FINDING PARAMETRIC EQUATIONS FOR 120 DEGREE AND 60 DEGREE TRIPLES.

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1. Notation

- $s \mid t$ means s divides t.
- $s \nmid t$ means s does not divide t.
- $\Rightarrow$ means "implies".
- If s and t are integers then $(s,t) = d$ means that d is their greatest common divisor. If $d = 1$ then s and t are relatively prime.
- In this paper the equation for a triangle will be called a triangle. That is, $\alpha^2 + \beta^2 + \alpha\beta = \gamma^2$ will be called a 120 degree triangle. And I will write "the 120 degree triangle $\alpha^2 + \beta^2 + \alpha\beta = \gamma^2$".

2. Introduction

A 120 degree triple (a, b, c) is a solution in positive integers to the equation

$$a^2 + b^2 - 2ab\cos 120\,^\circ = c^2 = a^2 + b^2 + ab. \tag{1}$$

If, additionally, a, b, and c are pairwise relatively prime then (a, b, c) is a primitive 120 degree triple, and $a^2 + b^2 + ab = c^2$ is a primitive 120 degree triangle. Clearly, if some integer k divides any two of the variables a, b, and c then it divides the remaining variable, therefore k can be divided out. Similarly, if (a, b, c) is primitive then multiply through by the appropriate integer to get your chosen non-primitive solution. Hence, to find all solutions, it's sufficient to find all primitive solutions.

3. Deriving parametric equations for finding all 120 degree triples

We will need a preliminary result.

Lemma 1. *If m, and n are positive integers such that $(m,n) = 1$ then $3 \nmid m-n \Rightarrow \big(m^2 - n^2, 2mn + n^2\big) = 1$, and $3 \mid m-n \Rightarrow \big(m^2 - n^2, 2mn + n^2\big) = 3$*

Proof. Let $\big(m^2 - n^2, 2mn + n^2\big) = d$ where d is a positive integer. We have, since $(m,n) = 1$, $(m+n, m-n) = (m \pm n, n) = (m \pm n, m) = 1$. So, $d \mid m^2 - n^2$ $= (m+n)(m-n)$ and $d \mid 2mn + n^2 = n(2m+n) \Rightarrow d \mid (m+n)(m-n)$ and $d \mid 2m + n$. Let $d_1 d_2 = d$ where d_1 and d_2 are positive integers.

- If $\big(m+n, 2m+n\big) = d_i$ where $i = 1$ or 2, then $d_i \mid 2m + n - m - n = m$ $\Rightarrow d_i = 1$. This implies that $d \mid m - n \Rightarrow d \mid m - n + 2m + n = 3m$ $\Rightarrow d = 3$ or 1. Therefore, if $3 \nmid m - n$ then $\big(m^2 - n^2, 2mn + n^2\big) = d = 1$, and if $3 \mid m - n$ then $\big(m^2 - n^2, 2mn + n^2\big) = d = 3$.

$\square$

Claim 1. *Let* $(a,b,c)=(b,a,c)$. *Then* (a,b,c) *is a 120 degree triple if and only if there exists relatively prime, positive integers,* m *and* n,
$3 \nmid m-n$, *and* $m>n$, *such that*

$$a=k\big(m^2-n^2\big),\quad b=k\big(2mn+n^2\big),\quad \textit{and } c=k\big(m^2+n^2+mn\big)$$

where k *is some positive integer.*

Proof. Let $a^2+b^2+ab=c^2$ be a primitive 120 degree triangle. From the triangle inequality, and since a, b and c are each non-zero, the sum of the lengths of any two sides is greater than the length of the remaining side. That is, $b<c+a$. Therefore, there exists relatively prime positive integers m and n, $m>n$, such that

$$\frac{m}{n}\,b=c+a.$$

Thus

$$\frac{m}{n}=\frac{c+a}{b}$$

reduced to lowest terms. From whence

$$\frac{m}{n}\,b-a=c=\sqrt{a^2+b^2+ab}. \tag{2}$$

Square both sides of (2), then cancel and rearrange terms to get the result,

$$\big(m^2-n^2\big)\,b=\big(2mn+n^2\big)\,a. \tag{3}$$

There are two cases, $3\nmid m-n$ and $3\mid m-n$.

Case I, $\mathbf{3\nmid m-n}$. From Lemma (1), $\big(m^2-n^2,2mn+n^2\big)=1$. Also $(a,b)=1$. Thus, from (3),

(4)
$$a=m^2-n^2\,,\quad b=2mn+n^2\,,\quad \text{and from (2)},c=\frac{m}{n}\big(2mn+n^2\big)-\big(m^2-n^2\big)=m^2+n^2+mn.$$

Case II, $\mathbf{3\mid m-n}$. From Lemma (1), $\left(\frac{m^2-n^2}{3},\frac{2mn+n^2}{3}\right)=1$. Then, from (3),

$$a=\frac{m^2-n^2}{3}\,,\quad b=\frac{2mn+n^2}{3}\,,\quad \text{and}\quad c=\frac{m^2+n^2+mn}{3}\,. \tag{5}$$

To show that (5) generates the same solutions as (4), note that $3\mid 2mn+n^2=n(2m+n)\Rightarrow 3\mid 2m+n$ and $3\mid m-n\Rightarrow 3\mid 2m+n-(m-n)=m+2n$. So let

$$u=\frac{m+2n}{3}\quad\text{and}\quad v=\frac{m-n}{3}\,.$$

Then

$$m=u+2v\quad\text{and}\quad n=u-v.$$

Substituting these values for m and n into (5) yields

$$a=2uv+v^2\,,\quad b=u^2-v^2\,,\quad\text{and}\quad c=u^2+v^2+uv \tag{6}$$

where u and v are relatively prime, positive integers, $u>v$ and $3\nmid u-v$. Hence (6), with the labels for a and b interchanged, is identical to (4). And in the other direction of the if and only if, indeed

$$\big(m^2-n^2\big)^2+\big(2mn+n^2\big)^2+\big(m^2-n^2\big)\big(2mn+n^2\big)=\big(m^2+n^2+mn\big)^2\,.$$

Therefore, all 120 degree triples where (a, b, c) is considered the same as (b, a, c) are given by the parametric equations,

$$a = k\left(m^2 - n^2\right), \quad b = k\left(2mn + n^2\right), \quad \text{and} \quad c = k\left(m^2 + n^2 + mn\right) \tag{7}$$

where m and n are relatively prime, positive integers, $m > n$, $3 \nmid m - n$, and k is some positive integer. $\square$

Alternatively, the triple (a, b, c) satisfies the integer-sided equation $a^2 + b^2 + ab = c^2$ if and only if there exists relatively prime, positive integers m and n, $m > n$, $3 \mid m - n$ such that

$$a = k\left(\frac{m^2 - n^2}{3}\right), \quad b = k\left(\frac{2mn + n^2}{3}\right), \quad \text{and} \quad c = k\left(\frac{m^2 + n^2 + mn}{3}\right)$$

for some positive integer k..

Another method (using Eisenstein integers).

Let

$$a^2 + b^2 + ab = c^2 \tag{8}$$

where a, b, and c are pairwise, relatively prime, positive integers. Set $i = \sqrt{-1}$, and let $z = a - b\bar{\omega}$ and $\bar{z} = a - b\omega$ where

$$\omega = \frac{-1}{2} + \frac{1}{2}\sqrt{3}\, i \quad \text{and} \quad \bar{\omega} = \frac{-1}{2} - \frac{1}{2}\sqrt{3}\, i.$$

Then z and $\bar{z}$ are Eisenstein Integers, and $\bar{z}$ is the conjugate of z. Note that $\omega\bar{\omega} = 1$, $\omega + \bar{\omega} = -1$, $\omega - \bar{\omega} = \sqrt{3}\, i$, $w^2 = \bar{\omega}$, and $\bar{\omega}^2 = \omega$. Thus

$$a = \frac{\omega z - \bar{\omega}\bar{z}}{\sqrt{3}\, i}, \quad b = \frac{z - \bar{z}}{\sqrt{3}\, i}, \quad \text{and} \quad c = \sqrt{z\bar{z}}. \tag{9}$$

Since $\gcd(a, b) = 1$ then $\gcd(z, \bar{z}) = 1$. Hence each of z and $\bar{z}$ is a square. That is, there exists integers m and n such that $(m - n\bar{\omega})^2 = z$ and $(m - n\omega)^2 = \bar{z}$. So, from equation (9),

$$\begin{aligned} a &= \frac{\omega(m - n\bar{\omega})^2 - \bar{\omega}(m - n\omega)^2}{\sqrt{3}\, i} = m^2 - n^2, \\ b &= \frac{(m - n\bar{\omega})^2 - (m - n\omega)^2}{\sqrt{3}\, i} = 2mn + n^2, \\ \text{and} \quad c &= \sqrt{(m - n\bar{\omega})^2(m - n\omega)^2} = m^2 + n^2 + mn. \end{aligned}$$

Since a is a positive integer, m and n must be positive integers, $m > n$. And since $\gcd(a, b) = 1$ then m and n must be relatively prime, and $3 \nmid m - n$.

4. Finding parametric equations for 60 degree triples

If u, v, and w are positive integers such that $u^2 + v^2 - uv = w^2$ then (u, v, w) is a 60 degree triple. If additionally u, v,and w are pairwise relatively prime then (u, v, w) is a primitive 60 degree triple. If (u, v, w) is a primitive 60 degree triple then $u^2 + v^2 - uv = w^2$ is a primitive 60 degree triangle, and vice versa.

Parametric equations for finding all 60 degree triples can be easily derived from the parametric equations for finding all 120 degree triples. (4)

First note that for any positive integer s, (s, s, s) is a 60 degree triple. That is, $s^2 + s^2 - ss = s^2$ where $s$is the length of each side of an equilateral triangle. Clearly, if (u, v, w) is a 60 degree triple where $u = v$ then $u = w$. So, since we already know how to write down the side lengths of an equilateral triangle, the triples such that $u = v$ need not be included in our derivation.

Claim 2. *(u, v, w) is a 60° triple such that $u \neq v$ if and only if there exists a 120° triple, (a, b, c), where either $u = a + b$, $v = b$, and $w = c$ or $u = a$, $v = a + b$, and $w = c$.*

Proof. As with 120° triples, it's sufficient to find all primitive solutions. There are two cases, $u > v$ and $v > u$. We have already covered the case where $u = v$.

Case I, $\mathbf{u} > \mathbf{v}$. We have $w^2 = u^2 + v^2 - uv = (u-v)^2 + v^2 + (u-v)v$. Hence $(u-v, v, w)$ is a 120° triple. Set $a = u - v$, $b = v$, and $c = w$. Then $u = a + b$, $v = b$, and $w = c$.

And in the other direction, $a^2 + b^2 + ab = c^2 = (a+b)^2 + b^2 - (a+b)b = u^2 + v^2 - uv = w^2$.

Case II, $\mathbf{v} > \mathbf{u}$. We have $w^2 = u^2 + v^2 - uv = u^2 + (v-u)^2 + u(v-u)$. Hence $(u, v-u, w)$ is a 120° triple. Set $a = u$, b=$v - u$, and $c = w$. Then $u = a$, $v = a + b$, and $w = c$.

And in the other direction, $a^2 + b^2 + ab = a^2 + (a+b)^2 - a(a+b) = u^2 + v^2 - uv = w^2$

□

From equation (7) and Claim (2), where $a = k\left(m^2 - n^2\right)$ and $b = k\left(2mn + n^2\right)$, if $u \neq v$, (u, v, w) is a 60 degree triple if and only if there exists relatively prime positive integers,m and n, $3 \nmid m - n$, and $m > n$ such that

$$u = k\left(2mn + m^2\right), \quad v = k\left(2mn + n^2\right), \quad \text{and} \quad w = k\left(m^2 + n^2 + mn\right),$$

or

$$u = k\left(m^2 - n^2\right), \quad v = k\left(2mn + m^2\right), \quad \text{and} \quad w = k\left(m^2 + n^2 + mn\right).$$

Where k is some positive integer.

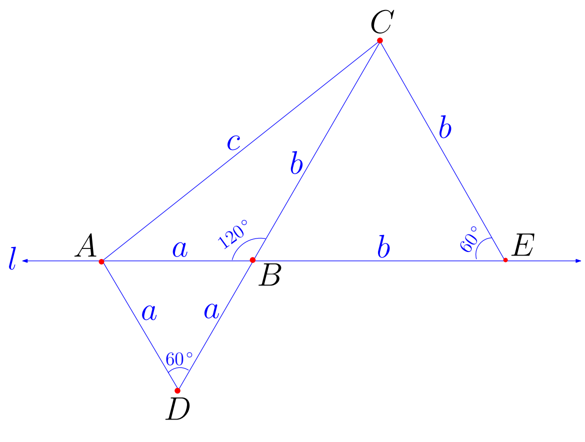

FIGURE 1. 120 and 60 degree triangles

Figure (1) illustrates Claim (2) where in $\triangle AEC$ $u = a + b$ and $v = b$, and in $\triangle ADC$ $u = a$ and $v = a + b$.

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