Problem Sheet PS03 (Kinematics of fluid flows)

Problems 1-4: Flow visualization

1. Consider the two-dimensional steady flowfield given by $\vec{V} = ax\,\hat{i} - ay\hat{j}$ where 'a' is a positive constant.
   a) Obtain the general equation for the family of streamlines for this flow.
   b) Sketch the streamline pattern in each quadrant in the x-y plane indicating the flow direction.
   c) Identify the stagnation streamlines (streamlines that intersect at the stagnation point).

2. Consider a two-dimensional unsteady flowfield given by $\vec{V} = 2x\,\hat{i} + 2y(1+t)\,\hat{j}$. Obtain the equations of
   a) streamline at t = 0 and at t = 1 that passes through point (1, 1)
   b) trajectory / path line of the fluid particle at (1, 1) at t = 0.
   c) streakline at t = 1 of fluid particles that have earlier passed through (1, 1). (Hint: Obtain the locations at t = 1 of all the particles that have passed (1, 1) at t = τ for $\tau \le 1$. The locations will be thus be a function of τ. Eliminate τ to obtain the locus of positions)

3. Consider the three dimensional steady velocity field u = Ax, v = Ay, w = Bz, where A, B are constants.
   a) Show that the velocity field represents a three-dimensional, irrotational flow for any choice of A, B.
   b) If the velocity field should also represent an incompressible flow, what relationship should exist between A and B.
   c) Since this is a three dimensional flow field, the streamlines would lie in a 3D space. However, the streamlines can be projected on the various coordinate planes to visualize the flow. Obtain the equation of the family of projected streamlines in the x-y and x-z planes. Are the streamlines defined at (0, 0, 0)?

4. Consider the steady two-dimensional fluid motion in a plane described by the velocity field in polar coordinates as v_r = func(r, θ), $v_\theta = B\,r$ where B is a constant and the radial coordinate 'r' is measured from some point.
   a) If the velocity field represents an incompressible flow, show that $v_r = f(\theta) / r$ is the most general form of v_r.
   b) Obtain the vorticity for the flow with $v_r = f(\theta) / r$, $v_\theta = B\,r$. Obtain the necessary conditions on f and B to render the flow irrotational
   c) Show that the reference point for the radial coordinate cannot lie in the flow domain for the above velocity field to physically represent a flow.
   d) Taking $f(\theta) = A$ (constant), obtain the general equation of the family of streamlines for this flow and interpret the flow pattern for A > 0 and B > 0.

Problems 5-11: Material derivative and particle acceleration

5. Obtain the expressions for radial and circumferential components of particle acceleration, a_r and a_θ, in polar coordinates. (Hint: Use the coordinate free form : $\vec{a} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$; and the fact that partial derivatives of unit vectors, $\hat{e}_r$ and $\hat{e}_\theta$ with respect to circumferential coordinate θ exist, i.e. ***see formula sheet***)

6. Many flow phenomena can be understood if one learns to approximately visualize the flow pattern via streamlines of the flow. An important step in this direction is to represent the kinematics of the flow using streamline coordinates as shown in fig. 1. The figure shows a streamline in a 2D flow (for simplicity) at a given instant 't'. Any point (say P) on the streamline can be located with the help of a streamwise distance 's' measured from some reference. Also the local normal (from concave towards convex side) coordinate at the point along with the corresponding unit vectors, $\hat{e}_s$ and $\hat{e}_n$ can be readily constructed as shown in fig. Let 'θ' be the local slope and 'R' the local radius of curvature of the streamline at the point P. Show that

   a) $\frac{\partial \hat{e}_s}{\partial t} = +\frac{\partial \theta}{\partial t}\hat{e}_n$, $\frac{\partial \hat{e}_s}{\partial s} = -\hat{e}_n / R$

b) Acceleration of the particle: $\vec{a} = \left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}\right)\hat{e}_s + \left(V\frac{\partial \theta}{\partial t} - \frac{V^2}{R}\right)\hat{e}_n$, where V is the instantaneous speed of the fluid particle at point P on the streamline

c) Simplify the answer obtained in (b) for a steady flow

d) Simplify the answer obtained in (b) for a steady flow with straight streamlines

7. The temperature 'T' is known to vary along the length of a long tunnel as $T = T_0 - \alpha\, e^{-x/L}\, \text{Sin}(2\pi t/\tau)$, where T_0, α, τ, L are constants and x is measured from the tunnel entrance. A fluid particle moves in the tunnel at a velocity given as $\vec{V} = U\hat{i}$, where U is a constant. Obtain the general expression for the time rate of change of temperature experienced by the particle.

8. Consider an inviscid flow between two circular disks of radii 'R' separated by a small gap 'h' as shown in fig.2. After a certain radius 'r_o', the flow can be taken to be radial.
   a) If the flow is steady and incompressible, simplify the continuity equation to show that $v_r = c / r$ where 'c' is a contant for r > 'r_o'.
   b) If the velocity at the exit to the disks is found to be 'V', obtain the value of the constant 'c'.
   c) Using the answers obtained in (a) and (b), estimate the fluid particle acceleration for r > 'r_o'. Is the flow accelerating or retarding.

9. The velocity of a fluid particle along the centerline of a reducing area passage of length 'L' is given as $\vec{V} = U(1 + k\frac{x}{L})\hat{i}$ where 'x' is the distance from the passage entrance and k is a positive constant. Obtain the expression for
   a) the acceleration of the fluid particle along the centerline of the passage
   b) for a particle that enters the passage at t = 0, the location 'x' of the same particle at a later time 't'
   c) Using the answer in (b), the relation governing the time 't_R' the particle spends in the passage.

10. Show that for a parallel incompressible flow with straight streamlines,
    a) The velocity does not vary along the flow direction (Use continuity)
    b) The convective acceleration in such a flow is zero.
    c) If the flow is steady also, the total acceleration is zero.

11. Consider the two-dimensional velocity field $v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$, $v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$. Show that
    a) The velocity field can represent an incompressible flow
    b) The velocity field can represent an inviscid flow past a circular cylinder of radius 'a' with center at origin. (Hint: In an inviscid flow the tangential velocity at a solid-fluid interface would be discontinuous)
    c) At large distances from the cylinder, show that the fluid velocity is directed along the positive x-axis with a magnitude equal to U.
    d) Obtain the general expression for the fluid particle acceleration for such a flow. Use it to find the acceleration on the cylinder surface .i.e. r = a

Problems 12-15: Kinematics of material lines (infinitesimal) in a fluid and vorticity

12. Consider the kinematics of infinitesimal material lines in a general two-dimensional flow i.e $\vec{V} = u(x,y)\hat{i} + v(x,y)\hat{j}$. Show that,
    i) the maximum and minimum instantaneous rotation rates experienced by infinitesimal material lines at any point in the flow is,

$$\omega_{max/min} = \frac{v,_x - u,_y}{2} \pm \frac{1}{2}\sqrt{(v,_x + u,_y)^2 + (v,_y - u,_x)^2} \text{ where } v,_x = \frac{\partial v}{\partial x} \text{ and so on.}$$

Also find the orientation of the material lines that rotate with these angular velocities.

ii) there exists a unique pair of orthogonal material lines at a given point in the flow domain that rotate at the same instantaneous angular velocity. Obtain the orientations of these material lines and their angular velocity.

(iii) The shear strain rate $\dot{\gamma}_{\theta,\theta+\frac{\pi}{2}}$ is zero if one of the lines is oriented at an angle 'θ' given by

$$\tan 2\theta = \frac{v,_x + u,_y}{u,_x - v,_y}$$

(iv) The orientations of the material lines at any point that experience the largest / smallest instantaneous linear strain rates ($\dot{\varepsilon}_\theta$) are given by,

$$\tan 2\theta = \frac{v,_x + u,_y}{u,_x - v,_y}$$

(v) The pair of material lines obtained in (ii), (iii) and (iv) are coincident.

13. Show that the fluid vorticity Ω is always divergence free. Further, for a finite volume in the flow domain, apply the Gauss theorem(s) to show that the nett flux of vorticity- $\oiint_S \Omega \cdot \hat{n} dS$ through the surface 'S' of the volume 'V' is always zero.

14. Consider a steady, incompressible flow with purely circular streamlines, $\vec{V} = v_\theta \hat{\varepsilon}_\theta$.
    a) Show that $v_\theta = f(r,z)$.
    b) If the variations along z are negligible, then show that
        i. for an irrotational flow, $v_\theta = k/r$ where $k = \text{const}$
        ii. for a rotational flow with uniform vorticity, $v_\theta = (\Omega r)/2$

15. Consider a closed vertical cylinder completely filled with fluid. If the cylinder is rotated at a steady speed 'ω', then after some time the fluid inside starts to rotate with the cylinder like a rigid body. Obtain the expression for vorticity in the fluid. (Hint: write the velocity field in cylindrical coordinates and then obtain vorticity)

Problems 16-22: Streamfunction, irrotational flow and potential function

16. Consider the velocity field of Prob. 11.
    a) Obtain the streamfunction ψ(r, θ) for the flow and plot the streamline for ψ = 0.
    b) Is the flow irrotational ? if yes then obtain the potential function ϕ(r, θ) for the flow.

17. Show that streamfunction can be defined for a two-dimensional compressible flow. For such a velocity field described by Cartesian components obtain the relation between velocity components and the streamfunction.

18. Consider the two-dimensional velocity field in polar coordinates $\vec{V} = nr^{n-1}[\cos(n\theta)\hat{\varepsilon}_r - \sin(n\theta)\hat{\varepsilon}_\theta]$.
    a) Is the flow incompressible? If yes, then find the appropriate stream function.
    b) Is the flow irrotational? If yes, then find the appropriate potential function.
    c) For n = 2, obtain the stagnation point(s) in the flow domain and plot the stagnation streamline(s). Interpret the various flow patterns that the streamfunction obtained in (i) for n = 2 can possibly represent.
    d) Show that for 1/2< n< 1, the streamfunction obtained in (i) represents a unidirectional flow past a convex corner of angle π / n OR a unidirectional flow in the concave corner of angle (2-1/n)π

e) Are the flow patterns for 1/2< n< 1 realistic? If no, then identify the cause that makes the flow patterns for n < 1 physically impossible even if the viscosity is ignored. (Hint: Obtain fluid velocities at the sharp corner or origin)

19. Consider the streamfunction $\psi = K\left(x^2 - y^2\right)$. Find the stagnation points in the flow and plot the streamlines in the x-y plane. Also show the direction of the flow on the streamlines in the full x-y plane.

20. An incompressible flow stream function is defined by $\psi = \frac{U}{L^2}\left(3x^2y - y^3\right)$ where U and L are positive constants. Use this streamfunction to find the volume flow rate through the rectangular surfaces AB and AC (fig. 3) having a width 'b' into the plane of figure. Also show the direction of flow through these surfaces.

21. Consider two large parallel plates separated by a small gap 'h' as shown in fig.4. The gap is initially filled by air at rest. The upper plate is imparted a velocity 'U' while the lower plate is kept fixed (fig.4). The air in the gap is set into motion through the combined action of no-slip and viscosity. Assuming steady parallel flow and neglecting variations in z-direction (as the plates are large), $\vec{V} = u(x,y)\hat{i}$.
    a) Show that for an incompressible flow u = u(y) only.
    b) For an incompressible flow, assuming linear variation of velocity 'u', obtain the general expression for the streamfunction ψ.

22. Consider a steady, viscous, incompressible, two-dimensional radial flow through a diverging passage formed by two plates of width 'w' into the paper as shown in fig. 5. Since the flow is radial, $\vec{V} = v_r(r,\theta)\hat{e}_r$.
    a) Using continuity, show that the steady radial velocity can be expressed in the general form $v_r = \frac{f(\theta)}{r}$, where f is some function of θ only.
    b) Using no-slip show that $f(\pm\alpha) = 0$
    c) From the geometry, one can argue that the distribution of the radial velocity must be symmetric about the passage centerline $\theta = 0$. Thus, prove that for the function $f(\theta)$, all odd derivatives at $\theta = 0$ must vanish. (Hint: Expand $f(\theta)$ about $\theta = 0$ in a Taylors series and utilize the fact that on either side of $\theta = 0$, the values of the function must be same). If the volume flow rate through the passage is 'Q', show that $\int_{-\alpha}^{+\alpha} f(\theta)d\theta = \frac{Q}{2\pi w}$

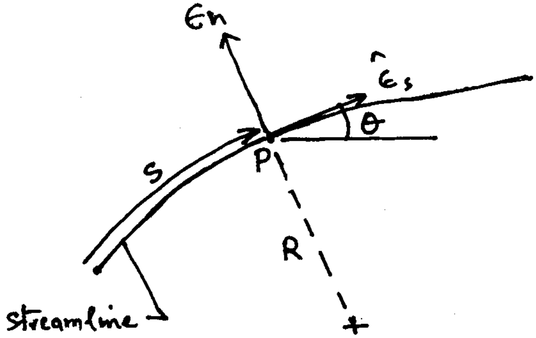

Fig 1

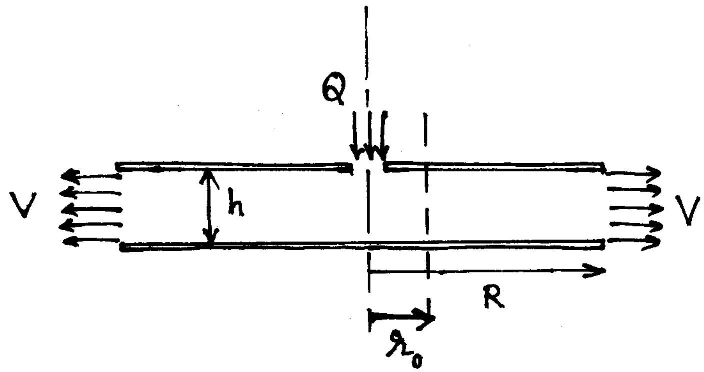

Fig. 2

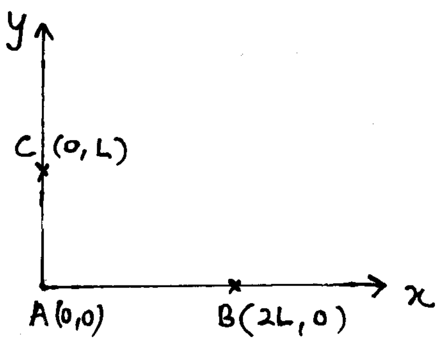

Fig. 3

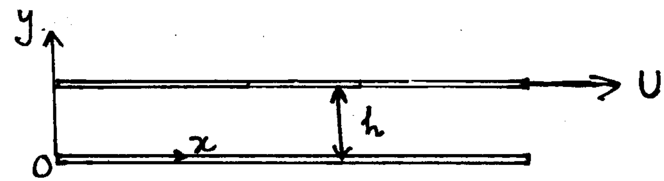

Fig. 4

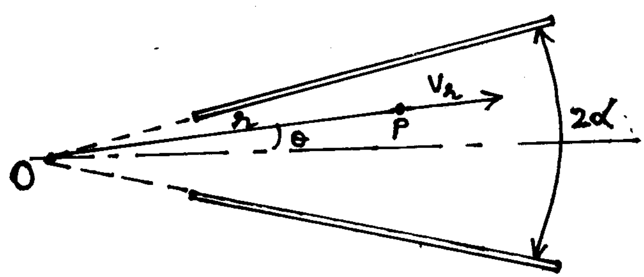

Fig. 5