

# Dynamics

## Chapter 2. Particles Kinematics

### 2-1 Cartesian Coordinates:

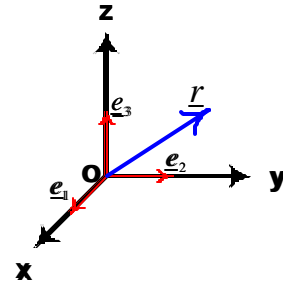
$\underline{e}_1, \underline{e}_2, \underline{e}_3$  are unit vectors. The orthogonal basis of unit vectors:

$$\underline{e}_1 \cdot \underline{e}_2 = \underline{e}_1 \cdot \underline{e}_3 = \underline{e}_2 \cdot \underline{e}_3 = 0$$

$$\underline{e}_1 \cdot \underline{e}_1 = \underline{e}_2 \cdot \underline{e}_2 = \underline{e}_3 \cdot \underline{e}_3 = 1$$

$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3, \underline{e}_2 \times \underline{e}_3 = \underline{e}_1, \underline{e}_3 \times \underline{e}_1 = \underline{e}_2$$

$$\underline{e}_2 \times \underline{e}_1 = -\underline{e}_3, \underline{e}_3 \times \underline{e}_2 = -\underline{e}_1, \underline{e}_1 \times \underline{e}_3 = -\underline{e}_2$$



There is a vector— $\underline{r}$  which is shown in follows.

$$\underline{r} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$$

$$\dot{\underline{r}} = x\dot{\underline{e}}_1 + y\dot{\underline{e}}_2 + z\dot{\underline{e}}_3 + \dot{x}\underline{e}_1 + \dot{y}\underline{e}_2 + \dot{z}\underline{e}_3$$

$$\text{note: } \dot{\underline{e}}_1 = \dot{\underline{e}}_2 = \dot{\underline{e}}_3 = 0$$

$$\ddot{\underline{r}} = \ddot{x}\underline{e}_1 + \ddot{y}\underline{e}_2 + \ddot{z}\underline{e}_3$$

The dynamics equations of motion for a particle subjected to a resultant force.

$$\underline{F} = m\ddot{\underline{r}} = m\ddot{x}\underline{e}_1 + m\ddot{y}\underline{e}_2 + m\ddot{z}\underline{e}_3$$

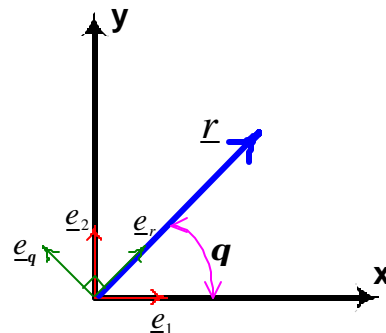
### 2-2 Polar Coordinates:

In the Cartesian Coordinates

$$\underline{r} = x\underline{e}_1 + y\underline{e}_2$$

In the Polar Coordinates

$$\underline{r} = r\underline{e}_r$$



The transformation of Cartesian and Polar Coordinates is:

$$\underline{e}_r = (\underline{e}_r \cdot \underline{e}_1)\underline{e}_1 + (\underline{e}_r \cdot \underline{e}_2)\underline{e}_2$$

$$= \cos q \underline{e}_1 + \sin q \underline{e}_2$$

$$\underline{e}_q = (\underline{e}_q \cdot \underline{e}_1)\underline{e}_1 + (\underline{e}_q \cdot \underline{e}_2)\underline{e}_2$$

$$= -\sin q \underline{e}_1 + \cos q \underline{e}_2$$

When  $\underline{e}_1, \underline{e}_2$  are fixed, the particle's velocity and acceleration expressions are shown below.

$$\begin{aligned}\dot{\underline{r}} &= x\dot{\underline{e}}_1 + y\dot{\underline{e}}_2 = \dot{r}\underline{e}_r + r\dot{\underline{e}}_q \\ \dot{\underline{e}}_r &= -(\sin \mathbf{q})\dot{\mathbf{q}}\underline{e}_1 + (\cos \mathbf{q})\dot{\mathbf{q}}\underline{e}_2 = \dot{\mathbf{q}}(-\sin \mathbf{q}\underline{e}_1 + \cos \mathbf{q}\underline{e}_2) = \dot{\mathbf{q}}\underline{e}_q \\ \dot{\underline{e}}_q &= -(\cos \mathbf{q})\dot{\mathbf{q}}\underline{e}_1 - (\sin \mathbf{q})\dot{\mathbf{q}}\underline{e}_2 = -\dot{\mathbf{q}}(\cos \mathbf{q}\underline{e}_1 + \sin \mathbf{q}\underline{e}_2) = -\dot{\mathbf{q}}\underline{e}_r \\ \therefore \dot{\underline{r}} &= \dot{r}\underline{e}_r + r\dot{\mathbf{q}}\underline{e}_q \\ \ddot{\underline{r}} &= \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\underline{e}}_q + r\ddot{\mathbf{q}}\underline{e}_q + r\dot{\mathbf{q}}\dot{\underline{e}}_q + r\dot{\mathbf{q}}\dot{\underline{e}}_q \\ &= \ddot{r}\underline{e}_r + \dot{r}\dot{\mathbf{q}}\underline{e}_q + \dot{r}\dot{\mathbf{q}}\underline{e}_q + r\ddot{\mathbf{q}}\underline{e}_q - r\dot{\mathbf{q}}^2\underline{e}_r \\ &= (\ddot{r} - r\dot{\mathbf{q}}^2)\underline{e}_r + (2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}})\underline{e}_q\end{aligned}$$

We define  $\ddot{r} - r\dot{\mathbf{q}}^2$  is centripetal acceleration, and  $2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}}$  is Coriolis acceleration. So, the balance of linear momentum for the motion of a particle is described as follows.

$$\begin{aligned}\underline{F} &= m\ddot{\underline{r}} \\ \underline{F} &= F_r\underline{e}_r + F_q\underline{e}_q \\ F_r &= \underline{F} \cdot \underline{e}_r = m(\ddot{r} - r\dot{\mathbf{q}}^2) \\ F_q &= \underline{F} \cdot \underline{e}_q = m(2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}})\end{aligned}$$

**Example 2-2-1.** Based on the figure, find the equations of motion.

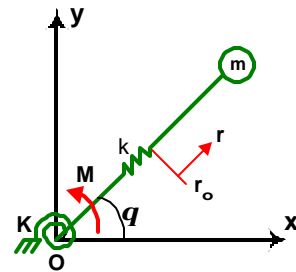
**Solution:**  $\underline{r} = r\underline{e}_r$

$$\begin{aligned}\dot{\underline{r}} &= \dot{r}\underline{e}_r + r\dot{\mathbf{q}}\underline{e}_q \\ \ddot{\underline{r}} &= (\ddot{r} - r\dot{\mathbf{q}}^2)\underline{e}_r + (2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}})\underline{e}_q\end{aligned}$$

The angular momentum is

$$\begin{aligned}\underline{H}^o &= m\underline{r} \times \dot{\underline{r}} \\ &= m(r\underline{e}_r) \times (\dot{r}\underline{e}_r + r\dot{\mathbf{q}}\underline{e}_q) \\ &= mr^2\dot{\mathbf{q}}\underline{e}_k\end{aligned}$$

The balance of angular momentum is



$$\dot{\underline{H}}^o = (mr^2 \ddot{\mathbf{q}} + 2mr\dot{r}\dot{\mathbf{q}})\underline{e}_k$$

Based on the free-body diagram, the governing equations of force and balance of momentum are.

$$\underline{F} = -k(r - r_o)\underline{e}_r + F_q \underline{e}_q$$

$$\underline{M}^o = (M - K\mathbf{q})\underline{e}_k$$

From Newton's second law of motion and law of balance of angular momentum, we get:

$$\begin{aligned} \underline{F} &= m\ddot{\underline{r}} \\ &= m[(\ddot{r} - r\dot{\mathbf{q}}^2)\underline{e}_r + (2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}})\underline{e}_q] \\ &= -k(r - r_o)\underline{e}_r + F_q \underline{e}_q \end{aligned}$$

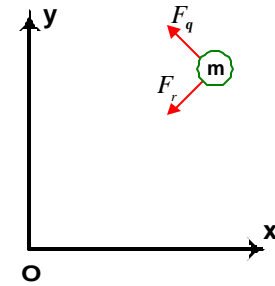
$$\begin{aligned} F_r &= -k(r - r_o) = m(\ddot{r} - r\dot{\mathbf{q}}^2) \\ m\ddot{r} - mr\dot{\mathbf{q}}^2 + kr &= kr_o \end{aligned}$$

$$F_q = 2\dot{r}\dot{\mathbf{q}} + r\ddot{\mathbf{q}}$$

$$\underline{M}^o = \dot{\underline{H}}^o$$

$$M - K\mathbf{q} = mr^2 \ddot{\mathbf{q}} + 2mr\dot{r}\dot{\mathbf{q}}$$

$$M = mr^2 \ddot{\mathbf{q}} + 2mr\dot{r}\dot{\mathbf{q}} + K\mathbf{q}$$



**Example 2-2-2.** A string wraps around a cylinder. Find the time— $t$  for the string to wrap up the cylinder? (note: there is a gravity force in  $-\underline{e}_3$  direction.)

**Solution:**  $\underline{r} = R\underline{e}_r + (l - R\mathbf{q})\underline{e}_q = R\underline{e}_r + \underline{r}\underline{e}_q$

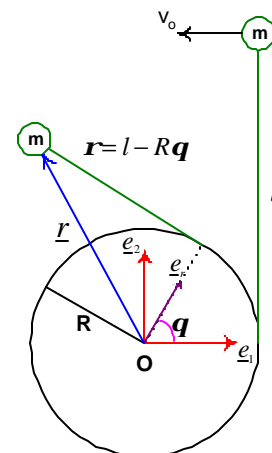
$$\underline{r} = l - R\mathbf{q}$$

$$\dot{\underline{r}} = -R\dot{\mathbf{q}}$$

$$\ddot{\underline{r}} = -R\ddot{\mathbf{q}}$$

$$\dot{\underline{r}} = R\dot{\underline{e}}_r + \dot{\underline{r}}\underline{e}_q + \underline{r}\dot{\underline{e}}_q$$

$$= -R\dot{\mathbf{q}}\underline{e}_r$$



$$\ddot{\underline{r}} = -(\dot{\underline{r}}\dot{\underline{q}} + \underline{r}\ddot{\underline{q}})\underline{e}_r - \underline{r}\dot{\underline{q}}^2 \underline{e}_q$$

$$\underline{F} = -T\underline{e}_q + (N - mg)\underline{e}_z$$

$$\underline{F} = m\ddot{\underline{r}}$$

$$= m[-(\dot{\underline{r}}\dot{\underline{q}} + \underline{r}\ddot{\underline{q}})\underline{e}_r - \underline{r}\dot{\underline{q}}^2 \underline{e}_q]$$

$$= -T\underline{e}_q + (N - mg)\underline{e}_z$$

$$\therefore T = m\underline{r}\dot{\underline{q}}^2$$

When  $t=0$ ,

$$\underline{r} = l, \quad v_o = l\dot{\underline{q}} = \underline{r}\dot{\underline{q}}$$

$$T = mv_o\dot{\underline{q}}$$

When  $\underline{r} \rightarrow 0$ ,  $\dot{\underline{q}} \rightarrow \infty$ , then  $T \rightarrow \infty$ .

$\underline{r}$  is a function of time as shown follows.

$$\dot{\underline{q}} = \frac{v_o}{\underline{r}} = -\frac{\dot{\underline{r}}}{R}$$

$$\underline{r}\dot{\underline{r}} = -v_o R$$

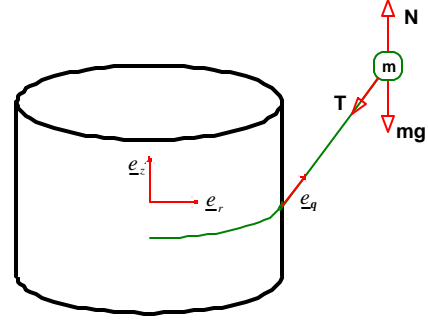
$$\underline{r}^2 = -2v_o R t + C$$

Apply the initial condition,  $C = l^2$ .

$$\underline{r} = (l^2 - 2v_o R t)^{\frac{1}{2}}$$

or

$$t = \frac{l^2}{2v_o R}$$



The angular velocity vector with the  $\underline{e}_r - \underline{e}_q$  rotating basis is  $\underline{w} = \dot{\underline{q}}\underline{e}_k$ .

$$\dot{\underline{e}}_r = \underline{w} \times \underline{e}_r = \dot{\underline{q}}\underline{e}_k \times \underline{e}_r = \dot{\underline{q}}\underline{e}_q$$

$$\dot{\underline{e}}_q = \underline{w} \times \underline{e}_q = \dot{\underline{q}}\underline{e}_k \times \underline{e}_q = -\dot{\underline{q}}\underline{e}_r$$

In transformation of coordinates, it can be shown as below.

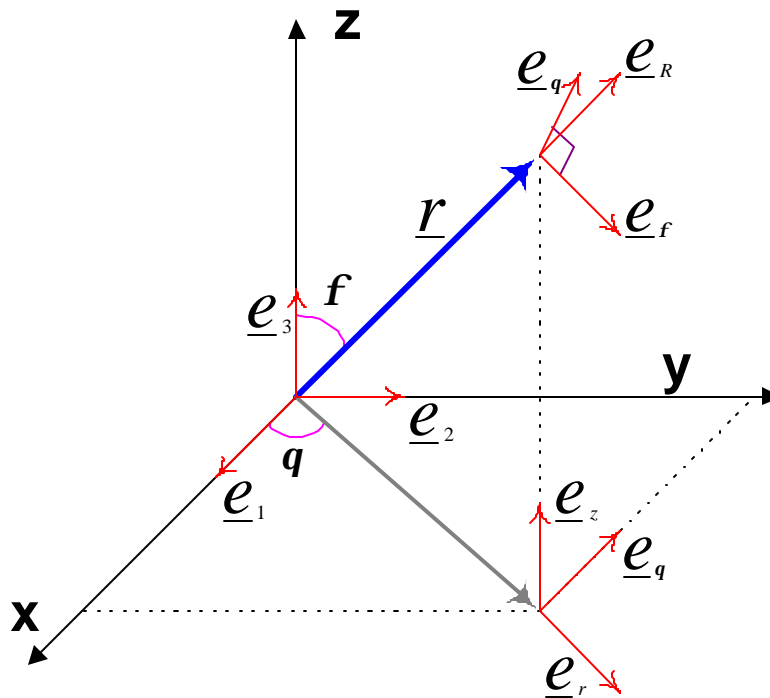
$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_q \\ \underline{e}_z \end{bmatrix} = \begin{bmatrix} \cos \mathbf{q} & \sin \mathbf{q} & 0 \\ -\sin \mathbf{q} & \cos \mathbf{q} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \cos \mathbf{q} & \sin \mathbf{q} & 0 \\ -\sin \mathbf{q} & \cos \mathbf{q} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### 2-3 Cylindrical Coordinate:

The transformation from Cartesian to Cylindrical Coordinate is discussed above. The figure of the transformation is shown in the next section.

### 2-4 Spherical Coordinates:



The sequence of transformation of these coordinates is Cartesian, Cylindrical, then Spherical Coordinates. The transformation from Cylindrical to Spherical Coordinate is:

$$\begin{bmatrix} \underline{e}_R \\ \underline{e}_f \\ \underline{e}_q \end{bmatrix} = \begin{bmatrix} \underline{e}_R \cdot \underline{e}_r & \underline{e}_R \cdot \underline{e}_q & \underline{e}_R \cdot \underline{e}_z \\ \underline{e}_f \cdot \underline{e}_r & \underline{e}_f \cdot \underline{e}_q & \underline{e}_f \cdot \underline{e}_z \\ \underline{e}_q \cdot \underline{e}_r & \underline{e}_q \cdot \underline{e}_q & \underline{e}_q \cdot \underline{e}_z \end{bmatrix} \begin{bmatrix} \underline{e}_r \\ \underline{e}_q \\ \underline{e}_z \end{bmatrix} = \begin{bmatrix} \sin \mathbf{f} & 0 & \cos \mathbf{f} \\ \cos \mathbf{f} & 0 & -\sin \mathbf{f} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \underline{e}_r \\ \underline{e}_q \\ \underline{e}_z \end{bmatrix}$$

The transformation from Cartesian to Spherical Coordinate is:

$$\begin{bmatrix} \underline{e}_R \\ \underline{e}_f \\ \underline{e}_q \end{bmatrix} = \begin{bmatrix} \sin f \cos q & \sin f \sin q & \cos f \\ \cos f \cos q & \cos f \sin q & -\sin f \\ -\sin q & \cos q & 0 \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

$$\dot{\underline{e}}_R = \dot{f} \underline{e}_f + \dot{q} \sin f \underline{e}_q$$

$$\dot{\underline{e}}_f = -\dot{f} \underline{e}_R + \dot{q} \cos f \underline{e}_q$$

$$\dot{\underline{e}}_q = -\dot{q} \cos f \underline{e}_f - \dot{q} \sin f \underline{e}_R$$

Based on the above figure, the general case in Spherical Coordinate will be:

$$\underline{r} = r \underline{e}_R$$

$$\dot{\underline{r}} = \dot{r} \underline{e}_R + r \dot{\underline{e}}_R = \dot{r} \underline{e}_R + r \dot{f} \underline{e}_f + r \dot{q} \sin f \underline{e}_q$$

$$\ddot{\underline{r}} = (\ddot{r} - r \dot{f}^2 - r \dot{q}^2 \sin^2 q) \underline{e}_R + (r \ddot{f} + 2 \dot{r} \dot{f} - r \dot{q}^2 \sin f \cos f) \underline{e}_f$$

$$+ (r \dot{q} \sin f + 2 \dot{r} \dot{q} \sin f + 2 r \dot{f} \dot{q} \cos f) \underline{e}_q$$

In Spherical Coordinate, two rotations exist. They are demonstrated in Cartesian and Cylindrical Coordinates.

$$\underline{w} = \dot{q} \underline{e}_3 + \dot{f} \underline{e}_q$$

Express the angular velocity vector in Spherical Coordinate.

$$\underline{w} = \dot{q} (-\sin f \underline{e}_f + \cos f \underline{e}_R) + \dot{f} \underline{e}_q$$

$$= \dot{q} \cos f \underline{e}_R - \dot{q} \sin f \underline{e}_f + \dot{f} \underline{e}_q$$

$$\dot{\underline{e}}_R = \underline{w} \times \underline{e}_R$$

$$= (\dot{q} \cos f \underline{e}_R - \dot{q} \sin f \underline{e}_f + \dot{f} \underline{e}_q) \times \underline{e}_R$$

$$= \dot{f} \underline{e}_f + \dot{q} \sin f \underline{e}_q$$

$$\dot{\underline{e}}_f = \underline{w} \times \underline{e}_f$$

$$= (\dot{q} \cos f \underline{e}_R - \dot{q} \sin f \underline{e}_f + \dot{f} \underline{e}_q) \times \underline{e}_f$$

$$= -\dot{f} \underline{e}_R + \dot{q} \cos f \underline{e}_q$$

