A 120 degree triple is a solution, \((a, b, c)\), in positive integers to the 120 degree triangle equation
\[ a^2 + b^2 - 2ab \cos 120^\circ = a^2 + b^2 + ab = c^2. \]
If additionally \(a, b,\) and \(c\) are pairwise relatively prime then \((a, b, c)\) is a primitive 120 degree triple.

\((a, b, c)\) is a primitive 120 degree triple if and only if there exists relative prime integers \(u\) and \(v\), \(u > v\) and \(3 \nmid u - v\) such that
\[ a = u^2 - v^2, \quad b = 2uv + v^2, \quad \text{and} \quad c = u^2 + v^2 + uv. \]
See (??) for a proof.

For \(n > 2\), the \(n^{th}\) Fibonacci number is given by \(F_n = F_{n-2} + F_{n-1}\) where \(F_1 = F_2 = 1\). The first few are 1, 1, 2, 3, 5, 8, 13, 21, 34.
Some notation

- \((s, t) = d\) means that the positive integer \(d\) is the greatest common divisor of the two integers \(s\) and \(t\). If \(d = 1\) then \(s\) and \(t\) are relatively prime.

- \(s \mid t\) means \(s\) divides \(t\).

- \(s \nmid t\) means \(s\) does not divide \(t\).

- \(\Rightarrow\) means implies.

Claim 1. If \(F_n, F_{n+1}, F_{n+2}, F_{n+3}, \) and \(F_{n+4}\) are 5 consecutive Fibonacci numbers then

\[
( F_n F_{n+3}, F_{n+1} F_{n+4}, F_{n+1} F_{n+4} + F_n F_{n+2} )
\]

is a 120 degree triple. And if \(3 \nmid F_n\) then it’s a primitive triple. That is

\[
( F_n F_{n+3})^2 + (F_{n+1} F_{n+4})^2 + (F_n F_{n+3})(F_{n+1} F_{n+4}) = (F_{n+1} F_{n+4} + F_n F_{n+2})^2
\]

where each side of the triangle is relatively prime to each of the other two sides.

Proof. First note that any two consecutive Fibonacci numbers are relatively prime. This can easily be proved by induction on \(n\).

\begin{enumerate}
  \item \((1, 1) = (2, 1) = (3, 2) = 1\). So it’s true for \(F_1, F_2, F_3,\) and \(F_4\).
  \item Assume that \((F_n, F_{n-1}) = 1\). We want to show that this implies that \((F_{n+1}, F_n) = 1\). To do so, let \(d = (F_{n+1}, F_n)\). This implies that \(d \mid F_{n+1} = F_{n-1} + F_n\) and \(d \mid F_n\). This implies that \(d \mid F_{n-1} + F_n = F_{n-1}\) \(\Rightarrow\) \(d = 1\) since \((F_n, F_{n-1}) = 1\).
\end{enumerate}

Hence \(F_{n+1}\) and \(F_n\) are relatively prime, and therefore, any two consecutive Fibonacci numbers are relatively prime.

Let \(u = F_{n+2}\) and \(v = F_{n+1}\). Then \(3 \nmid F_n \Rightarrow 3 \nmid F_n + F_{n+1} - F_{n+2} = F_{n+1} = u - v\). So, we have

\begin{enumerate}
  \item \(F_n = u - v\).
  \item \(F_{n+1} = v\).
  \item \(F_{n+2} = u\).
  \item \(F_{n+3} = v + u\).
  \item \(F_{n+4} = v + 2u\).
\end{enumerate}

From equation (1),

\[
\begin{align*}
a &= u^2 - v^2 = (u - v)(u + v) = F_n F_{n+3}, \\
b &= 2uv + v^2 = v(v + 2u) = F_{n+1} F_{n+4}, \\
c &= u^2 + v^2 + uv = v(v + 2u) + (u - v)u = F_{n+1} F_{n+4} + F_n F_{n+2}.
\end{align*}
\]

\(\square\)
Example

Let $F_n = 5$, then $F_{n+1} = 8$, $F_{n+2} = 13$, $F_{n+3} = 21$, and $F_{n+4} = 34$. So

\[ F_nF_{n+3} = 5 \cdot 21 = 105, \]
\[ F_{n+1}F_{n+4} = 8 \cdot 34 = 272, \]
and \[ F_{n+1}F_{n+4} + F_nF_{n+2} = 8 \cdot 34 + 5 \cdot 13 = 337. \]

Then
\[ 105^2 + 272^2 + 105 \cdot 272 = 337^2. \]

This works for generalized Fibonacci numbers also. That is, choose any two positive integers $N_0$ and $N_1$, then obtain integers $N_2$, $N_3$, and $N_4$ thusly,

1. $N_0 + N_1 = N_2$.
2. $N_1 + N_2 = N_3$.

Set $N_1 = v$ and $N_2 = u$. We have

1. $N_0 = u - v$.
2. $N_1 = v$.
3. $N_2 = u$.
4. $N_3 = v + u$.
5. $N_4 = v + 2u$.

Then $(N_0N_3, N_1N_4, N_1N_4 + N_0N_2)$ is a 120 degree triple.

**Example:** If $N_0 = 13$ and $N_1 = 1$ then $N_2 = 14$, $N_3 = 15$, and $N_4 = 29$. Therefore

\[ (13 \cdot 15)^2 + (1 \cdot 29)^2 + (13 \cdot 15)(1 \cdot 29) = (1 \cdot 29 + 13 \cdot 14)^2. \]

Sixty degree triangles

Construct equilateral triangles on each of the adjacent legs of the 120° triangle in Figure (1) creating the two 60° triangles $ABD$ and $ACD$ as shown in figure (2). Thus,

\[ (F_nF_{n+3}, F_nF_{n+3} + F_{n+1}F_{n+4}, F_{n+1}F_{n+4} + F_nF_{n+2}) \]
and \[ (F_nF_{n+3} + F_{n+1}F_{n+4}, F_{n+1}F_{n+4}, F_{n+1}F_{n+4} + F_nF_{n+2}) \]
are 60° triples. That is,

\[ (F_nF_{n+3})^2 + (F_nF_{n+3} + F_{n+1}F_{n+4})^2 = (F_nF_{n+3})(F_nF_{n+3} + F_{n+1}F_{n+4}) \]
\[ = (F_nF_{n+3} + F_{n+1}F_{n+4})^2 + (F_{n+1}F_{n+4})^2 - (F_nF_{n+3} + F_{n+1}F_{n+4})(F_{n+1}F_{n+4}) \]
\[ = (F_{n+1}F_{n+4} + F_nF_{n+2})^2. \]

E-mail address: fredlb@centurytel.net
Figure 2. 60 degree triangles