4.4.1 Applications of Bernoulli's Equation

Bernoulli's equation is very useful in enabling us to understand the behavior of many engineering flows. In this section, we give examples of both steady and unsteady flows of incompressible fluids that demonstrate the application of Bernoulli's equation to the flow of fluids.

Fluid Streams

One of the simplest inviscid fluid flows is that of a stream of fluid (such as water) flowing through a stationary fluid with which it does not mix (such as air). Consider the case of water flowing from a tap, as illustrated in figure 4.1. The water leaves the tap with a speed V_1 as a circular stream of diameter D_1. As it falls, it speeds up and contracts in diameter. Ultimately, the stream becomes so thin that surface tension forces break it up into droplets, but before this happens, the flow can be described by applying Bernoulli's equation 4.14 to the central streamline of the water stream:

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

On the central streamline, the pressure of the water will be the same as that in the atmosphere at the same height z because the radial acceleration of the water stream is negligible. (We also assume here that the surface tension is negligible so that the

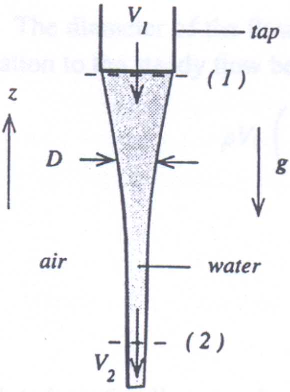

Figure 4.1 Water leaving a tap increases in speed as it falls downward through the stationary air, as described by Bernoulli's equation.

pressure inside the water column is the same as that outside.) The water pressure at 1 and 2 are thus related by the hydrostatic pressure distribution in the atmosphere:

$$p_2 + \rho_a g z_2 = p_1 + \rho_a g z_1$$

where ρ_a is the density of the ambient fluid (air) and is assumed to be constant. Substituting this expression into the previous one and solving for V_2^2 we find:

$$V_2^2 = V_1^2 + 2\left(1 - \frac{\rho_a}{\rho}\right) g(z_1 - z_2)$$

Bernoulli's equation demonstrates how the water velocity increases as the distance from the tap $z_1 - z_2$ increases. Because the density ρ_a of air is only about 10^{-3} times that of water, $\rho_a/\rho \ll 1$, and we may write Bernoulli's equation for this case as:

$$V_2^2 = V_1^2 + 2g(z_1 - z_2)$$

This is equivalent to assuming that the ambient air pressure is a constant and thus $p_1 = p_2$. However, this approximation would not be justifiable if the ambient fluid density were the same order of magnitude as that of the moving fluid stream, as would be the case of an oil stream injected into water or heated air rising into a colder atmosphere.[5] In such instances, the vertical acceleration of the stream is smaller than g, namely, $(1 - \rho_a/\rho)g$.

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[5] In the latter case, $1 - \rho_a/\rho$ is negative and $z_2 > z_1$, *i.e.*, the fluid stream flows upward.

The diameter of the flowing stream may next be found by applying mass conservation to the steady flow between 1 and 2:

$$\rho V_2\left(\frac{\pi D_2^2}{4}\right) = \rho V_1\left(\frac{\pi D_1^2}{4}\right)$$

$$D_2 = D_1\sqrt{\frac{V_1}{V_2}}$$

$$= D_1\left(\frac{V_1^2}{V_1^2 + 2g(z_1 - z_2)}\right)^{1/4}$$

Note how the diameter decreases quite slowly with increasing $z_1 - z_2$.

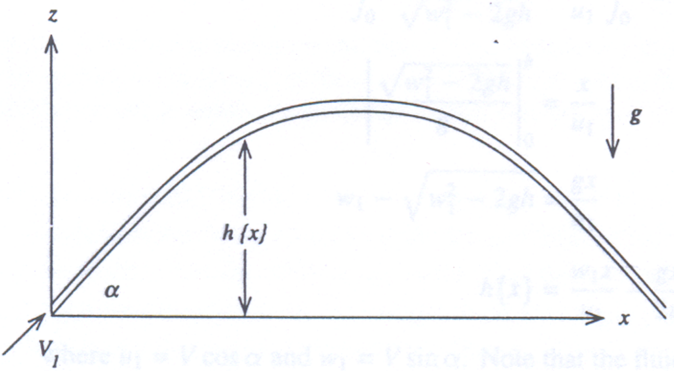

Figure E 4.4

Example 4.4

A fire hose directs a stream of water of velocity V_1 at an angle α above the horizontal, as illustrated in figure E 4.4. The stream rises initially but then eventually falls to the ground.

(a) Derive an expression for the height $h\{x\}$ of the stream above the hose nozzle as a function of the horizontal distance x from the nozzle. (b) Calculate the maximum value of h if $V_1 = 50\,m/s$ and $\alpha = 45°$.

Solution

(a) In terms of the vertical and horizontal components of **V**, w and u, Bernoulli's equation 4.14 is:

$$\frac{u_1^2 + w_1^2}{2} + gz_1 = \frac{u^2 + w^2}{2} + gz$$

assuming constant atmospheric pressure. The horizontal component of Euler's equation 4.6 is $Du/Dt = 0$; therefore u does not change along the fluid stream, and $u = u_1$. Solving Bernoulli's equation for w:

$$w = \sqrt{w_1^2 - 2gh}$$

where $h = z - z_1$. The slope of the fluid stream, dh/dx, must equal the ratio w/u:

$$\frac{dh}{dx} = \frac{w}{u} = \frac{\sqrt{w_1^2 - 2gh}}{u_1}$$

Integrating this differential equation for h from $x = 0$ to x:

$$\int_0^h \frac{dh}{\sqrt{w_1^2 - 2gh}} = \frac{1}{u_1} \int_0^x dx$$

$$-\left| \frac{\sqrt{w_1^2 - 2gh}}{g} \right|_0^h = \frac{x}{u_1}$$

$$w_1 - \sqrt{w_1^2 - 2gh} = \frac{gx}{u_1}$$

$$h\{x\} = \frac{w_1 x}{u_1} - \frac{gx^2}{2u_1^2}$$

where $u_1 = V \cos \alpha$ and $w_1 = V \sin \alpha$. Note that the fluid stream reaches the ground at $x = 2u_1 w_1/g$.

(b) The maximum value of h occurs when $dh/dx = 0$. From the differential equation for h, this occurs when $h = w_1^2/2g$. Calculating h:

$$h = \frac{(50\, m/s \times \sin 45°)^2}{2 \times 9.807\, m/s^2} = 63.73\, m$$

Flow Through an Orifice

Fluids may flow into or out of tanks or chambers through an opening that limits the rate of flow. Inviscid flow through such orifices moves at a speed that depends upon the pressure difference between the fluid inside and outside of the vessel. In figure 4.2, we show some streamlines of the steady inviscid incompressible flow of fluid passing through an orifice of area A from a chamber (on the left), having a pressure p_1, to a

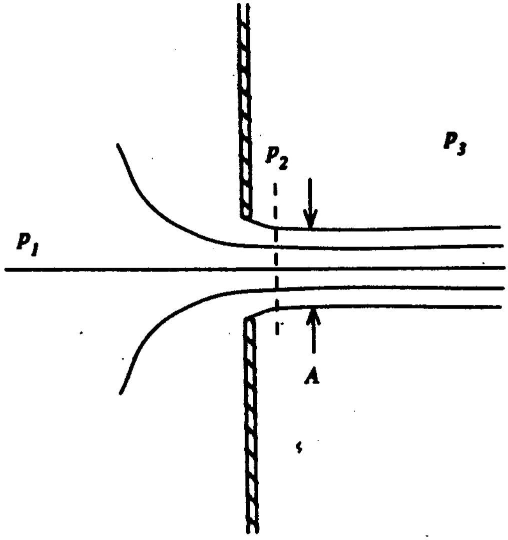

Figure 4.2 The inviscid incompressible flow of fluid through an orifice from a higher pressure chamber on the left to a lower pressure chamber on the right follows the sketched streamlines. The orifice pressure p_2 equals the receiving chamber pressure p_3.

chamber (on the right), having a lower pressure p_3. The pressure p_2 at the exit of the orifice is less than that in the chamber at the left so that the fluid accelerates as it flows toward the orifice. On the other hand, as the fluid emerges into the chamber at the right, it continues moving from left to right at an unchanging speed because there is no further change in pressure between the orifice exit (pressure p_2) and the stationary fluid in the chamber at the right (pressure p_3).[6] Writing Bernoulli's equation 4.14 for steady flow along the central streamline of the flow between a point far from the orifice, where $V_1 = 0$, and a point at the exit of the orifice, we find:

$$\frac{V_2^2}{2} + \frac{p_2}{\rho} + g z_2 = \frac{V_1^2}{2} + \frac{p_1}{\rho} + g z_1$$

$$\frac{V_2^2}{2} = \frac{p_1}{\rho} - \frac{p_2}{\rho}$$

$$V_2 = \sqrt{\frac{2(p_1 - p_3)}{\rho}} \tag{4.15}$$

where we have used the fact that $p_2 = p_3$. The volume flow rate Q and mass flow rate

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[6] In a supersonic compressible flow, the pressure p_2 can be greater than p_3.

$\dot{m}$ of fluid through the orifice become:[7]

$$Q = A\sqrt{\frac{2(p_1 - p_3)}{\rho}}$$

$$\dot{m} = A\sqrt{2\rho(p_1 - p_3)} \tag{4.16}$$

The constant-speed flow into the receiving chamber does not continue indefinitely in the downstream direction. Viscous forces between the moving and stationary fluid in the chamber cause the fluid jet to become unsteady, break up into eddies and dissipate its kinetic energy. Needless to say, this process is not an inviscid flow, and therefore Bernoulli's equation cannot be applied to it. Nevertheless, the flow upstream of the orifice can be accurately represented as inviscid.[8]

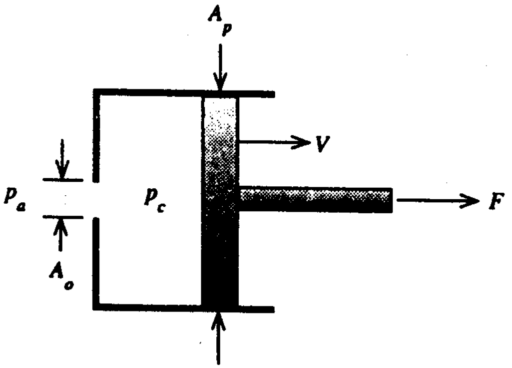

Figure E 4.5

Example 4.5

A circular cylinder of area A_p is fitted with a piston that is retracted at a constant speed V by a force F, as shown in figure E 4.5. As it is retracted, atmospheric air

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[7] Because of the viscous shear stress exerted on the fluid by the walls of the orifice, the actual volume and mass flow rates are slightly less than those of equation 4.16 for inviscid flow.

[8] It is possible to find a solution to Euler's equations yielding a flow with streamlines that are symmetric about the plane of the orifice. Such flows are not observed in practice, the fluid flow following the streamlines sketched in figure 4.2 instead. This asymmetry is caused by the viscous flow effects near the surface of the orifice that are not taken into account by Euler's equation.

of density ρ flows through an orifice of area A_o into the cylinder where the air has a lower pressure p_c than atmospheric pressure p_a.

(a) Assuming incompressible flow, derive expressions for the pressure difference $p_a - p_c$, the force F and the power P required to move the piston at the speed V.
(b) Assuming that figure E4.5 is a reasonable model for the flow of air into the human lung, calculate the power required to inhale 0.5 l of air in 2 s through a trachea "orifice" of area $A_o = 1.0\,cm^2$ when the air density $\rho = 1.225 kg/m^3$.

Solution

(a) By mass conservation, the volumetric flow rate into the cylinder given by equation 4.16 must equal the rate at which the cylinder volume is increasing, $A_p V$:

$$A_0 \sqrt{\frac{2(p_a - p_c)}{\rho}} = A_p V$$

$$p_a - p_c = \frac{\rho}{2}\left(\frac{A_p V}{A_o}\right)^2$$

The force F applied to the piston is equal to the pressure difference $p_a - p_c$ across the piston faces times the piston area A_p:

$$F = (p_a - p_c)A = \frac{\rho}{2}\left(\frac{A_p V}{A_o}\right)^2 A_p$$

and the power P is the product of the force F times the piston velocity V:

$$P = FV = \frac{\rho}{2}\left(\frac{(A_p V)^3}{A_o^2}\right)$$

(b) For the human lung, the volumetric flow rate $A_p V = 0.5\,l/2\,s = 2.5E(-4)\,m^3/s$. Thus the power P expended in inhaling (or exhaling) is:

$$P = \frac{1.225\,kg/m^3 \times (2.5E(-4)\,m^3/s)^3}{2 \times (1.0E(-4)\,m^2)^2} = 9.57E(-4)W$$

Flow from Pressurized Tanks

Fluids are often stored in containers under pressure greater than atmospheric: domestic water in home storage tanks, propane in home or camp fuel systems, compressed air in workshops, steam in power plant boilers, etc. Flow out of these storage vessels is usually regulated by valves delivering the fluid at a pressure lower than that at which the fluid is stored. Sometimes the fluid leaks to the atmosphere through a small hole

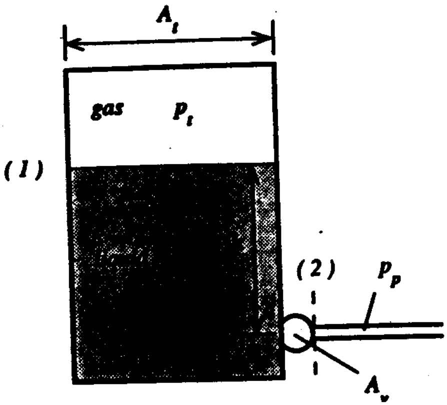

Figure 4.3 A tank holding liquid under pressure from a gas supplies a flow through a valve to a pipe of liquid at a lower pressure.

or crack. The rate of discharge of the fluid under these conditions may be found by applying Bernoulli's equation to the particular flow in question.

Consider a liquid stored in a tank under a pressure p_t maintained by a layer of gas at the top of the tank, as shown in Figure 4.3. A valve at the bottom of the tank regulates the outflow of liquid to a pipe conveying away the liquid, where the pressure is p_p. The speed of flow through the valve, V_v, may be found by applying Bernoulli's equation 4.14 to a streamline connecting the liquid/gas interface in the tank to the valve orifice:

$$\frac{V_2^2}{2} + \frac{p_2}{\rho} + g z_2 = \frac{V_1^2}{2} + \frac{p_1}{\rho} + g z_1$$

Usually, the area A_t of the free surface in the tank is much greater than the orifice area A_v of the valve. By mass conservation, the speed V_1 of the gas/liquid interface is much smaller than that of the liquid flowing through the valve:

$$A_t V_1 = A_v V_2$$

$$\frac{V_1}{V_2} = \frac{A_v}{A_t} \ll 1$$

Neglecting V_1 compared to V_2, Bernoulli's equation becomes:

$$\frac{p_t}{\rho} + g z_1 = \frac{V_v^2}{2} + \frac{p_p}{\rho} + g z_2$$

$$V_v = \sqrt{2gh + \frac{2(p_t - p_p)}{\rho}}$$

where we have replaced p_1 by p_t and $z_1 - z_2$ by the height h of the liquid/gas interface above the valve location. Note that the flow velocity through the valve is determined by the difference in the hydrostatic pressure $p_t + \rho g h$ at the level of the valve and the pressure p_p in the pipe downstream.

Example 4.6

An oil storage tank having a diameter $D_t = 30\,m$ is filled with oil to a depth $H = 5\,m$. The space above the oil is vented to the atmosphere. A pipe of inside diameter $D_p = 5\,cm$ leading from the base of the tank is accidentally broken, allowing the oil to spill onto the ground. Calculate how long it will take for the oil to drain completely from the tank.

Solution

Applying Bernoulli's equation, while noting that the pressure of the oil leaving the pipe and the pressure at the surface of the oil in the tank are both equal to atmospheric pressure and assuming that the velocity dh/dt at the oil surface is negligible, the speed V_p of flow out of the pipe is:

$$V_p = \sqrt{2gh\{t\}}$$

where $h\{t\}$ is the height of the oil surface above the bottom of the tank. By mass conservation, the rate of change of height dh/dt is related to V_p by:

$$\frac{d}{dt}\left(\frac{\pi D_t^2}{4}h\right) = -\frac{\pi D_p^2}{4}V_p$$

$$\frac{dh}{dt} = -\left(\frac{D_p}{D_t}\right)^2 V_p = -\left(\frac{D_p}{D_t}\right)^2 \sqrt{2gh}$$

Integrating this differential equation for h from $h = H$ to $h = 0$, we find:

$$\int_H^0 \frac{dh}{\sqrt{h}} = -\int_0^t \sqrt{2g}\left(\frac{D_p}{D_t}\right)^2 dt'$$

$$\left|2\sqrt{h}\right|_H^0 = -\sqrt{2g}\left(\frac{D_p}{D_t}\right)^2 |t'|_0^t$$

$$t = \left(\frac{D_t}{D_p}\right)^2 \sqrt{\frac{2H}{g}}$$

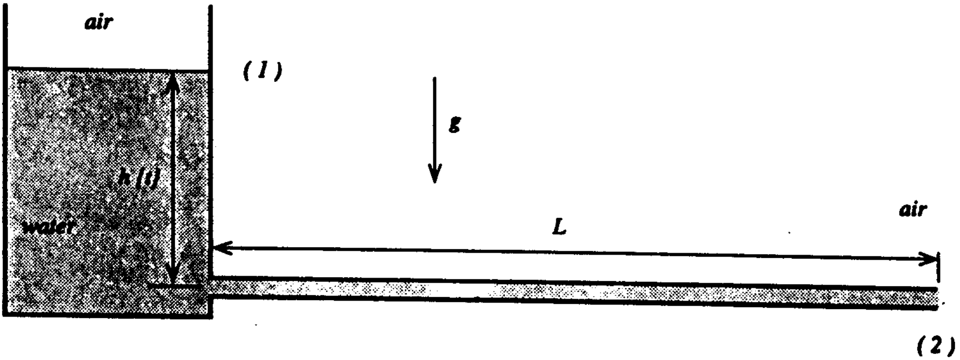

Figure 4.4 Water filling a tank and a long pipe is accelerated from rest when the end of the pipe is suddenly opened to the atmosphere.

$$= \left(\frac{30}{0.05}\right)^2 \sqrt{\frac{2 \times 5\,m}{9.807\,m/s^2}} = 3.635E(5)\,s = 101.0\,hr$$

Unsteady Flow

The examples considered so far are steady flows. When flows are started from rest or altered with time, the unsteady term in Bernoulli's equation 4.13 may be significant.

For example, consider the case of the starting of flow in a long pipe of length L. Supplied with water from a tank and discharging into the atmosphere through a valve at its end, the pipe is suddenly opened wide at time $t = 0$, as shown in figure 4.4. Writing Bernoulli's equation between the air/water interface in the tank and the exit of the pipe,

$$\int_1^2 \frac{\partial V}{\partial t} ds + \frac{V_2^2}{2} + \frac{p_2}{\rho} + g\,z_2 = \frac{V_1^2}{2} + \frac{p_1}{\rho} + g\,z_1$$

$$\int_1^2 \frac{\partial V}{\partial t} ds = g\,h - \frac{V_2^2}{2}$$

since $p_1 = p_2 = p_{atm}$. The only significant contribution to the integral comes from the fluid in the pipe whose velocity $V_2\{t\}$ varies with time. Evaluating this integral over the length L of the pipe,

$$L\frac{dV_2}{dt} = g\,h - \frac{V_2^2}{2}$$

$$\frac{dV_2}{2gh - V_2^2} = \frac{dt}{2L}$$

Integrating from $V_2 = 0$ at $t = 0$,

$$\tanh^{-1}\left\{\frac{V_2}{\sqrt{2gh}}\right\} = \frac{\sqrt{gh}\,t}{\sqrt{2}L}$$

$$V_2 = \sqrt{2gh}\tanh\left\{\frac{\sqrt{gh}\,t}{\sqrt{2}L}\right\}$$

At the very beginning, at times t that are very small compared with $L/\sqrt{gh}$ so that $\tanh x = x$, the velocity $V_2 = ght/L$ in agreement with the differential equation when $V_2 = 0$. During this period, the fluid acceleration is gh/L. For very long times $t \gg L/\sqrt{gh}$, V_2 approaches its steady value of $\sqrt{2gh}$. The magnitude of the time required to reach a steady flow is the time $L/\sqrt{2gh}$ required for a fluid particle to move the length of the pipe at the steady flow speed.

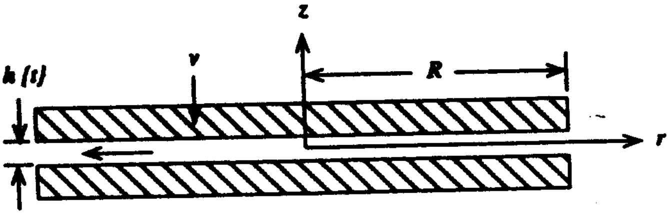

Figure E 4.7

Example 4.7

A pair of circular discs of radius R encloses a thin layer of liquid between their parallel faces, the thickness $h\{t\}$ of this layer decreasing with time at a constant speed v, *i.e.*, $dh/dt = -v$ (see figure E 4.7). The liquid is expelled radially outward at a speed V_r that varies with radius r and time t, but not with axial distance z. The liquid pressure at the exit radius R is atmospheric pressure p_a. Assuming an incompressible flow, derive expressions for (a) the radial velocity V_r, (b) the pressure p and (c) the force F that must be applied to each disc to maintain the given motion.

Solution

(a) Applying mass conservation equation 3.10 to a control volume enclosing the liquid layer out to a radius r,[9]

$$\frac{d}{dt}(\pi r^2 h) = -2\pi r h V_r$$

$$V_r = -\frac{r}{2h}\frac{dh}{dt} = \frac{vr}{2h}$$

(b) Applying Bernoulli's equation 4.14 to a horizontal streamline ($z = 0$) between the axis ($r = 0$) and a point at a radius r:

$$\int_0^r \frac{\partial V_r}{\partial t} ds + \frac{V_r^2}{2} + \frac{p}{\rho} - \frac{p_0}{\rho} = 0$$

where p_0 is the pressure on the axis. Noting that:

$$\frac{\partial V_r}{\partial t} = \frac{d}{dt}\left(\frac{vr}{2h}\right) = \frac{vr}{2}\left(-\frac{1}{h^2}\frac{dh}{dt}\right) = \frac{v^2 r}{2h^2}$$

then:

$$\int_0^r \frac{\partial V_r}{\partial t} dr = \frac{v^2 r^2}{4h^2}$$

Substituting this into Bernoulli's equation, we find:

$$p = p_0 - \rho\frac{V_r^2}{2} - \int_0^r \frac{\partial V_r}{\partial t} dr = p_0 - \frac{3}{8}\left(\frac{vr}{h}\right)^2 \rho$$

If we choose p_0 so that $p = p_a$ at $r = R$, then:

$$p = p_a + \frac{3v^2}{8h^2}(R^2 - r^2)\rho$$

(c) The force F is the integral of $p - p_a$ over the area of a disc:

$$F = \int_0^R \frac{3v^2}{8h^2}(R^2 - r^2)\rho(2\pi r)dr = \frac{3\pi}{16}\left(\frac{vR^2}{h}\right)^2 \rho$$

Metering Flow

It is sometimes desirable to be able to measure the rate at which fluids are flowing through a pipe or duct. By forcing the fluid to flow through a constriction inside the pipe or duct while measuring the pressure change accompanying this squeezing down

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[9] This value of V_r satisfies the mass conservation equation 3.15 in cylindrical coordinates when $V_z = -vz/h$.

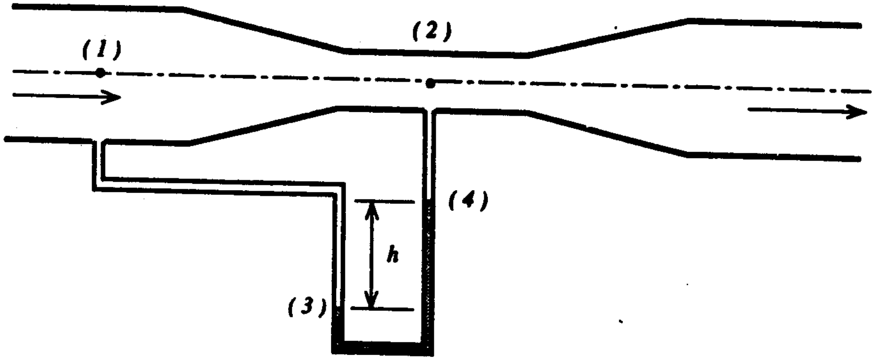

Figure 4.5 A venturi meter constricts the flow of a fluid so as to create a pressure difference that is related to the volumetric and mass flow rates. The manometer shown is used to measure the pressure difference.

of the flow, it is possible to compute the volumetric and mass flow rates of the flow.

A *venturi meter* is sketched in figure 4.5. It consists of a section of pipe that gradually reduces the flow area from A_1 upstream to A_2 at the point of minimum area. If the pressures are measured at these locations, then Bernoulli's equation 4.14 applied to the central streamline of the flow may be solved for the upstream flow speed in terms of the pressure change $p_1 - p_2$ and the velocity ratio V_2/V_1:

$$\frac{V_2^2}{2} + \frac{p_2}{\rho} + g z_2 = \frac{V_1^2}{2} + \frac{p_1}{\rho} + g z_1$$

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left((V_2/V_1)^2 - 1\right)}}$$

since $z_1 = z_2$. Mass conservation requires that $\rho V_1 A_1 = \rho V_2 A_2$, with the result that:

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left((A_1/A_2)^2 - 1\right)}}; \qquad Q = A_1\sqrt{\frac{2(p_1 - p_2)}{\rho\left((A_1/A_2)^2 - 1\right)}};$$

$$\dot{m} = A_1\sqrt{\frac{2\rho(p_1 - p_2)}{\left((A_1/A_2)^2 - 1\right)}} \tag{4.17}$$

The pressure difference may be measured by use of a manometer connected to the venturi meter as shown in figure 4.5 and filled with a liquid whose density ρ_m is greater than that of the working fluid. Because the moving fluid is not accelerating

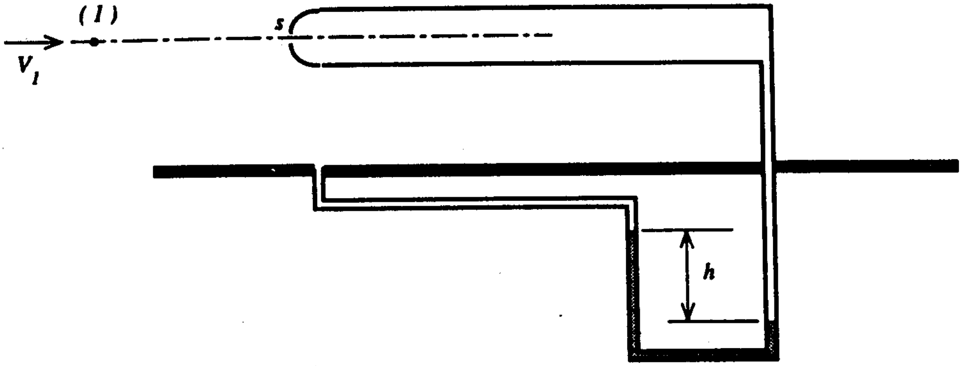

Figure 4.6 A pitot tube measures the flow speed directly upwind by means of the difference in pressure inside and outside the tube.

at 1 and 2, the pressure distribution is hydrostatic between 1 – 3 and 2 – 4:

$$p_1 + \rho g z_1 = p_3 + \rho g z_3$$

$$p_2 + \rho g z_2 = p_4 + \rho g z_4$$

By subtraction,

$$p_1 - p_2 = p_3 - p_4 + \rho g(z_3 - z_4)$$

since $z_1 = z_2$. For the manometer fluid,

$$p_3 + \rho_m g z_3 = p_4 + \rho_m g z_4$$

$$p_3 - p_4 = \rho_m g(z_4 - z_3)$$

Thus the pressure difference $p_1 - p_2$ becomes:

$$p_1 - p_2 = (\rho_m - \rho)gh$$

An instrument that can measure the velocity at a point in the flow, called the *pitot tube*, is illustrated in figure 4.6. It consists of a hollow tube aligned with the oncoming flow and closed at one end with a rounded plug containing a tiny hole at the tube centerline. The fluid inside the pitot tube is stationary, while the oncoming fluid flows around it. A fluid particle moving along the streamline that is coincident with the pitot tube axis comes to rest as it approaches the tip of the pitot tube (designated by s) because it must split and pass on either side of the tube. As it comes to rest momentarily, its pressure rises to a value p_s, called the *stagnation pressure*, that is related to the upstream flow speed V_1 by Bernoulli's equation:

$$\frac{V_1^2}{2} + \frac{p_1}{\rho} + g z_1 = \frac{V_s^2}{2} + \frac{p_s}{\rho} + g z_s$$

$$p_s = p_1 + \rho \frac{V_1^2}{2}$$

$$V_1 = \sqrt{\frac{2(p_s - p_1)}{\rho}}$$

because $V_s = 0$ and $z_1 = z_s$. The pressure of the stationary fluid inside the pitot tube equals the stagnation pressure of the external flow, with which it is in contact through the tiny hole at the *stagnation point s* of the tube. The pressure difference $p_s - p_1$ may be measured by a manometer arranged as shown in Figure 4.6. Following the analysis given above for the venturi meter, the pressure difference is $(\rho_m - \rho)gh$ and the flow speed of the oncoming flow is:

$$V_1 = \sqrt{\frac{2(\rho_m - \rho)gh}{\rho}}$$

To measure the volumetric flow rate Q in a pipe or duct, the pitot tube can be moved to all locations within the cross-section of the flow, and the velocity measurements integrated:

$$Q = \int \int V\{x, y\}\, dx\, dy$$

This type of measurement is often necessary when the fluid velocity varies noticeable within a pipe or duct.

4.5 Euler's Equation in Streamline Coordinates

It is sometimes convenient to choose an orthogonal coordinate system whose local directions are defined by a streamline of the flow. Called streamline coordinates, the three mutually perpendicular directions at a point in the flow are determined by the directions of the tangent, normal and binormal to the streamline passing through the point. This is illustrated in figure 4.7 for a point P on a streamline, where the unit vectors lying in these three directions are labeled $\mathbf{i}_s$, $\mathbf{i}_n$ and $\mathbf{i}_b$, respectively. The unit normal $\mathbf{i}_n$ points in the direction of the center of curvature O of the streamline at the point P, the distance OP being the radius of curvature R. To develop Euler's equation for this coordinate system, we will embed a cylindrical coordinate system with center