}_{N-k} = H_{N-k}\left(H^*_{N-k}H_{N-k}\right)^{-1}\mathbf{c}_{N-k} \tag{7}$$

Further simplification is possible with the QR decomposition. Then, $H_{N-k} = QR_{N-k}$, $R_{n-k} = [\mathbf{r}_1\ \mathbf{r}_2\ \ldots\ \mathbf{r}_{N-k}]$ and the upper triangular structure of R implies that:

$$R_{N-k} = \begin{bmatrix} I_{(N-k)\times(N-k)} \\ O_{k\times(N-k)} \end{bmatrix} R'_{N-k} \tag{8}$$

where R'_{N-k} (the top $N-k$ rows of R_{N-k}) is invertible. Equation (7) becomes:

$$\begin{aligned} &\mathbf{w}_{N-k} = \\ &Q\,[I\ O]^T R'_{N-k}\left(R'^*_{N-k}R'_{N-k}\right)^{-1}\mathbf{c}_{N-k} = \\ &[\mathbf{q}_1\ \mathbf{q}_2\ \ldots\ \mathbf{q}_{N-k}]\,R'^{-*}_{N-k}\mathbf{c}_{N-k} = \\ &\mathbf{q}_{N-k}\frac{1}{r^*_{N-k,N-k}} \end{aligned} \tag{9}$$

where $\mathbf{q}_i$ is the i^{th} column of Q.

B. MMSE V-BLAST

With the MMSE criterion, the objective is to minimize the variance of the error between the transmitted symbol and the receiver's estimate, which leads to the following orthogonality requirement:

$$E\left(x_{N-k}\mathbf{v}^*_{k+1}\right) = \mathbf{w}^*_{N-k}E\left(\mathbf{v}_{k+1}\mathbf{v}^*_{k+1}\right) \tag{10}$$

Assuming that decisions are correct with a very high probability, $\hat{x}_{N-k} = x_{N-k}$ and $\mathbf{v}_{k+1} = \sum_{j=1}^{N-k}\mathbf{h}_j x_j + \mathbf{n}'$. With the transmitted symbols being white with variance $\mathcal{E}_x$:

$$E\left(x_{N-k}\mathbf{v}^*_{k+1}\right) = \mathcal{E}_x\mathbf{h}^*_{N-k} \tag{11}$$

$$E\left(\mathbf{v}_{k+1}\mathbf{v}^*_{k+1}\right) = \mathcal{E}_x\sum_{j=1}^{N-k}\mathbf{h}_j\mathbf{h}^*_j + \frac{N_o}{2}I \tag{12}$$

and finally:

$$\mathbf{w}^*_{N-k} = \mathbf{h}^*_{N-k}\left(\sum_{j=1}^{N-k}\mathbf{h}_j\mathbf{h}^*_j + \frac{1}{SNR}I\right)^{-1} \tag{13}$$

where $SNR = \frac{\mathcal{E}_x}{No/2}$.

IV. Equivalence between V-BLAST and GDFE Receiver Operations

The equivalence of the V-BLAST and GDFE receiver processing is demonstrated by proving that the operations performed are identical, and assuming that the ordering of the received substreams is the same for both algorithms. The proofs differ depending on whether the ZF or the MMSE criterion is used.

A. ZF Case

Employing the fact $\mathbf{v}_{k+1} = \mathbf{y} - \sum_{i=1}^{k}\mathbf{h}_{N-k+i}\hat{x}_{N-k+i}$, the fundamental operation of the V-BLAST algorithm is:

$$\mathbf{w}^*_{N-k}\mathbf{v}_{k+1} = \mathbf{w}^*_{N-k}\left(\mathbf{y} - \sum_{i=1}^{k}\mathbf{h}_{N-k+i}\hat{x}_{N-k+i}\right) \tag{14}$$

Recalling that $\tilde{z}_{N-k} = \frac{1}{r_{N-k,N-k}}\mathbf{q}^*_{N-k}\mathbf{y}$, and substituting with (9), equation (14) becomes:

$$\begin{aligned} &\mathbf{w}^*_{N-k}\mathbf{v}_{k+1} = \\ &\tilde{z}_{N-k} - \textstyle\sum_{i=1}^{k}\frac{1}{r_{N-k,N-k}}\mathbf{q}^*_{N-k}\mathbf{h}_{N-k+i}\hat{x}_{N-k+i} \end{aligned} \tag{15}$$

Given the result:

$$\mathbf{q}_{N-k}^{*}\mathbf{h}_{N-k+i} = \mathbf{q}_{N-k}^{*}Q\mathbf{r}_{N-k+i} = r_{N-k,N-k+i} \tag{16}$$

and that $g_{ij} = \frac{r_{ij}}{r_{ii}}$, it is deduced that:

$$\mathbf{w}_{N-k}^{*}\mathbf{v}_{k+1} = \tilde{z}_{N-k} - \sum_{i=1}^{k} g_{N-k,N-k+i}\hat{x}_{N-k+i} \tag{17}$$

This equation proves that both ZF V-BLAST and ZF-GDFE make decisions with exactly the same inputs. Therefore, the methods are equivalent and the corresponding post-detection SNR's are equal.

B. MMSE Case

Starting with (13) and making use of $H_{N-k} = H\left[I_{N-k\times N-k}\ O_{k\times(N-k)}\right]^{T}$, one finds:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*} = \\
&\mathbf{h}_{N-k}^{*}\left(H_{N-k}H_{N-k}^{*} + (1/SNR)I\right)^{-1} = \\
&\mathbf{c}_{N-k}^{T}H^{*}\left\{H\left[I\ O\right]^{T}\left[I\ O\right]H^{*} + (1/SNR)I\right\}^{-1} = \\
&\mathbf{c}_{N-k}^{T}\left\{H^{*}H\left[I\ O\right]^{T}\left[I\ O\right] + (1/SNR)I\right\}^{-1}H^{*}
\end{aligned} \tag{18}$$

where $\mathbf{c}_{N-k}$ has the same interpretation as before, but now has size $N\times 1$. Utilizing the fact that $R_b^{-1} = R_f + \frac{1}{SNR}I \Rightarrow G^{*}S_oG = H^{*}H + \frac{1}{SNR}I$, (18) becomes:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*} = \\
&\mathbf{c}_{N-k}^{T}\left\{\left(G^{*}S_oG - \tfrac{1}{SNR}I\right)\begin{bmatrix} I & O \\ O & O\end{bmatrix} + \tfrac{1}{SNR}I\right\}^{-1}H^{*}
\end{aligned} \tag{19}$$

Define $S_o = diag(S_{o,1}, S_{o,2})$ and:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ O_{k\times(N-k)} & G_{22}\end{bmatrix} \tag{20}$$

where $S_{o,1}$, $S_{o,2}$ are diagonal with dimensions $(N-k)\times(N-k)$ and $k\times k$ respectively, G_{11}, G_{22} are upper triangular with dimensions $(N-k)\times(N-k)$ and $k\times k$ respectively, and G_{12} is $(N-k)\times k$. Then, the following is derived:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*} = \\
&\mathbf{c}_{N-k}^{T}\left\{G^{*}\begin{bmatrix} S_{o,1}G_{11} & O \\ O & O\end{bmatrix} + \begin{bmatrix} O & O \\ O & \frac{1}{SNR}I\end{bmatrix}\right\}^{-1}H^{*} = \\
&\mathbf{c}_{N-k}^{T}\begin{bmatrix} G_{11}^{*}S_{o,1}G_{11} & O \\ G_{12}^{*}S_{o,1}G_{11} & \frac{1}{SNR}I\end{bmatrix}^{-1}H^{*}
\end{aligned} \tag{21}$$

Employing the *block matrix inversion* formula [4, App. A]:

$$\begin{bmatrix} G_{11}^{*}S_{o,1}G_{11} & O \\ G_{12}^{*}S_{o,1}G_{11} & \frac{1}{SNR}I\end{bmatrix}^{-1} = \begin{bmatrix} G_{11}^{-1}S_{o,1}^{-1}G_{11}^{-*} & O \\ -SNR\cdot G_{12}^{*}G_{11}^{-*} & SNR\cdot I\end{bmatrix} \tag{22}$$

Equation (21) now becomes:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*} = \\
&\mathbf{c}_{N-k}^{T}\begin{bmatrix} G_{11}^{-1}S_{o,1}^{-1}G_{11}^{-*} & O \\ -SNR\cdot G_{12}^{*}G_{11}^{-*} & SNR\cdot I\end{bmatrix}H^{*} = \\
&\mathbf{c}_{N-k}^{T}\begin{bmatrix} G_{11}^{-1}S_{o,1}^{-1}G_{11}^{-*} & O \\ O & O\end{bmatrix}H^{*}
\end{aligned} \tag{23}$$

Since G_{11}^{-1} is monic, it is deduced that:

$$\mathbf{w}_{N-k}^{*} = \mathbf{c}_{N-k}^{T}S_o^{-1}G^{-*}H^{*} \tag{24}$$

The fundamental operation of MMSE V-BLAST is next examined. Knowing that $\tilde{\mathbf{z}} = S_o^{-1}G^{-*}H^{*}\mathbf{y}$:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*}\mathbf{v}_{k+1} = \\
&\mathbf{c}_{N-k}^{T}S_o^{-1}G^{-*}H^{*}\left(\mathbf{y} - \textstyle\sum_{i=1}^{k}\mathbf{h}_{N-k+i}\hat{x}_{N-k+i}\right) = \\
&\mathbf{c}_{N-k}^{T}\left(\tilde{\mathbf{z}} - \textstyle\sum_{i=1}^{k}S_o^{-1}G^{-*}H^{*}\mathbf{h}_{N-k+i}\hat{x}_{N-k+i}\right) = \\
&\mathbf{c}_{N-k}^{T}\left(\tilde{\mathbf{z}} - \textstyle\sum_{i=1}^{k}S_o^{-1}G^{-*}H^{*}H\mathbf{c}_{N-k+i}\hat{x}_{N-k+i}\right)
\end{aligned} \tag{25}$$

Substituting $H^{*}H$, one finds:

$$\begin{aligned}
&\mathbf{w}_{N-k}^{*}\mathbf{v}_{k+1} = \\
&\mathbf{c}_{N-k}^{T}\left[\tilde{\mathbf{z}} - \textstyle\sum_{i=1}^{k}S_o^{-1}G^{-*}\left(G^{*}S_oG - \frac{1}{SNR}I\right)\mathbf{c}_{N-k+i}\hat{x}_{N-k+i}\right] = \\
&\mathbf{c}_{N-k}^{T}\left[\tilde{\mathbf{z}} - \textstyle\sum_{i=1}^{k}\left(G - \frac{1}{SNR}S_o^{-1}G^{-*}\right)\mathbf{c}_{N-k+i}\hat{x}_{N-k+i}\right] = \\
&\tilde{z}_{N-k} - \textstyle\sum_{i=1}^{k}\mathbf{c}_{N-k}^{T}\left(G - \frac{1}{SNR}S_o^{-1}G^{-*}\right)\mathbf{c}_{N-k+i}\hat{x}_{N-k+i} = \\
&\tilde{z}_{N-k} - \textstyle\sum_{i=1}^{k}g_{N-k,N-k+i}\hat{x}_{N-k+i}
\end{aligned} \tag{26}$$

Therefore, it becomes evident that the decision inputs for both MMSE V-BLAST and MMSE-GDFE are exactly the same. The techniques are equivalent and result in the same post-detection SNR's.

It is not hard to show that both techniques (with either the ZF or the MMSE criterion) have approximately the same complexity. Assuming that all system parameters have been pre-computed (as for a stationary channel), the number of flops required for the detection of each block is $\mathcal{O}(N^2)$. If the system parameters need to be re-computed for each transmitted block (as for a block-stationary channel), then for both methods the complexity is $\mathcal{O}(N^3)$.

V. Conclusion

This paper proves that V-BLAST receiver processing is equivalent to the operations of the GDFE, thus providing a different perspective on this well-known high capacity multiple antenna system. It is worth stressing that *performance equivalency* rests on the assumption that the GDFE employs reordering of the received substreams (an inherent feature of V-BLAST), and that there is no transmission optimization.

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