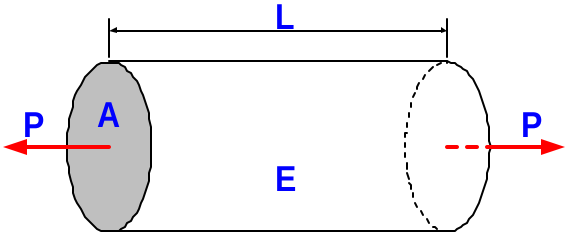
Finite Element Analysis

Chap 1. Introduction

1-1. 1-D Stiffness (Displacement) Method

From an axial deformation of the bar (as shown in the left figure), the displacement due to the load is:



$$u = \frac{PL}{AE}$$

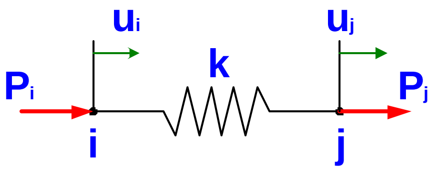
$$P = ku$$

where $k = \dfrac{AE}{L}$

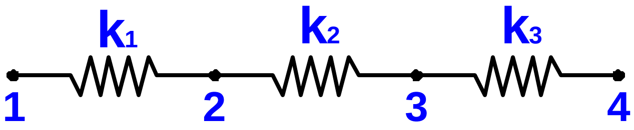
Since we consider the bar has a stiffness—k, springs will be the fundamental model objects. In a discrete model, there is a physical approach as shown in the following figure.

$$\underline{P} = \underline{k}\underline{u}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



In an assemble structures of 3 springs, it is shown in the following figure. The equation will be:

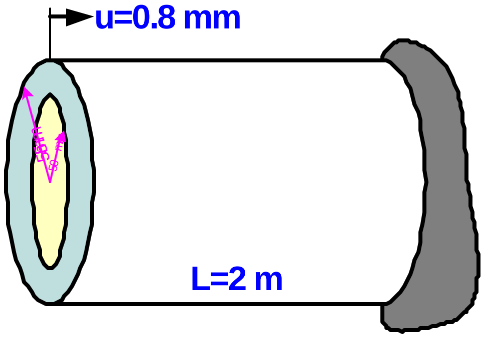


$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Example: Find the load which compresses the composite cylindrical bar as shown in the figure. The displacement of the bar is 0.8 mm, E_{core}=200 Gpa, E_{hollow}=100 Gpa.

Solution:

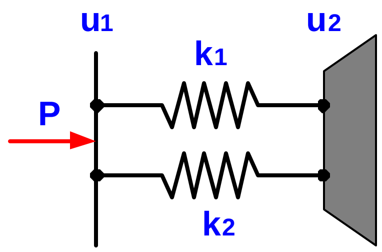
$$k_1 = 196\ MPa$$
$$k_2 = 43.2\ MPa$$



$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -(k_1 + k_2) \\ -(k_1 + k_2) & k_1 + k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

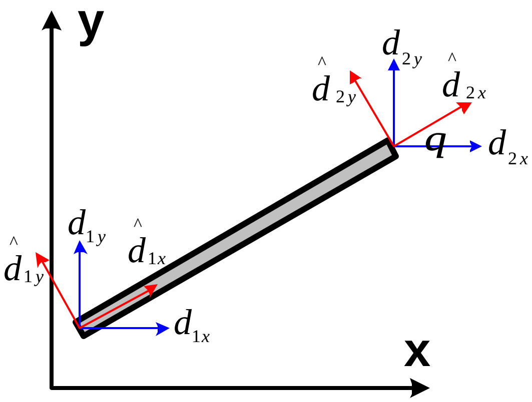
So, we get:

P_1=192 kN, and P_2=-192 kN



1-2. 1-D Stiffness (Displacement) Method

The transformation between global and local components is:



$$\begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{bmatrix}$$

We also can write:

$$\hat{\underline{d}} = \underline{T}\,\underline{d}$$

After both sides multiply stiffness matrix, we get:

$$\hat{\underline{f}} = \underline{T}\,\underline{f} = \hat{\underline{k}}\,\underline{T}\,\underline{d}$$

$$\underline{f} = \underline{T}^{-1}\hat{\underline{k}}\,\underline{T}\,\underline{d}, \qquad \underline{T}^{-1} = \underline{T}^{T}$$

$$\underline{f} = \underline{T}^T \hat{\underline{k}}\, \underline{T}\, \underline{d}$$

$$\underline{k} = \underline{T}^T \hat{\underline{k}}\, \underline{T}$$

$$\underline{k} = \begin{bmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & \cos q & \sin q \\ 0 & 0 & -\sin q & \cos q \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & \cos q & \sin q \\ 0 & 0 & -\sin q & \cos q \end{bmatrix}$$

$$\underline{k} = \frac{AE}{L} \begin{bmatrix} \cos^2 q & \cos q \sin q & -\cos^2 q & -\cos q \sin q \\ \cos q \sin q & \sin^2 q & -\cos q \sin q & -\sin^2 q \\ -\cos^2 q & -\cos q \sin q & \cos^2 q & \cos q \sin q \\ -\cos q \sin q & -\sin^2 q & \cos q \sin q & \sin^2 q \end{bmatrix}$$

The stiffness matrix for a bar in 3-D space is:

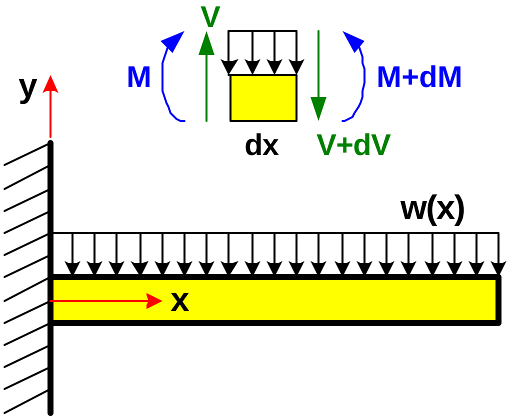
$$\underline{k} = \frac{AE}{L} \begin{bmatrix} \cos^2 q_x & \cos q_x \cos q_y & \cos q_x \cos q_z & -\cos^2 q_x & -\cos q_x \cos q_y & -\cos q_x \cos q_z \\ \cos q_x \cos q_y & \cos^2 q_y & \cos q_y \cos q_z & -\cos q_x \cos q_y & -\cos^2 q_y & -\cos q_y \cos q_z \\ \cos q_x \cos q_z & \cos q_y \cos q_z & \cos^2 q_z & -\cos q_x \cos q_z & -\cos q_y \cos q_z & -\cos^2 q_z \\ -\cos^2 q_x & -\cos q_x \cos q_y & -\cos q_x \cos q_z & \cos^2 q_x & \cos q_x \cos q_y & \cos q_x \cos q_z \\ -\cos q_x \cos q_y & -\cos^2 q_y & -\cos q_y \cos q_z & \cos q_x \cos q_y & \cos^2 q_y & \cos q_y \cos q_z \\ -\cos q_x \cos q_z & -\cos q_y \cos q_z & -\cos^2 q_z & \cos q_x \cos q_z & \cos q_y \cos q_z & \cos^2 q_z \end{bmatrix}$$

1.3. Development of Beam Equations

Based on the force and moment equilibrium of a differential element of the beam, we have:

$$-wdx + dV = 0$$

$$Vdx + dM = 0$$



The deflection of beam is $v(x)$, and the small slop of the curvature—$q = \dfrac{dv}{dx}$. So, the curvature of the beam is:

$$\frac{1}{r} = \frac{d^2 v}{dx^2} = \frac{M}{EI}$$

The weight function will be:

$$EI\frac{d^4v}{dx^4} = -w(x)$$

Assume the transverse displacement variation through the element length to be:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

The nodal degrees of freedom is shown as follows:

$$v(0) = d_{1y} = a_4$$
$$\frac{dv(0)}{dx} = f_1 = a_3$$
$$v(L) = d_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$$
$$\frac{dv(L)}{dx} = f_2 = 3a_1L^2 + 2a_2L + a_3$$

After solving a_1, a_2, a_3, a_4 , we get:

$$v = [\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(f_1 + f_2)]x^3 + [-\frac{3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2f_1 + f_2)]x^2 + f_1x + d_{1y}$$

In matrix, the equation can be expressed as:

$$v = \underline{N}\,\underline{d}$$

$$v = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ f_1 \\ d_{2y} \\ f_2 \end{Bmatrix}$$

where

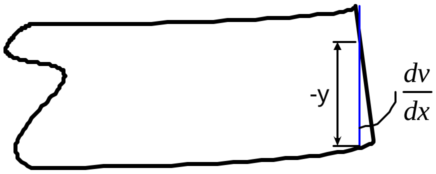
$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3), \qquad N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)$$
$$N_3 = \frac{1}{L^3}(-2x^3 + 3x^2L), \qquad N_4 = \frac{1}{L^3}(x^3L - x^2L^2)$$

In the axial strain/displacement relationship, we have:

$$e_x(x, y) = \frac{\partial u}{\partial x}$$

where u is the axial displacement function

$$u = -y\frac{dv}{dx}$$



$$e_x(x,y) = -y\frac{d^2v}{dx^2}$$

From beam theory, we get

$$m(x) = EI\frac{d^2v}{dx^2} \qquad\qquad V = EI\frac{d^3v}{d^3x}$$

After the boundary conditions are substituted into the above two equations, we have:

$$f_{1y} = V = EI\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}(12d_{1y} + 6Lf_1 - 12d_{2y} + 6Lf_2)$$

$$m_1 = -m = EI\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(6Ld_{1y} + 4L^2f_1 - 6Ld_{2y} + 2L^2f_2)$$

$$f_{2y} = -V = EI\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12d_{1y} - 6Lf_1 + 12d_{2y} - 6Lf_2)$$

$$m_2 = -m = EI\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(6Ld_{1y} + 2L^2f_1 - 6Ld_{2y} + 4L^2f_2)$$

Those equations can be expressed in matrix form:

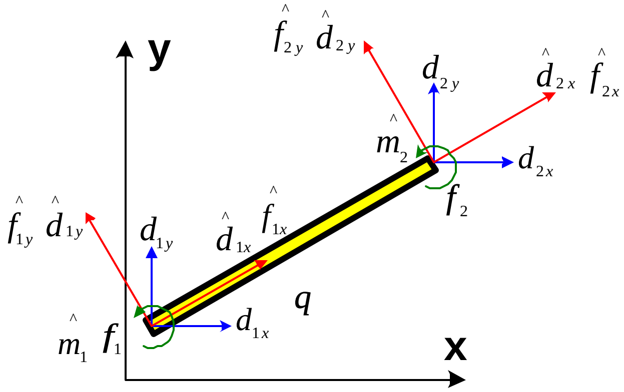
$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ f_1 \\ d_{2y} \\ f_2 \end{Bmatrix}$$

$$\text{where } k = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Note: in this section, the notations are indicating in local coordinate only.

1.4. Development of Plate Equations

In 2-D plane theory, the local forces and moments will be considered as shown in the following figure. The transformation of local and global coordinates is shown below.



$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \\ \hat{f}_{1} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{f}_{2} \end{Bmatrix} = \begin{bmatrix} \cos q & \sin q & 0 & 0 & 0 & 0 \\ -\sin q & \cos q & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos q & \sin q & 0 \\ 0 & 0 & 0 & -\sin q & \cos q & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ f_{1} \\ d_{2x} \\ d_{2y} \\ f_{2} \end{Bmatrix}$$

In the local stiffness matrix is the combination of axial effect and shear with bending moment effects. The expression is shown below:

$$\underline{\hat{k}} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

$$\underset{\sim}{\hat{k}} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 0 & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^3} & 2\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}$$

$$\underline{k} = \underline{T}^T \, \underline{\hat{k}} \, \underline{T}$$

1-5. Gauss-Jordan Method of Matrix Inversion

$$\begin{array}{cccc}
\text{Rows} & \text{Matrix } \mathbf{A} & \text{Matrix } \mathbf{I} & \text{Row Operation} \\
\begin{array}{c} <1> \\ <2> \\ <3> \end{array} & \begin{bmatrix} 2 & 1 & -2 \\ -2 & 3 & 3 \\ 4 & 2 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \\
\begin{array}{c} <4> \\ <5> \\ <6> \end{array} & \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} & \begin{array}{c} \frac{<1>}{2} \\ <2>+<4>\times 2 \\ <3>-<4>\times 4 \end{array} \\
\begin{array}{c} <7> \\ <8> \\ <9> \end{array} & \begin{bmatrix} 0 & 1 & \frac{1}{4} \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -2 & 0 & 1 \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix} & \begin{array}{c} \frac{<5>}{4} \\ <6>+<7>\times 0 \\ \frac{<8>}{3} \end{array}
\end{array}$$

$$\begin{array}{cccc}
\text{Rows} & \text{Matrix } \mathbf{U} & \text{Matrix } \mathbf{B} & \text{Row Operation} \\
\begin{array}{c} <4> \\ <7> \\ <9> \end{array} & \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix} & \\
\begin{array}{c} <10> \\ <11> \\ <12> \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{5}{12} & \frac{1}{4} & -\frac{1}{12} \\ \frac{7}{24} & -\frac{1}{8} & \frac{1}{24} \\ -\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \end{bmatrix} & \begin{array}{c} <7>-\frac{<9>}{4} \\ <4>-\frac{<10>}{2} \\ <11>+<9> \end{array}
\end{array}$$

Rows	Matrix **I**	Matrix $\mathbf{A}^{-1}$	Row Operation
$<12>$ $<10>$ $<9>$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{3}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{5}{12} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$	