Dynamics

Chapter 1. Introduction

There are three postulates of mechanics.

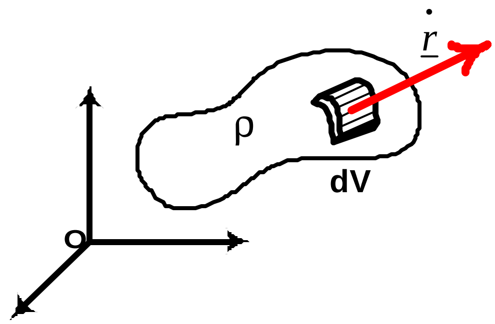
1. Conservation of Mass: $\dot{m} = \frac{\partial m}{\partial t}|_{body\ is\ fixed} = 0$.
2. Balance of Linear Momentum: $\underline{F} = \dot{\underline{G}}$.
3. Balance of Angular Momentum: $\underline{M}^o = \dot{\underline{H}}^o$.

Any of above formulas can be proved as follows:

a. Linear momentum:

$$dm = r dV$$

$$\underline{G} = \int_r \dot{\underline{r}}\, dm = \int_r r\, \dot{\underline{r}}\, dV$$



Linear momentum of system of particles:

$$\underline{G} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i$$

b. Angular momentum:

$$\underline{H}^o = \int_r \underline{r} \times (r\, \dot{\underline{r}}) dV = \int_r r\, \underline{r} \times \dot{\underline{r}} dV$$

note: $\underline{r}$ is a vector relative to a fixed point o.

Angular momentum of system of particles:

$$\underline{H}^o = \sum_{i=1}^{n} \underline{r}_i \times (m_i \dot{\underline{r}}_i) = \sum_{i=1}^{n} m_i \underline{r}_i \times \dot{\underline{r}}_i$$

c. Resultant force:

$$\underline{F} = \int_r r \underline{b} dV + \int_{\partial r} \underline{t} dA$$

note: $\underline{b} = -g\underline{e}_3$ body force per unit mass

$\underline{t}$ contact force per unit area

Resultant force of system of particles:

$$\underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

d. Resultant moment:

$$\underline{M}^o = \int_r r\, \underline{r} \times \underline{b} dV + \int_{\partial r} \underline{r} \times \underline{t} dA$$

Resultant moment of system of particles:

$$\underline{M}^o = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i$$

The three behaviors of mechanics are independent. For example, balance of linear momentum does not imply balance of angular momentum, except in the case of a single particle.

$$\underline{F} = m\ddot{\underline{r}}$$

$$\underline{M}^o = \underline{r} \times \underline{F} = \underline{r} \times m\ddot{\underline{r}} = \frac{d}{dt}(\underline{r} \times m\dot{\underline{r}}) = \dot{\underline{H}}^o$$

The postulates apply to the body or system of particles as a whole as well as to any arbitrary subset of the body or system of particles. It is useful to decompose the motion into one of translation of the center of mass and rotation about the center of mass.

1. Translation of the center mass.

   (i) Continuum.

   $$\underline{G} = \int_r r\,\dot{\underline{r}}\,dV$$

   $$\dot{\underline{G}} = \frac{d}{dt}\int_r r\,\dot{\underline{r}}dV = \int_r r\,\ddot{\underline{r}}dV$$

   Position of center mass: $\underline{r_m} = \frac{1}{m}\int_r r\,\underline{r}dV$

   $$m\underline{r_m} = \int_r r\,\underline{r}dV$$

   $$m\ddot{\underline{r_m}} = \int_r r\,\ddot{\underline{r}}\,dV$$

   $$\underline{F} = m\ddot{\underline{r_m}}$$

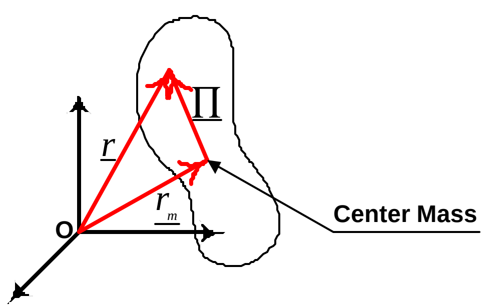
   (ii) System of particles.

   $$\dot{\underline{G}} = \frac{d}{dt}\sum_{i=1}^{n} m_i\,\dot{\underline{r}}_i = \sum_{i=1}^{n} m_i\,\ddot{\underline{r}}_i = \frac{m}{m}\sum_{i=1}^{n} m_i\,\ddot{\underline{r}}_i = m\ddot{\underline{r_m}}$$

   $$\dot{\underline{G}} = \underline{F} = m\ddot{\underline{r_m}}$$

2. Rotation about the center mass.

   (i) Continuum.



$$\dot{\underline{H}}^o = \frac{d}{dt}\int_r r\,\underline{r}\times\dot{\underline{r}}\,dV = \int_r (r\,\dot{\underline{r}}\times\dot{\underline{r}} + r\,\underline{r}\times\ddot{\underline{r}})dV \quad \left\{note : r\,\dot{\underline{r}}\times\dot{\underline{r}} = 0\right\}$$

$$= \int_r [r\,(\underline{r}_m + \underline{\Pi})\times(\ddot{\underline{r}}_m + \ddot{\underline{\Pi}})]dV$$

$$= \int_r r\,[\underline{r}_m\times\ddot{\underline{r}}_m + \underline{r}_m\times\ddot{\underline{\Pi}} + \underline{\Pi}\times\ddot{\underline{r}}_m + \underline{\Pi}\times\ddot{\underline{\Pi}})]dV$$

$$= (\int_r r\,dV)\underline{r}_m\times\ddot{\underline{r}}_m + \underline{r}_m\times(\int_r r\,\ddot{\underline{\Pi}}\,dV) + (\int_r r\,\underline{\Pi}dV)\times\ddot{\underline{r}}_m + \int_r (r\,\underline{\Pi}\times\ddot{\underline{\Pi}})dV$$

$$note : \int_r r\,\ddot{\underline{\Pi}}\,dV = \int_r r\,\underline{\Pi}dV = 0$$

$$\because m\underline{r}_m = \int_r r\,\underline{r}dV = \int_r r\,(\underline{r}_m + \underline{\Pi})dV = m\underline{r}_m + \int_r r\,\underline{\Pi}dV$$

$$\int_r r\,dV = m$$

$$= m\underline{r}_m\times\ddot{\underline{r}}_m + \int_r (r\,\underline{\Pi}\times\ddot{\underline{\Pi}})dV$$

$$= \underline{r}_m\times\underline{F} + \dot{\underline{H}}$$

$$note : H = \int_r (r\,\underline{\Pi}\times\dot{\underline{\Pi}})dV$$

$$\underline{M}^o = \int_r [r\,(\underline{r}_m + \underline{\Pi})\times\underline{b}]dV + \int_{\partial r}[(\underline{r}_m + \underline{\Pi})\times\underline{t}]dA$$

$$= \underline{r}_m\{\int_r r\,\underline{b}dV + \int_{\partial r}\underline{t}dA\} + \{\int_r [r\,\underline{\Pi}\times\underline{b}]dV + \int_{\partial r}[\underline{\Pi}\times\underline{t}]dA\}$$

$$= \underline{r}_m\times\underline{F} + \underline{M}$$

$$\Rightarrow \underline{M}^o = \dot{\underline{H}}^o \Rightarrow \underline{M} = \dot{\underline{H}}$$

(ii) System of particles.

$$\dot{\underline{H}}^o = \frac{d}{dt}\sum_{i=1}^{n} m_i\,\underline{r}_i\times\dot{\underline{r}}_i$$

$$= \sum_{i=1}^{n}[m_i\,\dot{\underline{r}}_i\times\dot{\underline{r}}_i + m_i\,\underline{r}_i\times\ddot{\underline{r}}_i]$$

$$= \sum_{i=1}^{n} m_i\,\underline{r}_i\times\ddot{\underline{r}}_i$$

$$= \sum_{i=1}^{n}[m_i\,\dot{\underline{r}}_i\times\dot{\underline{r}}_i + m_i\,\underline{r}_i\times\ddot{\underline{r}}_i]$$

$$= \sum_{i=1}^{n} m_i (\underline{r_m} + \underline{\prod}_i) \times (\ddot{\underline{r_m}} + \ddot{\underline{\prod}}_i)$$

$$= \{\underline{r_m} \times m \ddot{\underline{r_m}}\} + \{\sum_{i=1}^{n} m_i \underline{\prod}_i \times \ddot{\underline{r_m}}\} + \{\underline{r_m} \times \sum_{i=1}^{n} m_i \ddot{\underline{\prod}}_i\} + \{\sum_{i=1}^{n} m_i \underline{\prod}_i \times \ddot{\underline{\prod}}_i\}$$

$$note: \sum_{i=1}^{n} m_i \underline{\prod}_i = \sum_{i=1}^{n} m_i \ddot{\underline{\prod}}_i = 0$$

$$\because m\underline{r_m} = \sum_{i=1}^{n} m_i \underline{r}_i = \sum_{i=1}^{n} m_i (\underline{r_m} + \underline{\prod}_i) = m\underline{r_m} + \sum_{i=1}^{n} m_i \underline{\prod}_i$$

$$= \underline{r_m} \times m \ddot{\underline{r_m}} + \sum_{i=1}^{n} m_i \underline{\prod}_i \times \ddot{\underline{\prod}}_i$$

$$= \underline{r_m} \times \underline{F} + \dot{\underline{H}}$$

$$note: H = \sum_{i=1}^{n} m_i \underline{\prod}_i \times \dot{\underline{\prod}}_i$$

$$\underline{M}^o = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = \sum_{i=1}^{n} (\underline{r_m} + \underline{\prod}_i) \times \underline{F}_i$$

$$= \underline{r_m} \times \sum_{i=1}^{n} \underline{F}_i + \sum_{i=1}^{n} \underline{\prod}_i \times \underline{F}_i$$

$$= \underline{r_m} \times \underline{F} + \underline{M}$$

$$\underline{M}^o = \dot{\underline{H}}^o \Rightarrow \underline{M} = \dot{\underline{H}}$$