Important identities and theorems from Vector Calculus

Some important formulae, theorems and identities those are particularly useful in the analysis of fluid flows.

Notation

$\vec{\nabla} \equiv$ del operator, $A \equiv$ Any vector field (e.g velocity field, vorticity, grad (pressure) etc.)
$\phi, \psi \equiv$ Any scalar fields (e.g density, pressure, temperature etc)

1. Gradient, Divergence and Curl

	Cartesian	Cylindrical-Polar
$\vec{\nabla}\phi$	$\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$	$\hat{e}_r\frac{\partial\phi}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial\phi}{\partial\theta} + \hat{e}_z\frac{\partial\phi}{\partial z}$
$\vec{\nabla}.\vec{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial rA_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$
$\vec{\nabla}\times\vec{A}$	$\hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$	$\hat{e}_r(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}) + \hat{e}_\theta(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}) + \hat{e}_z(\frac{1}{r}\frac{\partial rA_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta})$

2. Laplacian operator ∇^2

a) Cartesian : $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ b) Cylindrical-polar : $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$

3. Spatial derivatives of unit vectors

The unit vectors in a coordinate frame may change their direction with the change in spatial coordinates. This happens in curvilinear coordinate systems like the cylindrical-polar and the spherical systems. The following relations are useful in the application of the $\vec{\nabla}$ operator in cylindrical-polar form:

Unit vectors	Spatial derivatives		
	$\frac{\partial}{\partial r}$	$\frac{\partial}{\partial\theta}$	$\frac{\partial}{\partial z}$
$\hat{e}_r$	0	$\hat{e}_\theta$	0
$\hat{e}_\theta$	0	$-\hat{e}_r$	0
$\hat{e}_z$	0	0	0

4. Some useful identities

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$$\nabla \times \nabla\phi = 0$$

$$\nabla.(\nabla \times \mathbf{A}) = 0$$

$$\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$$

$$\nabla \cdot (\phi\mathbf{A}) = \phi\nabla \cdot \mathbf{A} + \nabla\phi \cdot \mathbf{A}$$

$$\nabla \times (\phi\mathbf{A}) = \phi\nabla \times \mathbf{A} + \nabla\phi \times \mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

5. Important Theorems on line, surface and volume integrals

A. Theorems relating integrals in a region R (Volume) to integrals on the boundary S of the region (Surface)

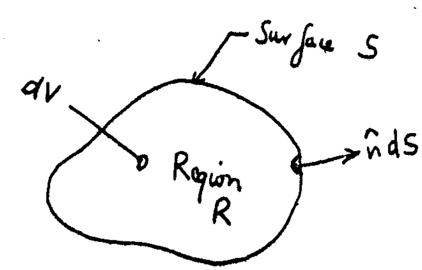
(i) Gauss theorems

$$\oiint_S \begin{pmatrix} \phi \\ \mathbf{A}\cdot \\ -\mathbf{A}\times \end{pmatrix} \hat{n}\, dS = \iiint_R \nabla \begin{pmatrix} \phi \\ \cdot\mathbf{A} \\ \times\mathbf{A} \end{pmatrix} dV$$

(ii) Green's theorems

First form ⟶ $\iiint_R (\psi\nabla^2\phi + \nabla\phi \cdot \nabla\psi)\,dV = \oiint_S \psi \frac{\partial\phi}{\partial n}\,dS$

Second form ⟶ $\iiint_R (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dV = \oiint_S (\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n})\,dS$



B. Theorems relating integrals on a closed curve (path) to integrals on the arbitrary surface bounded by the curve

(i) Stokes theorem: Theorem relating *circulation* of a vector field A around a closed circuit 'C' to the *flux* of Curl (A) through an arbitrary surface 'S' bounded by the closed circuit 'C'.

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{A}) \cdot \hat{n}\,dS$$