# CONTACT AND CRACK PROBLEMS IN LINEAR THEORY OF ELASTICITY

By **V.I. Fabrikant**

# TABLE OF CONTENTS

# INTRODUCTION

This book may be considered as a logical continuation of the two previously published books (V.I. Fabrikant, *Applications of Potential Theory in Mechanics*, Kluwer Academic, 1989) and (V.I. Fabrikant, *Mixed Boundary Value Problems of Potential Theory and Their Applications in Engineering*, Kluwer Academic, 1991), where a new and elementary method was described for solving mixed boundary value problems. The method can solve *non-axisymmetric problems* as easily as axisymmetric ones, *exactly and in closed form*. It enables us to treat *analytically* non-classical domains. The method also provides, as a bonus, a tool for exact evaluation of various two-dimensional integrals involving distances between two or more points.

The main emphasis of the first book was on solid mechanics problems. In the second book we described various applications of the new method to electromagnetics, acoustics and diffusion. Also included in the second book were some results in fracture mechanics and elastic contact problems, which were obtained later and could not be included in the first book.

There are numerous books on contact problems (Galin, 1951, Rvachev and Protsenko, 1977; Mossakovskii *et al.*, 1985), there are also books devoted to the crack problems (Cherepanov, 1974; Kassir and Sih, 1975). There seems to be no titles, giving exhaustive treatment of both contact and crack problems in one book. Why this is so, is beyond my comprehension. Indeed, from mathematical point of view, there is not much difference between them: a contact problem is characterized by a displacement, prescribed inside certain domain, with the stress being zero outside, while a crack problem is characterized by a stress prescribed inside a domain, with the displacement vanishing outside. In a way, one problem may be considered as inverse of the other.

This seems to be the first book, giving the state-of-the-art description of both contact and crack problems. The same mathematical apparatus, developed by the author, is used to solve both. Majority of presented solutions is exact and expressed in terms of elementary functions. One may argue, that these elementary solutions are not needed in the age of powerful computers. This is just not so: the most powerful computers are still quite bad in the cases of poor convergence and can not handle singularities directly. In addition, elementary solutions serve as excellent benchmark examples to verify the quality of new numerical methods or a new software.

The book is addressed to a wide audience ranging from engineers to mathematical physicists. While an engineer can find in the book some elementary, ready to use formulae for solving various practical problems, a mathematical physicist might become interested in new applications of the mathematical apparatus presented. The book should be of interest to specialists in electromagnetics, acoustics, diffusion, solid and fluid mechanics, etc.

The book is *accessible to anyone* with a background in university undergraduate calculus, but should be of interest to established scientists as well. Though the method is elementary, the transformations involved are sometimes very non-trivial and cumbersome, while the final result is usually very simple. The reader who is interested only in application of the general results to his/her particular problems may skip the long derivations and use the final formulae which requires little effort. The reader, who wants to master the method in order to solve new problems, has to repeat the derivations which are given in sufficient detail.

The book is based entirely on the author's results, and this is why the work of other scientists is mentioned only when such a quotation is inevitable for some reason, like numerical data needed to verify the accuracy of approximate results, comparison with existing results, or pointing out some errors in publications. There are several books and review articles presenting an adequate account of the state-of-the-art in the field. Appropriate references are given for the reader's convenience. The purpose of this book was neither to repeat nor to compete with them.

For the reader's convenience, it was attempted to make each chapter (and section, wherever possible) self-contained. The reader can skip several sections and continue reading, without loosing the ability to understand material. On the other hand, this resulted in repetition of some definitions and descriptions. The author thinks that the additional convenience is worth several extra pages in the book.

The book contains *global* variables, which denote the same quantity throughout the book, for example, $l_1(\cdot,\cdot,\cdot)$ $l_2(\cdot,\cdot,\cdot)$, $\Lambda$, etc. There is only limiting amount of characters in the Latin and Greek alphabets, while the number of parameters and notations used in the book is much greater, so inevitably some characters are used as *local* variables to denote several different quantities. For example, $\delta$ denotes in some sections the angle of inclination, while in other sections it denotes $\left(\rho\rho_0 e^{i(\phi-\phi_0)}\right)^{1/2}$. Hopefully, no character is used to denote 2 different quantities in the same section.

The book consists of 6 chapters. The first chapter describes the mathematical foundations of the method, with some applications. Two chapters are devoted to crack problems and 2 chapters describe the contact problems. Each subject is divided in two parts: the fundamental problems and the advanced ones. The most important results from the previous 2 books are repeated in a concise form, while the new results of the past 12 years are given in detail.

Several Appendices in the book contain various mathematical formulae (derivatives, integrals, etc.), which are not available in other mathematical reference books. Chapter 6 is completely devoted to such results, giving various derivatives, one- and two-dimensional integrals. The method developed in the book can be generalized for solving mixed boundary value problems for piezo-electric or piezo-magnito-electric bodies. Numerous publications in this field are available in the literature.

The book contains so much new material that some misprints and errors are inevitable, though every effort was made to eliminate them. The author would be grateful for every communication in this regard. All the readers' comments are welcome. The author's present mailing address is:

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# CHAPTER 1

## MATHEMATICAL FOUNDATIONS OF THE METHOD

The chapter gives the main mathematical results leading to the author's method of solving mixed boundary value problems, as well as new applications to the solution of biharmonic integral equation, computation of various integrals, involving Bessel functions, *etc.* The material in sections (1.1)–(1.7) follows relevant sections from (Fabrikant, 1991), while sections (1.8)–(1.11) follow the articles (Fabrikant, 1993a, 1994a, 2001a, 2001c, 2003).

### 1.1. The $\mathcal{L}$-operator and its properties

Let $f(r,\phi)$ be an arbitrary function which belongs to $L^1[0,2\pi]$ as a function of $\phi$ for any fixed $r \geq 0$. Let us associate with $f(r,\phi)$ the sequence

$$\{f_n(r)\} \qquad -\infty < n < \infty \tag{1.1.1}$$

of its Fourier coefficients. We consider $\mathcal{L}(k)$ as an operator on the linear space of sequences $\{f_n(r)\}$. We do not define any topology on this space. The algebraic operations are defined naturally as follows:

$$c_1\{f_n^{(1)}\} + c_2\{f_n^{(2)}\} := \{c_1 f_n^{(1)} + c_2 f_n^{(2)}\}, \tag{1.1.2}$$

$$\{f_n^{(1)}\} = \{f_n^{(2)}\} \;\Leftrightarrow\; f_n^{(1)} = f_n^{(2)} \quad \forall\, n \tag{1.1.3}$$

We define

$$\mathcal{L}(k)\{f_n\} = \{k^{|n|} f_n\}. \tag{1.1.4}$$

This definition makes sense for any $k \in \mathbb{C}$, and it implies that

$$\mathcal{L}(k_1)\,\mathcal{L}(k_2)\{f\} = \mathcal{L}(k_1 k_2)\,\{f\} \quad \forall \;\; k_1, k_2 \in \mathbb{C}, \tag{1.1.5}$$

$$\mathcal{L}(1)\,\{f\} = \{f\}. \tag{1.1.6}$$

Equation (6) is a particular case of (5) corresponding to $k_1 = k \neq 0$ and $k_2 = k^{-1}$.

We consider now the operator $\prod_{j=1}^{m} \mathcal{L}(k_j)$. It is well defined for any $k_j$, in particular, in the case when some of the $k_j$ are greater than 1. An obvious corollary is: if $\left| \prod_{j=1}^{m} k_j \right| < 1$ then $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ is a sequence of the Fourier coefficients of some function belonging to $L^1[0, 2\pi]$ if $\{f_n\}$ is a sequence of the Fourier coefficients of a function $f(\phi) \in L^1[0, 2\pi]$, and, moreover, the Fourier series corresponding to the sequence $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ converges absolutely and uniformly in $\phi \in [0, 2\pi]$.

In the case when $k<1$, formula (4) can be rewritten as

$$\mathcal{L}(k) f(\phi) = \sum_{n=-\infty}^{\infty} k^{|n|} f_n \mathrm{e}^{in\phi} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} k^{|n|} \mathrm{e}^{in\phi} \int_0^{2\pi} \mathrm{e}^{-in\phi_0} f(\phi_0)\, \mathrm{d}\phi_0 \,. \tag{1.1.7}$$

Summation in (7) yields

$$\mathcal{L}(k) f(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \lambda(k, \phi - \phi_0)\, f(\phi_0)\, \mathrm{d}\phi_0 \,, \tag{1.1.8}$$

where the notation was introduced

$$\lambda(k, \psi) = \frac{1 - k^2}{1 - 2k\cos\psi + k^2} \,. \tag{1.1.9}$$

Note that the $\mathcal{L}$-operator, as it is presented in (8), coincides with the one which was introduced by Poisson for solving the two-dimensional boundary value problem of potential theory for a circle. We are going to use it for solving the relevant three-dimensional problems. Whenever the operator $\mathcal{L}(k)$ is applied, with no limitation $k < 1$ assumed, its general definition (4) is valid, allowing the properties (5) and (6) to be used. As soon as it becomes clear that $k \leq 1$, the representation (8) becomes valid thus making it possible to write the final result in a closed and simplified form.

## 1.2. Integral representation for the reciprocal of the distance between two points

It was proven in (Fabrikant, 1971a) that

$$\frac{1}{R^{1+u}}=\frac{1}{\left(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\right)^{(1+u)/2}}=\frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{min(\rho_0,\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)x^u\mathrm{d}x}{\left((\rho^2-x^2)(\rho_0^2-x^2)\right)^{(1+u)/2}}. \tag{1.2.1}$$

Here $\lambda$ is defined by (1.1.9). The identity (1) can be verified by the introduction of a new variable

$$\eta(x)=[(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}/x, \tag{1.2.2}$$

Substitution of the identities

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=-\frac{\rho^2\rho_0^2-x^4}{x^3\eta}, \qquad \lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)=-\frac{x\eta}{R^2+\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}$$

in (1) transforms it into:

$$\frac{1}{R^{1+u}}=\frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^\infty\frac{\eta^{-u}\mathrm{d}\eta}{R^2+\eta^2}. \tag{1.2.3}$$

The integral in (3) can be evaluated by using formula (3.241.4) from (Gradshteyn and Ryzhik, 1963), thus proving the identity. All the results above are related to the distance between two points in the plane $z=0$. We need to generalize them to represent

$$\frac{1}{R_0^{1+u}}=\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}}. \tag{1.2.4}$$

One can observe that representation (1) remains valid if we formally substitute $\rho$ and $\rho_0$ by arbitrary quantities $l_1$ and $l_2$. We need to choose them so that

$$\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2=l_1^2+l_2^2-2l_1l_2\cos(\phi-\phi_0). \tag{1.2.5}$$

This leads to two equations

$$l_1^2 + l_2^2 = \rho^2 + \rho_0^2 + z^2, \qquad l_1 l_2 = \rho\rho_0 . \tag{1.2.6}$$

Their solution will take the form

$$l_1(\rho_0,\rho,z) = \frac{1}{2}\Big([(\rho+\rho_0)^2 + z^2]^{1/2} - [(\rho-\rho_0)^2 + z^2]^{1/2}\Big), \tag{1.2.7}$$

$$l_2(\rho_0,\rho,z) = \frac{1}{2}\Big([(\rho+\rho_0)^2 + z^2]^{1/2} + [(\rho-\rho_0)^2 + z^2]^{1/2}\Big). \tag{1.2.8}$$

Hereafter the following abbreviations will be used:

$$l_1(x) \equiv l_1(x,\rho,z), \qquad l_2(x) \equiv l_2(x,\rho,z) , \tag{1.2.9}$$

$$l_1 \equiv l_1(a,\rho,z), \qquad l_2 \equiv l_2(a,\rho,z) . \tag{1.2.10}$$

Note the limiting properties

$$\lim_{z\to 0} l_1(x) = \min(x,\rho) , \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho) . \tag{1.2.11}$$

In view of the properties above, the representation (1) can be generalized as follows:

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{l_1(\rho_0)} \frac{\lambda\left(\dfrac{x^2}{\rho\rho_0}, \phi-\phi_0\right) x^u \mathrm{d}x}{\big([l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\big)^{(1+u)/2}} . \tag{1.2.12}$$

Formula (12) simplifies when $u = 0$

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}} = \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{\lambda\left(\dfrac{x^2}{\rho\rho_0}, \phi-\phi_0\right)\mathrm{d}x}{\big([l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\big)^{1/2}} . \tag{1.2.13}$$

Again, one can notice that the integral in (13) may be evaluated as indefinite

$$\int \frac{\lambda\left(\dfrac{x^2}{\rho\rho_0}, \phi-\phi_0\right)\mathrm{d}x}{\big([l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\big)^{1/2}} = -\frac{1}{R_0}\tan^{-1}\left(\frac{\big([l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\big)^{1/2}}{xR_0}\right). \tag{1.2.14}$$

The last representation is very important and will be widely used throughout the book.

Another series of useful formulae can be obtained from those above by a simple change of variables, namely,

$$\int \frac{\lambda\left(\frac{\rho\rho_0}{x^2}, \phi-\phi_0\right)\mathrm{d}x}{\left([x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\right)^{1/2}} = \frac{1}{R_0}\tan^{-1}\left(\frac{\left([x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\right)^{1/2}}{xR_0}\right), \tag{1.2.15}$$

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2}, \phi-\phi_0\right)x^u\,\mathrm{d}x}{\left([x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\right)^{(1+u)/2}}, \tag{1.2.16}$$

$$\frac{1}{R_0} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}} = \frac{2}{\pi}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2}, \phi-\phi_0\right)\mathrm{d}x}{\left([x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\right)^{1/2}}, \tag{1.2.17}$$

$$\int \frac{\lambda\left(\frac{\rho\rho_0}{x^2}, \phi-\phi_0\right)\mathrm{d}x}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}} = \frac{1}{R}\tan^{-1}\left(\frac{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}{xR}\right). \tag{1.2.18}$$

The representations above are useful for solving external mixed boundary value problems.

Several modifications of (14) are available. For example, we can write

$$\int \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right)\mathrm{d}x}{\sqrt{\rho^2-x^2}\,[\rho_0^2-g^2(x)]^{1/2}} = -\frac{1}{R_0}\tan^{-1}\left(\frac{\sqrt{\rho^2-x^2}\,[\rho_0^2-g^2(x)]^{1/2}}{xR_0}\right). \tag{1.2.19}$$

Here

$$g(x) = x[1+z^2/(\rho^2-x^2)]^{1/2}\,. \tag{1.2.20}$$

It is important to notice that the function $g(x)$ is inverse to $l_1$ for $x^2<\rho^2$, and is inverse to $l_2$ for $x^2>\rho^2+z^2$. Introduction of a new variable $x=l_1(y)$, $y=g(x)$ transforms (19) into:

$$\int \frac{[l_2^2(y)-y^2]^{1/2}}{(\rho_0^2-y^2)^{1/2}[l_2^2(y)-l_1^2(y)]}\lambda\left(\frac{l_1^2(y)}{\rho\rho_0},\phi-\phi_0\right)dy=-\frac{1}{R_0}\tan^{-1}\left(\frac{(\rho_0^2-y^2)^{1/2}[l_2^2(y)-y^2]^{1/2}}{yR_0}\right). \tag{1.2.21}$$

A particular case of (13), when $z=0$, reads:

$$\frac{1}{R}=\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}=\frac{2}{\pi}\int_0^{min(\rho_0,\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)dx}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}. \tag{1.2.22}$$

The same result takes another form due to (17)

$$\frac{1}{R}=\frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\phi-\phi_0\right)dx}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}. \tag{1.2.23}$$

## 1.3. Internal mixed boundary value problem

The problem is called internal when the non-zero boundary conditions are prescribed inside a circle.

**Problem 1.** Let us consider a typical problem solved by our method. We need to find a harmonic function $V$, subject to the following boundary conditions at $z=0$:

$$V=v(\rho,\phi), \quad \text{if} \quad \rho\le a, \qquad \partial V/\partial z=0 \quad \text{if} \quad \rho>a. \tag{1.3.1}$$

$$V(\infty)=0. \tag{1.3.2}$$

Here $(\rho,\phi)$ are polar coordinates in the plane $z=0$; and $v$ is a given function.

The problem can be interpreted as an electrostatic one of a charged disc, with a certain potential prescribed on its surface, or it can be interpreted as an elastic contact problem of a circular punch pressed against an elastic half-space; other interpretations are also possible. We call the problem internal because the non-zero conditions are prescribed inside the disc. The potential function $V$ can be represented through a simple layer as follows:

$$V(\rho,\phi,z)=\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0,\phi_0)}{R_0}\rho_0\,d\rho_0\,d\phi_0. \tag{1.3.3}$$

Here

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}, \quad \text{and } \sigma = -\frac{1}{2\pi}\frac{\partial V}{\partial z}\Big|_{z=0}. \tag{1.3.4}$$

Substitution of (1.2.13) in (3) yields, after changing the order of integration

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^{a} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - g^2(x)}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) \tag{1.3.5}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}, \tag{1.3.6}$$

the $\mathcal{L}$-operator is defined by (1.1.4) and (1.1.8), the abbreviations $l_1$ and $l_2$ are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a d\rho_0 \int_0^{l_1(\rho_0)} dx = \int_0^{l_1} dx \int_{g(x)}^{a} d\rho_0 . \tag{1.3.7}$$

Substitution of the boundary condition (1) in (5) leads to the governing integral equation

$$4\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.3.8}$$

Expression (8) is now presented as a sequence of two Abel-type operators and one $\mathcal{L}$-operator. The detailed solution can be found in (Fabrikant, 1991, Sec. 1.3) and it reads:

$$\sigma(y,\phi) = -\frac{1}{\pi^2 y}\mathcal{L}(y)\frac{d}{dy}\int_y^a \frac{t\,dt}{(t^2 - y^2)^{1/2}}\mathcal{L}\left(\frac{1}{t^2}\right)\frac{d}{dt}\int_0^t \frac{\rho d\rho}{(t^2 - \rho^2)^{1/2}}\mathcal{L}(\rho)v(\rho,\phi). \tag{1.3.9}$$

Taking into consideration that $(\rho y/t^2) < 1$, the rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (9) as follows:

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{\sqrt{a^2 - y^2}} - \int_y^a \frac{dt}{(t^2 - y^2)^{1/2}}\frac{d}{dt}\Phi(t,y,\phi)\right]. \tag{1.3.10}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t}\int_0^t \frac{\rho\,\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\mathcal{L}\left(\frac{\rho y}{t^2}\right)v(\rho,\phi)\right]. \qquad (1.3.11)$$

Yet another form of solution is:

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left\{\frac{\Phi(a,y,\phi)}{\sqrt{a^2-y^2}}\right.$$

$$\left. -\frac{1}{2\pi}\int_0^{2\pi}\int_0^a \tan^{-1}\left[\frac{\sqrt{a^2-\rho^2}\sqrt{a^2-y^2}}{a\sqrt{\rho^2+y^2-2\rho y\cos(\phi-\psi)}}\right]\frac{\Delta v(\rho,\psi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi}{\sqrt{\rho^2+y^2-2\rho y\cos(\phi-\psi)}}\right\}. \qquad (1.3.12)$$

The solution obtained here consists of two parts: the first part is singular at the boundary while the second one vanishes at the boundary. In various applications it is required that the solution be nonsingular at the boundary. The necessary and sufficient condition then is $\Phi(a,a,\phi)=0$. In elastic contact problems this condition defines the radius of the contact domain. Notice also that in the case when $v$ is a two-dimensional harmonic function, the non-trivial solution is singular.

Now it is of interest to express the potential $V$ in the half-space directly through its value $v$ prescribed inside the disc $\rho=a$. Substitution of (9) in (5) yields, after subsequent integration:

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\,\mathcal{L}\left(\frac{x^2}{\rho g^2(x)}\right)\frac{\mathrm{d}}{\mathrm{d}g(x)}\int_0^{g(x)}\frac{r\,\mathrm{d}r}{[g^2(x)-r^2]^{1/2}}\,\mathcal{L}(r)v(r,\phi). \qquad (1.3.13)$$

Interchange of the order of integration in (13) and integration with respect to $x$ yields:

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]\frac{z}{R_0^3}\,v(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0. \qquad (1.3.14)$$

Here $R_0$ is defined by (4) and

$$h = \frac{\sqrt{a^2-l_1^2}\sqrt{a^2-\rho_0^2}}{a}. \qquad (1.3.15)$$

Formulae (13) and (14) define the potential function $V$ in the half-space $z \geq 0$, expressed directly through its value $v$ prescribed inside the disc $\rho = a$, $z = 0$. Expression (13) is useful when an explicit evaluation of the integrals is possible, while expression (14) is more convenient for numerical integration.

Note that in the limiting case, when $z = 0$, equation (14) transforms into a known result, namely,

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad \rho \leq a; \quad \text{and}$$

$$V(\rho,\phi,0) = \frac{\sqrt{\rho^2 - a^2}}{\pi^2} \int_0^{2\pi} \int_0^a \frac{v(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\sqrt{a^2 - \rho_0^2}[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]}, \quad \text{for} \quad \rho > a.$$

The solution of the first mixed boundary value problem is completed.

**Problem 2.** We consider now another internal problem, characterized by the following mixed conditions on the boundary $z = 0$:

$$\frac{\partial V}{\partial z} = -2\pi\sigma(\rho,\phi), \quad \text{for} \quad \rho \leq a, \quad \text{and} \quad 0 \leq \phi < 2\pi;$$

$$V = 0, \quad \text{for} \quad \rho > a, \quad \text{and} \quad 0 \leq \phi < 2\pi. \tag{1.3.16}$$

The problem (16) can be interpreted as an electrostatic one of a charged disc $\rho \leq a$ inside an infinite grounded diaphragm $\rho > a$. Mathematically similar problem arises in the consideration of a penny-shaped crack subjected to an arbitrary pressure $\sigma$.

The relationship between $\sigma$ outside the circle $\rho = a$ through its values inside takes the form:

$$\sigma(\rho,\phi) = -\frac{2}{\pi\sqrt{\rho^2 - a^2}} \int_0^a \frac{\sqrt{a^2 - \rho_0^2}\,\rho_0\,d\rho_0}{\rho^2 - \rho_0^2} \mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2\sqrt{\rho^2 - a^2}} \int_0^{2\pi} \int_0^a \frac{\sqrt{a^2 - \rho_0^2}\,\sigma(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}. \tag{1.3.17}$$

Formula (17) allows us to express the potential function $V$ directly through the prescribed value of $\sigma$ as follows:

$$V(\rho,\phi,z) = 4\int_0^a \frac{\mathrm{d}l_2(t)}{[l_2^2(t)-\rho^2]^{1/2}} \int_0^t \frac{\rho_0 \mathrm{d}\rho_0}{(t^2-\rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right)\sigma(\rho_0,\phi). \qquad (1.3.18)$$

An interchange of the order of integration in (18), and integration with respect to $t$ (see 1.2.15), yields:

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^a \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\sigma(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 . \qquad (1.3.19)$$

Numerous examples can be found in (Fabrikant, 1991).

## 1.4. External mixed boundary value problem

The problem is called external when the non-zero boundary conditions are prescribed outside the disc. As in the previous section, we consider two types of problem.

**Problem 1.** It is necessary to find a function, harmonic in the half-space $z \geq 0$, vanishing at infinity, and subject to the mixed boundary conditions on the plane $z = 0$, namely,

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = 0, \quad \text{for} \quad \rho < a, \quad 0 \leq \phi < 2\pi;$$

$$V = v(\rho,\phi), \quad \text{for} \quad \rho \geq a, \quad 0 \leq \phi < 2\pi . \qquad (1.4.1)$$

The problem (1) can be interpreted as an electrostatic one of a charged diaphragm, or as an external elastic contact problem. The governing integral equation is:

$$4\int_\rho^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}} \int_a^x \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \qquad (1.4.2)$$

Its solution is obtained:

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{x\mathrm{d}x}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \qquad (1.4.3)$$

The rules of differentiation allow us to rewrite (3) as follows

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{\sqrt{\rho^2-a^2}} + \int_a^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\frac{\partial}{\partial x}\chi(x,\rho,\phi)\right\}, \tag{1.4.4}$$

where

$$\chi(x,\rho,\phi) = x\int_x^\infty \frac{\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\frac{\partial}{\partial\rho_0}\left[\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)v(\rho_0\phi)\right]. \tag{1.4.5}$$

Further integration with respect to $x$ becomes possible after interchanging the order of integration, with the result:

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{\sqrt{\rho^2-a^2}}\right.$$

$$\left. + \frac{1}{2\pi}\int_0^{2\pi}\int_a^\infty \frac{\Delta v(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}\tan^{-1}\left(\frac{\sqrt{\rho^2-a^2}\sqrt{\rho_0^2-a^2}}{a\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}\right)\right\}. \tag{1.4.6}$$

Now we can express the potential function $V$ directly through its boundary value $v$. The result after the first integration takes the form:

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^\infty \frac{\mathrm{d}l_2(y)}{[l_2^2(y)-\rho^2]^{1/2}}\,\mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^\infty \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-y^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \tag{1.4.7}$$

Interchange of the order of integration and integration with respect to $y$ yields:

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty \frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right]v(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \tag{1.4.8}$$

where the notation was used $j(x)$, defined by

$$j(x) = \frac{\sqrt{\rho_0^2-x^2}\sqrt{l_2^2(x)-x^2}}{x}, \tag{1.4.9}$$

As before, we use the convention $j \equiv j(a)$. In the particular case, when $z = 0$, expression (8) simplifies to:

$$V(\rho,\phi,0)=\frac{1}{\pi^2}\sqrt{a^2-\rho^2}\int_0^{2\pi}\int_a^{\infty}\frac{v(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\sqrt{\rho_0^2-a^2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}, \qquad \text{for} \quad \rho<a;$$

$$V(\rho,\phi,0)=v(\rho,\phi), \quad \text{for} \quad \rho\ge a. \tag{1.4.10}$$

The general solution is completed.

**Problem 2.** We consider the problem of finding a harmonic function, vanishing at infinity, and subject to the mixed conditions on the plane $z=0$:

$$V=0, \quad \text{for} \quad \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=-2\pi\sigma(\rho,\phi), \quad \text{for} \quad \rho>a, \quad 0\le\phi<2\pi. \tag{1.4.11}$$

The problem may be interpreted as an electrostatic one of a charged infinite diaphragm, with a grounded disc inside, or as an external crack problem in elasticity. The values of $\sigma$ inside the circle $\rho=a$ through its values outside are defined by

$$\sigma(\rho,\phi)=-\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}\,\sigma(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{1.4.12}$$

Now one can express the potential $V$ directly through the prescribed $\sigma$. The result is:

$$V(\rho,\phi,z)=4\int_{l_1}^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_{g(x)}^{\infty}\frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \tag{1.4.13}$$

Interchange of the order of integration in (13), and integration with respect to $x$ results in

$$V(\rho,\phi,z)=\frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty}\frac{1}{R_0}\tan^{-1}\!\left(\frac{j}{R_0}\right)\sigma(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \tag{1.4.14}$$

where $R_0$ is defined by (1.3.4), and $j$ stands for $j(a)$, as defined by (9).

The second problem is now solved. Numerous examples can be found in

(Fabrikant, 1991, Sec. 1.4)

## 1.5. Some fundamental integrals

The integrals are called fundamental because of their primary importance to the new method, and also because almost all the integral representations, derived earlier, are just particular cases of the fundamental ones to be evaluated here. We consider three points in the system of cylindrical coordinates, namely, $M(\rho,\phi,z)$, $M_0(\rho_0,\phi_0,z_0)$, and $N(r,\psi,0)$. The following notation is introduced:

$$l_1(t) = \frac{1}{2}\left([(\rho+t)^2+z^2]^{1/2} - [(\rho-t)^2+z^2]^{1/2}\right), \tag{1.5.1}$$

$$l_2(t) = \frac{1}{2}\left([(\rho+t)^2+z^2]^{1/2} + [(\rho-t)^2+z^2]^{1/2}\right), \tag{1.5.2}$$

$$l_{10}(t) = \frac{1}{2}\left([(\rho_0+t)^2+z_0^2]^{1/2} - [(\rho_0-t)^2+z_0^2]^{1/2}\right), \tag{1.5.3}$$

$$l_{20}(t) = \frac{1}{2}\left([(\rho_0+t)^2+z_0^2]^{1/2} + [(\rho_0-t)^2+z_0^2]^{1/2}\right). \tag{1.5.4}$$

According to the earlier convention, $l_{10}$ stands as an abbreviation for $l_{10}(a)$, etc.; $R(\cdot,\cdot)$ denotes the distance between two points.

We consider the following integral:

$$I_1 = \int_0^{2\pi}\int_0^a \frac{1}{R(N,M_0)}\tan^{-1}\left(\frac{h_0}{R(N,M_0)}\right)\frac{z}{R^3(M,N)}\,r\,dr\,d\psi, \tag{1.5.5}$$

where

$$h_0 = \frac{\sqrt{a^2-l_{10}^2}\sqrt{a^2-r^2}}{a}. \tag{1.5.6}$$

The integral can be transformed to (see for details in Fabrikant, 1989, Sec. 1.6):

$$I_1 = 2\pi\int_0^a \frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\,\frac{[x^2-l_{10}^2(x)]^{1/2}}{l_{20}^2(x)-l_{10}^2(x)}\,\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)},\,\phi-\phi_0\right)dx. \tag{1.5.7}$$

It is noteworthy that the integrand in (1.5.7) is symmetric with respect to the points $M$ and $M_0$ while it did not look so in the original expression (1.5.5). The integrand in (1.5.7) is a perfect differential, so that the integral can be evaluated as indefinite:

$$\int \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi - \phi_0\right) \mathrm{d}x$$

$$= \frac{1}{2R_1} \tan^{-1}\left(\frac{\Theta_1(x)}{R_1}\right) + \frac{1}{2R_2} \tan^{-1}\left(\frac{\Theta_2(x)}{R_2}\right), \tag{1.5.8}$$

where

$$R_1 = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + (z - z_0)^2]^{1/2},$$

$$R_2 = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + (z + z_0)^2]^{1/2},$$

$$\Theta_1(x) = \theta(x) + zz_0/\theta(x), \qquad \Theta_2(x) = \theta(x) - zz_0/\theta(x),$$

$$\theta(x) = [x^2 - l_1^2(x)]^{1/2} [x^2 - l_{10}^2(x)]^{1/2}/x. \tag{1.5.9}$$

Correctness of the integral in (1.5.8) can be verified by differentiation. The algebra involved is not trivial. Here we present some intermediate transformations:

$$\frac{\partial}{\partial x}\theta(x) = \frac{\theta(x)\,[l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)]}{x\,[l_2^2(x) - l_1^2(x)]\,[l_{20}^2(x) - l_{10}^2(x)]}, \tag{1.5.10}$$

$$\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi - \phi_0\right) = \frac{l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)}{2x^2}\left[\frac{1}{R_1^2 + \Theta_1^2(x)} + \frac{1}{R_2^2 + \Theta_2^2(x)}\right] \tag{1.5.11}$$

Formula (1.5.8) allows us to evaluate the integral (1.5.5):

$$\int_0^{2\pi}\int_0^a \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\left(\frac{h_0}{R(N,M_0)}\right) r\,\mathrm{d}r\,\mathrm{d}\psi$$

$$= \pi\frac{|z|}{z}\left\{\frac{1}{R_1}\left[\tan^{-1}\left(\frac{\Theta_1}{R_1}\right) - \frac{\pi}{2}\frac{|zz_0|}{zz_0}\right] + \frac{1}{R_2}\left[\tan^{-1}\left(\frac{\Theta_2}{R_2}\right) + \frac{\pi}{2}\frac{|zz_0|}{zz_0}\right]\right\}, \tag{1.5.12}$$

where the contractions $\Theta_1$ and $\Theta_2$ stand for $\Theta_1(a)$ and $\Theta_2(a)$ respectively.

The second fundamental integral to be considered is:

$$I_2 = \int_0^{2\pi}\int_0^a \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{h_0} + \tan^{-1}\left(\frac{h_0}{R(N,M_0)}\right)\right]\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(M,N)}, \tag{1.5.13}$$

where $h_0$ is defined by (1.5.6).
As before, the integral can be reduced to:

$$I_2 = 2\pi\int_0^a \frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\frac{\sqrt{l_{20}^2(x)-x^2}}{l_{20}^2(x)-l_{10}^2(x)}\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right)\mathrm{d}x\,. \tag{1.5.14}$$

Note certain similarity between (1.5.14) and (1.5.7). The integrand in (1.5.14) is a perfect differential, and can be evaluated in elementary functions:

$$\int \frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\frac{\sqrt{l_{20}^2(x)-x^2}}{l_{20}^2(x)-l_{10}^2(x)}\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right)\mathrm{d}x$$

$$= -\frac{1}{2R_1}\tan^{-1}\left(\frac{\Xi_1(x)}{R_1}\right) - \frac{1}{2R_2}\tan^{-1}\left(\frac{\Xi_2(x)}{R_2}\right), \tag{1.5.15}$$

where $R_1$ and $R_2$ are defined by (1.5.9), and

$$\Xi_1(x) = \xi(x) + zz_0/\xi(x)\,, \quad \Xi_2(x) = \xi(x) - zz_0/\xi(x)\,,$$

$$\xi(x) = \sqrt{l_2^2(x)-x^2}\;\sqrt{l_{20}^2(x)-x^2}/x \tag{1.5.16}$$

Correctness of the integration can be verified by differentiation, using (1.5.11) and the property:

$$\frac{\partial}{\partial x}\xi(x) = -\frac{\xi(x)\,[l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)]}{x\,[l_2^2(x)-l_1^2(x)]\,[l_{20}^2(x)-l_{10}^2(x)]}, \tag{1.5.17}$$

Finally, the integral (1.5.13) can be evaluated as follows:

$$\int_0^{2\pi}\int_0^a \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{h_0} + \tan^{-1}\left(\frac{h_0}{R(N,M_0)}\right)\right]\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(M,N)}$$

$$= \pi \frac{|z_0|}{z_0} \left\{ \frac{1}{R_1} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{\Xi_1}{R_1} \right) \right] + \frac{1}{R_2} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{\Xi_2}{R_2} \right) \right] \right\}. \tag{1.5.18}$$

According to our convention, $\Xi_1$ and $\Xi_2$ denote $\Xi_1(a)$ and $\Xi_2(a)$ respectively.

The integrals evaluated above may be called internal because the domain of integration was the interior of a disc. We can also evaluate relevant external integrals. For example, we consider the integral:

$$I_3 = \int_0^{2\pi} \int_a^{\infty} \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\left( \frac{j_0}{R(N,M_0)} \right) r \, \mathrm{d}r \, \mathrm{d}\psi, \tag{1.5.19}$$

where

$$j_0 = (r^2 - a^2)^{1/2} (l_{20}^2 - a^2)^{1/2} / a \, . \tag{1.5.20}$$

The integral can be reduced to

$$I_3 = 2\pi \int_a^{\infty} \frac{\sqrt{l_2^2(x) - x^2}}{l_2^2(x) - l_1^2(x)} \frac{\sqrt{l_{20}^2(x) - x^2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left( \frac{l_1(x) l_{10}(x)}{l_2(x) l_{20}(x)}, \phi - \phi_0 \right) \mathrm{d}x \, . \tag{1.5.21}$$

This is the integral which was already evaluated in (1.5.15), so we may write the final result

$$\int_0^{2\pi} \int_a^{\infty} \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\left( \frac{j_0}{R(N,M_0)} \right) r \, \mathrm{d}r \, \mathrm{d}\psi$$

$$= \pi \frac{|z|}{z} \left\{ \frac{1}{R_1} \left[ \tan^{-1}\left( \frac{\Xi_1}{R_1} \right) - \frac{\pi}{2} \frac{|zz_0|}{zz_0} \right] + \frac{1}{R_2} \left[ \tan^{-1}\left( \frac{\Xi_2}{R_2} \right) + \frac{\pi}{2} \frac{|zz_0|}{zz_0} \right] \right\}, \tag{1.5.22}$$

Comparison with the relevant internal integral (1.5.12) indicates similarity, except for substitution of $\Theta$ by $\Xi$.

The second external integral is

$$I_4 = \int_0^{2\pi}\int_a^{\infty} \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{j_0} + \tan^{-1}\left(\frac{j_0}{R(N,M_0)}\right)\right]\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(M,N)}, \tag{1.5.23}$$

where $j_0$ is defined by (1.5.20).
The integral can be reduced to:

$$I_4 = 2\pi\int_a^{\infty} \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)}\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right)\mathrm{d}x\,. \tag{1.5.24}$$

This integral was evaluated in (1.5.8), and the final result is:

$$\int_0^{2\pi}\int_a^{\infty} \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{j_0} + \tan^{-1}\left(\frac{j_0}{R(N,M_0)}\right)\right]\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(M,N)}$$

$$= \pi\frac{|z_0|}{z_0}\left\{\frac{1}{R_1}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\Theta_1}{R_1}\right)\right] + \frac{1}{R_2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\Theta_2}{R_2}\right)\right]\right\}. \tag{1.5.25}$$

One can notice the same similarity between the internal (1.5.18) and the external (1.5.25) integrals. The similarity goes further. By using the property

$$[l_2^2(x) - x^2][x^2 - l_1^2(x)] = x^2 z^2,$$

we deduce, that for $zz_0 > 0$,

$$\Xi_1(x) = \xi(x) + \theta(x), \qquad \Xi_2(x) = \xi(x) - \theta(x),$$

$$\Theta_1(x) = \theta(x) + \xi(x), \qquad \Theta_2(x) = \theta(x) - \xi(x). \tag{1.5.26}$$

This means that $\Xi_1 = \Theta_1$ and $\Xi_2 = -\Theta_2$ for $zz_0 > 0$. In the case, when $zz_0 < 0$, the relationships change, namely, $\Xi_1 = -\Theta_1$ and $\Xi_2 = \Theta_2$ .

## 1.6. Mixed boundary value problems in spherical coordinates

Exact solution in closed form is obtained to the following mixed problem for a charged sphere: an arbitrary charge density distribution is prescribed at the surface of a spherical cap while an arbitrary potential is given at the rest of the sphere. The new method makes the solution straightforward and elementary, with no special functions or integral transforms involved. A new type of solution is obtained for the Dirichlet problem with discontinuous boundary conditions.

**Integral representation for the reciprocal of the distance between two points in spherical coordinates.** We consider two points in spherical coordinates $M(r,\theta,\phi)$ and $N(a,\theta_0,\phi_0)$. The parameters $l_1$ and $l_2$, which were introduced in section 1.2, have the geometrical interpretation as the difference and the sum of the shortest and the longest distance from a point to the edge of a circular disk. In spherical coordinates the same quantities with respect to a spherical cap can be expressed as

$$m_1(\theta_0,\theta,a,r) = \frac{1}{2}\left(\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} - \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right),$$

$$m_2(\theta_0,\theta,a,r) = \frac{1}{2}\left(\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} + \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right). \quad (1.6.1)$$

The following properties can be easily established:

$$m_1 m_2 = ra\sin\theta\sin\theta_0, \qquad m_1{}^2 + m_2{}^2 = r^2 + a^2 - 2ar\cos\theta\cos\theta_0, \quad (1.6.2)$$

so that the distance between two points $M$ and $N$ can be expressed as $R_0^2 = m_1^2 + m_2^2 - 2m_1m_2\cos(\phi-\phi_0)$. This property allows us to use formulae from section 1.2. For example, we can derive the following integral representations:

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_0^{t_1(\theta_0)} \frac{\lambda\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\tau-\cos\theta}\sqrt{\cos\gamma(\tau)-\cos\theta_0}}, \quad (1.6.3)$$

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_{t_2(\theta_0)}^{\pi} \frac{\lambda\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta-\cos\tau}\sqrt{\cos\theta_0-\cos\gamma(\tau)}}, \quad (1.6.4)$$

where

$$t_1 \equiv t_1(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_1(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_1(\theta_0)}{m_2(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2},$$

$$t_2 \equiv t_2(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_2(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_2(\theta_0)}{m_1(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2}, \qquad (1.6.5)$$

$$\cos\gamma(\tau) = \cos\tau - \frac{(r-a)^2}{4ar}\frac{\sin^2\tau}{\cos\tau-\cos\theta}, \qquad (1.6.6)$$

and hereafter $m_1(x)$ and $m_2(x)$ are understood as abbreviations for $m_1(x,\theta,a,r)$ and $m_2(x,\theta,a,r)$ respectively. It is possible to show that both $t_1$ and $t_2$ are inverse to $\gamma$, i.e. $\gamma[t_{1,2}(\theta_0)]=\theta_0$. Note also that $t_1 \le \min(\theta,\theta_0)$ and $t_2 \ge \max(\theta,\theta_0)$. We can see certain analogy between the notations and their properties used in cylindrical coordinates and those in spherical coordinates: $l$ corresponds to $t$, $g$ corresponds to $\gamma$, etc.

By using analogy with section 1.2, we can derive the following indefinite integrals:

$$\int \frac{\lambda\left(\dfrac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\tau-\cos\theta}\sqrt{\cos\gamma(\tau)-\cos\theta_0}} = -\frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_1(\tau)}{R_0}\right), \qquad (1.6.7)$$

$$\int \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta-\cos\tau}\sqrt{\cos\theta_0-\cos\gamma(\tau)}} = \frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_2(\tau)}{R_0}\right), \qquad (1.6.8)$$

where

$$y_1(\tau) = 2\sqrt{ar}\sqrt{\cos\tau-\cos\theta}\sqrt{\cos\gamma(\tau)-\cos\theta_0}/\sin\tau,$$

$$y_2(\tau) = 2\sqrt{ar}\sqrt{\cos\theta-\cos\tau}\sqrt{\cos\theta_0-\cos\gamma(\tau)}/\sin\tau, \qquad (1.6.9)$$

$$R_0^2 = r^2 + a^2 - 2ar[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)]. \qquad (1.6.10)$$

The integrals in (7) and (8) can be verified by using the identity

$$\lambda\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right) = \frac{\sin\tau\; y_1(\tau)}{R_0^2+y_1^2(\tau)}\frac{dy_1(\tau)}{d\tau}. \tag{1.6.11}$$

A similar relationship can be established to verify (8).

**Formulation of the problem.** We consider the following general problem: it is necessary to find the electrostatic field of a charged sphere of radius $a$ when an arbitrary potential $v$ is prescribed over a spherical cap $0\le\theta\le\alpha$, while an arbitrary charge distribution $\sigma$ is given at the rest of the sphere. As before, we represent the potential through a simple layer

$$V(r,\theta,\phi) = \int_0^{2\pi} d\phi_0 \int_0^{\alpha} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, d\theta_0}{R_0} + \int_0^{2\pi} d\phi_0 \int_{\alpha}^{\pi} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, d\theta_0}{R_0}, \tag{1.6.12}$$

where $R_0$ is defined by (10).

Substitution of (3) and (4) in (12) yields, after interchanging the order of integration

$$V(r,\theta,\phi) = \frac{2a^2}{\sqrt{ar}}\left\{\int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau-\cos\theta}} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\gamma(\tau)-\cos\theta_0}}\, \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi)\right.$$

$$\left. + \int_{t_2(\alpha)}^{\pi} \frac{d\tau}{\sqrt{\cos\theta-\cos\tau}} \int_{\alpha}^{\gamma(\tau)} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0-\cos\gamma(\tau)}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)\right\}. \tag{1.6.13}$$

It is convenient at this stage to split our problem in two: *i*) to find the electrostatic potential of a charged sphere when an arbitrary charge density is given at a spherical cap, and the zero potential is prescribed elsewhere; *ii*) to find the potential when the zero charge density is prescribed at a spherical cap, and an arbitrary potential is given elsewhere. Both problems are treated separately.

**Problem 1.** We consider the boundary value problem, with the following mixed conditions at $r=a$:

$$\sigma = \sigma(\theta,\phi), \quad \text{for} \quad 0\le\phi<2\pi, \quad 0\le\theta<\alpha;$$

$$V(a,\theta,\phi) = 0, \quad \text{for} \quad 0\le\phi<2\pi, \quad \alpha<\theta\le\pi. \tag{1.6.14}$$

In this case, the value of $\sigma$ in the domain $\alpha<\theta\le\pi$ can be expressed through its values over $0 \le \theta < \alpha$ as follows:

$$\sigma(\theta,\phi) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}\ \sin\theta_0\, d\theta_0}{\cos\theta_0 - \cos\theta}\, \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta/2)}\right)\sigma(\theta_0,\phi). \tag{1.6.15}$$

Expression (15) can also be rewritten in the form

$$\sigma(\theta,\phi) = -\frac{1}{2\pi^2\sqrt{\cos\alpha - \cos\theta}} \int_0^{2\pi}\int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}\ \sigma(\theta_0,\phi_0)\sin\theta_0\, d\theta_0\, d\phi_0}{1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\phi - \phi_0)}, \tag{1.6.16}$$

which corresponds to the Green's function found in a geometric form by Lord Kelvin who used his method of images. Certain simplification occurs in the case of axial symmetry, namely,

$$\sigma(\theta) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}}{\cos\theta_0 - \cos\theta}\,\sigma(\theta_0)\sin\theta_0\, d\theta_0\,. \tag{1.6.17}$$

The charge density is now known all over the sphere and we can find the value of potential:

$$V(r,\theta,\phi) = \frac{2a^2}{\sqrt{ar}} \int_{t_2(0)}^{t_2(\alpha)} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_0^{\gamma(\tau)} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\gamma(\tau)}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)\,. \tag{1.6.18}$$

Interchange in the order of integration in (18) and subsequent integration with respect to $\tau$ (see (8)) results in

$$V(r,\theta,\phi) = \frac{2}{\pi}\int_0^{2\pi} d\phi_0 \int_0^\alpha \frac{\sigma(\theta_0,\phi_0)}{R_0}\tan^{-1}\left(\frac{\xi}{R_0}\right) a^2\sin\theta_0\, d\theta_0\,, \tag{1.6.19}$$

where $R_0$ is defined by (10) and

$$\xi = \frac{\sqrt{2}\sqrt{\cos\theta_0 - \cos\alpha}\sqrt{m_2^2(\alpha) - \cos^2(\alpha/2)\, m_2^2(0)}}{\sin\alpha}\,. \tag{1.6.20}$$

Expressions (18) and (19) give two equivalent solutions to the problem 1, the first one being more convenient for the exact evaluation of the integrals involved, while the second one has certain advantages when a numerical integration is required.

**Problem 2.** We consider a charged sphere with the following boundary conditions at its surface $r=a$:

$$\sigma(\theta,\phi)=0, \quad \text{for} \quad 0\le\phi<2\pi, \quad 0\le\theta<\alpha;$$

$$V(a,\theta,\phi)=v(\theta,\phi), \quad \text{for} \quad 0\le\phi<2\pi, \quad \alpha\le\theta\le\pi. \tag{1.6.21}$$

The following integral equation results after substituting (21) in (13):

$$2a\int_{\theta}^{\pi}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{\alpha}^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.6.22}$$

Its solution is:

$$\sigma(\theta_2,\phi)=-\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a\sin\theta_2}\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.6.23}$$

Expression (23) can be simplified as follows:

$$\sigma(\theta_2,\phi)=-\frac{1}{2\pi^2 a}\left\{\frac{\Phi(\alpha,\theta_2,\phi)}{\sqrt{\cos\alpha-\cos\theta_2}}+\int_{\alpha}^{\theta_2}\frac{d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\frac{\partial}{\partial\theta_1}\Phi(\theta_1,\theta_2,\phi)\right\}. \tag{1.6.24}$$

Here

$$\Phi(\theta_1,\theta_2,\phi)=2\tan\frac{\theta_1}{2}\int_{\theta_1}^{\pi}\frac{\cos(\theta/2)\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\frac{d}{d\theta}\left[\cos\left(\frac{\theta}{2}\right)\mathcal{L}\left(\frac{\tan^2(\theta_1/2)}{\tan(\theta/2)\tan(\theta_2/2)}\right)v(\theta,\phi)\right]. \tag{1.6.25}$$

Formulae (23) and (24)–(25) give two equivalent forms of solution of the integral equation (22). We note two different terms in (24): the first one is singular at $\theta_2\to\alpha$, while the second one tends to zero at $\theta_2\to\alpha$.

The potential in space due to a charged sphere can be obtained by substitution of (23) into (13). The result is:

$$V(r,\theta,\phi) = -\frac{\sqrt{a}}{\pi\sqrt{r}} \int_{t_2(\alpha)}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \mathcal{L}\left(\frac{\tan^2[\gamma(\tau)/2]\tan(\theta/2)}{\tan^2(\tau/2)}\right)$$

$$\times \frac{\partial}{\partial\gamma(\tau)} \int_{\gamma(\tau)}^{\pi} \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \tag{1.6.26}$$

Interchanging the order of integration in (26), we obtain, after subsequent integration with respect to $\tau$,

$$V(r,\theta,\phi) = \frac{a|r^2 - a^2|}{2\pi^2} \int_0^{2\pi} d\phi_0 \int_\alpha^{\pi} \left[\frac{R_0}{\chi} + \tan^{-1}\left(\frac{\chi}{R_0}\right)\right] \frac{v(\theta_0,\phi_0)}{R_0^3} \sin\theta_0 \, d\theta_0 \,, \tag{1.6.27}$$

where

$$\chi = y_1[t_1(\alpha)] = \frac{2\sqrt{ar}\sqrt{\cos t_1(\alpha) - \cos\theta}\sqrt{\cos\alpha - \cos\theta_0}}{\sin t_1(\alpha)} . \tag{1.6.28}$$

It can be proven that (28) can be obtained from (20) by a formal substitution of $\theta_0$, $\theta$, and $\alpha$ by $\pi - \theta_0$, $\pi - \theta$, and $\pi - \alpha$ respectively.

The following identities can be useful in transformations:

$$[\cos\theta - \cos t_2(x)][\cos t_1(x) - \cos\theta] = \left(\frac{a-r}{a+r}\right)^2 \sin^2\theta,$$

$$[\cos x - \cos t_2(x)][\cos t_1(x) - \cos x] = \left(\frac{a-r}{a+r}\right)^2 \sin^2 x \,,$$

$$\frac{\partial t_2(x)}{\partial x} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos t_1(x) - \cos\theta]} \frac{\partial t_1(x)}{\partial x} = \frac{\sin x\,[\cos\theta - \cos t_2(x)]}{\sin\theta\,[\cos x - \cos t_2(x)]} \frac{\partial t_1(x)}{\partial x} \,,$$

$$\sin t_1(x) \sin t_2(x) = \frac{4ar}{(a+r)^2} \sin x \sin\theta,$$

$$\frac{\sin t_1(x)}{\sin t_2(x)} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos\theta - \cos t_2(x)]} = \frac{\sin x\,[\cos t_1(x) - \cos\theta]}{\sin\theta\,[\cos x - \cos t_2(x)]} \,,$$

$$\cos t_1(x) \cos t_2(x) = \frac{4ar\cos\theta\cos x - (r-a)^2}{(r+a)^2} \,,$$

$$\cos t_1(x) + \cos t_2(x) = \frac{4ar}{(r+a)^2}(\cos x + \cos\theta),$$

$$\frac{[\cos t_1(x) - \cos x]\,[\cos t_1(x) - \cos\theta]}{\sin^2 t_1(x)} = \frac{[\cos x - \cos t_2(x)]\,[\cos\theta - \cos t_2(x)]}{\sin^2 t_2(x)} = \frac{(a-r)^2}{4ar}.$$

Certain integral characteristics can be evaluated without solving any integral equation. For example, the total charge in Problem 1 can be found by integration of both sides of (15), with the result:

$$Q_1 = \frac{2}{\pi} a^2 \int_0^{2\pi} d\phi \int_0^{\alpha} \sigma(\theta,\phi) \cos^{-1}\left(\frac{\cos(\alpha/2)}{\cos(\theta/2)}\right) \sin\theta \, d\theta. \qquad (1.6.29)$$

The total charge in Problem 2 can be obtained from (23) as

$$Q_2 = \frac{a}{2\pi^2} \int_0^{2\pi} d\phi \int_{\alpha}^{\pi} \left[\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha - \cos\theta}} + \tan^{-1}\left(\frac{\sqrt{\cos\alpha - \cos\theta}}{\sqrt{1-\cos\alpha}}\right)\right] v(\theta,\phi) \sin\theta \, d\theta. \qquad (1.6.30)$$

**Dirichlet problem with discontinuous boundary conditions.** In many practical cases, the boundary conditions for Dirichlet problem are changing so rapidly that they can be modeled as discontinuous. The spherical harmonic expansion solution converges very badly in those cases, and it is usually divergent on the surface of the sphere, thus making the solution unfit for practical purposes. On the other hand, the closed form solution, given by Poisson, is very inconvenient for practical evaluation of the integrals. The new method allows us to obtain an alternative solution, which is equivalent to the one obtained by Poisson, but is easy amenable for the exact evaluations of the integrals involved.

We consider a charged sphere with the following conditions at its surface $r = a$:

$$V(a,\theta,\phi) = v(\theta,\phi), \quad \text{for} \quad 0 \le \theta \le \alpha, \quad 0 \le \phi < 2\pi;$$

$$V(a,\theta,\phi) = 0, \quad \text{for} \quad \alpha \le \theta \le \pi, \quad 0 \le \phi < 2\pi. \qquad (1.6.31)$$

The problem, in a sense, is inverse to problem 1, therefore, substitution of (31) in (18) leads to the governing integral equation

$$2a\int_{\theta}^{\alpha}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{0}^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.6.32}$$

The exact solution of (32) can be obtained in the form:

$$\sigma(\theta_2,\phi)=\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a\sin\theta_2}\frac{d}{d\theta_2}\int_{0}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.6.33}$$

Expression (33) is valid in the interval $0\le\theta_2\le\alpha$. The charge density distribution at the rest of the sphere can be obtained by substitution of (33) in (15), with the result for $\alpha<\theta<\pi$:

$$\sigma(\theta,\phi)=\frac{\mathcal{L}[\cot(\theta/2)]}{4\pi^2 a}\int_{0}^{\alpha}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta_1-\cos\theta)^{3/2}}\,\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_1-\cos\theta_0}}\,\mathcal{L}\left(\cot\frac{\theta_0}{2}\right)v(\theta_0,\phi). \tag{1.6.34}$$

The elementary analysis of (33) and (34) shows that both charge density distributions have non-integrable singularities of opposite sign at $\theta\to\alpha$, when $v(\alpha-0,\phi)\neq 0$, otherwise expression (33) has no singularities, and formula (34) can give an integrable singularity. The total charge can be obtained by integration (33) and (34), with the result:

$$Q=\frac{a}{4\pi}\int_{0}^{2\pi}d\phi\int_{0}^{\alpha}v(\theta,\phi)\sin\theta\,d\theta\,. \tag{1.6.35}$$

The potential in space due to a charged sphere can be obtained by substitution of (33) in (18) which yields, after simplification,

$$V(r,\theta,\phi) = -\frac{\sqrt{a}}{\pi\sqrt{r}} \int_{t_2(0)}^{t_2(\alpha)} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan^2[\gamma(\tau)/2]}{\tan^2(\tau/2)}\right)$$

$$\times \frac{\partial}{\partial\gamma(\tau)} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}}\, \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \qquad (1.6.36)$$

Interchange of the order of integration in (36) and subsequent integration with respect to $\tau$ result in the well-known Poisson formula, namely,

$$V(r,\theta,\phi) = -\frac{a|r^2 - a^2|}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^{\alpha} \frac{v(\theta_0,\phi_0)}{R_0^3} \sin\theta_0\, d\theta_0\,. \qquad (1.6.37)$$

Expression (36) is equivalent to (37) and has definite advantages when an exact evaluation of the integrals is possible.

**Influence of a point charge.** We consider the interaction between a point charge $q$ located at the point with spherical coordinates $(r_0,\theta_0,\phi_0)$ and a grounded spherical cap $\alpha \le \theta \le \pi$ of radius $a$. The induced charge density distribution is given for $\alpha < \theta < \pi$ as

$$\sigma(\theta,\phi) = -q\, \frac{\mathcal{L}[\cot(\theta/2)]}{2\pi^2 a \sin\theta} \frac{d}{d\theta} \int_{\alpha}^{\theta} \frac{\sin\theta_1\, d\theta_1}{\sqrt{\cos\theta_1 - \cos\theta}}$$

$$\times \lambda\left(\frac{\tan^2(\theta_1/2)\tan(\theta_0/2)}{\tan^2[t_{20}(\theta_1)/2]}, \phi - \phi_0\right) \frac{\partial t_{20}(\theta_1)/\partial\theta_1}{\sqrt{ar_0}\sqrt{\cos\theta_0 - \cos t_{20}(\theta_1)}}\,. \qquad (1.6.38)$$

The integral in (38) can be evaluated exactly in the same manner as before, with the result

$$\sigma(\theta,\phi) = -\frac{q|r^2 - a^2|}{2\pi^2 a R^3} \left[\frac{R}{\chi_0} + \tan^{-1}\left(\frac{\chi_0}{R}\right)\right], \qquad (1.6.39)$$

where

$$\chi_0 = \frac{2\sqrt{ar_0}\sqrt{\cos t_{10}(\alpha) - \cos\theta_0}\sqrt{\cos\alpha - \cos\theta}}{\sin t_{10}(\alpha)}\,, \qquad (1.6.40)$$

$$t_{10}(x) \equiv t_1(x, \theta_0, a, r_0), \qquad t_{20}(x) \equiv t_2(x, \theta_0, a, r_0),$$

In the particular case $r_0 \to a$ and $\theta_0 < \alpha$, formula (39) simplifies as

$$\sigma(\theta, \phi) = -q \frac{\sqrt{\cos\theta_0 - \cos\alpha}}{2\pi^2 \sqrt{\cos\alpha - \cos\theta}} \frac{1}{R^2}, \tag{1.6.41}$$

which is in agreement with (16). Expression (39) is convenient for a direct evaluation of the induced charge density distribution but, if some further mathematical transformations are needed, then the equivalent formula (38) has definite advantages. For example, to evaluate the total charge $Q$ using (39) would be quite difficult, while (38) gives:

$$Q = -\frac{q}{\pi r_0}[(a + r_0)\sin^{-1}A_{10} - |a - r_0|\sin^{-1}A_{20}] \tag{1.6.42}$$

where

$$A_{10} = \frac{(r_0 + a)\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha) + 4ar_0\cos^2(\theta/2)\cos^2(\alpha/2)}},$$

$$A_{20} = \frac{|r_0 - a|\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha) - 4ar_0\sin^2(\theta/2)\cos^2(\alpha/2)}}, \qquad m_{20}(\alpha) = m_2(\alpha, \theta, a, r_0). \tag{1.6.43}$$

When the point charge is located at the axis ($\theta_0 = \pi$), formula (42) simplifies as

$$Q = \frac{q}{\pi r_0}\left[|r_0 - a|\left(\frac{\pi - \alpha}{2}\right) - (r_0 + a)\tan^{-1}\left(\frac{r_0 + a}{|r_0 - a|}\cot\frac{\alpha}{2}\right)\right]. \tag{1.6.44}$$

In the case of a complete sphere we have $\alpha = 0$, and formula (44) simplifies further

$$Q = \frac{q}{2r_0}[|r_0 - a| - (r_0 + a)].$$

Similar formulae can be obtained for $\theta_0 = 0$.

The potential $V_c$ due to the induced charge can be obtained by utilization of (4) in (26) which gives, after the first integration

$$V_c(r,\theta,\phi) = -\frac{q}{\pi\sqrt{rr_0}} \int_\alpha^\pi \frac{\lambda\left(\dfrac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi-\phi_0\right) t_2' t_{20}' \, dx}{\sqrt{\cos\theta - \cos t_2}\sqrt{\cos\theta_0 - \cos t_{20}}} . \qquad (1.6.45)$$

Here the abbreviations $t_1$, $t_2$, $t_{10}$, and $t_{20}$ are understood as $t_1(x)$, $t_2(x)$, $t_{10}(x)$, $t_{20}(x)$ respectively, the prime signs indicate the partial derivatives with respect to $x$. The integral in (45) can be evaluated exactly, and the final result is:

$$V(r,\theta,\phi) = V_q + V_c = q\left\{\frac{1}{2R_1}\left[1+\frac{2}{\pi}\tan^{-1}\left(\frac{\eta_1(\alpha)}{R_1}\right)\right] - \frac{1}{2R_2}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{\eta_2(\alpha)}{R_2}\right)\right]\right\}, \qquad (1.6.46)$$

which is in agreement with a similar result of Hobson (1900) in toroidal coordinates. The notations are

$$R_1^2 = r^2 + r_0^2 - 2rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)] ,$$

$$R_2^2 = \frac{r^2 r_0^2}{a^2} + a^2 - 2rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)] ,$$

$$\eta_{1,2} = \frac{(r+a)(r_0+a)}{2a} S(x) \pm \frac{(r-a)(r_0-a)}{2aS(x)} ,$$

$$S(x) = \frac{\sqrt{\cos t_1 - \cos x}\sqrt{\cos t_{10} - \cos x}}{\sin x} . \qquad (1.6.47)$$

The following identities were used to perform integration in (45)

$$\lambda\left(\frac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi-\phi_0\right) = \frac{(a+r)^2(a+r_0)^2}{16a^2\sin^2 x}[(1-\cos t_1\cos t_2)(\cos t_{10} - \cos t_{20})$$

$$+ (1-\cos t_{10}\cos t_{20})(\cos t_1 - \cos t_2)]\left[\frac{1}{R_1^2+\eta_1^2(x)} + \frac{1}{R_2^2+\eta_2^2(x)}\right], \qquad (1.6.48)$$

$$R_1^2 + \eta_1^2(x) = R_2^2 + \eta_2^2(x), \qquad (1.6.49)$$

$$\frac{\partial}{\partial x} S(x) = \frac{S(x)}{2\sin x}\left[\frac{1-\cos t_1\cos t_2}{\cos t_1 - \cos t_2} + \frac{1-\cos t_{10}\cos t_{20}}{\cos t_{10} - \cos t_{20}}\right], \qquad (1.6.50)$$

$$\frac{\partial t_2}{\partial x} = \frac{2\sqrt{ar}}{a+r} \frac{\sqrt{\cos t_1 - \cos x}\sqrt{\cos\theta - \cos t_2}}{\cos t_1 - \cos t_2}. \tag{1.6.51}$$

An expression similar to (51) can be written for the derivative of $t_{20}$. Substitution of (48)–(51) in (45) makes the procedure of integration very simple.

## 1.7. Mixed boundary value problems in toroidal coordinates

Further extension of previously obtained results to the case of toroidal coordinates is presented here. It is based on a special integral representation for the reciprocal of the distance between two points. Its substitution in the governing integral equation reduces the problem to sequence of two consecutive Abel type operators combined with the $\mathcal{L}$-operator. Each can be inverted exactly and in closed form, thus giving the solution. Some integrals of fundamental value, involving distances between several points, are established. The complete set of systems of coordinates, where the new method can be applied, is not known at this time and can constitute a subject for a separate investigation.

**Mathematical preliminaries.** The following relationships exist between the Cartesian $(x, y, z)$ and toroidal $(v, u, \phi)$ coordinates

$$x = \frac{c\sinh v\cos\phi}{\cosh v - \cos u}, \quad y = \frac{c\sinh v\sin\phi}{\cosh v - \cos u}, \quad z = \frac{c\sin u}{\cosh v - \cos u}. \tag{1.7.1}$$

Here $c$ is a dimensional parameter. The surfaces $u=$constant are spherical caps

$$x^2 + y^2 + (z - c\cot u)^2 = \left(\frac{c}{\sin u}\right)^2, \tag{1.7.2}$$

with the common line of intersection along the circle $\rho = c$, $z = 0$. The surfaces $v=$constant are tori

$$(\sqrt{x^2+y^2} - c\coth v)^2 + z^2 = \left(\frac{c}{\sinh v}\right)^2. \tag{1.7.3}$$

The properties of toroidal coordinates allow us to use this system of coordinates for solving mixed boundary value problems for various geometries including the case of several spherical caps.

Let the toroidal coordinates of points $M$ and $N$ be $(v, u, \phi)$ and $(x, \beta, \psi)$ respectively. The distance between two points in toroidal coordinates is

$$R_0 \equiv R(M,N) = \frac{\sqrt{2}c\sqrt{\cosh v\cosh x - \sinh v\sinh x\cos(\phi-\psi) - \cos(u-\beta)}}{\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}. \tag{1.7.4}$$

We can establish validity of the following integral:

$$\int \frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}, \phi-\psi\right)d\tau}{\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma(\tau)}} = -\frac{2c}{R_0\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}$$

$$\times\tan^{-1}\left[\frac{2c\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma(\tau)}}{R_0\sinh\tau\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}\right]. \tag{1.7.5}$$

Here

$$\cosh\gamma \equiv \cosh\gamma(\tau) \equiv \cosh\gamma(\tau,\ \beta,\ v,\ u) = \cosh\tau + \sin^2\left(\frac{u-\beta}{2}\right)\frac{\sinh^2\tau}{\cosh v - \cosh\tau}, \tag{1.7.6}$$

We intentionally use in this section the same notation $t_1$, $t_2$, and $\gamma$ in order to demonstrate certain analogy between the toroidal and spherical coordinates. We hope the reader will not be confused. Introduce the following notation:

$$t_1 \equiv t_1(x,\beta,v,u) = 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)} - \sqrt{\cosh(x-v)-\cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\}, \tag{1.7.7}$$

$$t_2 \equiv t_2(x,\beta,v,u) = 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)} + \sqrt{\cosh(x-v)-\cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\}, \tag{1.7.8}$$

For the brevity sake, we use the following conventions: the parameters of $\gamma$, $t_1$ and $t_2$, given respectively in (6), (7), and (8), are considered as the default parameters. This would allow us to write, for example, $\gamma\ (y,\delta)$ instead of $\gamma\ (y,\delta,v,u)$. The rule is rather simple: the parameters which are not given explicitly assumed to be the default ones.

Notice that both $t_1$ and $t_2$ are inverse to $\gamma$. This means that $\gamma(t_1)=x$ and $\gamma(t_2)=x$. The following properties are valid: $t_1 \le \min(v,x)$, $t_2 \ge \max(v,x)$; the equality sign holds for $u=\beta$. By using previous results we can obtain the following integral representation for the reciprocal of the distance between two points:

$$\frac{1}{R_0} = \frac{\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}{\pi c} \int_0^{t_1} \frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}, \phi - \psi\right) d\tau}{\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma}}. \tag{1.7.9}$$

We can derive several variations of (9). For example, introducing a new variable $\tau = t_1(y)$, expression (9) will take the form

$$\frac{1}{R_0} = \frac{\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}{\pi c} \int_0^{x} \frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)}, \phi - \psi\right) t_1'(y)\, dy}{\sqrt{\cosh v - \cosh t_1(y)}\sqrt{\cosh x - \cosh y}}. \tag{1.7.10}$$

Here the symbol (′) stands for the partial derivative with respect to the parameter in brackets. By using (A12), one can rewrite (10) as

$$\frac{1}{R_0} = \frac{\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}{\pi c\, |\cos[(u-\beta)/2]|}$$

$$\times \int_0^{x} \frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)}, \phi - \psi\right)\sqrt{\cosh t_2(y) - \cosh y}\; dy}{[\cosh t_2(y) - \cosh t_1(y)]\sqrt{\cosh x - \cosh y}}. \tag{1.7.11}$$

We can also compute a more general indefinite integral, namely,

$$I_1 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$

$$\times \int \frac{\sqrt{\cosh t_2 - \cosh x}\sqrt{\cosh t_{20} - \cosh x}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})}\lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) dx\,. \tag{1.7.12}$$

Here $t_{10} = t_1(x, \beta, v_0, u_0)$ and $t_{20} = t_2(x, \beta, v_0, u_0)$ respectively. Introduce new variables

$$\eta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)S(x) \pm \frac{1}{S(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.7.13}$$

where

$$S(x)=\frac{\sqrt{\cosh t_2-\cosh x}\sqrt{\cosh t_{20}-\cosh x}}{\sinh x} \tag{1.7.14}$$

The following identities may be established by using formulae from Appendix:

$$\frac{\mathrm{d}S(x)}{\mathrm{d}x}=-\frac{S(x)}{2\sinh x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right], \tag{1.7.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{S(x)}\right)=\frac{1}{2\,S(x)\sinh x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right], \tag{1.7.16}$$

$$\lambda\left(\frac{\tanh(t_1/2\,\tanh(t_{10}/2))}{\tanh(t_2/2)\,\tanh(t_{20}/2)},\,\phi-\phi_0\right)=\frac{\cos^2[(u-\beta)/2]\,\cos^2[(u_0-\beta)/2]}{2\sinh^2 x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}\right.$$

$$\left.+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right](\cosh t_2-\cosh t_1)\,(\cosh t_{20}-\cosh t_{10})\left[\frac{1}{\cosh w-\cos(u-u_0)+2\eta_1^2}\right.$$

$$\left.+\frac{1}{\cosh w-\cos(u+u_0-2\beta)+2\eta_2^2}\right]. \tag{1.7.17}$$

Here

$$\cosh w=\cosh v\,\cosh v_0-\sinh v\,\sinh v_0\cos(\phi-\phi_0). \tag{1.7.18}$$

The transformations leading to (15)–(17) are very non-trivial. One has to use the appropriate formulae from Appendix in an ingenious way in order to repeat the results. Taking into consideration that

$$\frac{\mathrm{d}\eta_1(x)}{\mathrm{d}x}=\frac{\eta_2(x)}{S(x)}\frac{\mathrm{d}S(x)}{\mathrm{d}x},\qquad \frac{\mathrm{d}\eta_2(x)}{\mathrm{d}x}=\frac{\eta_1(x)}{S(x)}\frac{\mathrm{d}S(x)}{\mathrm{d}x},$$

$$\cosh w-\cos(u-u_0)+2\eta_1^2=\cosh w-\cos(u+u_0-2\beta)+2\eta_2^2, \tag{1.7.19}$$

the substitution of (15)–(17) and (19) in (12) leads to

$$I_1=-\int\left[\frac{\mathrm{d}\eta_1}{\cosh w-\cos(u-u_0)+2\eta_1^2}+\frac{\mathrm{d}\eta_2}{\cosh w-\cos(u+u_0-2\beta)+2\eta_2^2}\right]. \tag{1.7.20}$$

The last integral can be computed in an elementary way, and the final result is:

$$I_1 = -\frac{1}{\sqrt{2[\cosh w - \cos(u-u_0)]}} \tan^{-1}\left[\frac{\sqrt{2}\eta_1(x)}{\sqrt{\cosh w - \cos(u-u_0)}}\right]$$

$$-\frac{1}{\sqrt{2[\cosh w - \cos(u+u_0-2\beta)]}} \tan^{-1}\left[\frac{\sqrt{2}\eta_2(x)}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\right]. \tag{1.7.21}$$

Here the reader may ask us two questions. First, why have we decided that the integral (12) is computable, and second, how did we come up with expressions (13) and the properties (15)–(17)? The integral (12) was encountered in solving the problem of influence of a point charge on a spherical bowl which, as we know, has an elementary solution. This meant that the integral (12) has to be computable. The hints on how to compute it can be taken from similar integral in section 1.6. One has just to replace the appropriate trigonometric functions by the hyperbolic ones.

Yet another integral can be computed in a similar manner, namely,

$$I_2 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$

$$\times \int \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh x - \cosh t_{10}}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})} \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) \mathrm{d}x . \tag{1.7.22}$$

The same integral (22) can be rewritten as

$$I_2 = \int \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) \frac{t_2'(x) t_{20}'(x)\,\mathrm{d}x}{\sqrt{\cosh t_2 - \cosh v}\sqrt{\cosh t_{20} - \cosh v_0}} . \tag{1.7.23}$$

Introduction of new variables

$$\Theta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)T(x) \pm \frac{1}{T(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.7.24}$$

with

$$T(x) = \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh x - \cosh t_{10}}}{\sinh x},$$

and use of the identities (17) and

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = \frac{T(x)}{2\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.7.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{T(x)}\right)=-\frac{1}{2\,T(x)\sinh x}\left[\frac{\cosh t_1\cosh t_2-1}{\cosh t_2-\cosh t_1}+\frac{\cosh t_{10}\cosh t_{20}-1}{\cosh t_{20}-\cosh t_{10}}\right], \qquad (1.7.26)$$

allow us to compute the integral

$$I_2=\frac{1}{\sqrt{2[\cosh w-\cos(u-u_0)]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_1(x)}{\sqrt{\cosh w-\cos(u-u_0)}}\right]$$

$$+\frac{1}{\sqrt{2[\cosh w-\cos(u+u_0-2\beta)]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_2(x)}{\sqrt{\cosh w-\cos(u+u_0-2\beta)}}\right]. \qquad (1.7.27)$$

One may deduce from (A2) that

$$T(x)=\left|\tan\left(\frac{u-\beta}{2}\right)\tan\left(\frac{u_0-\beta}{2}\right)\right|\frac{1}{S(x)}.$$

This property gives us various relationships between $\eta$ and $\Theta$ depending on the signs of the trigonometric functions. For example, when $\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]>0$ and $\sin[(u-\beta)/2]\sin[(u_0-\beta)/2]>0$, we have $\eta_1=\Theta_1$ and $\eta_2=-\Theta_2$. The derived integrals will be used in solving various mixed boundary value problems.

**Problem description.** We consider two spherical caps defined in the toroidal coordinates $(v,u,\phi)$ as follows:

$$0\le v\le b_0, \qquad u=u_0, \qquad 0\le\phi\le 2\pi;$$

$$0\le v\le b, \qquad u=\beta, \qquad 0\le\phi\le 2\pi. \qquad (1.7.28)$$

In the limiting case $b\to\infty$ and $b_0\to\infty$ the spherical caps intersect along a circle of radius $c$ which is the basic circle of the system of coordinates. We consider an electrostatic problem when an arbitrary charge distribution $\sigma$ is prescribed on the first spherical cap ($u=u_0$), and an arbitrary potential distribution $V$ is given on the surface of the second cap. It is then necessary to find the electrostatic field in the whole space. It is convenient to split the problem in two: the first problem assumes that $\sigma=0$ and $V\neq 0$, while in the second problem we take $V=0$ and $\sigma\neq 0$. The linear superposition of the two solutions would give us the general solution to the problem.

**Problem 1.** Since the first cap is not charged, we have to solve the Dirichlet problem for a spherical cap with the following condition on its surface:

$$V = V(v,\phi)\,, \qquad \text{for} \quad 0 \le v \le b, \qquad u = \beta\,, \qquad 0 \le \phi \le 2\pi\,. \tag{1.7.29}$$

The governing integral equation is:

$$V(v,\phi) = 2c\sqrt{\cosh v - \cos\beta}\int_0^v \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}\int_\tau^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}\,. \tag{1.7.30}$$

Here $\sigma$ is the charge distribution. The integral equation (30) represents a sequence of two Abel type operators and the $\mathcal{L}$-operator. Each can be inverted in a manner similar to the one employed in previous sections. The final result is:

$$\sigma(s,\phi) = -\frac{(\cosh s - \cos\beta)^{3/2}}{2\pi^2 c\,\sinh s}\,\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y - \cosh s}}\,\mathcal{L}\left(\coth^2\frac{y}{2}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{\sqrt{\cosh v - \cos\beta}\sqrt{\cosh y - \cosh v}}\,. \tag{1.7.31}$$

Formula (31) gives the expression for the charge density in terms of the prescribed potential $V$ .

We can now obtain the potential in space through its value on the spherical cap. The following result can be obtained:

$$V(v,u,\phi) = \frac{1}{\pi}\sqrt{\cosh v - \cos u}\int_0^b \frac{\mathrm{d}t_1}{\sqrt{\cosh v - \cosh t_1}}\,\mathcal{L}\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\sinh y\,\mathcal{L}[\tanh(y/2)]\,V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y - \cos\beta}\sqrt{\cosh x - \cosh y}}\,. \tag{1.7.32}$$

We interchange the order of integration in (32) and integrate with respect to $x$. We obtain:

$$V(v,u,\phi) = \frac{c^3\,|\sin(u-\beta)|}{\pi^2(\cosh v - \cos u)}\int_0^{2\pi}\int_0^b \frac{1}{R_y^3}\left[\frac{R_y}{\chi(b)} + \tan^{-1}\left(\frac{\chi(b)}{R_y}\right)\right]\frac{V(y,\psi)\sinh y\,\mathrm{d}y\,\mathrm{d}\psi}{(\cosh y - \cos\beta)^2}\,. \tag{1.7.33}$$

The last formula is in agreement with the classical result of Hobson (1900).

**Problem 2.** The boundary conditions in this case take the form:

$$\sigma = \sigma_0(v,\phi), \qquad \text{for} \quad 0 \le v \le b_0, \qquad u = u_0, \qquad 0 \le \phi \le 2\pi. \tag{1.7.34}$$

$$V = 0, \qquad \text{for} \quad 0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi, \tag{1.7.35}$$

Denote the surface of the first cap as $S_0$, and the surface of the second cap as $S$. Introduce the following points, with their toroidal coordinates: $M(v,u,\phi)$, $N(x,\beta,\psi)$, $N_0(v_0,u_0,\phi_0)$, and $K(v,\beta,\phi)$. The potential in the space can be presented again through the simple layer distributions

$$V(M) = \int\!\!\int_S \frac{\sigma(N)\,\mathrm{d}S}{R(M,N)} + \int\!\!\int_{S_0} \frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(M,N_0)}. \tag{1.7.36}$$

We note that $\sigma_0$ in (36) is known from (34) while $\sigma$ is not yet known. It can be found from the integral equation which results from substitution of the second boundary condition (35) in (36), namely,

$$0 = \int\!\!\int_S \frac{\sigma(N)\,\mathrm{d}S}{R(K,N)} + \int\!\!\int_{S_0} \frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \tag{1.7.37}$$

We can rewrite (37) as

$$2c\sqrt{\cosh v - \cos\beta}\int_0^v \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh\tau}}\int_\tau^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}$$

$$= -\int\!\!\int_{S_0} \frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \tag{1.7.38}$$

The general solution to (38) in the form:

$$\sigma(N) = -\int\!\!\int_{S_0} G(N,N_0)\,\sigma(N_0)\,\mathrm{d}S_0, \tag{1.7.39}$$

where the Green's function $G$ is defined by

$$G(N,N_0)=\frac{c\,|\sin(u_0-\beta)|}{\pi^2(\cosh v_0-\cos u_0)\,R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)}+\tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right], \tag{1.7.40}$$

with

$$\chi_0(y)=\frac{2c\sqrt{\cosh t_{20}(y)-\cosh v_0}\sqrt{\cosh y-\cosh x}}{\sinh t_{20}(y)\sqrt{\cosh v_0-\cos u_0}\sqrt{\cosh x-\cos\beta}}. \tag{1.7.41}$$

Now we can express the potential in space directly in terms of the prescribed charge distribution $\sigma_0$. The integrals involved, though looking quite formidable, can be computed in terms of elementary functions.

$$V(M)=\int\!\!\int_{S_0}\left\{1+\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_1(b)}{\sqrt{\cosh w-\cos(u-u_0)}}\right)-\frac{\sqrt{\cosh w-\cos(u-u_0)}}{\sqrt{\cosh w-\cos(u+u_0-2\beta)}}\left[1\right.\right.$$

$$\left.\left.-\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_2(b)}{\sqrt{\cosh w-\cos(u+u_0-2\beta)}}\right)\right]\right\}\frac{\sigma_0(N_0)\,dS_0}{2R(M,N_0)}. \tag{1.7.42}$$

In the particular case of $b\to\infty$ formula (42) simplifies as follows:

$$V(M)=\int\!\!\int_{S_0}\left\{1+\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u-u_0)/2]}{\sqrt{\cosh w-\cos(u-u_0)}}\right)-\frac{\sqrt{\cosh w-\cos(u-u_0)}}{\sqrt{\cosh w-\cos(u+u_0-2\beta)}}\left[1\right.\right.$$

$$\left.\left.+\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u+u_0-2\beta)/2]}{\sqrt{\cosh w-\cos(u+u_0-2\beta)}}\right)\right]\right\}\frac{\sigma_0(N_0)\,dS_0}{2R(M,N_0)}. \tag{1.7.43}$$

The last formula is in agreement with the long standing result of Hobson (1900). Several examples are considered in (Fabrikant, 1991).

**Appendix.** Some essential formulae used in the main body of this section are presented here.

$$\tanh\left(\frac{t_1}{2}\right)\tanh\left(\frac{t_2}{2}\right)=\tanh\left(\frac{x}{2}\right)\tanh\left(\frac{v}{2}\right), \tag{1.7.A1}$$

$$(\cosh t_2-\cosh v)(\cosh v-\cosh t_1)=\sinh^2 v\,\tan^2\left(\frac{u-\beta}{2}\right), \tag{1.7.A2}$$

$$\cos^2\left(\frac{u-\beta}{2}\right)(\cosh t_1-\cosh\tau)(\cosh t_2-\cosh\tau)=(\cosh v-\cosh\tau)[\cosh x-\cosh\gamma(\tau)]\,, \quad (1.7.A3)$$

$$\cosh t_1\cosh t_2=\frac{\sin^2[(u-\beta)/2]+\cosh v\ \cosh x}{\cos^2[(u-\beta)/2]}\,, \quad (1.7.A4)$$

$$\cosh t_1+\cosh t_2=\frac{\cosh v+\cosh x}{\cos^2[(u-\beta)/2]}\,, \quad (1.7.A5)$$

$$(\cosh x-\cosh t_1)(\cosh v-\cosh t_1)=\sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_1\,, \quad (1.7.A6)$$

$$(\cosh t_2-\cosh x)(\cosh t_2-\cosh v)=\sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_2\,, \quad (1.7.A7)$$

$$\sinh t_1\sinh t_2=\frac{\sinh x\sinh v}{\cos^2[(u-\beta)/2]}\,, \quad (1.7.A8)$$

$$\frac{\sinh t_1}{\sinh t_2}=\frac{\sinh v(\cosh x-\cosh t_1)}{\sinh x(\cosh t_2-\cosh v)}=\frac{\sinh x(\cosh v-\cosh t_1)}{\sinh v(\cosh t_2-\cosh x)}\,, \quad (1.7.A9)$$

$$(\cosh t_1-1)(\cosh t_2-1)=\frac{(\cosh v-1)(\cosh x-1)}{\cos^2[(u-\beta)/2]}\,, \quad (1.7.A10)$$

$$\cosh t_2-\cosh t_1=\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)}\sqrt{\cosh(x-v)-\cos(u-\beta)}}{\cos^2[(u-\beta)/2]}\,, \quad (1.7.A11)$$

$$\frac{\partial t_1}{\partial x}=\frac{\sqrt{\cosh v-\cosh t_1}\sqrt{\cosh t_2-\cosh x}}{|\cos[(u-\beta)/2]|\,(\cosh t_2-\cosh t_1)}=\frac{\sinh x(\cosh v-\cosh t_1)}{\sinh t_1(\cosh t_2-\cosh t_1)\cos^2[(u-\beta)/2]}$$

$$=\frac{|\sin[(u-\beta)/2]|\sinh x\sqrt{\cosh v-\cosh t_1}}{\cos^2[(u-\beta)/2]\,(\cosh t_2-\cosh t_1)\sqrt{\cosh x-\cosh t_1}}\,, \quad (1.7.A12)$$

$$\frac{\partial t_2}{\partial x}=\frac{\sinh x(\cosh t_2-\cosh v)}{\cos^2[(u-\beta)/2]\sinh t_2(\cosh t_2-\cosh t_1)}=\frac{\sqrt{\cosh x-\cosh t_1}\sqrt{\cosh t_2-\cosh v}}{|\cos[(u-\beta)/2]|\,(\cosh t_2-\cosh t_1)}$$

$$=\frac{|\sin[(u-\beta)/2]|\sinh x\sqrt{\cosh t_2-\cosh v}}{\cos^2[(u-\beta)/2]\,(\cosh t_2-\cosh t_1)\sqrt{\cosh t_2-\cosh x}}\,, \quad (1.7.A13)$$

$$\frac{t_2'}{t_1'} = \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh t_2 - \cosh v}}{\sqrt{\cosh v - \cosh t_1}\sqrt{\cosh t_2 - \cosh x}} = \frac{\sinh v(\cosh x - \cosh t_1)}{\sinh x(\cosh v - \cosh t_1)}$$

$$= \frac{\sinh x(\cosh t_2 - \cosh v)}{\sinh v(\cosh t_2 - \cosh x)} = \frac{\sinh t_1}{\sinh t_2} \sinh^2 v \tan^2[(u-\beta)/2] \,. \qquad (1.7.\text{A}14)$$

## 1.8. Continuations of some solutions in potential theory

**Introduction.** In mixed boundary value problems of potential theory, the value of potential is given on a part of the boundary, while its derivative is given on the rest of the boundary. We show here that it is possible to express, for example, the potential on the rest of the boundary directly through its known values on the part of the boundary. It is well known (see Fabrikant, 1989, section 1.4), that for an integral equation

$$v(\rho,\phi) = \int_0^{2\pi}\int_0^{a} \frac{\sigma(\rho_0,\phi_0)\,\rho_0\, d\rho_0\, d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}\,, \quad \text{for } 0 \le \rho \le a\,, \quad 0 \le \phi < 2\pi\,. \qquad (1.8.1)$$

where $v$ is a known function inside a circle $\rho \le a$, and the unknown function $\sigma$ is zero outside the circle, the following continuation relationship exists

$$v(\rho,\phi) = \frac{\sqrt{\rho^2-a^2}}{\pi^2}\int_0^{2\pi}\int_0^{a} \frac{v(\rho_0,\phi_0)\,\rho_0\, d\rho_0\, d\phi_0}{\sqrt{a^2-\rho_0^2}\,R^2}\,, \quad \text{for } \rho > a\,, \qquad (1.8.2)$$

where $R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)$.

In the case of an integral equation for an exterior of the circle, namely,

$$v(\rho,\phi) = \int_0^{2\pi}\int_a^{\infty} \frac{\sigma(\rho_0,\phi_0)\,\rho_0\, d\rho_0\, d\phi_0}{R}\,, \quad \text{for } a \le \rho < \infty\,, \quad 0 \le \phi < 2\pi\,, \qquad (1.8.3)$$

and $\sigma = 0$ inside the circle $\rho = a$, the following relationship is valid (Fabrikant, 1989, formula 1.5.19):

$$v(\rho,\phi) = \frac{\sqrt{a^2-\rho^2}}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{v(\rho_0,\phi_0)\,\rho_0\, d\rho_0\, d\phi_0}{\sqrt{\rho_0^2-a^2}\,R^2}\,, \quad \text{for } \rho < a\,. \qquad (1.8.4)$$

Yet another type of integral equation is encountered in elastic crack problems:

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2}\Delta\int_0^{2\pi}\int_0^a \frac{v(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{R}, \quad \text{for} \quad 0 \le \rho \le a, \quad 0 \le \phi < 2\pi. \tag{1.8.5}$$

Here $\sigma$ is a known function, $v$ is zero outside the circle $\rho = a$ and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.8.6}$$

In this case, the following continuation solution exists (Fabrikant, 1989, formula 1.4.27)

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\sqrt{\rho^2 - a^2}}\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2 - \rho_0^2}\,\sigma(\rho_0,\phi_0)}{R^2}\rho_0\,d\rho_0\,d\phi_0, \quad \text{for} \quad \rho > a. \tag{1.8.7}$$

The external equation

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2}\Delta\int_0^{2\pi}\int_a^\infty \frac{v(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{R}, \quad \text{for} \quad a \le \rho < \infty, \quad 0 \le \phi < 2\pi. \tag{1.8.8}$$

has the following continuation solution (Fabrikant, 1989, formula 1.5.24):

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\sqrt{a^2 - \rho^2}}\int_0^{2\pi}\int_a^\infty \frac{\sqrt{\rho_0^2 - a^2}\,\sigma(\rho_0,\phi_0)}{R^2}\rho_0\,d\rho_0\,d\phi_0, \quad \text{for} \quad \rho < a. \tag{1.8.9}$$

We note, that the reciprocal theorem allows us to derive (9) from (2) and (7) from (4). Indeed, we can define 2 potential fields. The first one is defined in the plane $z = 0$ as follows: the potential $v_1^-$ is prescribed inside a circle $\rho = a$, and the charge distribution $\sigma_1^+ = 0$ outside the circle. The second field in the plane $z = 0$ is defined by the charge $\sigma_2^+$ prescribed outside the circle, and the potential $v_2^- = 0$ inside the circle. The reciprocal theorem states that

$$\int\!\!\int_{S^-} v_1^-\sigma_2^-\,dS^- + \int\!\!\int_{S^+} v_1^+\sigma_2^+\,dS^+ = \int\!\!\int_{S^-} v_2^-\sigma_1^-\,dS^- + \int\!\!\int_{S^+} v_2^+\sigma_1^+\,dS^+. \tag{1.8.10}$$

Due to the boundary conditions, equation (10) simplifies as follows:

$$\iint_{S^-} v_1^- \sigma_2^- \, dS^- + \iint_{S^+} v_1^+ \sigma_2^+ \, dS^+ = 0 \,. \tag{1.8.11}$$

Assume that we have the relationships:

$$v_1^+(t_e) = \iint_{S^-} M(t_i, t_e)\, v_1^-(t_i)\, dS^- \,, \qquad \sigma_2^-(t_i) = \iint_{S^+} N(t_i, t_e)\, \sigma_2^+(t_e)\, dS^+ \,. \tag{1.8.12}$$

The first expression corresponds to (2) and the second to (9). Here $t_i$ is a point inside a circle $S^-$, $t_e$ is a point external to the circle. Substitution of (12) into (11) gives

$$\iint_{S^-} v_1^-(t_i)\left(\iint_{S^+} N(t_i, t_e)\, \sigma_2^+(t_e)\, dS^+\right) dS^- = = -\iint_{S^+} \sigma_2^+(t_e)\left(\iint_{S^-} M(t_i, t_e)\, v_1^-(t_i)\, dS^-\right) dS^+ \,. \tag{1.8.13}$$

From (13) it is evident that

$$N(t_i, t_e) = -M(t_i, t_e) \,. \tag{1.8.14}$$

In (2) the internal point is $t_i(\rho_0, \phi_0)$ and the external point is $t_e(\rho, \phi)$. In (9) the notation is opposite, namely, $t_i(\rho, \phi)$ and $t_e(\rho_0, \phi_0)$. Taking this into consideration, direct comparison of (2) and (9) proves correctness of (14). The reciprocity of (4) and (7) can be verified in the same manner.

In general, we should note that if the potential $v$ is given all over the plane $z=0$, then the charge density distribution is

$$\sigma = -\frac{1}{4\pi^2}\Delta \iint_{S} \frac{v}{R}\, dS \,. \tag{1.8.15}$$

Here $S$ is the whole plane $z=0$. The inverse of (15) is

$$v = \iint_{S} \frac{\sigma}{R}\, dS \,. \tag{1.8.16}$$

In the next section, we shall consider more complicated integral equation and show that the continuity condition exists there too.

**Formulation of the problem and its solution.** In (Fabrikant, 1989, section 2.6), it is shown that in the case of tangential displacement $u=u_x+iu_y$ prescribed inside a circle $\rho=a$, and tangential stresses vanishing outside the circle, with normal stresses equal zero all over the plane $z=0$, the governing integral equation takes the form

$$\frac{1}{2}G_1\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)}{R}\rho_0\,d\rho_0\,d\phi_0+\frac{1}{2}G_2\int_0^{2\pi}\int_0^a \frac{q\bar{\tau}(\rho_0,\phi_0)}{\bar{q}R}\rho_0\,d\rho_0\,d\phi_0=u(\rho,\phi),$$
$$\text{for}\quad 0\le\rho\le a,\qquad 0\le\phi<2\pi. \qquad (1.8.17)$$

Here $\tau=\tau_{zx}+i\tau_{yz}$ is the complex tangential stress, overbar indicates the complex conjugate value, $G_1$ and $G_2$ are the elastic constants, defined in (Fabrikant, 1989, formula 2.1.14) and

$$q=\rho e^{i\phi}-\rho_0 e^{i\phi_0},\qquad R^2=q\bar{q}.$$

Since $\tau=0$ for $\rho>a$, we can show that we can define the displacements $u$ outside the circle directly in terms of the displacements inside, without having to find $\tau$.

Assuming that the expansions exist

$$\tau(\rho,\phi)=\sum_{n=-\infty}^{\infty}\tau_n(\rho)e^{in\phi},\qquad u(\rho,\phi)=\sum_{n=-\infty}^{\infty}u_n(\rho)e^{in\phi}, \qquad (1.8.18)$$

equation (17) can be presented as an infinite set of equations (Fabrikant, 1989, formulae 2.6.5 and 2.6.6)

$$\frac{2G_1}{\rho^{n+1}}\int_0^\rho\frac{x^{2n+2}\,dx}{\sqrt{\rho^2-x^2}}\int_x^a\frac{\tau_{n+1}(\rho_0)\,d\rho_0}{\rho_0^n\sqrt{\rho_0^2-x^2}}+\frac{2G_2}{\rho^{n+1}}\int_0^\rho\frac{x^{2n-2}\,dx}{\sqrt{\rho^2-x^2}}\int_x^a\frac{(2n-1)\rho^2-2nx^2}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0=u_{n+1}(\rho),$$
$$(1.8.19)$$

$$\frac{2G_1}{\rho^{n-1}}\int_0^\rho\frac{x^{2n-2}\,dx}{\sqrt{\rho^2-x^2}}\int_x^a\frac{\tau_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}+\frac{2G_2}{\rho^{n-1}}\int_0^\rho\frac{x^{2n-2}\,dx}{\sqrt{\rho^2-x^2}}\int_x^a\frac{(2n-1)\rho_0^2-2nx^2}{\rho_0^{n}\sqrt{\rho_0^2-x^2}}\bar{\tau}_{n+1}(\rho_0)\,d\rho_0$$
$$=u_{-n+1}(\rho), \qquad (1.8.20)$$

Equations (19) and (20) are valid for $n=1,2,3,\ ...$ and $\rho\le a$. In the case of axial symmetry, $n=0$ and the relevant equation is

$$\frac{2}{\rho}\int_0^{\rho}\frac{x^2\,\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^a\frac{G_1\tau_1(\rho_0)-G_2\bar{\tau}_1(\rho_0)}{\sqrt{\rho_0^2-x^2}}\,\mathrm{d}\rho_0=u_1(\rho)\,,\quad\text{for}\ \ \rho\le a\,. \tag{1.8.21}$$

The interesting feature of equations (19–21) is the fact that in the case of $\rho > a$, the only thing, which changes, is the upper limit of the first integral in each term, namely, $\rho$ should be changed to $a$, the rest of equations remains unchanged. This means, that we can express the displacements $u$ outside the circle $\rho = a$ directly through the displacements inside, if we find an operator, which would produce the desired change in the limit of integration. Such an operator does exist, as we can illustrate by an example. We have an integral

$$I_0^{\rho}=\int_0^{\rho}\frac{f(x)\,\mathrm{d}x}{\sqrt{\rho^2-x^2}}\,. \tag{1.8.22}$$

Let us apply to both sides of (22) the operator

$$\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a\frac{\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}(r^2-\rho^2)}\,,\qquad\text{with}\ r>a\,.$$

The result is

$$\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a\frac{\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}(r^2-\rho^2)}I_0^{\rho}$$

$$=\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a\frac{\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}(r^2-\rho^2)}\int_0^{\rho}\frac{f(x)\,\mathrm{d}x}{\sqrt{\rho^2-x^2}}$$

$$=\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a f(x)\,\mathrm{d}x\int_x^a\frac{\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}\sqrt{\rho^2-x^2}(r^2-\rho^2)}$$

$$=\int_0^a\frac{f(x)\,\mathrm{d}x}{\sqrt{r^2-x^2}}=I_0^a\,. \tag{1.8.23}$$

We have indeed obtained in (23) exactly the same integral as in (22), but with the upper limit changed to $a$. All the terms in (20) and (21) are of the same nature as (22), so application of the operator (23) yields immediately

$$u_{-n+1}(\rho)=\frac{2\sqrt{\rho^2-a^2}}{\pi\rho^{n-1}}\int_0^a\frac{u_{-n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{a^2-\rho_0^2}\,(\rho^2-\rho_0^2)},\quad \text{for}\quad \rho>a\ \text{ and }\ n=1,2,3,... \qquad (1.8.24)$$

Equation (19) does not conform completely to the model (23) due to the presence in the second term of the factor $(2n-1)\rho^2$. We present transformations related to this term separately:

$$\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a\frac{\rho\,d\rho}{\sqrt{a^2-\rho^2}(r^2-\rho^2)}\int_0^\rho\frac{(2n-1)\rho^2-2nx^2}{\sqrt{\rho^2-x^2}}x^{2n-2}\,dx\int_x^a\frac{\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}$$

$$=\frac{2}{\pi}\sqrt{r^2-a^2}\int_0^a x^{2n-2}\,dx\int_x^a\frac{\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}\int_x^a\frac{(2n-1)\rho^2-2nx^2}{\sqrt{a^2-\rho^2}\sqrt{\rho^2-x^2}(r^2-\rho^2)}\rho\,d\rho$$

$$=\int_0^a\frac{x^{2n-2}\,dx}{\sqrt{r^2-x^2}}\int_x^a\frac{(2n-1)r^2-2nx^2}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0$$

$$-(2n-1)\sqrt{r^2-a^2}\int_0^a x^{2n-2}\,dx\int_x^a\frac{\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}. \qquad (1.8.25)$$

The first term in (25) is exactly what we need, but we also obtained some additional term. The general result at this stage is

$$u_{n+1}(\rho)=\frac{2\sqrt{\rho^2-a^2}}{\pi\rho^{n+1}}\int_0^a\frac{u_{n+1}(\rho_0)\rho_0^{n+2}\,d\rho_0}{\sqrt{a^2-\rho_0^2}(\rho^2-\rho_0^2)}$$

$$+2G_2(2n-1)\frac{\sqrt{\rho^2-a^2}}{\rho^{n+1}}\int_0^a x^{2n-2}\,dx\int_x^a\frac{\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-x^2}}. \qquad (1.8.26)$$

Now we need to express the last term in (26) through known displacements

$$\int_0^a x^{2n-2}\,dx \int_x^a \frac{\bar{\tau}_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2 - x^2}} = \int_0^a \frac{\bar{\tau}_{-n+1}(\rho_0)}{\rho_0^{n-2}}\,d\rho_0 \int_0^{\rho_0} \frac{x^{2n-2}\,dx}{\sqrt{\rho_0^2 - x^2}}$$

$$= \frac{\sqrt{\pi}\,\Gamma(n-1/2)}{2\Gamma(n)} \int_0^a \bar{\tau}_{-n+1}(\rho_0)\,\rho_0^n\,d\rho_0 \equiv J\,. \qquad (1.8.27)$$

Now we recall from (Fabrikant, 1989, section 2.6)

$$\tau_{-n+1}(\rho) = \rho^{n-1} \int_\rho^a \frac{f_{-n+1}(t)\,dt}{\sqrt{t^2 - \rho^2}} + \left(\frac{G_2}{G_1}\overline{C}_n + D_n\right)\frac{\rho^{n-1}}{\sqrt{a^2 - \rho^2}}, \quad \text{for } n = 1, 2, 3, \ldots \qquad (1.8.28)$$

where

$$C_n = -a^{-2n+1}\int_0^a t^{2n-1} f_{n+1}(t)\,dt\,,$$

$$D_n = \frac{2}{\pi^2 G_1 a^{2n-2}}\frac{d}{da}\int_0^a \frac{u_{-n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{a^2 - \rho_0^2}}\,, \qquad (1.8.29)$$

$$f_{-n+1}(r) = -\frac{G_1\psi_{-n+1}(r) + G_2\overline{\psi}_{n+1}(r)}{G_1^2 - G_2^2}$$

$$f_{n+1}(r) = -\frac{G_1\psi_{n+1}(r) + G_2\overline{\psi}_{-n+1}(r)}{G_1^2 - G_2^2}\,, \qquad (1.8.30)$$

$$\psi_{-n+1}(r) = \frac{2}{\pi^2}\frac{d}{dr}\left[r^{-2n+2}\frac{d}{dr}\int_0^r \frac{\rho^n u_{-n+1}(\rho)\,d\rho}{\sqrt{r^2 - \rho^2}}\right],$$

$$\psi_{n+1}(r) = \frac{2}{\pi^2 r^{2n-1}}\frac{d}{dr}\int_0^r \frac{d[\rho^{n+1}u_{n+1}(\rho)]}{\sqrt{r^2 - \rho^2}}\,. \qquad (1.8.31)$$

Substitution of (28) in (27) yields

$$J=\frac{\pi}{2(2n-1)}\left[\int_0^a \bar{f}_{-n+1}(t)t^{2n-1}\,\mathrm{d}t+\left(\frac{G_2}{G_1}C_n+\overline{D}_n\right)a^{2n-1}\right]. \qquad (1.8.32)$$

Further substitution of (29), (30) and (31) in (32) gives

$$J=\frac{\pi}{2(2n-1)}\left[\int_0^a \left(\bar{f}_{-n+1}(t)-\frac{G_2}{G_1}f_{n+1}(t)\right)t^{2n-1}\,\mathrm{d}t+\frac{2a}{\pi^2}\frac{\mathrm{d}V(a)}{\mathrm{d}a}\right]. \qquad (1.8.33)$$

Here

$$V(t)=\int_0^t \frac{\overline{u}_{-n+1}(\rho)\,\rho^n\,\mathrm{d}\rho}{\sqrt{t^2-\rho^2}}. \qquad (1.8.34)$$

Substitution of (30) in (33) gives additional simplifications:

$$J=\frac{\pi}{2(2n-1)}\left[-\int_0^a \frac{\overline{\psi}_{-n+1}(t)t^{2n-1}\,\mathrm{d}t}{G_1}+\frac{2a}{\pi^2 G_1}\frac{\mathrm{d}V(a)}{\mathrm{d}a}\right]$$

$$=\frac{1}{\pi G_1(2n-1)}\left[-\int_0^a t^{2n-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-2n+2}\,\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right)\mathrm{d}t+a\,\frac{\mathrm{d}V(a)}{\mathrm{d}a}\right]$$

$$=\frac{V(a)}{\pi G_1}. \qquad (1.8.35)$$

Finally substitution of (35) in (26) gives

$$u_{n+1}(\rho)=\frac{2}{\pi}\frac{\sqrt{\rho^2-a^2}}{\rho^{n+1}}\left[\int_0^a \frac{u_{n+1}(\rho_0)\,\rho_0^{n+2}\,\mathrm{d}\rho_0}{\sqrt{a^2-\rho_0^2}(\rho^2-\rho_0^2)}+(2n-1)\frac{G_2}{G_1}\int_0^a \frac{\overline{u}_{-n+1}(\rho_0)\rho_0^n\,\mathrm{d}\rho_0}{\sqrt{a^2-\rho_0^2}}\right],\quad \text{for } \rho>a. \qquad (1.8.36)$$

In the case of $n=0$, only the first term of (36) is used, since $u_{-n+1}$ is defined only for $n\geq 1$. Equations (24) and (46) give the Fourier series representation of the displacements outside the circle $\rho=a$ through the prescribed displacements inside.

The summation of (24) and (36), according to (18), can be performed and the final result is:

$$u(\rho,\phi)=\frac{\sqrt{\rho^2-a^2}}{\pi^2}\left[\int_0^{2\pi}\int_0^a \frac{u(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{R^2\sqrt{a^2-\rho_0^2}}\right.$$

$$\left.+\frac{G_2}{G_1}e^{2i\phi}\int_0^a\int_0^{2\pi}\frac{\overline{u}(\rho_0,\phi_0)(\rho^2+\delta^2)}{\sqrt{a^2-\rho_0^2}\,(\rho^2-\delta^2)^2}\rho_0\,d\rho_0\,d\phi_0\right],\quad \text{for } \rho>a. \tag{1.8.37}$$

Here

$$\delta^2=\rho\rho_0 e^{i(\phi-\phi_0)}. \tag{1.8.38}$$

**Solution of the external problem.** In this case we have an arbitrary tangential displacement $u$ prescribed outside a circle $\rho=a$, tangential stress vanishing inside and no normal stresses all over the boundary $z=0$. We need to express the tangential displacement inside the circle through its value outside.

The governing integral equations (Fabrikant, 1989, formulae 2.7.6 and 2.7.7) are:

$$2G_1\rho^{n+1}\int_\rho^\infty\frac{dx}{x^{2n+2}\sqrt{x^2-\rho^2}}\int_a^x\frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}\,d\rho_0}{\sqrt{x^2-\rho_0^2}}+2G_2\rho^{n+1}\int_\rho^\infty\frac{dx}{x^{2n+2}\sqrt{x^2-\rho^2}}$$

$$\times\int_a^x\frac{2nx^2-(2n+1)\rho_0^2}{\sqrt{x^2-\rho_0^2}}\overline{\tau}_{-n+1}(\rho_0)\rho_0^n\,d\rho_0=u_{n+1}(\rho),\quad\text{for}\quad\rho\geq a,\quad n=0,1,2,\ldots \tag{1.8.39}$$

$$2G_1\rho^{n-1}\int_\rho^\infty\frac{dx}{x^{2n-2}\sqrt{x^2-\rho^2}}\int_a^x\frac{\tau_{-n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}+2G_2\rho^{n-1}\int_\rho^\infty\frac{(2n-1)x^2-2n\rho^2}{x^{2n}\sqrt{x^2-\rho^2}}dx$$

$$\times\int_a^x\frac{\overline{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}=u_{-n+1}(\rho),\quad\text{for}\quad\rho\geq a,\quad n=1,2,3,\ldots \tag{1.8.40}$$

The relevant expressions, valid for $\rho\leq a$, differ from (39–40) only by the lower limit of integration in the first integral of each term, namely, instead of $\rho$ there should be $a$. We consider the integral

$$I_\rho^\infty = \int_\rho^\infty \frac{f(x)\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}. \tag{1.8.41}$$

Let us make the following transformation with (41):

$$\frac{2}{\pi}\sqrt{a^2-r^2}\int_a^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-a^2}(\rho^2-r^2)}\,I_\rho^\infty$$

$$=\frac{2}{\pi}\sqrt{a^2-r^2}\int_a^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-a^2}(\rho^2-r^2)}\int_\rho^\infty \frac{f(x)\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}$$

$$=\frac{2}{\pi}\sqrt{a^2-r^2}\int_a^\infty f(x)\,\mathrm{d}x\int_a^x \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-a^2}\sqrt{x^2-\rho^2}(\rho^2-r^2)}$$

$$=\int_a^\infty \frac{f(x)\,\mathrm{d}x}{\sqrt{x^2-r^2}} = I_a^\infty\,. \tag{1.8.42}$$

As we see, the required transformation is indeed (42). Applying it to both sides of (39), divided by $\rho^{\mathrm{n}}$, we obtain:

$$u_{\mathrm{n+1}}(\rho) = \frac{2}{\pi}\rho^{\mathrm{n+1}}\sqrt{a^2-\rho^2}\int_a^\infty \frac{u_{\mathrm{n+1}}(\rho_0)\,\mathrm{d}\rho_0}{\rho_0^{\mathrm{n}}\sqrt{\rho_0^2-a^2}(\rho_0^2-\rho^2)}\,,\quad \text{for } \rho<a\,,\quad n=0,1,2,\ldots \tag{1.8.43}$$

Equation (40) is not immediately amenable to a similar transformation due to the existence of $2n\rho^2$ in the second term. We proceed:

$$\frac{2}{\pi}r^{\mathrm{n}-1}\sqrt{a^2-r^2}\int_a^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-a^2}(\rho^2-r^2)}\int_\rho^\infty \frac{(2n-1)x^2-2n\rho^2}{x^{2\mathrm{n}}\sqrt{x^2-\rho^2}}\,\mathrm{d}x\int_a^x \frac{\bar{\tau}_{\mathrm{n+1}}(\rho_0)\rho_0^{\mathrm{n}}\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}$$

$$=\frac{2}{\pi}r^{\mathrm{n}-1}\sqrt{a^2-r^2}\int_a^\infty\left(\int_a^x \frac{\bar{\tau}_{\mathrm{n+1}}(\rho_0)\rho_0^{\mathrm{n}}\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\right)\frac{\mathrm{d}x}{x^{2\mathrm{n}}}\int_a^x \frac{(2n-1)x^2-2n\rho^2}{\sqrt{\rho^2-a^2}\sqrt{x^2-\rho^2}(\rho^2-r^2)}\,\rho\,\mathrm{d}\rho$$

$$= r^{n-1}\int_a^\infty \frac{(2n-1)x^2-2nr^2}{x^{2n}\sqrt{x^2-r^2}}\,dx\int_a^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}$$

$$-2nr^{n-1}\sqrt{a^2-r^2}\int_a^\infty \frac{dx}{x^{2n}}\int_a^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}. \quad (1.8.44)$$

Again, the first term in (44) is exactly what we need and the additional term appeared too. The general expression will take the form:

$$u_{-n+1}(\rho)=\frac{2}{\pi}\rho^{n-1}\sqrt{a^2-\rho^2}\int_a^\infty \frac{u_{-n+1}(\rho_0)\rho_0\,d\rho_0}{\rho_0^{n-1}\sqrt{\rho_0^2-a^2}(\rho_0^2-\rho^2)}+4nG_2\rho^{n-1}\sqrt{a^2-\rho^2}\int_a^\infty \frac{dx}{x^{2n}}\int_a^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}. \quad (1.8.45)$$

Let us transform the second term in (45).

$$\int_a^\infty \frac{dx}{x^{2n}}\int_a^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{\sqrt{x^2-\rho_0^2}}=\int_a^\infty \bar{\tau}_{n+1}(\rho_0)\rho_0^n\,d\rho_0\int_{\rho_0}^\infty \frac{dx}{x^{2n}\sqrt{x^2-\rho_0^2}}$$

$$=\frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+1/2)}\int_a^\infty \frac{\bar{\tau}_{n+1}(\rho_0)\,d\rho_0}{\rho_0^n}. \quad (1.8.46)$$

Now we have to recall from (Fabrikant, 1989, section 2.7):

$$\tau_{n+1}=\frac{1}{\rho^{n+1}}\left[\int_a^\rho \frac{f_{n+1}(t)\,dt}{\sqrt{\rho^2-t^2}}+\frac{D_n}{\sqrt{\rho^2-a^2}}\right], \quad \text{for} \quad n=0,1,2,\ldots \quad (1.8.47)$$

Here

$$D_n=\frac{1}{G_1}\left[G_2\bar{C}_n-\frac{2}{\pi^2}a^{2n+2}\frac{dW(a)}{da}\right],$$

$$W(t)=\int_t^\infty \frac{u_{n+1}(\rho)\,d\rho}{\rho^n\sqrt{\rho^2-t^2}}, \quad (1.8.48)$$

$$C_{\rm n} = -a^{2n+1}\int_a^\infty f_{-n+1}(t)t^{-2n-1}\,\mathrm{d}t, \tag{1.8.49}$$

$$f_{-n+1}(r) = \frac{G_1\psi_{-n+1}(r) + G_2\overline{\psi}_{n+1}(r)}{G_1^2 - G_2^2}$$

$$f_{n+1}(r) = \frac{G_1\psi_{n+1}(r) + G_2\overline{\psi}_{-n+1}(r)}{G_1^2 - G_2^2}. \tag{1.8.50}$$

$$\psi_{-n+1}(r) = -\frac{2}{\pi^2}r^{2n+1}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{\mathrm{d}\rho}{\sqrt{\rho^2 - r^2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{u_{-n+1}(\rho)}{\rho^{n-1}}\right],$$

$$\psi_{n+1}(r) = \frac{2}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{2n+2}\frac{\mathrm{d}W(r)}{\mathrm{d}r}\right]. \tag{1.8.51}$$

Substitution of (47) in (46) yields:

$$\frac{\sqrt{\pi}\Gamma(n)}{2\Gamma(n+1/2)}\int_a^\infty \frac{\mathrm{d}\rho}{\rho^{2n+1}}\left[\int_a^\rho \frac{\bar{f}_{n+1}(t)\,\mathrm{d}t}{\sqrt{\rho^2 - t^2}} + \frac{\overline{D}_n}{\sqrt{\rho^2 - a^2}}\right] = \frac{\pi}{4n}\left[\int_a^\infty \frac{\bar{f}_{n+1}(t)}{t^{2n+1}}\,\mathrm{d}t + \frac{\overline{D}_n}{a^{2n+1}}\right]. \tag{1.8.52}$$

Further utilization of (49–51) in (52) gives:

$$\frac{\pi}{4n}\left[\int_a^\infty \frac{\bar{f}_{n+1}(t)}{t^{2n+1}}\,\mathrm{d}t - \frac{G_2}{G_1}\int_a^\infty \frac{f_{-n+1}(t)}{t^{2n+1}}\,\mathrm{d}t - \frac{2}{\pi^2}\frac{a}{G_1}\frac{\mathrm{d}\overline{W}(a)}{\mathrm{d}a}\right]$$

$$= \frac{\pi}{4n}\left[\frac{1}{G_1}\int_a^\infty \frac{\overline{\psi}_{n+1}(t)}{t^{2n+1}}\,\mathrm{d}t - \frac{2}{\pi^2}\frac{a}{G_1}\frac{\mathrm{d}\overline{W}(a)}{\mathrm{d}a}\right]$$

$$= -\frac{1}{2\pi nG_1}\left[\int_a^\infty \frac{\mathrm{d}t}{t^{2n+1}}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{2n+2}\frac{\mathrm{d}\overline{W}(t)}{\mathrm{d}t}\right) + a\frac{\mathrm{d}\overline{W}(a)}{\mathrm{d}a}\right] = \frac{2n+1}{2\pi nG_1}\overline{W}(a). \tag{1.8.53}$$

Now (45) will take the form:

$$u_{-n+1}(\rho)=\frac{2}{\pi}\rho^{n-1}\sqrt{a^2-\rho^2}\left[\int_a^\infty \frac{u_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-a^2}(\rho_0^2-\rho^2)}\right.$$

$$\left.+(2n+1)\frac{G_2}{G_1}\int_a^\infty \frac{\bar{u}_{n+1}(\rho_0)\,d\rho_0}{\rho_0^{n}\sqrt{\rho_0^2-a^2}}\right],\quad \text{for}\quad \rho\le a \quad\text{and}\quad n=1,2,3,... \tag{1.8.54}$$

Summation of (43) and (54) gives the closed form solution

$$u(\rho,\phi)=\frac{\sqrt{a^2-\rho^2}}{\pi^2}\left[\int_0^{2\pi}\int_a^\infty \frac{u(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{R^2\sqrt{\rho_0^2-a^2}}\right.$$

$$\left.+\frac{G_2}{G_1}\int_0^{2\pi}\int_a^\infty \frac{\bar{u}(\rho_0,\phi_0)\,e^{2i\phi_0}(3\rho_0^2-\bar{\delta}^2)}{\sqrt{\rho_0^2-a^2}\,(\rho_0^2-\bar{\delta}^2)^2}\,\rho_0\,d\rho_0\,d\phi_0\right],\quad \text{for}\ \rho<a. \tag{1.8.55}$$

Here $\delta^2$ is defined by (38).

**Application of the reciprocal theorem for solving relevant crack problems.** We consider a transversely isotropic elastic half–space $z\ge 0$, subject to 2 different sets of boundary conditions. The first set is characterized by tangential displacement $u_1^-$ prescribed inside a circle $\rho=a$, tangential stress $\tau_1^+=0$ outside the circle and the normal stress vanishing all over the plane $z=0$. The second state is characterized by tangential stress $\tau_2^+$ prescribed outside the circle and the tangential displacement $u_2^-=0$ inside the circle.

We know the relationship between $u_1^+$ outside the circle and $u_1^-$ inside (37). We need to infer from it the relationship between $\tau_2^+$ and $\tau_2^-$ in the second state. By using the same logic, as described in (10–14), we see that the relationship sought is of sign opposite to (37) and with interchange of $\rho$ by $\rho_0$ and $\phi$ by $\phi_0$, namely,

$$\tau(\rho,\phi)=-\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\left\{\int_0^{2\pi}\int_a^\infty \frac{\sqrt{\rho_0^2-a^2}}{R^2}\,\tau(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0\right.$$

$$\left.+\frac{G_2}{G_1}\int_0^{2\pi}\int_a^\infty \frac{\sqrt{\rho_0^2-a^2}\,\bar{\tau}(\rho_0,\phi_0)\,(\rho_0^2+\bar{\delta}^2)\,e^{2i\phi_0}}{(\rho_0^2-\bar{\delta}^2)^2}\,\rho_0\,d\rho_0\,d\phi_0\right\}, \tag{1.8.56}$$

Equation (56) gives the tangential stress in the crack neck in terms of the tangential stress applied to an external circular crack.

The Fourier series expansion of (56) takes the form:

$$\tau_{-n+1}(\rho) = -\frac{2}{\pi}\frac{\rho^{n+1}}{\sqrt{a^2-\rho^2}}\int_a^\infty \frac{\sqrt{\rho_0^2-a^2}\,\tau_{n+1}(\rho_0)\rho_0\,d\rho_0}{\rho_0^{n+1}(\rho_0^2-\rho^2)}, \quad \text{for} \quad n = 0, 1, 2, ...$$

$$\tau_{n+1}(\rho) = -\frac{2}{\pi}\frac{\rho^{n-1}}{\sqrt{a^2-\rho^2}}\left[\int_a^\infty \frac{\sqrt{\rho_0^2-a^2}\,\tau_{-n+1}(\rho_0)\rho_0\,d\rho_0}{\rho_0^{n-1}(\rho_0^2-\rho^2)}\right.$$

$$\left. +(2n-1)\frac{G_2}{G_1}\int_a^\infty \frac{\sqrt{\rho_0^2-a^2}\,\bar{\tau}_{n+1}(\rho_0)}{\rho_0^{n+1}}\rho_0\,d\rho_0\right], \quad \text{for} \quad n = 1, 2, 3, ... \qquad (1.8.57)$$

Reciprocal theorem can be used in exactly the same manner to obtain the continuation solution for tangential stresses in a penny-shaped crack. For this purpose, formula (55) can be used. Application of the reciprocal theorem yields:

$$\tau(\rho,\phi) = -\frac{1}{\pi^2\sqrt{\rho^2-a^2}}\left\{\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2-\rho_0^2}}{R^2}\tau(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0\right.$$

$$\left. +\frac{G_2}{G_1}e^{2i\phi}\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,(3\rho^2-\delta^2)}{(\rho^2-\delta^2)^2}\bar{\tau}(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0\right\}. \qquad (1.8.58)$$

Formula (58) was originally derived in a different manner by (Fabrikant, 1989, section 2.7).

The Fourier series expansion of (58) is

$$\tau_{n+1}(\rho) = -\frac{2}{\pi\rho^{n+1}\sqrt{\rho^2-a^2}}\left[\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,\tau_{n+1}(\rho_0)\rho_0^{n+2}\,d\rho_0}{\rho^2-\rho_0^2}\right.$$

$$\left. +(2n+1)\frac{G_2}{G_1}\int_0^a \bar{\tau}_{-n+1}(\rho_0)\sqrt{a^2-\rho_0^2}\,d\rho_0\right],$$

$$\tau_{-n+1}(\rho) = -\frac{2}{\pi\rho^{n-1}\sqrt{\rho^2 - a^2}} \int_0^a \frac{\sqrt{a^2 - \rho_0^2}\,\tau_{-n+1}(\rho_0)\rho_0^n\, d\rho_0}{\rho^2 - \rho_0^2}. \tag{1.8.59}$$

**Discussion.** It is of interest to notice that the governing integral equation of the contact problem (17) can be rewritten as

$$\frac{1}{2}G_1\Delta\int\int_S R\tau\, dS - \frac{1}{2}G_2\Lambda^2\int\int_S R\bar{\tau}\, dS = u. \tag{1.8.60}$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

The governing integral equation of the crack problem has been derived in (Fabrikant, 1989) as

$$-\frac{1}{2\pi^2(G_1^2 - G_2^2)}\left[G_1\Delta\int\int_S \frac{u}{R}\, dS + G_2\Lambda^2\int\int_S \frac{\bar{u}}{R}\, dS\right] = \tau. \tag{1.8.61}$$

Here $S$ is domain of the crack. As we can see, the structure of both equations (60)–(61) is very similar, in the first one the distance between 2 points is in the numerator, while in the second it is in the denominator. It is obvious that when $S$ denotes the whole plane $z=0$, equations (60) and (61) are mutually inverse.

## 1.9. Exact method for solving mixed boundary value problems with application to half-plane contact and crack problems

**Abstract.** A new method is presented for the exact solution in closed form of a mixed boundary value problem of potential theory when the potential is prescribed on one half-plane (say, $y \geq 0$, $z=0$) and the charge density distribution is prescribed on the other half-plane ($y<0$, $z=0$). The method is based on a new integral representation for the reciprocal of the distance between two points. Its substitution into the simple layer distribution leads to an integral equation, which can be solved exactly, with no integral transforms or series expansions involved. The general results are applied to solving a punch problem and a half-plane crack problem. A complete solution for the fields of stresses

and displacements is given in closed form and in terms of elementary functions.

**Introduction.** We consider the following problem: find a harmonic function $V(x,y,z)$, vanishing at infinity, subject to the boundary conditions on the plane $z=0$:

$$V(x,y,0)=v(x,y)\,,\qquad \text{for}\quad y\geq 0\,,\quad -\infty<x<\infty\,,$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0}=-2\pi\sigma(x,y)\,,\quad \text{for}\quad y<0\,,\quad -\infty<x<\infty\,. \tag{1.9.1}$$

This kind of problem was solved by Ufliand (1967), who used the integral transform techniques. Kit and Khai (1989) obtained similar results by a combination of integral transform and Riemann-Hilbert techniques. Ufliand (1967) applied his results to solving half-plane crack problems for the case of isotropy, with explicit computation of the main potential functions for the case of normal and tangential concentrated loading of crack faces. He considered the cases of symmetric as well as antisymmetric loading, namely, symmetric and antisymmetric normal load, symmetric and antisymmetric shear loading, directed normally to the crack edge. The solution of these problems was mainly limited to computation of stresses in the plane of the crack. In the case of symmetric and antisymmetric shear loading directed parallel to the crack edge, only a general outline was presented for determination of the main potential functions. The case of a transversely isotropic body, to the best of our knowledge, was not considered before.

The main advantage of the new method described below is its simplicity: no integral transforms or special functions are needed. All the analysis, including computation of three-dimensional potentials, is performed in closed form and in terms of elementary functions. All the parameters used have physical significance, thus simplifying further investigation of the properties of solutions. Our method is based on the following representation for the reciprocal of the distance between two points $N(x,y)$, and $N_0(x_0,y_0)$:

$$\frac{1}{R}=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=\frac{2}{\pi}\int_{\max(y,y_0)}^{\infty}\frac{\lambda^*(2u-y-y_0,\,x-x_0)\,\mathrm{d}u}{\sqrt{(u-y)(u-y_0)}}\,. \tag{1.9.2}$$

Here

$$\lambda^*(a,b)=\frac{a}{a^2+b^2}\,. \tag{1.9.3}$$

The idea of such an integral representation came to our mind after reading the work of Rubin (1988) on fractional integrals.

We introduce a new variable

$$\eta^* = 2\sqrt{u-y}\ \sqrt{u-y_0}\,. \tag{1.9.4}$$

It is easy to show that

$$\lambda^*(2u-y-y_0, x-x_0) = \frac{2u-y-y_0}{(2u-y-y_0)^2+(x-x_0)^2} = \frac{\eta^*}{2[R^2+(\eta^*)^2]}\frac{\mathrm{d}\eta^*}{\mathrm{d}u}\,. \tag{1.9.5}$$

Substitution of (5) into (2) yields:

$$\frac{1}{R} = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}\eta^*}{R^2+(\eta^*)^2}\,, \tag{1.9.6}$$

thus proving (2). We use asterisks for two purposes: first, to emphasize the analogy between our apparatus used for the geometry of a circle; second, to show, that in the case of a half-plane there are certain differences as well.

The integral representation (2) is convenient for solving problems in the upper half-plane $y \geq 0$. In the half-plane $y<0$, the following equivalent can be established:

$$\frac{1}{R} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \frac{2}{\pi}\int_{-\infty}^{\min(y,y_0)} \frac{\lambda^*(y+y_0-2u, x-x_0)\,\mathrm{d}u}{\sqrt{(y-u)(y_0-u)}}\,. \tag{1.9.7}$$

Now we need to establish an integral representation for the reciprocal of the distance $R_0$ between $M(x,y,z)$ and $N_0(x_0,y_0)$, namely,

$$\frac{1}{R_0} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}}\,. \tag{1.9.8}$$

Since quantities $y$ and $y_0$ in (2) are arbitrary, they can be formally replaced by, say, $l_1^*(y_0,y,z)$ and $l_2^*(y_0,y,z)$, such that

$$\frac{1}{R_0} = \frac{1}{\sqrt{(x-x_0)^2+[l_2^*(y_0,y,z)-l_1^*(y_0,y,z)]^2}}\,, \tag{1.9.9}$$

which requires that

$$l_2^*(y_0,y,z) - l_1^*(y_0,y,z) = \sqrt{(y-y_0)^2 + z^2}\,. \tag{1.9.10}$$

According to (2) and (7), we need to define $l_1^*(y_0,y,z)$ and $l_2^*(y_0,y,z)$ in such a way, that

$$l_1^*(y_0,y,0) = \min(y,y_0)\,, \qquad l_2^*(y_0,y,0) = \max(y,y_0)\,. \tag{1.9.11}$$

We can also require, that $\lambda^*$ in (5) be invariant, namely,

$$l_1^*(y_0,y,z) + l_2^*(y_0,y,z) = y_0 + y\,. \tag{1.9.12}$$

All these requirements can be satisfied by putting

$$l_1^*(y_0) \equiv l_1^*(y_0,y,z) = \frac{1}{2}\left(y + y_0 - \sqrt{(y-y_0)^2 + z^2}\right),$$

$$l_2^*(y_0) \equiv l_2^*(y_0,y,z) = \frac{1}{2}\left(y + y_0 + \sqrt{(y-y_0)^2 + z^2}\right). \tag{1.9.13}$$

We can also define $g^*(u)$, which is inverse to both $l_1^*(y_0)$ and $l_2^*(y_0)$ in such a way, that $g^*[l_1^*(y_0)] = g^*[l_2^*(y_0)] = y_0$, namely,

$$g^*(u) = u - \frac{z^2}{4(u-y)}\,. \tag{1.9.14}$$

Hereafter $l_1^*$ is understood as $l_1^*(0,y,z)$ and $l_1^*(y_0)$ denotes $l_1^*(y_0,y,z)$; similar remarks are valid for $l_2^*$, as defined in (13).

As before, it can be easily verified, that

$$\frac{1}{R_0} = \frac{2}{\pi}\int_{l_2^*(y_0)}^{\infty} \frac{\lambda^*(2u - y - y_0,\, x - x_0)\,\mathrm{d}u}{\sqrt{u - l_1^*(y_0)}\,\sqrt{u - l_2^*(y_0)}} = \frac{2}{\pi}\int_{-\infty}^{l_1^*(y_0)} \frac{\lambda^*(y + y_0 - 2u,\, x - x_0)\,\mathrm{d}u}{\sqrt{l_1^*(y_0) - u}\,\sqrt{l_2^*(y_0) - u}}\,. \tag{1.9.15}$$

In fact, the integrals in (15) can be computed as indefinite, thus:

$$\int \frac{\lambda^*(2u - y - y_0,\, x - x_0)\,\mathrm{d}u}{\sqrt{u - l_1^*(y_0)}\,\sqrt{u - l_2^*(y_0)}} = \frac{1}{R_0}\tan^{-1}\left(\frac{h^*(u)}{R_0}\right), \tag{1.9.16}$$

where

$$h^*(u) = 2\sqrt{u - l_1^*(y_0)}\ \sqrt{u - l_2^*(y_0)}\,. \tag{1.9.17}$$

The integral representations (2), (7) and (15) allow us to formulate and solve various problems, as it is shown below.

**Problem of the first type.** Let the boundary conditions on the plane $z=0$ be:

$$V(x,y,0) = v(x,y)\,, \qquad \text{for} \quad y \geq 0\,, \quad -\infty < x < \infty\,,$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = 0\,, \qquad \text{for} \quad y < 0\,, \quad -\infty < x < \infty\,. \tag{1.9.18}$$

We need to find the charge density distribution $\sigma$ for $y>0$, and the potential $V(x,y,z)$ in the whole space.

Let us represent the potential as a simple layer:

$$V(x,y,z) = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \mathrm{d}y_0 \frac{\sigma(x_0,y_0)}{R_0}\,. \tag{1.9.19}$$

Substitution of (15) in (19) yields:

$$V(x,y,z) = 2\int_{l_2^*}^{\infty} \frac{\mathrm{d}u}{\sqrt{u-y}} \int_0^{g^*(u)} \mathrm{d}y_0 \frac{\mathcal{L}^*(2u-y-y_0)\,\sigma(x,y_0)}{\sqrt{g^*(u)-y_0}}\,. \tag{1.9.20}$$

Here the $\mathcal{L}^*$-operator was introduced as

$$\mathcal{L}^*(k)\sigma(x,\cdot) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k\sigma(x_0,\cdot)\,\mathrm{d}x_0}{k^2 + (x-x_0)^2}\,, \qquad k > 0\,, \tag{1.9.21}$$

and the following rule for change of the order of integration was used:

$$\int_0^{\infty} \mathrm{d}y_0 \int_{l_2^*(y_0)}^{\infty} \mathrm{d}u = \int_{l_2^*}^{\infty} \mathrm{d}u \int_0^{g^*(u)} \mathrm{d}y_0\,. \tag{1.9.22}$$

It is of interest to note, that the $\mathcal{L}$-operator, introduced in (Fabrikant, 1989), had the property, that $\mathcal{L}(k_1)\,\mathcal{L}(k_2)=\mathcal{L}(k_1 k_2)$ and $\mathcal{L}(1)=1$; the new operator $\mathcal{L}^*$ is different:

$$\mathcal{L}^*(k_1)\,\mathcal{L}^*(k_2)=\mathcal{L}^*(k_1+k_2), \qquad \mathcal{L}^*(0)=1\,. \tag{1.9.23}$$

Substitution of boundary conditions (18) into (20) leads to the governing integral equation:

$$2\int_y^\infty \frac{\mathrm{d}u}{\sqrt{u-y}}\int_0^u \frac{\mathcal{L}^*(2u-y-y_0)\,\sigma(x,y_0)\,\mathrm{d}y_0}{\sqrt{u-y_0}}=v(x,y)\,. \tag{1.9.24}$$

Application of the operator

$$\frac{\mathrm{d}}{\mathrm{d}y_1}\int_{y_1}^\infty \frac{\mathrm{d}y}{\sqrt{y-y_1}}\,\mathcal{L}^*(y) \tag{1.9.25}$$

to both sides of (24) gives:

$$-2\pi\int_0^{y_1} \frac{\mathcal{L}^*(2y_1-y_0)\,\sigma(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_1-y_0}}=\frac{\mathrm{d}}{\mathrm{d}y_1}\int_{y_1}^\infty \frac{\mathrm{d}y}{\sqrt{y-y_1}}\,\mathcal{L}^*(y)\,v(x,y)\,. \tag{1.9.26}$$

Here the following integral was used:

$$\int_{y_1}^u \frac{\mathrm{d}y}{\sqrt{u-y}\,\sqrt{y-y_1}}=\pi\,. \tag{1.9.27}$$

The next operator to apply is:

$$\frac{\mathrm{d}}{\mathrm{d}y_2}\int_0^{y_2} \frac{\mathrm{d}y_1}{\sqrt{y_2-y_1}}\,\mathcal{L}^*(-2y_1) \tag{1.9.28}$$

with the result:

$$-2\pi^2 \mathcal{L}^*(-y_2)\,\sigma(x,y_2) = \frac{d}{dy_2}\int_0^{y_2} \frac{dy_1}{\sqrt{y_2-y_1}}\,\mathcal{L}^*(-2y_1)\,\frac{d}{dy_1}\int_{y_1}^{\infty} \frac{dy}{\sqrt{y-y_1}}\,\mathcal{L}^*(y)\,v(x,y)\,. \qquad (1.9.29)$$

The final result is:

$$\sigma(x,y) = -\frac{\mathcal{L}^*(y)}{2\pi^2}\frac{d}{dy}\int_0^{y} \frac{dy_1}{\sqrt{y-y_1}}\,\mathcal{L}^*(-2y_1)\,\frac{d}{dy_1}\int_{y_1}^{\infty} \frac{dy_0}{\sqrt{y_0-y_1}}\,\mathcal{L}^*(y_0)v(x,y_0)\,. \qquad (1.9.30)$$

The back substitution of (30) in (20) allows us to express the potential in the whole space through its boundary values. The first simplification yields:

$$V(x,y,z) = -\frac{1}{\pi}\int_{l_2^*}^{\infty} \frac{du}{\sqrt{u-y}}\,\mathcal{L}^*\big(2u-y-2g^*(u)\big)\,\frac{\partial}{\partial g^*(u)}\int_{g^*(u)}^{\infty} \frac{\mathcal{L}^*(y_0)v(x,y_0)\,dy_0}{\sqrt{y_0-g^*(u)}}\,. \qquad (1.9.31)$$

On introducing a new variable $t$ by the relationship $u = l_2^*(t,y,z)$, namely, $t = g^*(u)$, expression (31) will take the form:

$$V(x,y,z) = -\frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{l_2^*(t)-y}\,dt}{l_2^*(t)-l_1^*(t)}\,\mathcal{L}^*\big(2l_2^*(t)-y-2t\big)\,\frac{d}{dt}\int_t^{\infty} \frac{\mathcal{L}^*(y_1)v(x,y_0)\,dy_0}{\sqrt{y_0-t}}\,. \qquad (1.9.32)$$

Here the abbreviation $l_2^*(t)$ is used for $l_2^*(t,y,z)$ and the following formula of differentiation was used:

$$\frac{dl_2^*(t)}{dy} = \frac{l_2^*(t)-y}{l_2^*(t)-l_1^*(t)}\,. \qquad (1.9.33)$$

The change of the order of integration in (32) yields:

$$V(x,y,z) = \frac{1}{\pi}\int_0^{\infty} dy_0 \left\{ \mathcal{L}^*(y_0)\,\frac{d}{dy_0}\int_0^{y_0} \frac{dt}{\sqrt{y_0-t}}\left[\frac{\mathcal{L}^*\big(2l_2^*(t)-y-2t\big)}{l_2^*(t)-l_1^*(t)}\sqrt{l_2^*(t)-y}\,\right]\right\} v(x,y_0)\,. \qquad (1.9.34)$$

The integral in curly brackets can be computed exactly (see Appendix A), and the final result is:

$$V(x,y,z) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{z}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] v(x_0,y_0)\,\mathrm{d}y_0\,. \tag{1.9.35}$$

where

$$h^* = 2\sqrt{y_0 l_2^*}\,. \tag{1.9.36}$$

In the case of $y<0$ and $z\to 0$, the formula (35) gives the potential in the negative half-plane, through its values in the positive half-plane as follows:

$$V(x,y,0) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \sqrt{-\frac{y}{y_0}}\;\frac{v(x_0,y_0)\,\mathrm{d}y_0}{R^2} \quad \text{for} \quad y<0\,. \tag{1.9.37}$$

**Problem of the second kind.** Let the boundary conditions on the plane $z=0$ be:

$$-\frac{1}{2\pi}\frac{\partial V}{\partial z}\bigg|_{z=0} = \sigma^+(x,y)\,, \qquad \text{for} \quad y>0\,, \quad -\infty<x<\infty\,,$$

$$V(x,y,0) = 0\,, \qquad \text{for} \quad y\le 0\,, \quad -\infty<x<\infty\,. \tag{1.9.38}$$

It is necessary to find the function $V(x,y,z)$ in the whole space and $\sigma^-$ for $y<0$. In order to derive the governing integral equation, we use the second condition (38). Repeating the derivation of (24) for $y<0$, we obtain:

$$v^+(x,y)\,\Big|_{y<0} = 2\int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u-y}} \int_0^{u} \frac{\mathcal{L}^*(2u-y-y_0)\,\sigma^+(x,y_0)\,\mathrm{d}y_0}{\sqrt{u-y_0}}\,. \tag{1.9.39}$$

By using the integral representation (7), the potential in the lower half-plane due to the charge distribution there will take the form:

$$v^-(x,y)\,\Big|_{y<0} = 2\int_{-\infty}^{y} \frac{\mathrm{d}u}{\sqrt{y-u}} \int_u^{0} \frac{\mathcal{L}^*(y+y_0-2u)\,\sigma^-(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}\,. \tag{1.9.40}$$

Since the second condition (38) implies that $v^{+}+v^{-}=0$, the governing integral equation will be:

$$2\int_{0}^{\infty}\frac{\mathrm{d}u}{\sqrt{u-y}}\int_{0}^{u}\frac{\mathcal{L}^{*}(2u-y-y_0)\,\sigma^{+}(x,y_0)\,\mathrm{d}y_0}{\sqrt{u-y_0}}=-2\int_{-\infty}^{y}\frac{\mathrm{d}u}{\sqrt{y-u}}\int_{u}^{0}\frac{\mathcal{L}^{*}(y+y_0-2u)\,\sigma^{-}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}\,. \tag{1.9.41}$$

The left-hand side of (41) can be transformed by using two representations (2) and (7) as follows:

$$\int_{0}^{\infty}\frac{\mathrm{d}u}{\sqrt{u-y}}\int_{0}^{u}\frac{\mathcal{L}^{*}(2u-y-y_0)\sigma^{+}(x,y_0)\,\mathrm{d}y_0}{\sqrt{u-y_0}}=\int_{0}^{\infty}\mathrm{d}y_0\left\{\int_{y_0}^{\infty}\frac{\mathcal{L}^{*}(2u-y-y_0)\,\mathrm{d}u}{\sqrt{u-y}\,\sqrt{u-y_0}}\right\}\sigma^{+}(x,y_0)$$

$$=\int_{0}^{\infty}\mathrm{d}y_0\left\{\int_{-\infty}^{y_0}\frac{\mathcal{L}^{*}(y+y_0-2u)\,\mathrm{d}u}{\sqrt{y-u}\,\sqrt{y_0-u}}\right\}\sigma^{+}(x,y_0)=\int_{-\infty}^{y}\frac{\mathrm{d}u}{\sqrt{y-u}}\int_{0}^{\infty}\frac{\mathcal{L}^{*}(y+y_0-2u)\,\sigma^{+}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}$$

$$=-\int_{-\infty}^{y}\frac{\mathrm{d}u}{\sqrt{y-u}}\int_{u}^{0}\frac{\mathcal{L}^{*}(y+y_0-2u)\,\sigma^{-}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}\,. \tag{1.9.42}$$

Comparison of the last two expressions of (42) leads to:

$$\int_{0}^{\infty}\frac{\mathcal{L}^{*}(y_0-2u)\sigma^{+}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}=-\int_{u}^{0}\frac{\mathcal{L}^{*}(y_0-2u)\sigma^{-}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}\,. \tag{1.9.43}$$

Application of the operator

$$\frac{\mathrm{d}}{\mathrm{d}v}\int_{v}^{0}\frac{\mathrm{d}u}{\sqrt{u-v}}\,\mathcal{L}^{*}(2u) \tag{1.9.44}$$

to both sides of (43) yields:

$$\pi\mathcal{L}^{*}(v)\sigma^{-}(x,v)=-\int_{0}^{\infty}\sqrt{-\frac{y_0}{v}}\;\frac{\mathcal{L}^{*}(y_0)}{y_0-v}\,\sigma^{+}(x,y_0)\,\mathrm{d}y_0\,, \tag{1.9.45}$$

and finally

$$\sigma^{-}(x,y)\Big|_{y<0} = -\frac{1}{\pi}\int_0^\infty \sqrt{-\frac{y_0}{y}}\;\frac{\mathcal{L}^*(y_0-y)}{y_0-y}\,\sigma^{+}(x,y_0)\,\mathrm{d}y_0\,. \tag{1.9.46}$$

Expression (46) gives us the direct relationship between the charge distribution $\sigma^+$ in the upper half-plane and $\sigma^-$ in the lower one. It can be rewritten without the $\mathcal{L}^*$-operator as follows:

$$\sigma^{-}(x,y) = -\frac{1}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}x_0\int_0^\infty \sqrt{-\frac{y_0}{y}}\;\frac{\sigma^{+}(x_0,y_0)\,\mathrm{d}y_0}{(x-x_0)^2+(y-y_0)^2}\,. \tag{1.9.47}$$

Now the charge distribution $\sigma$ is known all over the plane $z=0$ and we can find the potential directly in terms of the prescribed density $\sigma^+$. By utilizing (7), we obtain:

$$V^{-}(x,y,z) = 2\int_{-\infty}^{l_1^*}\frac{\mathrm{d}u}{\sqrt{y-u}}\int_{g^*(u)}^{0}\frac{\mathcal{L}^*(y+y_0-2u)\sigma^{-}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}\,. \tag{1.9.48}$$

Substitution of (47) in (48) gives after simplification:

$$V^{-}(x,y,z) = -2\int_{-\infty}^{l_1^*}\frac{\mathrm{d}u}{\sqrt{y-u}}\int_0^\infty\frac{\mathcal{L}^*(y+y_0-2u)\sigma^{+}(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-g^*(u)}}\,. \tag{1.9.49}$$

The positive counterpart, according to (20), will take the form:

$$V^{+}(x,y,z) = 2\int_0^\infty \mathrm{d}y_0\left\{\int_{l_2^*(y_0)}^{\infty}\frac{\mathcal{L}^*(2u-y_0-y)\,\mathrm{d}u}{\sqrt{u-y}\,\sqrt{g^*(u)-y_0}}\right\}\sigma^{+}(x,y_0)$$

$$= 2\int_0^\infty \mathrm{d}y_0\left\{\int_{-\infty}^{l_1^*(y_0)}\frac{\mathcal{L}^*(y+y_0-2u)\,\mathrm{d}u}{\sqrt{y-u}\,\sqrt{y_0-g^*(u)}}\right\}\sigma^{+}(x,y_0)\,. \tag{1.9.50}$$

We change the order of integration in (50) according to the scheme:

$$\int_0^{\infty} \mathrm{d}y_0 \int_{-\infty}^{l_1^*(y_0)} \mathrm{d}u = \int_{l_1^*}^{y} \mathrm{d}u \int_{g^*(u)}^{\infty} \mathrm{d}y_0 + \int_{-\infty}^{l_1^*} \mathrm{d}u \int_0^{\infty} \mathrm{d}y_0 \,. \tag{1.9.51}$$

Taking the complete potential as a superposition of $V^-$ and $V^+$, we see, that (49) cancels out with the second term in (51), so the only term left is:

$$V(x,y,z) = 2\int_{l_1^*}^{y} \frac{\mathrm{d}u}{\sqrt{y-u}} \int_{g^*(u)}^{\infty} \frac{\mathcal{L}^*(y+y_0-2u)\,\sigma^+(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-g^*(u)}} \,. \tag{1.9.52}$$

We can also change the order of integration in (52) and perform the integration with respect to $u$:

$$V(x,y,z) = 2\int_0^{\infty} \mathrm{d}y_0 \left\{ \int_{l_1^*}^{l_1^*(y_0)} \frac{\mathcal{L}^*(y+y_0-2u)\,\mathrm{d}u}{\sqrt{y-u}\,\sqrt{y-g^*(u)}} \right\} \sigma^+(x,y_0)$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{1}{R_0} \tan^{-1}\left(\frac{2\sqrt{y_0 l_2^*}}{R_0}\right) \sigma(x_0,y_0)\,\mathrm{d}y_0 \,. \tag{1.9.53}$$

We discard the superscript + on $\sigma$ in (53), because from the limits of integration it is obvious, that $\sigma$ is related to the half-plane $y>0$. In the plane $z=0$, (52) and (53) simplify to:

$$V(x,y,0) = 2\int_0^{y} \frac{\mathrm{d}u}{\sqrt{y-u}} \int_u^{\infty} \frac{\mathcal{L}^*(y+y_0-2u)\,\sigma(x,y_0)\,\mathrm{d}y_0}{\sqrt{y_0-u}}$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{1}{R} \tan^{-1}\left(\frac{2\sqrt{y y_0}}{R}\right) \sigma(x_0,y_0)\,\mathrm{d}y_0 \,. \tag{1.9.54}$$

**Application to elastic contact problems.** We consider a transversely isotropic elastic half-space $z \geq 0$, characterized by five elastic constants $A_{ik}$, as described in (Fabrikant, 1989). Let a semi-infinite smooth punch act on the boundary $z=0$, $y \geq 0$, while the rest of the boundary, namely, $z=0$, $y<0$, is stress-free. Assume, that the punch produces the normal displacement:

$$w = w(x,y) \qquad \text{for} \quad y \geq 0, \quad -\infty < x < \infty . \tag{1.9.55}$$

The other boundary conditions on the plane $z=0$ are:

$$\sigma_z = 0, \qquad \text{for} \quad y < 0, \quad -\infty < x < \infty ,$$

$$\tau_z = 0, \qquad \text{for} \quad -\infty < x, y < \infty . \tag{1.9.56}$$

It is known from (Fabrikant, 1989), that the governing integral equation is:

$$H \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{\sigma(x_0,y_0)}{R} \mathrm{d}y_0 = w(x,y), \quad y \geq 0, \quad -\infty < x < \infty . \tag{1.9.57}$$

Here $H$ is the elastic constant, defined in (Fabrikant, 1989) and $R^2 = (x-x_0)^2 + (y-y_0)^2$. According to (30), the solution of (57) will take the form:

$$\sigma(x,y) = -\frac{\mathcal{L}^*(y)}{2\pi^2 H} \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \frac{\mathrm{d}y_1}{\sqrt{y-y_1}} \mathcal{L}^*(-2y_1) \frac{\mathrm{d}}{\mathrm{d}y_1} \int_{y_1}^{\infty} \frac{\mathrm{d}y_0}{\sqrt{y_0-y_1}} \mathcal{L}^*(y_0) w(x,y_0) . \tag{1.9.58}$$

The complete solution to the problem can be expressed through two potential functions:

$$F_1(z) = \frac{H\gamma_1}{m_1 - 1} F(z_1), \qquad F_2(z) = \frac{H\gamma_2}{m_2 - 1} F(z_2) . \tag{1.9.59}$$

Here $z_k = z/\gamma_k$, $k=1,2$; $m_k$ and $\gamma_k$ are the elastic constants defined in (Fabrikant, 1989); the main potential function $F(z)$ is defined as

$$F(z) \equiv F(x,y,z) = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \ln\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2} + z\right) \sigma(x_0,y_0)\, \mathrm{d}y_0 . \tag{1.9.60}$$

By substitution of (58) in (60), we can easily compute $\partial F/\partial z$, which will coincide with (35) and further integration with respect to $z$ (see Appendix B) will give:

$$F(x,y,z) = \frac{1}{\pi H}\int_{-\infty}^{\infty} dx_0 \int_0^{\infty} K(x,y,z;x_0,y_0)\, w(x_0,y_0)\, dy_0\,, \tag{1.9.61}$$

where

$$K(x,y,z;x_0,y_0) = -\frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \frac{\sqrt{2}}{\sqrt{y_0}}\,\Re\left[\frac{1}{\sqrt{iq}}\tan^{-1}\sqrt{\frac{iq}{2l_2^*}}\right],$$

$$q = (x-x_0) + i(y-y_0)\,, \tag{1.9.62}$$

and $\Re$ stands for the real part of a complex expression. The displacements and stresses are given through the main potential function as follows (Fabrikant, 1989):

$$u = \Lambda(F_1 + F_2 + iF_3), \qquad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z}\,,$$

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\left[[\gamma_1^2 - (1+m_1)\gamma_3^2]F_1 + [\gamma_2^2 - (1+m_2)\gamma_3^2]F_2\right],$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + iF_3)\,,$$

$$\sigma_z = A_{44}\frac{\partial^2}{\partial z^2}\left[(1+m_1)\gamma_1^2F_1 + (1+m_2)\gamma_2^2F_2\right],$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}\left[(1+m_1)F_1 + (1+m_2)F_2 + iF_3\right]. \tag{1.9.63}$$

Here $u = u_x + iu_y$ is the complex tangential displacement and

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\phi}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right), \qquad i = \sqrt{-1}\,, \qquad \Delta = \Lambda\bar{\Lambda}.$$

$$\sigma_1 = \sigma_x + \sigma_y, \qquad \sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy}, \qquad \tau_z = \tau_{zx} + i\tau_{yz}\,. \tag{1.9.64}$$

Hence, to find a complete solution, we need the following derivatives of $K$:

$$\frac{\partial K}{\partial z} = \frac{z}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right], \tag{1.9.65}$$

$$\Lambda K = \frac{q}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] - \frac{1}{\overline{q}h^*}\left[1 - \sqrt{\frac{2il_2^*}{\overline{q}}}\ \tan^{-1}\sqrt{\frac{\overline{q}}{2il_2^*}}\right], \tag{1.9.66}$$

$$\frac{\partial^2 K}{\partial z^2} = \frac{1}{R_0^3}\left(1 - \frac{3z^2}{R_0^2}\right)\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] + \frac{1}{h^*[R_0^2 + (h^*)^2]}\left(\frac{z^2}{R_0^2} + \frac{l_1^*}{l_2^* - l_1^*}\right), \tag{1.9.67}$$

$$\frac{\partial}{\partial z}\Lambda K = -\frac{3zq}{R_0^5}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] + \frac{z}{h^*[R_0^2 + (h^*)^2]}\left(\frac{q}{R_0^2} - \frac{i}{2(l_2^* - l_1^*)}\right), \tag{1.9.68}$$

$$\Lambda^2 K = -\frac{3q^2}{R_0^5}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] + \frac{1}{h^*[R_0^2 + (h^*)^2]}\left(\frac{q^2}{R_0^2} + \frac{y - l_1^*}{l_2^* - l_1^*}\right)$$
$$+ \frac{3}{\overline{q}^2}\left(\frac{1}{h^*} - \sqrt{\frac{i}{2y_0\overline{q}}}\ \tan^{-1}\sqrt{\frac{\overline{q}}{2il_2^*}}\right) - \frac{1}{\overline{q}h^*(2il_2^* + \overline{q})}. \tag{1.9.69}$$

The substitution of (65)–(69) and (61) in (63) gives the complete solution.

**Application to the half-plane crack problem.** We consider a transversely isotropic elastic space weakened in the plane $z=0$ by a crack $y \geq 0$. The crack is opened by normal stress $p$, applied to the crack faces in opposite directions. The boundary conditions on the plane $z=0$ are:

$$\sigma_z = -p(x,y), \quad \text{for} \quad -\infty < x < \infty, \quad y > 0,$$

$$w = 0, \quad \text{for} \quad -\infty < x < \infty, \quad y \leq 0,$$

$$\tau_z = 0, \quad \text{for} \quad -\infty < x, y < \infty. \tag{1.9.70}$$

The main potential functions in this case are:

$$\Phi_1(x,y,z) \equiv \Phi_1(z) = -\frac{\gamma_1}{2\pi(m_1 - 1)}\Phi(z_1),$$

$$\Phi_2(x,y,z) \equiv \Phi_2(z) = -\frac{\gamma_2}{2\pi(m_2 - 1)}\Phi(z_2) \tag{1.9.71}$$

where

$$\Phi(z) \equiv \Phi(x,y,z) = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{w(x_0,y_0)\,\mathrm{d}y_0}{R_0} \tag{1.9.72}$$

and $R_0^2 = (x-x_0)^2 + (y-y_0)^2 + z^2$. The governing integral equation will take the form:

$$p(x,y) = -\frac{1}{4\pi^2 H}\Delta \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{w(x_0,y_0)\,\mathrm{d}y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \tag{1.9.73}$$

where $\Delta$ is defined in (64). The integral equation (73) is inverse to (54), therefore its solution is:

$$w(x,y) = \frac{2}{\pi} H \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \frac{p(x_0,y_0)}{R} \tan^{-1}\left(\frac{2\sqrt{yy_0}}{R}\right) \mathrm{d}y_0 \,. \tag{1.9.74}$$

Substitution of (74) in (72) allows us to express the main potential function in terms of prescribed pressure as

$$\Phi(x,y,z) = \frac{2}{\pi} H \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_0^{\infty} \mathcal{K}(x,y,z;x_0,y_0)\, p(x_0,y_0)\,\mathrm{d}y_0 \,, \tag{1.9.75}$$

where

$$\mathcal{K}(x,y,z;x_0,y_0) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_0^{\infty} \frac{1}{R_{10}} \tan^{-1}\left(\frac{2\sqrt{yy_1}}{R_{10}}\right) \frac{\mathrm{d}y_1}{R_1} \,. \tag{1.9.76}$$

Here

$$R_{10} = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2} \,,$$

$$R_1 = \sqrt{(x_1-x)^2 + (y_1-y)^2 + z^2} \,. \tag{1.9.77}$$

For the field of displacements and stresses we mainly need various derivatives of $\mathcal{K}$. They are:

$$\frac{\partial \mathcal{K}}{\partial z} = -\frac{2\pi}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right), \tag{1.9.78}$$

$$\Lambda\mathcal{K} = \frac{2\pi}{\bar{q}}\left[\frac{z}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) - c\tan^{-1}\sqrt{\frac{\bar{s}^*}{-2l_1^*}}\right], \tag{1.9.79}$$

$$\frac{\partial^2 \mathcal{K}}{\partial z^2} = 2\pi\left[\frac{z}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{h^*}{z[R_0^2+(h^*)^2]}\left(\frac{l_1^*}{l_2^*-l_1^*} + \frac{z^2}{R_0^2}\right)\right], \tag{1.9.80}$$

$$\frac{\partial}{\partial z}\Lambda\mathcal{K} = 2\pi\left[\frac{q}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{h^*}{[R_0^2+(h^*)^2]}\left(\frac{q}{R_0^2} - \frac{i}{2(l_2^*-l_1^*)}\right)\right], \tag{1.9.81}$$

$$\Lambda^2\mathcal{K} = 2\pi\left[\frac{c}{\bar{q}}\left(\frac{i}{\bar{s}^*} + \frac{2}{\bar{q}}\right)\tan^{-1}\sqrt{\frac{\bar{s}^*}{-2l_1^*}} - \frac{z(3R_0^2-z^2)}{\bar{q}^2 R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right.$$

$$\left. - \frac{2\sqrt{y_0 l_1^*}}{\bar{q}\,\bar{s}^*(\bar{s}^*-2l_1^*)} - \frac{z\,h^*}{R_0^2+(h^*)^2}\left(\frac{q}{\bar{q}R_0^2} + \frac{1}{4l_2^*(l_2^*-l_1^*)}\right)\right]. \tag{1.9.82}$$

In expressions (65)–(69) and (78)–(82) the values for $l_1^*$, $l_2^*$ and $h^*$ are defined by formulae (13) and (36). The other parameters are defined as

$$s^* = (y+y_0) - i(x-x_0)\,, \qquad q = (x-x_0) + i(y-y_0)\,,$$

$$c = \sqrt{\frac{2y_0}{\bar{s}^*}} \tag{1.9.83}$$

and the overbar everywhere indicates the complex conjugate quantity.

We consider, as an example, the action of a pair of normal concentrated forces $P\delta(x-x_0)\,\delta(y-y_0)$ applied to the crack faces in the opposite directions at the points $(x_0,y_0,0^{\pm})$, $y_0>0$. According to (63), (71), (75) and (78)–(82), the complete solution for the field of stresses and displacements in a transversely isotropic elastic space is:

$$u = \frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1^*(z_1) + \frac{\gamma_2}{m_2-1}f_1^*(z_2)\right], \tag{1.9.84}$$

$$w = \frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2^*(z_1) + \frac{m_2}{m_2-1}f_2^*(z_2)\right], \tag{1.9.85}$$

$$\sigma_1 = \frac{2P}{\pi^2(\gamma_1 - \gamma_2)}\left[\left(\frac{\gamma_1}{(m_1+1)\gamma_3^2} - \frac{1}{\gamma_1}\right) f_3^*(z_1) - \left(\frac{\gamma_2}{(m_2+1)\gamma_3^2} - \frac{1}{\gamma_2}\right) f_3^*(z_2)\right], \tag{1.9.86}$$

$$\sigma_2 = \frac{4}{\pi} H A_{66} P \left[\frac{\gamma_1}{m_1 - 1} f_4^*(z_1) + \frac{\gamma_2}{m_2 - 1} f_4^*(z_2)\right], \tag{1.9.87}$$

$$\sigma_z = \frac{P}{\pi^2(\gamma_1 - \gamma_2)}\left(\gamma_1 f_3^*(z_1) - \gamma_2 f_3^*(z_2)\right), \tag{1.9.88}$$

$$\tau_z = \frac{P}{\pi^2(\gamma_1 - \gamma_2)}\left(f_5^*(z_1) - f_5^*(z_2)\right), \tag{1.9.89}$$

where

$$f_1^*(z) = \frac{1}{q}\left[c \tan^{-1}\sqrt{\frac{\bar{s}^*}{-2l_1^*}} - \frac{z}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right], \tag{1.9.90}$$

$$f_2^*(z) = \frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right), \tag{1.9.91}$$

$$f_3^*(z) = -\left[\frac{z}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{h^*}{z[R_0^2 + (h^*)^2]}\left(\frac{l_1^*}{l_2^* - l_1^*} + \frac{z^2}{R_0^2}\right)\right], \tag{1.9.92}$$

$$f_4^*(z) = -\left[\frac{c}{\bar{q}}\left(\frac{i}{\bar{s}^*} + \frac{2}{\bar{q}}\right)\tan^{-1}\sqrt{\frac{\bar{s}^*}{-2l_1^*}} - \frac{z(3R_0^2 - z^2)}{\bar{q}^2 R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right.$$

$$\left. - \frac{2\sqrt{y_0 l_1^*}}{\bar{q}\,\bar{s}^*(\bar{s}^* - 2l_1^*)} - \frac{z h^*}{R_0^2 + (h^*)^2}\left(\frac{q}{\bar{q} R_0^2} + \frac{1}{4l_2^*(l_2^* - l_1^*)}\right)\right]. \tag{1.9.93}$$

$$f_5^*(z) = -\left[\frac{q}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{h^*}{[R_0^2 + (h^*)^2]}\left(\frac{q}{R_0^2} - \frac{1}{2(l_2^* - l_1^*)}\right)\right], \tag{1.9.94}$$

The results, obtained in (84)–(89) are valid for isotropic bodies as well, provided that we take

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \qquad H = \frac{1 - \nu^2}{\pi E}, \qquad A_{44} = A_{66} = \frac{E}{2(1+\nu)}, \tag{1.9.95}$$

where $E$ is the elastic modulus and $\nu$ is the Poisson's coefficient. The relevant limits can be computed according to the L'Hôpital rule. We leave the actual computations for the case of isotropy to the reader.

**Appendix A.** Derivation of (35). By using the rule of differentiation under the integral sign, the expression in curly brackets of (34) can be rewritten as

$$\frac{\mathcal{L}^*(y_0+2l_2^*-y)\sqrt{l_2^*-y}}{(l_2^*-l_1^*)\sqrt{y_0}}+\int_0^{y_0}\frac{\mathrm{d}t}{\sqrt{y_0-t}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathcal{L}^*\big(2l_2^*(t)-y+y_0-2t\big)}{l_2^*(t)-l_1^*(t)}\sqrt{l_2^*(t)-y}\right]. \tag{1.9.A1}$$

We introduce $h^*[l_1^*(t)]$, as defined in (17), and transform it as follows:

$$\begin{aligned} h^*[l_1^*(t)] &= 2\sqrt{l_2^*(y_0)-l_1^*(t)}\ \ \sqrt{l_1^*(y_0)-l_1^*(t)} \\ &= 2\sqrt{yy_0-(z^2/4)-l_1^*(t)(y+y_0)+[l_1^*(t)]^2} \\ &= 2\sqrt{y-l_1^*(t)}\ \ \sqrt{y_0-l_1^*(t)-z^2/\{4[y-l_1^*(t)]\}} \\ &= 2\sqrt{l_2^*(t)-t}\ \ \sqrt{y_0-t}\equiv\hat{h}(t)\,. \end{aligned} \tag{1.9.A2}$$

Here we used the identities:

$$l_2^*(t)-t=y-l_1^*(t)\,,\qquad l_1^*(t)\,l_2^*(t)=yt-\frac{1}{4}z^2\,,\qquad g^*[l_1^*(t)]=t\,. \tag{1.9.A3}$$

The derivative of $\hat{h}$ can be computed as

$$\frac{\mathrm{d}\hat{h}}{\mathrm{d}t}=-\frac{\sqrt{l_2^*(t)-t}}{\sqrt{y_0-t}}\,\frac{l_2^*(t)-l_1^*(t)+y_0-t}{l_2^*(t)-l_1^*(t)}\,. \tag{1.9.A4}$$

By using the identity

$$2l_2^*(t)-y+y_0-2t=l_2^*(t)-l_1^*(t)+y_0-t\,, \tag{1.9.A5}$$

we can present the expression in square brackets in (A1) as

$$\frac{[l_2^*(t)-l_1^*(t)+y_0-t]\sqrt{l_2^*(t)-y}}{[l_2^*(t)-l_1^*(t)]\,\{[l_2^*(t)-l_1^*(t)+y_0-t]^2+(x-x_0)^2\}}$$

$$= -\frac{\sqrt{l_2^*(t)-y}\,\sqrt{y_0-t}}{\sqrt{l_2^*(t)-t}\,[R_0^2+\hat{h}^2(t)]} = -\frac{2z(y_0-t)^{3/2}}{\hat{h}^2(t)\,[R_0^2+\hat{h}^2(t)]}\,\frac{\mathrm{d}\hat{h}(t)}{\mathrm{d}t}\,. \tag{1.9.A6}$$

Substitution of (A6) in (A1) and integration by parts yields (we replace the $\mathcal{L}^*$ operator by $\lambda^*$)

$$\frac{\lambda^*(y_0+2l_2^*-y,\,x-x_0)}{(l_2^*-l_1^*)\sqrt{y_0}} - 2z\int_0^{y_0}\frac{\mathrm{d}t}{\sqrt{y_0-t}}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(y_0-t)^{3/2}\,\hat{h}'(t)}{\hat{h}^2(t)\,[R_0^2+\hat{h}^2(t)]}\right]$$

$$= -2z\lim_{t\to y_0}\left[\frac{(y_0-t)\,\hat{h}'(t)}{\hat{h}^2(t)\,[R_0^2+\hat{h}^2(t)]}\right] + z\int_0^{y_0}\frac{\hat{h}'(t)\,\mathrm{d}t}{\hat{h}^2(t)\,[R_0^2+\hat{h}^2(t)]}$$

$$= -2z\lim_{t\to y_0}\left[\frac{(y_0-t)\,\hat{h}'(t)}{\hat{h}^2(t)\,[R_0^2+\hat{h}^2(t)]}\right] + \frac{z}{R_0^2}\int_{\hat{h}(0)}^{\hat{h}(y_0)}\left(\frac{1}{\hat{h}^2(t)} - \frac{1}{R_0^2+\hat{h}^2(t)}\right)\mathrm{d}\hat{h}(t)$$

$$= \frac{z}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right]. \tag{1.9.A7}$$

Here, for the sake of brevity, we introduced the notation $h^* = \hat{h}(0)$, as defined in (A2). Substitution (A7) back into (34) leads to (35). Note, that the limit $t\to y_0$ in (A7) is infinite and cancels out with the next term $z/[R_0^2\hat{h}(t)]$ for $t\to y_0$. We note also other useful identities:

$$\sqrt{l_2^*(t)-y}\;\sqrt{y-l_1^*(t)} = \sqrt{l_2^*(t)-t}\;\sqrt{t-l_1^*} = \sqrt{l_1^*(t)-t}\;\sqrt{l_1^*(t)-y}$$

$$= \sqrt{l_2^*(t)-y}\;\sqrt{l_2^*(t)-t} = \frac{z}{2}\,,$$

$$y-l_1^*(t) = l_2^*(t)-t\,, \qquad t-l_1^*(t) = l_2^*(t)-y\,,$$

$$\frac{\partial l_1^*(t)}{\partial t} = \frac{y-l_1^*(t)}{l_2^*(t)-l_1^*(t)} = \frac{l_2^*(t)-t}{l_2^*(t)-l_1^*(t)}\,. \tag{1.9.A8}$$

**Appendix B.** Here we present the details of the integration resulting in the expression, given (62). The integral to be computed is:

$$I = \int \frac{z}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right] \mathrm{d}z\,. \tag{1.9.B1}$$

Here we note some useful derivatives and identities:

$$\frac{\partial l_1^*}{\partial z} = -\frac{z}{2(l_2^* - l_1^*)}\,, \qquad \frac{\partial l_2^*}{\partial z} = \frac{z}{2(l_2^* - l_1^*)}\,, \qquad \frac{\partial h^*}{\partial z} = \frac{y_0 z}{h^*(l_2^* - l_1^*)}$$

$$\frac{\partial}{\partial z}\tan^{-1}\left(\frac{h^*}{R_0}\right) = \frac{R_0}{R_0^2 + (h^*)^2}\left(\frac{y_0 z}{h^*(l_2^* - l_1^*)} + \frac{z}{h^*}\right) - \frac{z}{R_0 h^*}\,,$$

$$R_0^2 + (h^*)^2 = (2l_1^* - s^*)(2l_1^* - \overline{s}^*) = (2l_2^* - i\overline{q})(2l_2^* + iq)\,. \tag{1.9.B2}$$

We Integrate by parts in (B1) and obtain:

$$I = \int \frac{z\,\mathrm{d}z}{R_0^2 h^*} - \frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \int \frac{\mathrm{d}z}{R_0}\frac{\partial}{\partial z}\tan^{-1}\left(\frac{h^*}{R_0}\right). \tag{1.9.B3}$$

Using (B2), we can rewrite (B3) as

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \int \frac{y_0 z\,\mathrm{d}z}{h^*(l_2^* - l_1^*)[R_0^2 + (h^*)^2]} + \int \frac{z\,\mathrm{d}z}{h^*[R_0^2 + (h^*)^2]}\,. \tag{1.9.B4}$$

The change of variables of integration with help of the expressions in (B2) will allow us to transform the integrals in (B4) respectively into the following:

$$\int \frac{y_0 z\,\mathrm{d}z}{h^*(l_2^* - l_1^*)[R_0^2 + (h^*)^2]} = \sqrt{y_0}\int \frac{\mathrm{d}l_2^*}{\sqrt{l_2^*}\,(2l_2^* - i\overline{q})\,(2l_2^* + iq)}\,,$$

$$\int \frac{z\,\mathrm{d}z}{h^*[R_0^2 + (h^*)^2]} = \frac{2}{\sqrt{y_0}}\int \frac{\sqrt{l_2^*}\,\mathrm{d}l_2^*}{(2l_2^* - i\overline{q})\,(2l_2^* + iq)} - \frac{y}{\sqrt{y_0}}\int \frac{\mathrm{d}l_2^*}{\sqrt{l_2^*}\,(2l_2^* - i\overline{q})\,(2l_2^* + iq)}\,. \tag{1.9.B5}$$

After substitution of (B5) in (B4) and some simplifications we obtain:

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{1}{2\sqrt{y_0}}\left[\int \frac{\mathrm{d}l_2^*}{\sqrt{l_2^*}\,(2l_2^* - i\overline{q})} + \int \frac{\mathrm{d}l_2^*}{\sqrt{l_2^*}\,(2l_2^* + iq)}\right]. \tag{1.9.B6}$$

The integrals in (B6) are elementary and we have:

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{1}{\sqrt{2y_0}}\,2\Re\left[\frac{1}{\sqrt{iq}}\tan^{-1}\sqrt{\frac{2l_2^*}{iq}}\right]. \qquad (1.9.B7)$$

Finally, (B7) allows us to write the result of the definite integral:

$$\int_\infty^z \frac{z}{R_0^3}\left[\frac{R_0}{h^*} + \tan^{-1}\left(\frac{h^*}{R_0}\right)\right]\mathrm{d}z = -\frac{1}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \frac{1}{\sqrt{2y_0}}\,2\Re\left[\frac{1}{\sqrt{iq}}\tan^{-1}\sqrt{\frac{iq}{2l_2^*}}\right]. \qquad (1.9.B8)$$

## 1.10. Exact solution of the biharmonic integral equation and its applications

**Abstract.** A new type of integral equation, which is called here biharmonic, is studied in detail. An exact closed form solution is obtained for a circular domain by using a new integral representation for the distance between two points, combined with the properties of the Abel type integrals and the $\mathcal{L}$-operators, introduced by the author earlier. The necessary and sufficient conditions have been established for existence of an integrable solution in the case of a circular domain. The results are illustrated by several examples.

**Introduction.** The study, presented here, was prompted by a letter from Ciavarella (Oxford, as he then was), where he was asking me the following question: given an arbitrary finite domain $S$ at the boundary $z=0$ of an elastic half-space $z\geq 0$, is it possible to find a unidirectional tangential stress distribution, such that it would produce a constant tangential displacement in the same direction and would not produce any displacement in the perpendicular direction. The rest of the boundary is assumed to be stress-free, and the normal stress vanishes all over the boundary. It is well known (see, for example, Mindlin, 1949), that such a distribution does exist in the case of an elliptical domain. The question now is whether this will also be true in the case of a general domain. Partial response to this question was given in (Fabrikant, 1989), where the governing integral equation for such a problem (we call it *tangential contact problem*) in the case of a transversely isotropic elastic body was presented in the form:

$$\frac{1}{2}G_1\Delta\int\!\!\int_S R(J,J_0)\,\tau(J_0)\,\mathrm{d}S_{J_0} - \frac{1}{2}G_2\Lambda^2\int\!\!\int_S R(J,J_0)\,\bar{\tau}(J_0)\,\mathrm{d}S_{J_0} = u(J)\,. \qquad (1.10.1)$$

Here $G_1$ and $G_2$ are the elastic constants introduced in (Fabrikant, 1989); $R(\cdot,\cdot)$ stands for the distance between two points, the points $J$ and $J_0$ have Cartesian coordinates $(x,y,0)$ and $(x_0,y_0,0)$ or the polar cylindrical coordinates $(\rho,\varphi,0)$ and $(\rho_0,\phi_0,0)$ respectively; the complex tangential stress $\tau$ was introduced as $\tau=\tau_{zx}+i\tau_{yz}$, and the relevant complex tangential displacement was introduced as $u=u_x+iu_y$; the overbar denotes a complex conjugate quantity; $\Delta$ is a two-dimensional Laplacian, and the differential operator $\Lambda$ is

$$\Lambda = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} = \mathrm{e}^{i\phi}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right), \qquad \Delta = \Lambda\overline{\Lambda}\,. \tag{1.10.2}$$

From the form of equation (1), one may conclude that Ciavarella's question can be answered in affirmative, provided that there exists a real function $\tau_0$, such that

$$\iint_S R(J,J_0)\,\tau_0(J_0)\,\mathrm{d}S_{J_0} = c_1x^2 + c_2y^2 + c_3x + c_4y + c_5\,, \tag{1.10.3}$$

where $c_1$, ..., $c_5$ are real constants. Since application of $\Delta$ and $\Lambda$ to both sides of (3) gives us real constants, this proves that existence of $\tau_0$ effectively guarantees, that such a unidirectional tangential stress will result in a unidirectional constant tangential displacement over domain $S$. Now the problem is reduced to investigation of the integral equation (3). There is huge literature on the integral equation, where the distance between two points is in the denominator (a simple layer in potential theory), but I could not recall any reference study of the integral equation (3), where the distance is in the numerator. It seems that such an equation does not even have a name. I have decided to call it *biharmonic integral equation* for the following reason: a general biharmonic function $f$ can be presented in the following form:

$$f(M) = \iint_S R(M,J_0)\,\tau(J_0)\,\mathrm{d}S_{J_0}\,. \tag{1.10.4}$$

Here point $M$ has the cylindrical polar coordinates $(\rho,\phi,z)$. Comparison of (3) and (4) explains the logic behind the name given to the integral equation.

One can notice a small "cloud" over the existence of solution of integral equation (3), namely, if we assume existence of solution in the case, where $c_1=c_2=0$, this would mean in physical terms that there exists a non-zero tangential stress distribution, such that it does not produce any tangential displacement in any direction, which does not make sense, so we have to assume that such a solution does not exist, even in the case of a circular domain $S$. On the other hand, application of Laplacian to both sides of (3) results in

$$\int\!\!\int_S \frac{\tau(J_0)}{R(J,J_0)}\,\mathrm{d}S_{J_0} = 2(c_1+c_2)\,. \tag{1.10.5}$$

Equation (5) is the well known integral equation of the normal contact problem, which is known to have a solution for arbitrary domain. Of course, there is no guarantee, that the solution of (5) will also be a solution for the integral equation (3): they may differ by an arbitrary harmonic function.

So, the purpose of this study can now be outlined as follows: to find a solution to the biharmonic integral equation:

$$\int\!\!\int_S R(J,J_0)\,\tau(J_0)\,\mathrm{d}S_{J_0} = w(J)\,, \tag{1.10.6}$$

where $w$ is a known function and $\tau$ is unknown, as well as the necessary and sufficient conditions imposed on function $w$, so that the integral equation (6) be solvable. Since this equation has never been considered before, we limit ourselves here to the case of domain $S$ being a circle of radius $a$. The case of elliptical and more complicated domains will be considered separately.

The analysis goes along the following route. First, an integral representation for the distance between two points is established. This representation is used then to reformulate the problem in terms of the $\mathcal{L}$-operators, introduced in (Fabrikant, 1989), and the Abel-type integrals. Process of finding the inverse operators leads to a closed form solution, as well as to the mathematical conditions imposed on function $w$, which would guarantee existence of an integrable solution. Several examples are considered as illustrations of effectiveness of the obtained solution. Additional applications of the developed mathematical apparatus to computation of various integrals, involving distances between two points, is also presented. Majority of these integrals seem to be new and have not been computed before.

**Integral representation for the distance between two points.** The idea for such a representation comes from the general formula, derived in (Fabrikant, 1989):

$$\frac{1}{R^{1+k}} = \frac{2}{\pi}\cos\left(\frac{\pi k}{2}\right)\int_0^{min(\rho_0,\rho)} \frac{\lambda\left(\dfrac{x^2}{\rho\rho_0},\,\phi-\phi_0\right)x^k\,\mathrm{d}x}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+k)/2}}\,, \quad \text{for} \quad k>-1\,. \tag{1.10.7}$$

Here $R$ is an abbreviation for $R(J,J_0)$, and

$$\lambda(m,\psi) = \frac{1-m^2}{1+m^2-2m\cos\psi}, \qquad m<1\,. \tag{1.10.8}$$

Though the value of $k$ in (7) is assumed to be greater than $-1$, we may try to use it for $k=-2$. The formal result reads:

$$R(J,J_0) = -\frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}{x^2}\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\mathrm{d}x\,. \tag{1.10.9}$$

The integral in (9) is divergent. Rephrasing Heaviside, who has allegedly said: "This series is divergent; therefore we may be able to do something with it", we can say: "This integral is divergent, therefore it might be quite useful." Indeed, we can add and subtract an obvious term eliminating singularity, with the result:

$$R(J,J_0) = \frac{2}{\pi}\left\{\max(\rho,\rho_0) + \int_0^{min(\rho_0,\rho)}\left[\frac{\rho\rho_0}{x^2} - \frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}{x^2}\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\right]\mathrm{d}x\right\}. \tag{1.10.10}$$

Expression (10) is a mathematically correct integral representation for the distance between two points. Its correctness can be verified by a direct computation of the integral in (10). Indeed, such an integral can be computed as indefinite:

$$\int\left[\frac{\rho\rho_0}{x^2} - \frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}{x^2}\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\right]\mathrm{d}x = \frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}-\rho\rho_0}{x}$$

$$-R\tan^{-1}\left(\frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}{xR}\right). \tag{1.10.11}$$

In order to obtain the integral representation for the distance between two points in the case where the $z$-coordinate is non-zero, one can introduce the quantities (Fabrikant, 1989):

$$l_1(x) = \frac{1}{2}\left(\sqrt{(\rho+x)^2+z^2} - \sqrt{(\rho-x)^2+z^2}\right), \tag{1.10.12}$$

$$l_2(x) = \frac{1}{2}\left(\sqrt{(\rho+x)^2+z^2} + \sqrt{(\rho-x)^2+z^2}\right). \tag{1.10.13}$$

One can verify the following properties of $l_1$ and $l_2$:

$$l_1(x)l_2(x) = x\rho\,, \qquad l_1^2(x) + l_2^2(x) = x^2 + \rho^2 + z^2\,,$$

$$\lim_{z\to 0} l_1(x) = \min(x,\rho)\,, \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho)\,, \tag{1.10.14}$$

The formal substitution of $\rho$ and $\rho_0$ by $l_1(\rho_0)$ and $l_2(\rho_0)$ in (10) leads to the representation:

$$R(M,J_0) = \frac{2}{\pi}\left\{ l_2(\rho_0) + \int_0^{l_1(\rho_0)} \left[\frac{\rho\rho_0}{x^2} - \frac{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}{x^2}\,\lambda\!\left(\frac{x^2}{\rho\rho_0},\,\phi-\phi_0\right)\right] \mathrm{d}x \right\}. \tag{1.10.15}$$

Substitution of (15) into (4) allows us to transform it as follows:

$$\int_0^{2\pi}\int_0^a R(M,J_0)\,\tau(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = \frac{2}{\pi}\left\{\int_0^{2\pi}\int_0^a l_2(\rho_0)\,\tau(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0\right.$$

$$-\rho\int_0^{2\pi}\int_0^a \tau(\rho_0,\phi_0)\rho_0^2\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 \left.\frac{1}{x}\right|_0^{l_1(\rho_0)} - 2\pi\int_0^{l_1}\frac{\sqrt{\rho^2-x^2}}{x^2}\,\mathrm{d}x\int_{g(x)}^a \sqrt{\rho_0^2-g^2(x)}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)$$

$$\left.\times\,\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0\right\} = -4\int_0^{l_1}\frac{\mathrm{d}x}{x}\frac{\partial}{\partial x}\left[\sqrt{\rho^2-x^2}\int_{g(x)}^a \sqrt{\rho_0^2-g^2(x)}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0\right]. \tag{1.10.16}$$

Here the notation $l_1$ stands as an abbreviation for $l_1(a)$, the $\mathcal{L}$-operator was introduced as (Fabrikant, 1989):

$$\mathcal{L}(m)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi_0)\,\lambda(m,\,\phi-\phi_0)\,\mathrm{d}\phi_0\,, \tag{1.10.17}$$

the quantity $g(x)$ is defined as

$$g^2(x) = x^2\left(1 + \frac{z^2}{\rho^2-x^2}\right), \tag{1.10.18}$$

and the following rule of interchange of the order of integration was used:

$$\int_0^a \mathrm{d}\rho_0 \int_0^{l_1(\rho_0)} \mathrm{d}x = \int_0^{l_1} \mathrm{d}x \int_{g(x)}^a \mathrm{d}\rho_0 \,. \tag{1.10.19}$$

In order to verify (19), one has to use the fact that function $g$ is inverse to both $l_1$ and $l_2$.

In the case of $z=0$ and $\rho<a$, utilization of (16) in (6) allows us to rewrite the biharmonic integral equation in the form:

$$-4\int_0^\rho \frac{\mathrm{d}x}{x}\frac{\partial}{\partial x}\left[\sqrt{\rho^2-x^2}\int_x^a \sqrt{\rho_0^2-x^2}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0\right] = w(\rho,\phi)\,. \tag{1.10.20}$$

Now the biharmonic integral equation has been transformed into the form involving Abel-operators and the $\mathcal{L}$-operators, and we can proceed with its solution.

**Solution of the biharmonic integral equation.** We assume that the known function $w$ and the unknown function $\tau$ can be expanded in a Fourier series:

$$w(\rho,\phi) = \sum_{n=-\infty}^{\infty} w_n(\rho)\,\mathrm{e}^{in\phi}\,, \qquad \tau(\rho,\phi) = \sum_{n=-\infty}^{\infty} \tau_n(\rho)\,\mathrm{e}^{in\phi}\,. \tag{1.10.21}$$

From the structure of (20) it is obvious that the $n$-th harmonic of $\tau$ will be related to the $n$-th harmonic of $w$ only, therefore, in order to solve (20) we can consider the integral equation for the $n$-th harmonic, to solve it, and then to do the proper summation. The integral equation for the $n$-th harmonic will take the form:

$$-4\int_0^\rho \frac{\mathrm{d}x}{x}\frac{\partial}{\partial x}\left[\sqrt{\rho^2-x^2}\int_x^a \sqrt{\rho_0^2-x^2}\left(\frac{x^2}{\rho\rho_0}\right)^{\mathrm{n}}\tau_n(\rho_0)\,\rho_0\,\mathrm{d}\rho_0\right] = w_n(\rho)\,. \tag{1.10.22}$$

What follows is the procedure of solving (22), which mainly consists of application of various integral and differential operators. If the reader asks, why we are using this or that operator at certain step in solution, the answer is very simple: since there is no established general procedure, the solution had to be found by the ”trial and error” method, and it was. The process leading to the solution is presented below. The reader interested in the final result only can skip the derivation.

Application of the operator

$$\frac{\partial}{\partial r}\int_0^r \frac{r^{n+1}\,dr}{\sqrt{r^2-\rho^2}}$$

to both sides of (22) yields, after interchange of the order of integration:

$$2\pi\left\{\int_r^a \sqrt{\rho_0^2-r^2}\;\left(\frac{r^2}{\rho_0}\right)^n \tau_n(\rho_0)\,\rho_0\,d\rho_0 - r\int_0^r \frac{dx}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x^2}{\rho_0}\right)^n\right]\tau_n(\rho_0)\,\rho_0\,d\rho_0\right\}$$

$$=\frac{\partial}{\partial r}\int_0^r \frac{w_n(\rho)\,\rho^{n+1}\,d\rho}{\sqrt{r^2-\rho^2}}\,. \qquad (1.10.23)$$

The next operator to apply is:

$$\int_y^a \frac{dr}{\sqrt{r^2-y^2}\,r^{2n+1}}\,,$$

with the result:

$$\pi^2\int_y^a\left(\frac{\rho_0}{y}-1\right)\tau_n(\rho_0)\rho_0^{1-n}\,d\rho_0 - 2\pi\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\int_0^r \frac{dx}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x^2}{r^2\rho_0}\right)^n\right]\tau_n(\rho_0)\,\rho_0\,d\rho_0$$

$$=\int_y^a \frac{dr}{\sqrt{r^2-y^2}\,r^{2n+1}}\frac{\partial}{\partial r}\int_0^r \frac{w_n(\rho)\,\rho^{n+1}\,d\rho}{\sqrt{r^2-\rho^2}}\,. \qquad (1.10.24)$$

Now, multiplying both sides of (24) by $y$ and differentiating twice with respect to $y$, we obtain an explicit expression for $\tau_n$:

$$\tau_n(y)=\frac{y^{n-1}}{\pi^2}\frac{\partial^2}{\partial y^2}\left\{y\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\left[\frac{W_n(r)}{r^{2n}}+2\pi\int_0^r \frac{dx}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x^2}{r^2\rho_0}\right)^n\right]\tau_n(\rho_0)\,\rho_0\,d\rho_0\right]\right\}, \qquad (1.10.25)$$

which leaves us with the task of dealing with the last term in (25) which also contains $\tau_n$. Here the notation was introduced:

$$W_n(r) = \frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{w_n(\rho)\,\rho^{n+1}\,d\rho}{\sqrt{r^2-\rho^2}}\,. \tag{1.10.26}$$

First, we establish the following useful identity:

$$y\frac{\partial^2}{\partial y^2}\left(y\int_y^a \frac{F(r)\,dr}{\sqrt{r^2-y^2}}\right) = \int_y^a \frac{[F'(r)\,r^2]'}{\sqrt{r^2-y^2}}\,dr - \frac{a^3F(a)}{(a^2-y^2)^{3/2}} - \frac{a^2F'(a)}{\sqrt{a^2-y^2}}\,. \tag{1.10.27}$$

Here $F$ is an arbitrary function, which is finite and differentiable at $a$, the prime sign denotes derivative with respect to the argument. The identity (27) is derived by using twice the rule of differentiation under the integral sign:

$$\frac{\partial}{\partial y}\int_y^a \frac{F(r)\,dr}{\sqrt{r^2-y^2}} = -\frac{a\,F(a)}{y\sqrt{a^2-y^2}} + \frac{1}{y}\int_y^a \frac{F'(r)\,r\,dr}{\sqrt{r^2-y^2}} = -\frac{y\,F(a)}{a\sqrt{a^2-y^2}} + y\int_y^a \frac{[F(r)/r]'}{\sqrt{r^2-y^2}}\,dr\,.$$

For the purpose of further transformations we denote:

$$F_n(r) = \frac{2}{\pi}\int_0^r \frac{dx}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x}{r}\right)^{2n}\right]\tau_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0\,. \tag{1.10.28}$$

In dealing with the last term of (25), according to (27), we need first to compute:

$$\int_0^r \frac{dx}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x}{r}\right)^{2n}\right]\tau_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0 = \frac{1}{r^{2n}}\left[\int_0^r [\Phi(\rho_0,\rho_0)\right.$$

$$\left. - \Phi(0,\rho_0)]\,\tau_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0 + \int_r^a [\Phi(r,\rho_0) - \Phi(0,\rho_0)]\,\tau_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0\right]. \tag{1.10.29}$$

Here the notation was introduced:

$$\Phi(t,\rho_0) = -\int_t^{\rho_0} \frac{x^{2n-2}\,[2n\rho_0^2 - (2n+1)x^2]}{\sqrt{\rho_0^2-x^2}}\,dx\,. \tag{1.10.30}$$

The integral in (30) is computable in terms of elementary functions, but we do not really need to do it. Since $\Phi(\rho_0,\rho_0)=0$, expression (29) can be simplified further as

$$\int_0^r \frac{\mathrm{d}x}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x}{r}\right)^{2n}\right]\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0=\frac{1}{r^{2n}}\left[-\int_0^a \Phi(0,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right.$$

$$\left.+\int_r^a \Phi(r,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right]. \qquad (1.10.31)$$

The differentiation of (31) with respect to $r$ yields:

$$\frac{\partial}{\partial r}\int_0^r \frac{\mathrm{d}x}{x}\int_x^a \frac{\partial}{\partial x}\left[\sqrt{\rho_0^2-x^2}\left(\frac{x}{r}\right)^{2n}\right]\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0=\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}\,r^2}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0$$

$$-\frac{2n}{r^{2n+1}}\left[-\int_0^a \Phi(0,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0+\int_r^a \Phi(r,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right]. \qquad (1.10.32)$$

Now, according to (27), we need to multiply both sides of (32) by $r^2$ and differentiate the result with respect to $r$. This procedure gives:

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\int_0^r \frac{\mathrm{d}x}{x}\int_x^a \frac{\partial}{\partial x}\left(\sqrt{\rho_0^2-x^2}\left(\frac{x}{r}\right)^{2n}\right)\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right]$$

$$=\frac{\partial}{\partial r}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0-\frac{2n}{r}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0$$

$$-\frac{2n(2n-1)}{r^{2n}}\left[-\int_0^a \Phi(0,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0+\int_r^a \Phi(r,\rho_0)\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right]. \qquad (1.10.33)$$

The next step is the substitution of (33) in (27) and computation of the integrals. It is shown here step by step. The first integral to be computed is:

$$\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0 = \frac{\pi}{2}\left(y^{2-n}\tau_n(y)-(2n+1)\int_y^a \tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right)$$

$$+\frac{1}{\sqrt{a^2-y^2}}\lim_{r\to a}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\,. \qquad (1.10.34)$$

Here the following general rules were used:

$$\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\left(r^2\int_r^a \frac{f(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}\right)=\frac{\pi}{2}\left(\int_y^a f(\rho_0)\,\rho_0\,\mathrm{d}\rho_0 - y^2 f(y)\right)$$

$$+\frac{a^2}{\sqrt{a^2-y^2}}\lim_{r\to a}\int_r^a \frac{f(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}\,,$$

$$\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\int_r^a \frac{f(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}} = -\frac{\pi}{2}f(y)+\frac{1}{\sqrt{a^2-y^2}}\lim_{r\to a}\int_r^a \frac{f(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}\,. \qquad (1.10.35)$$

The last term in (35) is not necessarily zero, since function $f$ might have singularity at $a$.

The next integral to be computed is:

$$\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}\,r}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0 = \frac{\pi}{2}\left(\frac{2n}{y}\int_a^y \tau_n(\rho_0)\,\rho_0^{2-n}\,\mathrm{d}\rho_0\right.$$

$$\left.-(2n+1)\int_a^y \tau_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\right). \qquad (1.10.36)$$

The integral to follow is slightly more involved:

$$\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\int_r^a \frac{\Phi(r,\rho_0)}{r^{2n}}\,f_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0 = \int_y^a f_n(\rho_0)\,\rho_0^{1-n}\,\mathrm{d}\rho_0\int_y^{\rho_0}\frac{\Phi(r,\rho_0)\,\mathrm{d}r}{\sqrt{r^2-y^2}\,r^{2n}}\,. \qquad (1.10.37)$$

Since, according to (30), $\Phi(r,\rho_0)$ is defined as:

$$\Phi(r,\rho_0) = -\int_r^{\rho_0} \frac{x^{2n-2}\,[2n\rho_0^2-(2n+1)x^2]}{\sqrt{\rho_0^2-x^2}}\,dx\,, \tag{1.10.38}$$

we can introduce a new variable $t = r\rho_0/x$, so that

$$\Phi(r,\rho_0) = -\frac{\rho_0^{2n}}{r^{1-2n}}\int_r^{\rho_0}\frac{[2nt^2-(2n+1)r^2]\,dt}{t^{2n+1}\sqrt{t^2-r^2}}\,. \tag{1.10.39}$$

Substitution of (39) into (37) gives, after simplification:

$$\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\int_r^a \frac{\Phi(r,\rho_0)}{r^{2n}}\,f_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0 = -\frac{\pi}{2y}\int_y^a\left(\frac{(4n^2-1)y-4n^2\rho_0}{2n(2n-1)\rho_0^{2n}}\right.$$

$$\left.+\frac{1}{2n(2n-1)y^{2n-1}}\right)f_n(\rho_0)\,\rho_0^{n+1}\,d\rho_0\,. \tag{1.10.40}$$

From (30) we may compute:

$$\Phi(0,\rho_0) = -\frac{\sqrt{\pi}\,\Gamma(n-1/2)}{4\Gamma(n+1)}\,\rho_0^{2n}\,. \tag{1.10.41}$$

Now utilization of (41), (40), (36), (34) and (27) in (25) leads to:

$$\tau_n(y) = \frac{y^{n-1}}{\pi^2}\frac{\partial^2}{\partial y^2}\left\{y\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\left(\frac{1}{r^2}\right)^n\left[\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{w_n(\rho)\,\rho^{n+1}\,d\rho}{\sqrt{r^2-\rho^2}}\right]\right\} + \tau_n(y)$$

$$-\frac{1}{y^{n+2}}\int_y^a \tau_n(\rho_0)\,\rho_0^{n+1}\,d\rho_0 + y^{n-2}\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\frac{2\Gamma(n+1/2)}{\sqrt{\pi}\,r^{2n}\,\Gamma(n)}\int_0^a \tau_n(\rho_0)\,\rho_0^{n+1}\,d\rho_0$$

$$-y^{n-2}\left(\frac{a^3F_n(a)}{(a^2-y^2)^{3/2}}+\frac{a^2F_n'(a)}{\sqrt{a^2-y^2}}\right)+\frac{2y^{n-2}}{\pi\sqrt{a^2-y^2}}\lim_{r\to a}\int_r^a \frac{2n\rho_0^2-(2n+1)r^2}{\sqrt{\rho_0^2-r^2}}\,\tau_n(\rho_0)\,\rho_0^{1-n}\,d\rho_0\,. \tag{1.10.42}$$

We can get from (28) and (29):

$$F_n(a) = -\frac{2}{\pi a^{2n}} \int_0^a \Phi(0,\rho_0)\, \tau_n(\rho_0)\, \rho_0^{1-n}\, d\rho_0 = \frac{\sqrt{\pi}\,\Gamma(n-1/2)}{2\pi\Gamma(n+1)\, a^{2n}} \int_0^a \tau_n(\rho_0)\, \rho_0^{n+1}\, d\rho_0\,, \qquad (1.10.43)$$

and from (32) we get:

$$F_n'(a) = \frac{2}{\pi} \lim_{r\to a} \int_r^a \frac{2n\rho_0^2 - (2n+1)r^2}{\sqrt{\rho_0^2 - r^2}\, r^2}\, \tau_n(\rho_0)\, \rho_0^{1-n}\, d\rho_0 - \frac{\Gamma(n-1/2)}{a^{2n+1}\sqrt{\pi}\,\Gamma(n)} \int_0^a \tau_n(\rho_0)\, \rho_0^{n+1}\, d\rho_0\,.$$

(1.10.44)

Simplifications in (42) due to (43) and (44) finally lead to a solution:

$$\tau_n(y) = -\frac{1}{y^{n+1}} \frac{\partial}{\partial y} \left\{ \frac{y^{2n+1}}{\pi^2} \frac{\partial^2}{\partial y^2} \left[ y \int_y^a \frac{W_n(r)\, dr}{r^{2n}\sqrt{r^2 - y^2}} \right] \right\} + F_n(a) \frac{3a^3 y^n}{(a^2 - y^2)^{5/2}}\,. \qquad (1.10.45)$$

There is though one problem with this solution: we have terms with singularities of the order 5/2, which are not integrable. In order to eliminate these singularities, certain conditions should be imposed on $W_n$. These conditions can be derived by performing differentiation under the integral sign in (45), using the rule (27), with the result:

$$\tau_n(y) = \frac{y^{n-2}}{\pi^2} \left[ \frac{a[a^{2-2n} W_n'(a)]'}{\sqrt{a^2 - y^2}} - \int_y^a \frac{r\, dr}{\sqrt{r^2 - y^2}} \frac{\partial^2}{\partial r^2} \left( \frac{W_n'(r)}{r^{2n-2}} \right) dr \right.$$

$$\left. + \frac{y^2 a^{2-2n} W_n'(a)}{(a^2 - y^2)^{3/2}} \right] + \frac{3a^3 y^n}{(a^2 - y^2)^{5/2}} \left( F_n(a) + \frac{W_n(a)}{\pi^2 a^{2n}} \right). \qquad (1.10.46)$$

It is obvious from (46) that, in order to eliminate the non-integrable singularities, the following conditions should hold:

$$W_n(a) = -\pi^2 a^{2n} F_n(a)\,, \qquad W_n'(a) = 0\,. \qquad (1.10.47)$$

In addition, one may conclude from (46) that in the case where $W_n''(a) = 0$ the solution $\tau$ will be zero at the boundary $\rho = a$.

Assuming that (47) holds, the remaining terms in (46) can be rearranged to give:

$$\tau_n(y) = -\frac{y^{n-1}}{\pi^2}\frac{\partial}{\partial y}\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\left(\frac{W_n'(r)}{r^{2n-2}}\right). \tag{1.10.48}$$

It is reminded that $W_n$ is defined in (26).

Now we need to consider in more detail the first condition (47). Taking into consideration (43), it can be rewritten as

$$W_n(a) = -\frac{\pi^{3/2}\,\Gamma(n-1/2)}{2\Gamma(n+1)}\int_0^a \tau_n(\rho_0)\,\rho_0^{n+1}\,\mathrm{d}\rho_0 \tag{1.10.49}$$

Now we have to check whether a substitution of (48) into (49) leads to an identity. We have:

$$W_n(a) = -\frac{\pi^{3/2}\,\Gamma(n-1/2)}{2\Gamma(n+1)}\int_0^a \frac{y^{2n}\,\mathrm{d}y}{\pi^2}\frac{\partial}{\partial y}\int_y^a \frac{\mathrm{d}r}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\left(\frac{W_n'(r)}{r^{2n-2}}\right). \tag{1.10.50}$$

Differentiation under the integral sign and integration by parts in (50) gives us:

$$W_n(a) = W_n(a) - \frac{1}{4n^2-1}\lim_{r\to 0}\left(r^2 W_n''(r) - (4n-1)r\,W_n'(r) + (4n^2-1)\,W_n(r)\right). \tag{1.10.51}$$

This means that (49) is an identity if, and only if

$$\lim_{r\to 0}\left(r^2 W_n''(r) - (4n-1)r\,W_n'(r) + (4n^2-1)\,W_n(r)\right) = 0\,. \tag{1.10.52}$$

Elementary analysis shows that (52) will be satisfied for any $r$ when

$$W_n(r) = cr^{2n+1}\,, \quad \text{or} \quad W_n(r) = cr^{2n-1}\,, \quad c = \text{const.} \tag{1.10.53}$$

It is also obvious that (52) is never satisfied when $W_n = \text{const.} \neq 0$ This means that the second condition of (47) requires that the parameter $a$ enter the expression for $w$ explicitly. A more detailed analysis can be performed assuming that $w_n$ can be presented as a series

$$w_n(\rho) = \sum_{k=-m}^{\infty} c_k \rho^k \tag{1.10.54}$$

Here $c_k$ are constants and at least one of them depends on the parameter $a$. Negative powers of $r$ are admissible for higher harmonics; the lower limit of summation $m$ in (54) depends on $n$, and will be determined from the further analysis. Substitution of (54) into (26) allows us to compute:

$$W_n(r) = \sum_{k=-m}^{\infty} \frac{2\sqrt{\pi}\,\Gamma[1+(k+n)/2]}{\Gamma[(1+k+n)/2]} c_k r^{k+n-1},$$

$$W_n'(r) = \sum_{k=-m}^{\infty} \frac{4\sqrt{\pi}\,\Gamma[1+(k+n)/2]}{\Gamma[(k+n-1)/2]} c_k r^{k+n-2}. \tag{1.10.55}$$

Since $W_n$ can not be a constant and negative powers in $W_n$ are not allowed, except for the zero harmonic [see (53)], we may conclude that $m=0$ when $n=0$ and $m=n-2$ otherwise and the term with $k=1-n$ should vanish, which means that $c_{1-n}=0$. Since $k\geq 2-n$, the last limitation is only applicable to the zero harmonic: it can not contain a linear term $c_1\rho$.

Performing summation in (48) and returning back to original notation, the final solution will take the form:

$$\tau(y,\phi) = -\frac{1}{\pi^2 y}\mathcal{L}(y)\frac{\partial}{\partial y}\int_y^a \frac{dr}{\sqrt{r^2-y^2}}\frac{\partial}{\partial r}\left[r^2\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)\, w(\rho,\phi)\right)\right]. \tag{1.10.56}$$

The summation of the conditions (47) yields:

$$\lim_{r\to a}\left[\mathcal{L}\left(\frac{1}{r}\right)\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)\, w(\rho,\phi)\right] = -\frac{\pi}{2}\int_0^{2\pi}\int_0^a\left[1-\left(1-\frac{\rho_0}{a}e^{i(\phi-\phi_0)}\right)^{1/2}\right.$$

$$\left.-\left(1-\frac{\rho_0}{a}e^{-i(\phi-\phi_0)}\right)^{1/2}\right]\tau(\rho_0,\phi_0)\,\rho_0\, d\rho_0\, d\phi_0, \tag{1.10.57}$$

$$\lim_{r\to a}\left\{\mathcal{L}\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w(\rho,\phi)\right]\right\}=0\,. \tag{1.10.58}$$

Expression (57) looks quite cumbersome. It can be replaced by the equivalent expression (52) which, in turn, can be rewritten as

$$\lim_{r\to 0}\left\{r^{2n+2}\,\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{W_n(r)}{r^{2n-1}}\right)\right]\right\}=0\,,$$

or

$$\lim_{r\to 0}\left\{r^{2n-2}\,\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{W_n(r)}{r^{2n+1}}\right)\right]\right\}=0\,. \tag{1.10.59}$$

Formula (56), combined with conditions (58) and (59), are the main results of this section.

**Discussion.** After the solution in the form (56) has been derived, one can find a simpler way to obtain it. Indeed, the biharmonic integral equation (20) can be presented in the form:

$$4\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^a \frac{\rho^2+\rho_0^2-2x^2}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0$$

$$-4\int_0^\rho \sqrt{\rho^2-x^2}\,\frac{\mathrm{d}x}{x}\int_x^a \sqrt{\rho_0^2-x^2}\;\frac{\partial}{\partial x}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0 = w(\rho,\phi)\,. \tag{1.10.60}$$

Application to both sides of (60) of the operator

$$\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)$$

yields:

$$\pi\int_0^r (r^2+x^2+2\rho_0^2-4x^2)\,\mathrm{d}x\int_x^a \frac{1}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\left(\frac{x^2}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0-\pi\int_0^r (r^2-x^2)\,\frac{\mathrm{d}x}{x}$$

$$\times \int_x^a \sqrt{\rho_0^2 - x^2}\ \frac{\partial}{\partial x} \mathcal{L}\left(\frac{x^2}{\rho_0}\right) \tau(\rho_0,\phi)\, \rho_0\, d\rho_0 = \int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}} \mathcal{L}(\rho)\, w(\rho,\phi)\,. \qquad (1.10.61)$$

Differentiation of both sides of (61) with respect to $r$ gives us:

$$2\pi\left\{ r\int_0^r dx \int_x^a \frac{1}{\sqrt{\rho_0^2-x^2}} \mathcal{L}\left(\frac{x^2}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\, d\rho_0 - r\int_0^r \frac{dx}{x}\int_x^a \sqrt{\rho_0^2-x^2}\ \frac{\partial}{\partial x}\mathcal{L}\left(\frac{x^2}{\rho_0}\right)\tau(\rho_0,\phi)\rho_0 d\rho_0 \right.$$

$$\left. + \int_r^a \sqrt{\rho_0^2 - r^2}\ \mathcal{L}\left(\frac{r^2}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\, d\rho_0 \right\} = \frac{\partial}{\partial r}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}} \mathcal{L}(\rho)\, w(\rho,\phi)\,. \qquad (1.10.62)$$

Yet another differentiation with respect to $r$ of (62) divided by $r$ results in

$$-2\pi\int_r^a \frac{\sqrt{\rho_0^2-r^2}}{r^2}\mathcal{L}\left(\frac{r^2}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\, d\rho_0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)\, w(\rho,\phi)\right). \qquad (1.10.63)$$

The next operator to apply is:

$$\frac{\partial}{\partial r}\left[r^2\,\mathcal{L}\left(\frac{1}{r^2}\right)\right],$$

with the result:

$$2\pi\int_r^a \frac{r}{\sqrt{\rho_0^2-r^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\, d\rho_0 = \frac{\partial}{\partial r}\left[r^2\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)\, w(\rho,\phi)\right)\right]. \qquad (1.10.64)$$

And finally, the application of the operator

$$\mathcal{L}(y)\frac{\partial}{\partial y}\int_y^a \frac{dr}{\sqrt{r^2-y^2}}$$

to both sides of (64) leads to (56). The condition (58) can be recovered from (63) by taking limit $r \to a$. We were unable though to recover the condition (52) and this was the main reason, why we did not present this derivation in the main part of this section. The solution (56) is not the only one available. For example, application of the operator

$$\int_y^a \frac{r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\,\mathcal{L}\left(\frac{1}{r^2}\right)$$

to both sides of (63) gives:

$$\pi^2\int_y^a\left(\frac{\rho_0}{y}-1\right)\mathcal{L}\left(\frac{1}{\rho_0}\right)\tau(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0=-\int_y^a\frac{r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\,\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\int_0^r\frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w(\rho,\phi)\right). \tag{1.10.65}$$

Now multiplication of both sides of (65) by $y$ and double differentiation with respect to $y$ gives yet another equivalent solution:

$$\tau(y,\phi)=-\frac{1}{\pi^2 y}\,\mathcal{L}(y)\,\frac{\partial^2}{\partial y^2}\left[y\int_y^a\frac{r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\,\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\int_0^r\frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w(\rho,\phi)\right)\right]. \tag{1.10.66}$$

Yet another application of the developed apparatus is for computation of various integrals involving distance between two points. As an example, here is the case, where

$$\tau(\rho_0,\phi_0)=\sqrt{a^2-\rho_0^2}\,\rho_0^{\mathrm{n}}\,\mathrm{e}^{i(\mathrm{n}-2\mathrm{k})\phi_0}, \tag{1.10.67}$$

the relevant integral is:

$$\int_0^{2\pi}\int_0^a\sqrt{a^2-\rho_0^2}\,\rho_0^{\mathrm{n}}\,\mathrm{e}^{i(\mathrm{n}-2\mathrm{k})\phi_0}\,R(J,J_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0$$

$$=-\frac{\mathrm{e}^{i(\mathrm{n}-2\mathrm{k})\phi}}{\rho^{|\,\mathrm{n}-2\mathrm{k}|}}\,4\int_0^\rho\frac{\mathrm{d}x}{x}\frac{\partial}{\partial x}\left[\sqrt{\rho^2-x^2}\,x^{|\,2\mathrm{n}-4\mathrm{k}|}\int_x^a\sqrt{\rho_0^2-x^2}\sqrt{a^2-\rho_0^2}\;\;\rho_0^{2\mathrm{k}+1}\,\mathrm{d}\rho_0\right]$$

$$=\frac{\mathrm{e}^{i(\mathrm{n}-2\mathrm{k})\phi}\rho^{|\,\mathrm{n}-2\mathrm{k}|}}{2(k+1)(k+2)a^{2\mathrm{k}}}\sum_{\mathrm{m}=0}^{\mathrm{k}}\frac{\Gamma(k+3/2-m)\,\Gamma(m+3/2)}{\Gamma(k+1-m)\,\Gamma(m+1)}\left(\frac{\rho}{a}\right)^{2\mathrm{m}}\frac{\Gamma(|n-2k|+m-1/2)}{\Gamma(|n-2k|+m+1)}$$

$$\times\left[-a^4+\frac{2(|n-2k|+m-1/2)}{(|n-2k|+m+1)}\,a^2\rho^2-\frac{(|n-2k|+m)^2-1/4}{(|n-2k|+m+1)(|n-2k|+m+2)}\,\rho^4\right].$$

(1.10.68)

Here the following integral was used:

$$\int_x^a \sqrt{\rho_0^2 - x^2}\,\sqrt{a^2-\rho_0^2}\,\rho_0^{2k+1}\,d\rho_0 = \frac{a^{2k}(a^2-x^2)^2}{2(k+1)(k+2)}\sum_{m=0}^{k}\frac{\Gamma(k+3/2-m)\,\Gamma(m+3/2)}{\Gamma(k+1-m)\,\Gamma(m+1)}\left(\frac{x}{a}\right)^{2m}. \tag{1.10.69}$$

The result in (68) shows that the well known in potential theory polynomial theorem is valid in this case too. A significant simplification of (68) takes place when $k=0$:

$$\int_0^{2\pi}\int_0^a \sqrt{a^2-\rho_0^2}\,\rho_0^{n}\,e^{in\phi_0} R(J,J_0)\,\rho_0\,d\rho_0\,d\phi_0 = -\frac{\pi^{3/2}\,\Gamma(n-1/2)}{16\Gamma(n+1)}(\rho e^{i\phi})^{n}\Bigg[a^4$$

$$-\frac{2n-1}{n+1}a^2\rho^2 + \frac{n^2-1/4}{(n+1)(n+2)}\rho^4\Bigg]. \tag{1.10.70}$$

In the case $n=2k$, formula (68) gives:

$$\int_0^{2\pi}\int_0^a \sqrt{a^2-\rho_0^2}\,R(J,J_0)\,\rho_0^{2k+1}\,d\rho_0\,d\phi_0 = \frac{\sqrt{\pi}}{2(k+1)(k+2)a^{2k}}$$

$$\times\sum_{m=0}^{k}\frac{\Gamma(k+3/2-m)\,\Gamma(m+3/2)}{\Gamma(k+1-m)\,\Gamma(m+1)}\left(\frac{\rho}{a}\right)^{2m}\frac{\Gamma(m-1/2)}{\Gamma(m+1)}$$

$$\times\Bigg[-a^4 + \frac{2k-1}{k+1}a^2\rho^2 - \frac{k^2-1/4}{(k+1)(k+2)}\rho^4\Bigg].$$

And one more integral which might be useful in practice:

$$\int_0^{2\pi}\int_0^a \frac{R(J,J_0)\,\rho_0^{2n+1}\,d\rho_0\,d\phi_0}{\sqrt{a^2-\rho_0^2}} = \sqrt{\pi}\,a^{2n+2}\sum_{k=0}^{n+1}\frac{\Gamma(n+3/2-k)\,\Gamma^2(k-1/2)}{4\Gamma(n+2-k)\,\Gamma^2(k+1)}\left(\frac{\rho}{a}\right)^{2k}.$$

Several particular cases of the last integral:

$$\int_0^{2\pi}\int_0^a \frac{R(J,J_0)}{\sqrt{a^2-\rho_0^2}}\,\rho_0^3\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = \frac{\pi^2}{4}\left(\frac{3}{2}a^4+\frac{1}{2}a^2\rho^2+\frac{1}{16}\rho^4\right),$$

$$\int_0^{2\pi}\int_0^a \frac{R(J,J_0)}{\sqrt{a^2-\rho_0^2}}\,\rho_0^5\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = \frac{\pi^2}{4}\left(\frac{5}{4}a^6+\frac{3}{8}a^4\rho^2+\frac{1}{32}a^2\rho^4+\frac{1}{64}\rho^6\right).$$

And here is an example of computation of an integral in the case where the $z$-coordinate is non-zero:

$$\int_0^{2\pi}\int_0^a \sqrt{a^2-\rho_0^2}\,R(M,J_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = -4\int_0^{l_1}\frac{\mathrm{d}x}{x}\frac{\partial}{\partial x}\left[\sqrt{\rho^2-x^2}\int_{g(x)}^a \sqrt{\rho_0^2-g^2(x)}\right.$$

$$\left.\times\sqrt{a^2-\rho_0^2}\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0\right] = \frac{\pi}{4}\left[\sqrt{l_2^2-a^2}\left(a(l_2^2-2z^2)+\frac{l_1^2}{8a}(14l_1^2+8l_2^2-15\rho^2)\right)\right.$$

$$\left.+\sin^{-1}\left(\frac{a}{l_2}\right)\left(\rho^2(a^2-z^2)-\frac{1}{8}\rho^4+(a^2+z^2)^2\right)\right]. \qquad (1.10.71)$$

Various more complicated integrals can now be computed just by integrating both sides of (71) with respect to $z$.

And one more integral, which does not seem to have been computed before:

$$\int_0^{2\pi}\int_0^a \sqrt{a^2-\rho_0^2}\,\big(z\ln[R(M,J_0)+z]-R(M,J_0)\big)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0$$

$$=\frac{\pi}{2}\left[\left(\frac{1}{6}z^4-\frac{1}{2}a^4+a^2z^2-\frac{1}{2}\rho^2z^2-\frac{1}{2}a^2\rho^2+\frac{1}{16}\rho^4\right)\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{4}{3}a^3z\ln[l_2+(l_2^2-\rho^2)^{1/2}]\right.$$

$$\left.+\frac{\sqrt{l_2^2-a^2}}{a}\left(\frac{5}{6}a^2\rho^2-\frac{19}{9}a^4+\frac{8}{9}a^2l_1^2-\frac{35}{48}\rho^2l_1^2-\frac{1}{6}a^2l_2^2+\frac{25}{72}l_1^4\right)\right].$$

The reader can find additional integrals of similar kind in Chapter 6.

**Examples.** As the first example, we consider the case where

$$w(\rho,\phi)=\frac{\pi^2}{8}\left(a^4+a^2\rho^2-\frac{1}{8}\rho^4\right). \tag{1.10.72}$$

Substitution of (72) into (26) yields:

$$W_0(r)=\frac{\pi^2}{8r}\left(a^4+2a^2r^2-\frac{1}{3}r^4\right), \qquad W_0'(r)=\frac{\pi^2}{8r^2}\left(-a^4+2a^2r^2-r^4\right),$$

$$W_0''(r)=\frac{\pi^2}{4}\left(\frac{a^4}{r^3}-r\right). \tag{1.10.73}$$

Substitution of (73) in (52) and in the second condition of (47) shows, that they are satisfied. One can see that not only $W_0'(a)=0$ here, but also $W_0''(a)=0$ as well. By observing expression (46), one can conclude, that in the case where the second derivative is equal to zero, the solution will be zero at the circle boundary. Indeed, substitution of (73) into (56) yields:

$$\tau_0(y)=\sqrt{a^2-y^2}\,. \tag{1.10.74}$$

We consider as the second example the case:

$$w(\rho,\phi)=\frac{\pi^2}{64}\left(\rho^4+8a^2\rho^2+24a^4\right).$$

The relevant computations in this case give:

$$W_0(r)=\frac{\pi^2}{8}\left(\frac{1}{3}r^3+2a^2r+\frac{3a^4}{r}\right), \qquad W_0'(r)=\frac{\pi^2}{8}\left(r^2+2a^2-\frac{3a^4}{r^2}\right),$$

$$W_0''(r)=\frac{\pi^2}{4}\left(r+\frac{3a^4}{r^3}\right). \tag{1.10.75}$$

In this case also the existence conditions are satisfied, but $W_0''(a)$ is non-zero, which means, that the solution will have an integrable singularity at the boundary. Indeed, the solution is:

$$\tau_0(y)=\frac{y^2}{\sqrt{a^2-y^2}}\,.$$

We consider next the case where the first harmonic is prescribed, namely,

$$w(\rho,\phi) = \frac{\pi^2}{16}\left(4a^2 - \rho^2\right)\rho e^{i\phi}\,.$$

The relevant computations here are:

$$W_1(r) = \frac{\pi^2}{2}\left(a^2 - \frac{1}{3}r^2\right) r\,, \qquad W_1'(r) = \frac{\pi^2}{2}\,(a^2 - r^2)\,,$$

$$\tau_1(y) = -\frac{y}{\sqrt{a^2 - y^2}}\,.$$

And as the last example, here is the case of a second harmonic:

$$w(\rho,\phi) = 2\pi e^{2i\phi}\left(\frac{a\rho}{3} - \frac{\pi}{16}\rho^2\right),$$

$$W_2(r) = \pi^2 r^2\left(\frac{a}{2} - \frac{r}{3}\right), \qquad W_2'(r) = \pi^2 r(a - r)\,, \qquad \tau_2(y) = \frac{y^2 - 2a^2}{y^2\sqrt{a^2 - y^2}}\,.$$

It is important to note that in the case of higher harmonics the solution may have a very strong singularity at zero.

**Conclusion.** An exact closed form solution has been found to the biharmonic equation (6) in the case where domain $S$ is a circle of radius $a$. The solution has the form (56) or (66). It has been found that the solution with an integrable singularity at the boundary exists only in the cases where the right-hand side satisfies specific conditions (58) and (59). Yet another condition has to be satisfied, namely, $W_n''(a)=0$, if one need to obtain a solution vanishing at the boundary.

As a bonus, numerous new integrals, involving distances between points are computed in terms of elementary functions.

Looking back at equation (3) which started the whole investigation, we may safely conclude that its solution exists only if $c_1=c_2$, $c_3=c_4=0$ and $c_5/(c_1+c_2)=a^2$. In all cases, for the solution to exist, the radius of the circle must be explicitly present in the prescribed function $w$.

## 1.11. Computation of infinite integrals involving three Bessel functions by introduction of a new formalism

It has been accepted by many scientists, that whatever can be done in the field of evaluation of integrals of Bessel functions, had already been done and that nothing new or important can be introduced at this stage. This section is written to disprove such a notion. Introduction of a new and elegant formalism not only allows simple computations of various integrals in terms of elementary functions, it also allows to discover errors (or misprints) in long-standing results, which by now have entered practically all respectable tables of integrals.

**Introduction.** This section represents logical continuation of (Fabrikant and Dôme, 2001c), where the new formalism was first utilized for computation of various infinite integrals, involving Bessel functions. At first, I was the single author of the article. I have sent it to several journals and got quite a negative reaction. Rosedale from Proceedings of Cambridge Philosophical Society wrote to me, that my material can not be published in any research journal and advised me to submit it to some newspaper for math teachers. Freiberger from Quarterly of Applied Mathematics advised me to read Watson’s treatise, to which I responded with a list of literature on Bessel functions in four languages, which I have already read. I guess, this list convinced him to submit my article for review and the reviewer (Dôme) was so impressed by the idea, that he decided to become a co-author.

Though it is known theoretically that certain integrals involving Bessel functions can be expressed in terms of elementary functions, practical evaluation of such integrals is very difficult. For example, in Gradshteyn and Ryzhik, 1994, formula 6.751.3 (formula contains a misprint), one can see

$$\int_0^\infty \mathrm{e}^{-cx}\cos(ax)\,J_0(bx)\,\mathrm{d}x = \frac{\left(\sqrt{(c^2+b^2-a^2)+4a^2c^2}+b^2+c^2-a^2\right)^{1/2}}{\sqrt{2}\,\sqrt{(c^2+b^2-a^2)+4a^2c^2}}, \quad c>0. \quad (1.11.1)$$

If one tries to differentiate (1) with respect to $c$ in order to obtain the integral

$$\int_0^\infty \mathrm{e}^{-cx}\cos(ax)\,J_0(bx)\,x\,\mathrm{d}x, \quad (1.11.2)$$

the result would be unwieldy.

In order to overcome all these difficulties, the following notation from (Fabrikant, 1989) is introduced here:

$$l_1(x,b,c) \equiv l_1(x) = \frac{1}{2}\left(\sqrt{(x+b)^2+c^2} - \sqrt{(x-b)^2+c^2}\right), \qquad (1.11.3)$$

$$l_2(x,b,c) \equiv l_2(x) = \frac{1}{2}\left(\sqrt{(x+b)^2+c^2} + \sqrt{(x-b)^2+c^2}\right). \qquad (1.11.4)$$

The geometric interpretation of $l_1(x,\rho,z)$ and $l_2(x,\rho,z)$ is quite obvious: if one considers a circle of radius $x$ in the plane $z=0$ and a point with the polar cylindrical coordinates $(\rho,\phi,z)$, then $l_2(x,\rho,z)$ and $l_1(x,\rho,z)$ represent half of the sum and the difference respectively of the longest and the shortest distance from the point to the circle.

The following properties of $l_1(x,b,c)$ and $l_2(x,b,c)$ can be verified directly:

$$\lim_{c\to 0} l_1(x) = \min(x,b), \qquad l_1(x) \le \min(x,b),$$

$$\lim_{c\to 0} l_2(x) = \max(x,b), \qquad l_2(x) \ge \max(x,b), \qquad (1.11.5)$$

$$l_1(x)l_2(x) = xb, \qquad l_1^2(x) + l_2^2(x) = x^2 + b^2 + c^2, \qquad (1.11.6)$$

$$\sqrt{x^2 - l_1^2(x)}\sqrt{l_2^2(x) - x^2} = xc, \qquad \sqrt{b^2 - l_1^2(x)}\sqrt{l_2^2(x) - b^2} = bc, \qquad (1.11.7)$$

$$\sqrt{x^2 - l_1^2(x)}\sqrt{b^2 - l_1^2(x)} = c\,l_1(x), \qquad \sqrt{l_2^2(x) - x^2}\sqrt{l_2^2(x) - b^2} = c\,l_2(x). \qquad (1.11.8)$$

The differentiation can be performed by using the main formulae:

$$\frac{\partial}{\partial b} l_1(x) = \frac{b[x^2 - l_1^2(x)]}{l_1(x)[l_2^2(x) - l_1^2(x)]} = \frac{\partial}{\partial x} l_2(x),$$

$$\frac{\partial}{\partial b} l_2(x) = \frac{b[l_2^2(x) - x^2]}{l_2(x)[l_2^2(x) - l_1^2(x)]} = \frac{\partial}{\partial x} l_1(x),$$

$$\frac{\partial}{\partial c} l_1(x) = -\frac{c\,l_1(x)}{l_2^2(x) - l_1^2(x)}, \qquad \frac{\partial}{\partial c} l_2(x) = \frac{c\,l_2(x)}{l_2^2(x) - l_1^2(x)}. \qquad (1.11.9)$$

Numerous additional useful formulae can be found in Chapter 6. For the sake of brevity, we shall use the notation $l_1$ and $l_2$ to denote $l_1(a,b,c)$ and $l_2(a,b,c)$ respectively.

Utilization of $l_1$ and $l_2$ in (1) allows to simplify it significantly, namely,

$$\int_0^\infty e^{-cx}\cos(ax)\,J_0(bx)\,dx = \frac{\sqrt{l_2^2-a^2}}{l_2^2-l_1^2}\,. \tag{1.11.10}$$

Now differentiation of (10) with respect to $c$ can be performed and the integral (2) can be computed as follows:

$$\int_0^\infty e^{-cx}\cos(ax)\,J_0(bx)\,x\,dx = \frac{c[l_2^4-a^2(2a^2+2c^2-b^2)]}{\sqrt{l_2^2-a^2}\,(l_2^2-l_1^2)^3}\,. \tag{1.11.11}$$

which is quite manageable, as compared to the result, one would get by differentiation of (1) with respect to $c$.

In the main body of this section, we derive new representations for the integrals involving a product of three Bessel functions. Certain particular cases, where these integrals are computable in terms of elementary functions, are discussed. Comparison with known results is made. We found that a long standing result of Bailey (1935), which can be found in almost every respectable table of integrals, namely,

$$\int_0^\infty t\,J_\mu(ct\sin\phi\cos\Phi)\,J_\nu(ct\cos\phi\sin\Phi)\,J_{\nu-\mu}(ct)\,dt$$

$$= -\frac{2\sin(\pi\mu)\sin^\mu\phi\,\sin^\nu\Phi}{\pi c^2\cos^\nu\phi\,\cos^\mu\Phi\,\cos(\phi+\Phi)\cos(\phi-\Phi)}\,, \qquad c>0 \tag{1.11.12}$$

is correct for $0<\phi+\Phi<\pi/2$, but is incorrect for $\pi>\phi+\Phi>\pi/2$. Our representation allowed to obtain the result which is correct for $\phi<\pi/2$ and $\Phi<\pi/2$ and $\pi>\phi+\Phi>0$. A small table of integrals involving Bessel functions is given in Appendix in order to illustrate advantages of the new formalism.

**Derivation of the main representations.** The following general result was established by Bailey (1935):

$$\int_0^\infty t^{\gamma-1}J_\mu(at)\,J_\nu(bt)\,K_\rho(ct)\,dt = \frac{2^{\gamma-2}a^\mu b^\nu\,\Gamma[(\gamma+\mu+\nu-\rho)/2]\,\Gamma[(\gamma+\mu+\nu+\rho)/2]}{c^{\gamma+\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$\times F_4\Big(\frac{\gamma+\mu+\nu-\rho}{2},\frac{\gamma+\mu+\nu+\rho}{2};\mu+1,\nu+1;-\frac{a^2}{c^2},-\frac{b^2}{c^2}\Big)\,. \tag{1.11.13}$$

Here $F_4$ is the hypergeometric function of 2 variables. The result is valid when $\Re(\gamma+\mu+\nu)>|\Re(\rho)|$ and $\Re(c\pm ia\pm ib)>0$. The symbol $\Re$ denotes the real part of the expression to follow.

Bailey (1935) also established the following property:

$$F_4\left(\alpha,\beta;\gamma,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right)$$

$$=(1-x)^{\alpha}(1-y)^{\alpha}F_1\left(\alpha;\gamma-\beta,1+\alpha-\gamma;\gamma;x,xy\right),\qquad x<1,\quad xy<1. \tag{1.11.14}$$

In order to use (14), we presume $\gamma=\nu-\mu-\rho+2$ and

$$\frac{x}{(1-x)(1-y)}=\frac{a^2}{c^2},\qquad \frac{y}{(1-x)(1-y)}=\frac{b^2}{c^2}. \tag{1.11.15}$$

Expressing $x$ from (15) as $x=ya^2/b^2$ and substituting it into the second equation (15), we get:

$$a^2y^2-(a^2+b^2+c^2)y+b^2=0. \tag{1.11.16}$$

Using properties (6), the solutions of (15) can be written in the form:

$$y_1=\frac{l_1^2}{a^2},\qquad y_2=\frac{l_2^2}{a^2},\qquad x_1=\frac{l_1^2}{b^2},\qquad x_2=\frac{l_2^2}{b^2}. \tag{1.11.17}$$

Convergence of (14) requires that $x<1$ and $xy<1$, so we have to choose the solution of (15) as

$$y=\frac{l_1^2}{a^2},\qquad x=\frac{l_1^2}{b^2}. \tag{1.11.18}$$

One can easily deduce that the second solution in (17) gives the values greater than 1.

Substitution of (18) and (14) in (13) yields:

$$\int_0^{\infty} t^{\nu-\mu-\rho+1}J_{\mu}(at)\,J_{\nu}(bt)\,K_{\rho}(ct)\,\mathrm{d}t=\frac{2^{\nu-\mu-\rho}a^{\mu}b^{\nu}\,\Gamma(\nu-\rho+1)}{c^{2\nu-\rho+2}\,\Gamma(\mu+1)}\left(\frac{c^2}{l_2^2}\right)^{\nu-\rho+1}$$

$$\times F_1\left(\nu-\rho+1;\mu-\nu,\nu-\rho-\mu+1;\mu+1;\frac{l_1^2}{b^2},\frac{l_1^2}{l_2^2}\right). \tag{1.11.19}$$

Here we used the property (8).

We recall a well known integral representation (Bateman et al, 1953a, formula 5.8.5)

$$F_1(a,b,c,d;x,y) = \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)} \int_0^1 \frac{u^{a-1}\,(1-u)^{d-a-1}\,\mathrm{d}u}{(1-ux)^b\,(1-uy)^c}\,, \qquad \Re(a) > 0\,, \quad \Re(d-a) > 0\,. \tag{1.11.20}$$

Utilization of (20) in (19) gives, after simplification:

$$\int_0^\infty t^{\nu-\mu-\rho+1}\,J_\mu(at)\,J_\nu(bt)\,K_\rho(ct)\,\mathrm{d}t = \frac{2^{1-\mu+\nu-\rho}}{a^\mu\,b^\nu\,c^\rho\,\Gamma(\mu-\nu+\rho)}$$

$$\times \int_0^{l_1} \frac{x^{1+2\nu-2\rho}[(l_2^2-x^2)(l_1^2-x^2)]^{\mu-\nu+\rho-1}\,\mathrm{d}x}{(b^2-x^2)^{\mu-\nu}}\,, \quad \mu-\nu+\rho \ge 0\,, \quad \nu-\rho > -1\,. \tag{1.11.21}$$

Yet another case where (14) can be used is $\gamma = \mu-\nu+\rho+2$. The repetition of the above derivation yields:

$$\int_0^\infty t^{\mu-\nu+\rho+1}\,J_\mu(at)\,J_\nu(bt)\,K_\rho(ct)\,\mathrm{d}t = \frac{2^{1+\mu-\nu+\rho}\,c^\rho}{a^\mu\,b^\nu\,\Gamma(\nu-\mu-\rho)}$$

$$\times \int_0^{l_1} \frac{x^{1+2\mu+2\rho}[(l_2^2-x^2)(l_1^2-x^2)]^{\nu-\mu-\rho-1}\,\mathrm{d}x}{(a^2-x^2)^{\nu-\mu}}\,, \quad \nu-\mu-\rho \ge 0\,, \quad \mu+\rho > -1\,. \tag{1.11.22}$$

Taking into consideration that $K_\rho(\cdot) = K_{-\rho}(\cdot)$, we may conclude that (22) can be obtained directly from (21) by replacing $\rho$ by $-\rho$ and interchanging places of $\mu$ and $\nu$, $a$ and $b$.

Now we recall that

$$J_\rho(z) = [\exp(-\rho\pi i/2)\,K_\rho(-iz) - \exp(\rho\pi i/2)\,K_\rho(iz)]/\pi i\,. \tag{1.11.23}$$

Utilization of (23) in (13) gives yet another result of Bailey (1935):

$$\int_0^\infty t^{\gamma-1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt = \frac{2^{\gamma-1} a^\mu b^\nu\, \Gamma[(\gamma+\mu+\nu+\rho)/2]}{c^{\gamma+\mu+\nu}\, \Gamma(\mu+1)\, \Gamma(\nu+1)\, \Gamma[1-(\gamma+\mu+\nu-\rho)/2]}$$

$$\times F_4\Bigl(\frac{\gamma+\mu+\nu-\rho}{2}, \frac{\gamma+\mu+\nu+\rho}{2}; \mu+1, \nu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\Bigr). \qquad (1.11.24)$$

Formula (24) is valid for $c>a+b$, $\Re(\gamma+\mu+\nu+\rho)>0$, $\Re(\gamma)<5/2$.

As before, in order to use (14), we presume $\gamma=\nu-\mu-\rho+2$ and

$$\frac{x}{(1-x)(1-y)} = -\frac{a^2}{c^2}, \qquad \frac{y}{(1-x)(1-y)} = -\frac{b^2}{c^2}. \qquad (1.11.25)$$

Expressing $x$ from (25) as $x=ya^2/b^2$ and substituting it into the second equation (25), we get:

$$a^2y^2-(a^2+b^2-c^2)y+b^2=0. \qquad (1.11.26)$$

In order to make (26) formally the same as (16), we have two choices: one, to formally replace $c$ by $ic$; the second choice is to formally replace $a$ by $ia$ and $b$ by $ib$. The first choice would lead to both $l_1$ and $l_2$ purely imaginary and complex conjugate; the second choice leaves both $l_1$ and $l_2$ real, but $l_1$ will become negative.

The first choice gives the solution:

$$y = \frac{l_{1c}^2}{a^2}, \qquad x = \frac{l_{1c}^2}{b^2}, \quad \text{where } l_{1c} = l_1(a,b,ic), \quad a>0, \quad b>0, \quad c>a+b. \qquad (1.11.27)$$

The second choice gives the solution:

$$y = \frac{l_{1a}^2}{a^2}, \qquad x = \frac{l_{1a}^2}{b^2}, \quad \text{where } l_{1a} = l_1(ia,ib,c), \quad a>0, \quad b>0, \quad c>a+b. \qquad (1.11.28)$$

Repeating the transformations similar to those in (19)–(21) and using (27), we arrive at the integral representation:

$$\int_0^\infty t^{\nu-\mu-\rho+1} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, dt = \frac{2^{\nu-\mu-\rho+2} \sin[\pi(\rho-\nu)]}{\pi(-1)^{\nu-\rho+1}\, \Gamma(\mu-\nu+\rho)\, a^\mu b^\nu c^\rho}$$

$$\times \int_0^{l_{1c}} \frac{x^{1+2\nu-2\rho}\,[(l_{2c}^2-x^2)(l_{1c}^2-x^2)]^{\mu-\nu+\rho-1}\,\mathrm{d}x}{(b^2-x^2)^{\mu-\nu}}, \quad \mu-\nu+\rho\geq 0\,, \quad \rho-\nu\leq 1\,. \tag{1.11.29}$$

Since $l_{1c}$ is imaginary, one can see certain difficulty in the practical use of (29) when a numerical evaluation of the integral is required. The expressions of the type $(-1)^{\mu}$ are understood everywhere as $\exp(i\pi\mu)$.

Utilization of (28) leads to yet another integral representation:

$$\int_0^{\infty} t^{\nu-\mu-\rho+1}\,J_{\mu}(at)\,J_{\nu}(bt)\,J_{\rho}(ct)\,\mathrm{d}t = \frac{2^{\nu-\mu-\rho+2}\sin[\pi(\rho-\nu)]}{\pi\,\Gamma(\mu-\nu+\rho)\,a^{\mu}\,b^{\nu}\,c^{\rho}}$$

$$\times \int_0^{|l_{1a}|} \frac{x^{1+2\nu-2\rho}\,[(l_{2a}^2-x^2)(l_{1a}^2-x^2)]^{\mu-\nu+\rho-1}\,\mathrm{d}x}{(b^2+x^2)^{\mu-\nu}}, \quad \mu-\nu+\rho\geq 0\,, \quad \rho-\nu\leq 1\,. \tag{1.11.30}$$

One may immediately conclude that the integral in (30) vanishes when $\rho-\nu$ is an integer and $c>a+b$.

Formally replacing $a$ by $ia$ and $b$ by $ib$ in (13), we obtain yet another result due to Bailey (1936):

$$\int_0^{\infty} t^{\gamma-1}\,I_{\mu}(at)\,I_{\nu}(bt)\,K_{\rho}(ct)\,\mathrm{d}t = \frac{2^{\gamma-2}\,a^{\mu}\,b^{\nu}\,\Gamma[(\gamma+\mu+\nu-\rho)/2]\,\Gamma[(\gamma+\mu+\nu+\rho)/2]}{c^{\gamma+\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$\times F_4\Big(\frac{\gamma+\mu+\nu-\rho}{2}, \frac{\gamma+\mu+\nu+\rho}{2}; \mu+1\,, \nu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\Big), \quad \Re(\gamma+\mu+\nu) > \big|\Re(\rho)\big|\,. \tag{1.11.31}$$

Once again, by taking $\gamma=\nu-\mu-\rho+2$ and using the procedure described above, we arrive at:

$$\int_0^{\infty} t^{\nu-\mu-\rho+1}\,I_{\mu}(at)\,I_{\nu}(bt)\,K_{\rho}(ct)\,\mathrm{d}t = \frac{2^{\nu-\mu-\rho}\,a^{\mu}\,b^{\nu}\,\Gamma(\nu-\rho+1)}{c^{2\nu-\rho+2}\,\Gamma(\mu+1)}\left(\frac{c^2}{l_{2a}^2}\right)^{\nu-\rho+1}$$

$$\times F_1\Big(\nu-\rho+1; \mu-\nu, \nu-\mu-\rho+1; \mu+1; -\frac{l_{1a}^2}{b^2}, -\frac{l_{1a}^2}{l_{2a}^2}\Big)\,.$$

Now utilization of (20) yields:

$$\int_0^\infty t^{\nu-\mu-\rho+1} I_\mu(at)\, I_\nu(bt)\, K_\rho(ct)\, \mathrm{d}t = \frac{2^{\nu-\mu-\rho+1}}{\Gamma(\mu-\nu+\rho)\, a^\mu\, b^\nu\, c^\rho}$$

$$\times \int_0^{|l_{1a}|} \frac{x^{1+2\nu-2\rho}\, [(l_{2a}^2 - x^2)(l_{1a}^2 - x^2)]^{\mu-\nu+\rho-1}\, \mathrm{d}x}{(b^2+x^2)^{\mu-\nu}}, \quad \mu-\nu+\rho \geq 0, \quad \rho-\nu < 1\,. \tag{1.11.32}$$

Formally replacing $b$ by $ib$ in (13), we obtain a result:

$$\int_0^\infty t^{\gamma-1} J_\mu(at)\, I_\nu(bt)\, K_\rho(ct)\, \mathrm{d}t = \frac{2^{\gamma-2}\, a^\mu\, b^\nu\, \Gamma[(\gamma+\mu+\nu-\rho)/2]\, \Gamma[(\gamma+\mu+\nu+\rho)/2]}{c^{\gamma+\mu+\nu}\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$$

$$\times F_4\Big(\frac{\gamma+\mu+\nu-\rho}{2}, \frac{\gamma+\mu+\nu+\rho}{2}; \mu+1, \nu+1; -\frac{a^2}{c^2}, \frac{b^2}{c^2}\Big)\,. \tag{1.11.33}$$

Taking, as before, $\gamma = \nu-\mu-\rho+2$, we arrive at the equation:

$$a^2y^2 - (c^2+a^2-b^2)y - b^2 = 0\,. \tag{1.11.34}$$

We can formally solve it introducing $l_1(a,ib,c)$, but this would lead us into complex domain. We can stay in the real domain by introducing $l_1(c,b,a) = \tilde{l}_1$ and $l_2(c,b,a) = \tilde{l}_2$.

By using the identities:

$$c^2+a^2-b^2 = [(\tilde{l}_2)^2 - b^2] - [b^2 - (\tilde{l}_1)]\,, \qquad a^2b^2 = [(\tilde{l}_2)^2 - b^2][b^2 - (\tilde{l}_1)]\,,$$

we can find the necessary roots of (34) as

$$y_1 = \frac{(\tilde{l}_2)^2 - b^2}{a^2}\,, \qquad y_2 = -\frac{b^2 - (\tilde{l}_1)^2}{a^2}\,, \tag{1.11.35}$$

which in turn leads to:

$$x_1 = -\frac{(\tilde{l}_2)^2 - b^2}{b^2}\,, \qquad x_2 = \frac{b^2 - (\tilde{l}_1)^2}{b^2}\,. \tag{1.11.36}$$

The conditions $x<1$ and $xy<1$ narrows our choice to:

$$x=\frac{b^2-(\tilde{l}_1)^2}{b^2}, \qquad y=-\frac{b^2-(\tilde{l}_1)^2}{a^2}. \tag{1.11.37}$$

And using the procedure, similar to those outlined above, we arrive at

$$\int_0^\infty t^{\nu-\mu-\rho+1} J_\mu(at)\, I_\nu(bt)\, K_\rho(ct)\,\mathrm{d}t=\frac{2^{\nu-\mu-\rho}\, a^\mu\, b^\nu\, \Gamma(\nu-\rho+1)}{c^{2\nu-\rho+2}\,\Gamma(\mu+1)}\left(\frac{c^2[b^2-(\tilde{l}_1)^2]}{a^2b^2}\right)^{\nu-\rho+1}$$

$$\times F_1\left(\nu-\rho+1;\mu-\nu,\nu-\mu-\rho+1;\mu+1;\frac{b^2-(\tilde{l}_1)^2}{b^2},-\frac{b^2-(\tilde{l}_1)^2}{(\tilde{l}_2)^2-b^2}\right),$$

which can be represented as

$$\int_0^\infty t^{\nu-\mu-\rho+1} J_\mu(at)\, I_\nu(bt)\, K_\rho(ct)\,\mathrm{d}t=\frac{2^{1+\nu-\mu-\rho}}{\Gamma(\mu-\nu+\rho)\, a^\mu\, b^\nu\, c^\rho}$$

$$\times\int_0^{s_1}\frac{x^{1+2\nu-2\rho}\,[(s_1^2-x^2)(s_2^2+x^2)]^{\mu-\nu+\rho-1}\,\mathrm{d}x}{(b^2-x^2)^{\mu-\nu}}, \qquad \mu-\nu+\rho\ge 0, \quad \rho-\nu<1. \tag{1.11.38}$$

Here

$$s_1^2=b^2-(\tilde{l}_1)^2, \qquad s_2^2=(\tilde{l}_2)^2-b^2. \tag{1.11.39}$$

Of course, one can rewrite (38) interchanging parameters $a$ and $c$, thus returning to the regular notations $l_1$ and $l_2$ instead of $\tilde{l}_1$ and $\tilde{l}_2$.

Formulae (21), (30), (32), (38), are the main new results of this section. One can derive similar representations for other combinations of the products of 3 Bessel functions by using essentially the same mathematical apparatus, as described above. This is left to the interested reader.

**Discussion.** We consider now various particular cases of the main results, derived above and, where possible, compare them with the existing results.

First, we discuss the integral representation (21). In the particular case of $\rho=\nu-\mu$, the gamma-function in the denominator grows to infinity and so does the integral. This indeterminate expression of the type $\infty/\infty$ can be computed in elementary fashion, with the result:

$$\int_0^\infty t\, J_\mu(at)\, J_\nu(bt)\, K_{\nu-\mu}(ct)\, dt = \frac{(b^2 - l_1^2)^{\nu-\mu}\, l_1^{2\mu}}{a^\mu\, b^\nu\, c^{\nu-\mu}(l_2^2 - l_1^2)} . \tag{1.11.40}$$

A similar formula is given in Bailey (1935) in the form:

$$\int_0^\infty t\, J_\mu(ct \sin\phi)\, J_\nu(ct \sin\Phi)\, K_{\nu-\mu}(ct \cos\phi \cos\Phi)\, dt = \frac{\sin^\mu\phi \sin^\nu\Phi \cos^{\nu-\mu}\phi \cos^{\mu-\nu}\Phi}{c^2(1 - \sin^2\phi \sin^2\Phi)} . \tag{1.11.41}$$

The correspondence between (40) and (41) can be proven easily, as soon as one can deduce that

$$l_1 = c \sin\phi \sin\Phi , \qquad l_2 = c , \qquad (b^2 - l_1^2) = c^2 \sin^2\Phi \cos^2\phi . \tag{1.11.42}$$

Clearly, our formula (40) is more convenient to use, because it does not require the introduction of artificial parameters $\phi$ and $\Phi$.

In the particular case of $\mu = 0$ and $\nu = 1/2$, formula (40) gives:

$$\int_0^\infty e^{-cx}\, J_0(ax) \sin(bx)\, dx = \frac{\sqrt{b^2 - l_1^2}}{l_2^2 - l_1^2} .$$

Integration of the last result with respect to $c$ yields:

$$\int_0^\infty e^{-cx}\, J_0(ax) \sin(bx) \frac{dx}{x} = \sin^{-1}\left(\frac{b}{l_2}\right) ,$$

which is in agreement with Gradshteyn and Ryzhik, 1994, formula 6.752.1.

In the particular case of $\mu = -1/2$ and $\nu = 0$, formula (40) gives (10). In the particular case of $\mu = 1/2$ and $\nu = 1$, formula (40) gives:

$$\int_0^\infty e^{-cx}\, J_1(bx) \sin(ax)\, dx = \frac{l_1 \sqrt{b^2 - l_1^2}}{b(l_2^2 - l_1^2)} .$$

Integration of the last result with respect to $c$ yields:

$$\int_0^\infty e^{-cx} J_1(bx) \sin(ax) \frac{dx}{x} = \frac{a - \sqrt{a^2 - l_1^2}}{b},$$

which is in agreement with Gradshteyn and Ryzhik, 1994, formula 6.752.2. The integrations above were performed according to the table of integrals given in Section 6.2.

By using the property of Bessel functions

$$\frac{\partial}{\partial a}\left(a^\mu J_\mu(at)\right) = t\, a^\mu J_{\mu-1}(at), \tag{1.11.43}$$

we can multiply both sides of (40) by $a$ and differentiate with respect to $a$. The result is:

$$\int_0^\infty t^2 J_{\mu-1}(at)\, J_\nu(bt)\, K_{\nu-\mu}(ct)\, dt = \frac{2(b^2 - l_1^2)^{\nu-\mu}\, l_1^{2\mu-2}}{a^{\mu-1}\, b^\nu\, c^{\nu-\mu}(l_2^2 - l_1^2)^2}\left(\mu b^2 - \nu l_1^2 - \frac{l_1^2(a^2 + c^2 - b^2)}{l_2^2 - l_1^2}\right). \tag{1.11.44}$$

This result does not seem to have been reported before. We have found in the existing tables only one formula to verify (44) for $\mu = \nu = 1$ (Bateman and Erdélyi, Vol.2, 1954, formula 8.3.26), which in our notation reads:

$$\int_0^\infty t^2 J_0(at)\, J_1(bt)\, K_0(ct)\, dt = \frac{2b(b^2 + c^2 - a^2)}{(l_2^2 - l_1^2)^3}. \tag{1.11.45}$$

One can also use the property of Bessel functions

$$J_{\mu+1}(z) = 2\mu J_\mu(z)/z - J_{\mu-1}(z) \tag{1.11.46}$$

and to find the integral:

$$\int_0^\infty t^2 J_{\mu+1}(at)\, J_\nu(bt)\, K_{\nu-\mu}(ct)\, dt = \frac{2(b^2 - l_1^2)^{\nu-\mu}\, l_1^{2\mu-2}}{a^{\mu-1}\, b^\nu\, c^{\nu-\mu}\,(l_2^2 - l_1^2)}\left(\frac{b^2(\nu a^2 - \mu l_1^2)}{l_2^2\,(l_2^2 - l_1^2)} + \frac{l_1^2(a^2 + c^2 - b^2)}{(l_2^2 - l_1^2)^2}\right). \tag{1.11.47}$$

This result also seems to be new. It can be checked by (45) if we take $\mu = \nu = 0$; one can notice that $a$ and $b$ would change places.

In a similar manner, we can increase and decrease the order of any of the Bessel functions in (40), which would be much more difficult to do using Bailey result (41).

Various new results can be obtained by consideration of limiting cases in (40), for example, the case of $a=0$ leads to:

$$\int_0^\infty t^{\mu+1} J_\nu(bt)\, K_{\nu-\mu}(ct)\, \mathrm{d}t = \frac{2^\mu b^\nu \Gamma(\mu+1)}{(b^2+c^2)^{\mu+1} c^{\nu-\mu}}. \tag{1.11.48}$$

which is essentially in agreement with (Bateman and Erdélyi, Vol. 2, 1954, formula 8.13.3)

If we take $\rho=1/2$ in (21), the result is:

$$\int_0^\infty \mathrm{e}^{-ct} t^{\nu-\mu} J_\mu(at)\, J_\nu(bt)\, \mathrm{d}t = \frac{2^{1+\nu-\mu}}{\sqrt{\pi}\, a^\mu b^\nu \Gamma(\mu-\nu+1/2)} \int_0^{l_1} \frac{x^{2\nu} [(l_2^2-x^2)(l_1^2-x^2)]^{\mu-\nu-1/2}\, \mathrm{d}x}{(b^2-x^2)^{\mu-\nu}}. \tag{1.11.49}$$

Formula (49) corresponds to the result which was first derived in (Fabrikant and Dôme, 2001c) by a much more complicated method and where several particular cases of (49) were studied. We consider yet another particular case, namely, $\mu=\nu$,

$$\int_0^\infty \mathrm{e}^{-ct} J_\mu(at)\, J_\mu(bt)\, \mathrm{d}t = \frac{2}{\pi a^\mu b^\mu} \int_0^{l_1} \frac{x^{2\mu}\, \mathrm{d}x}{\sqrt{l_2^2-x^2}\sqrt{l_1^2-x^2}}. \tag{1.11.50}$$

It is well known from (Bateman and Erdélyi, Vol. 2, formula 8.11.16), that (50) can be computed in terms of Legendre function $Q_{\mu-1/2}$ of the argument $(l_1^2+l_2^2)/(2l_1l_2)$, so we can consider the right-hand side of (50) as a new and elementary integral representation for the Legendre function.

In (Bateman and Erdélyi, Vol. 2, formula 8.11.20), one can see:

$$\int_0^\infty \mathrm{e}^{-cx} J_\mu(ax)\, J_\nu(bx)\, \mathrm{d}x = \frac{(2c)^{\nu+\mu+1}}{\pi} a^\mu b^\nu \int_0^{\pi/2} \frac{(\cos\theta)^{\mu+\nu}}{[c^2+\cos^2\theta(b^2-a^2+u)]^\mu}$$

$$\times \frac{\cos[(\mu-\nu)\theta]\, \mathrm{d}\theta}{[c^2+\cos^2\theta(a^2-b^2+u)]^\nu u \cos^2\theta}. \tag{1.11.51}$$

where

$$u^2 \cos^4\theta = [c^2 + \cos^2\theta(a^2 + b^2)]^2 - 4a^2b^2\cos^4\theta\,. \tag{1.11.52}$$

The integral representation (51)–(52) is algebraically very cumbersome. Here the introduction of the new formalism also helps to simplify and bring some mathematical beauty. We denote

$$l_{1\theta} = \frac{1}{2}\left(\sqrt{(a+b)^2\cos^2\theta + c^2} - \sqrt{(a-b)^2\cos^2\theta + c^2}\right),$$

$$l_{2\theta} = \frac{1}{2}\left(\sqrt{(a+b)^2\cos^2\theta + c^2} + \sqrt{(a-b)^2\cos^2\theta + c^2}\right). \tag{1.11.53}$$

This notation allows us to write:

$$c^2 + \cos^2\theta(a^2 - b^2 + u) = l_{2\theta}^2 + l_{1\theta}^2 - 2b^2\cos^2\theta + l_{2\theta}^2 - l_{1\theta}^2 = 2(l_{2\theta}^2 - b^2\cos^2\theta)\,,$$

$$u^2\cos^4\theta = [c^2 + \cos^2\theta(a^2 + b^2)]^2 - 4a^2b^2\cos^4\theta = (l_{2\theta}^2 - l_{1\theta}^2)^2\,. \tag{1.11.54}$$

Substitution of (54) into (51) allows to simplify it as follows:

$$\int_0^\infty e^{-cx} J_\mu(ax)\, J_\nu(bx)\, dx = \frac{2c^{\mu-\nu+1}}{\pi}\, a^\mu\, b^\nu \int_0^{\pi/2} \frac{(\cos\theta)^{\mu+\nu}\cos[(\mu-\nu)\theta]\, d\theta}{(l_{2\theta}^2 - a^2\cos^2\theta)^{\mu-\nu}\,(l_{2\theta}^2 - l_{1\theta}^2)\, l_{2\theta}^{2\nu}}\,. \tag{1.11.55}$$

Clearly, (55) is more simple and elegant, as compared to (51), so the newly introduced formalism helped here as well. There is though difficulty in actual computation of (51) or (55) for $c \to 0$ since the product of $c$-term and the integrand will be close to zero everywhere, except a very small interval near $\theta = \pi/2$, where it will behave like a $\delta$-function. We can use the integral representation for the product of two Bessel functions (Bateman et al, 1953b, formula 7.7.2.12)

$$J_\mu(az)\, J_\nu(bz) = \frac{(2a)^\mu\,(2b)^\nu}{\pi} \int_{-\pi/2}^{\pi/2} e^{i\theta(\mu-\nu)}\,(\cos\theta)^{\mu+\nu}\,(\lambda)^{-\mu-\nu}\, J_{\mu+\nu}(\lambda z)\, d\theta\,, \tag{1.11.56}$$

where

$$\lambda = \sqrt{2\cos\theta\,(a^2 e^{i\theta} + b^2 e^{-i\theta})} \tag{1.11.57}$$

and to derive a more convenient representation for the integral

$$\int_0^\infty \mathrm{e}^{-cx} J_\mu(ax)\, J_\nu(bx)\, \mathrm{d}x = \frac{(2a)^\mu (2b)^\nu}{\pi} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{i\theta(\mu-\nu)} (\cos\theta)^{\mu+\nu} (\lambda)^{-\mu-\nu}\, \mathrm{d}\theta \int_0^\infty \mathrm{e}^{-cx} J_{\mu+\nu}(\lambda x)\, \mathrm{d}x$$

$$= \frac{(2a)^\mu (2b)^\nu}{\pi} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{i\theta(\mu-\nu)} (\cos\theta)^{\mu+\nu} (\lambda)^{-\mu-\nu} \frac{\left(\sqrt{c^2+\lambda^2}-c\right)^{\mu+\nu}}{\sqrt{c^2+\lambda^2}\,\lambda^{\mu+\nu}}\, \mathrm{d}\theta$$

$$= \frac{(2a)^\mu (2b)^\nu}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{e}^{i\theta(\mu-\nu)} (\cos\theta)^{\mu+\nu}\, \mathrm{d}\theta}{\sqrt{c^2+\lambda^2}\left(\sqrt{c^2+\lambda^2}+c\right)^{\mu+\nu}}\,. \tag{1.11.58}$$

Formula (58) seems to be new and has a good convergence everywhere.

It is useful to indicate a particular case of (40), when $\mu=\nu$:

$$\int_0^\infty t\, J_\nu(at)\, J_\nu(bt)\, K_0(ct)\, \mathrm{d}t = \frac{l_1^\nu}{l_2^\nu (l_2^2 - l_1^2)}\,. \tag{1.11.59}$$

Formula (59) is in agreement with (Gradshteyn and Ryzhik, 1994, formula 6.522.3 and 6.522.5).

We can also set $\mu=1/2$ in (49) and obtain an integral representation for the product of exponential, trigonometric and Bessel function. This is left to an interested reader.

Now we consider several particular cases of (30). The integral is computable in terms of elementary functions for the case of $\rho=\nu-\mu$:

$$\int_0^\infty t\, J_\mu(at)\, J_\nu(bt)\, J_{\nu-\mu}(ct)\, \mathrm{d}t = \frac{2\sin(\pi\mu)\, (l_{1a}^2)^\mu\, (b^2 + l_{1a}^2)^{\nu-\mu}}{\pi a^\mu\, b^\nu\, c^{\nu-\mu} (l_{2a}^2 - l_{1a}^2)}\,. \tag{1.11.60}$$

A similar result of Bailey (1935), which entered practically every table of integrals, reads:

$$\int_0^\infty t\, J_\mu(ct\sin\phi\cos\Phi)\, J_\nu(ct\cos\phi\sin\Phi)\, J_{\nu-\mu}(ct)\, \mathrm{d}t$$

$$= \frac{2\sin(\pi\mu)\,\sin^{\mu}\phi\,\sin^{\nu}\Phi}{\pi c^2\cos^{\nu}\phi\,\cos^{\mu}\Phi\,\cos(\phi+\Phi)\,\cos(\phi-\Phi)}\,. \tag{1.11.61}$$

By using the definition of $l_{1a} = l_1(ia, ib, c)$, one can show that (60) is equivalent to (61) for the case $\phi+\Phi<\pi/2$ and is not equivalent to (61) in the case $\pi>\phi+\Phi>\pi/2$. Direct numerical computations also showed that Bailey (1935) result (61) is incorrect for $\pi>\phi+\Phi>\pi/2$. The right formula in this case is:

$$\int_0^\infty t\,J_\mu(ct\sin\phi\cos\Phi)\,J_\nu(ct\cos\phi\sin\Phi)\,J_{\nu-\mu}(ct)\,\mathrm{d}t$$

$$= \frac{2\sin(\pi\mu)\,\cos^{\nu}\phi\,\cos^{\mu}\Phi}{\pi c^2\sin^{\mu}\phi\,\sin^{\nu}\Phi\,\cos(\phi+\Phi)\,\cos(\phi-\Phi)}\,,\qquad \text{for}\quad \pi>\phi+\Phi>\pi/2\,. \tag{1.11.62}$$

Our formula (60) is correct in the whole range $0<\phi+\Phi<\pi$, $\phi<\pi/2$ and $\Phi<\pi/2$. Strictly speaking, formula (60) is valid for $c>a+b$, but we may notice an interesting bonus: in the case of semi-integer $\mu$ and integer $\nu$, formula (60) seems to be valid for any positive $a$, $b$ and $c$. Here are some examples. If we take $\mu=1/2$ and $\nu=1$, we can write using (60):

$$\int_0^\infty t\,J_{1/2}(at)\,J_1(bt)\,J_{1/2}(ct)\,\mathrm{d}t = \frac{2\sqrt{l_{1a}^2(b^2+l_{1a}^2)}}{\pi\sqrt{ac}\,b\,(l_{2a}^2-l_{1a}^2)}\,. \tag{1.11.63}$$

The same integral can be computed directly, with the result:

$$\int_0^\infty t\,J_{1/2}(at)\,J_1(bt)\,J_{1/2}(ct)\,\mathrm{d}t = \frac{1}{\pi\sqrt{ac}\,b}\,\Re\!\left(\frac{a+c}{\sqrt{(a+c)^2-b^2}} - \frac{|c-a|}{\sqrt{(a-c)^2-b^2}}\right). \tag{1.11.64}$$

Though the results in (63) and (64) look different, they are, in fact, the same. One can establish, for example, that

$$l_{2a}^2 - l_{1a}^2 = \sqrt{c^2-(a-b)^2}\ \sqrt{c^2-(a+b)^2} = \sqrt{(a+c)^2-b^2}\ \sqrt{(a-c)^2-b^2}\,.$$

Next, we can prove, that

$$(a+c)\sqrt{(a-c)^2-b^2} - |c-a|\sqrt{(a+c)^2-b^2} = 2\sqrt{l_{1a}^2(b^2+l_{1a}^2)}$$

by taking square of both sides and using the properties (6)–(8). Thus, we have proven that (63) gives the correct value for the integral not just for $c > a + b$, but for any positive values of its parameters $a$, $b$ and $c$, provided that we take the real part of the result.

Yet another example is the case of $\mu = 3/2$ and $\nu = 1$. The result due to (60) is:

$$\int_0^\infty t\, J_{3/2}(at)\, J_1(bt)\, J_{-1/2}(ct)\, \mathrm{d}t = -\Re\left(\frac{2(l_{1a}^2)^{3/2}\sqrt{c}}{\pi a^{3/2}\, b(l_{2a}^2 - l_{1a}^2)\sqrt{b^2 + l_{1a}^2}}\right). \quad (1.11.65)$$

The same integral computed in a regular way gives:

$$\int_0^\infty t\, J_{3/2}(at)\, J_1(bt)\, J_{-1/2}(ct)\, \mathrm{d}t = \frac{1}{\pi\sqrt{ac}\, ab}\Re\left(\mathrm{sign}(c-a)\frac{c^2 - ac - b^2}{\sqrt{(c-a)^2 - b^2}} + \frac{b^2 - ac - c^2}{\sqrt{(c+a)^2 - b^2}}\right). \quad (1.11.66)$$

Again, one can prove that the results in (65) and (66) are identical. On the other hand, when $\mu$ is an integer and $\nu$ is a half-integer, formula (60) gives only the result valid for $c > a + b$. For example, direct evaluation of the integral:

$$\int_0^\infty t\, J_1(at)\, J_{3/2}(bt)\, J_{1/2}(ct)\, \mathrm{d}t = \frac{1}{\pi\sqrt{bc}\, ab}\Re\left(\frac{a^2 - c^2 + bc}{\sqrt{a^2 - (c-b)^2}} - \frac{a^2 - c^2 - bc}{\sqrt{a^2 - (c+b)^2}}\right) \quad (1.11.67)$$

shows that it is zero for $c > a + b$.

Now we discuss the properties of formula (32). As before, it is easily computable in the case of $\rho = \nu - \mu$:

$$\int_0^\infty t\, I_\mu(at)\, I_\nu(bt)\, K_{\nu-\mu}(ct)\, \mathrm{d}t = \frac{(l_{1a}^2)^\mu (b^2 + l_{1a}^2)^{\nu-\mu}}{a^\mu b^\nu c^{\nu-\mu}(l_{2a}^2 - l_{1a}^2)}. \quad (1.11.68)$$

Formula (68) seems to be new. It is of interest to notice that (68) is similar to (60), except for the factor $2\sin(\pi\mu)/\pi$. In the case of $\mu = \nu$, formula (68) simplifies as

$$\int_0^\infty t\, I_\nu(at)\, I_\nu(bt)\, K_0(ct)\, \mathrm{d}t = \frac{(l_{1a}^2)^\nu}{a^\nu b^\nu (l_{2a}^2 - l_{1a}^2)},$$

which is in agreement with (Gradshteyn and Ryzhik, 1994, formula 6.522.3)

The integral in (32) is also computable in the case of $a \to 0$, with the result:

$$\int_0^\infty t^{\nu-\rho+1} I_\nu(bt) K_\rho(ct)\,\mathrm{d}t = \frac{2^{\nu-\rho}\,\Gamma(\nu-\rho+1)\,b^\nu}{c^\rho (c^2-b^2)^{\nu-\rho+1}}\,. \qquad (1.11.69)$$

Though we did not find (69) in the tables, there exists formula 8.13.3 in (Bateman and Erdélyi, 1954, Vol. 2), with $J_\nu$ instead of $I_\nu$, and from which (69) can be deduced.

If we take $\rho = 1/2$ in (32), we obtain:

$$\int_0^\infty \mathrm{e}^{-ct} t^{\nu-\mu} I_\mu(at) I_\nu(bt)\,\mathrm{d}t = \frac{2^{\nu-\mu+1}}{\sqrt{\pi}\,\Gamma(\mu-\nu+1/2)\,a^\mu b^\nu} \int_0^{|l_{1a}|} \frac{x^{2\nu}\,[(l_{2a}^2-x^2)(l_{1a}^2-x^2)]^{\mu-\nu-1/2}\,\mathrm{d}x}{(b^2+x^2)^{\mu-\nu}}\,. \qquad (1.11.70)$$

Further simplification of (70) takes place for $\mu = \nu$:

$$\int_0^\infty \mathrm{e}^{-ct} I_\nu(at) I_\nu(bt)\,\mathrm{d}t = \frac{2}{\pi\, a^\nu b^\nu} \int_0^{|l_{1a}|} \frac{x^{2\nu}\,\mathrm{d}x}{\sqrt{(l_{2a}^2-x^2)(l_{1a}^2-x^2)}}\,. \qquad (1.11.71)$$

From (71) one may conclude that in the case of semi-integer $\nu$, the integral is computable in terms of elementary functions; in the case of integer $\nu$, the result can be expressed in terms of complete elliptic integrals. For example, in the case of $\nu = 0$, we get:

$$\int_0^\infty \mathrm{e}^{-ct} I_0(at) I_0(bt)\,\mathrm{d}t = \frac{2\mathbf{K}(-l_{1a}/l_{2a})}{\pi l_{2a}}\,.$$

Here $\mathbf{K}$ is a complete elliptic integral of the first kind. In the case of $a = b$, the last result is in agreement with (Bateman and Erdélyi, 1953, Vol. 1, formula 4.16.10)

Yet another simplification of (70) takes place for $\mu = \nu + 1/2$:

$$\int_0^\infty \mathrm{e}^{-ct} I_{\nu+1/2}(at) I_\nu(bt)\,\frac{\mathrm{d}t}{\sqrt{t}} = \frac{\sqrt{2}}{\sqrt{\pi}\,a^{\nu+1/2}\,b^\nu} \int_0^{|l_{1a}|} \frac{x^{2\nu}\,\mathrm{d}x}{\sqrt{b^2+x^2}}\,. \qquad (1.11.72)$$

Differentiation of both sides of (72) with respect to $c$ yields:

$$\int_0^\infty e^{-ct} I_{\nu+1/2}(at)\, I_\nu(bt)\sqrt{t}\,dt = \frac{\sqrt{2}\, c(l_{1a}^2)^{\nu+1/2}}{\sqrt{\pi a}\,(ab)^\nu \sqrt{b^2+l_{1a}^2}(l_{2a}^2-l_{1a}^2)}\,. \tag{1.11.73}$$

All these formulae seem to be new.

And finally, we discuss the properties of (38). As in previous cases, the integral is computable when $\rho=\nu-\mu$, with the result:

$$\int_0^\infty t\, J_\mu(ct)\, I_\nu(bt)\, K_{\nu-\mu}(at)\,dt = \frac{(b^2-l_1^2)^\mu\,(l_1^2)^{\nu-\mu}}{a^{\nu-\mu}\, b^\nu\, c^\mu\,(l_2^2-l_1^2)}\,. \tag{1.11.74}$$

Here we already switched from $l^{\sim}$ to $l$ by interchanging places $c$ and $a$.

In the case of $\rho=\nu$ and $\mu=0$, we get:

$$\int_0^\infty t\, J_0(ct)\, I_\nu(bt)\, K_\nu(at)\,dt = \frac{l_1^\nu}{l_2^\nu\,(l_2^2-l_1^2)}\,, \tag{1.11.75}$$

which is in agreement with (Bateman and Erdélyi, Vol. 2, 1954, formula 8.3.30)

Taking $\nu=0$ in (75) and differentiating both sides with respect to $a$, we arrive at:

$$\int_0^\infty t^2\, J_0(ct)\, I_0(bt)\, K_1(at)\,dt = \frac{2a(c^2+a^2-b^2)}{(l_2^2-l_1^2)^3}\,, \tag{1.11.76}$$

which is in agreement with (Bateman and Erdélyi, Vol. 2, 1954, formula 8.3.28)

By taking in (38) $\mu=\rho=0$ and $\nu=-1/2$, we arrive at the integral:

$$\int_0^\infty J_0(ct)\cosh(bt)\, K_0(at)\,dt = b\int_0^{t_1} \frac{dx}{\sqrt{(t_1^2-x^2)(t_2^2+x^2)(b^2-x^2)}}\,. \tag{1.11.77}$$

Here

$$t_1^2 = b^2-l_1^2\,, \qquad t_2^2 = l_2^2-b^2\,. \tag{1.11.78}$$

The reader may notice that we have interchanged $a$ and $c$, so that we are now dealing with regular $l_1$ and $l_2$ instead of $\tilde{l}_1$ and $\tilde{l}_2$. The integral in the right-hand side of (77) can be computed according to (Gradshteyn and Ryzhik, 1994, formula 3.147.4) as:

$$\int_0^\infty J_0(ct)\cosh(bt)\,K_0(at)\,\mathrm{d}t = \frac{\mathbf{K}(k)}{\sqrt{l_2^2-l_1^2}}, \qquad k^2 = \frac{l_2^2-a^2}{l_2^2-l_1^2}, \tag{1.11.79}$$

where $K$ is the complete elliptic integral of the first kind. Formula (79) is in agreement with (Gradshteyn and Ryzhik, 1994, formula 6.662.1)

By taking in (38) $\mu=0$, $\rho=1$ and $\nu=1/2$, we arrive at the integral:

$$\int_0^\infty J_0(ct)\sinh(bt)\,K_1(at)\,\mathrm{d}t = \frac{1}{a}\int_0^{t_1} \frac{\sqrt{b^2-x^2}\,\mathrm{d}x}{\sqrt{(t_1^2-x^2)(t_2^2+x^2)}}. \tag{1.11.80}$$

The integral in the right-hand side of (80) can be computed according to (Gradshteyn and Ryzhik, 1994, formula 3.167.28) as

$$\int_0^\infty J_0(ct)\sinh(bt)\,K_1(at)\,\mathrm{d}t = \frac{1}{ab\sqrt{l_2^2-l_1^2}}\left[l_2^2\mathbf{K}(k) - t_2^2\,\Pi\left(\frac{b^2-l_1^2}{l_2^2-l_1^2}, k\right)\right]. \tag{1.11.81}$$

Here $\Pi$ is the complete elliptic integral of the third kind and $k$ is defined by (79). The complete elliptic integral of the third kind can be presented via complete and incomplete elliptic integrals of the first and second kind. We use for this purpose formula 13.8.23 from (Bateman and Erdélyi, 1955, Vol. 3). The final result is:

$$\int_0^\infty J_0(ct)\sinh(bt)\,K_1(at)\,\mathrm{d}t = \frac{1}{a}\left[\frac{b\,\mathbf{K}(k)}{\sqrt{l_2^2-l_1^2}} + \mathbf{E}(k)\,F(\theta,k) - \mathbf{K}(k)\,E(\theta,k)\right]. \tag{1.11.82}$$

where $\mathbf{E}$ is the complete elliptic integral of second kind, $F$ and $E$ are incomplete elliptic integrals of the first and second kind respectively, $k$ is defined by (79) and

$$\sin\theta = \frac{b}{l_2}. \tag{1.11.83}$$

In order to demonstrate the advantages of the new formalism, we reprint below the same integral, as it is given in (Bateman and Erdélyi, 1954, Vol. 2, formula 8.3.24)

$$\int_0^\infty J_0(ct)\sinh(bt)\,K_1(at)\,\mathrm{d}t = \frac{1}{a}\left[u\,E(k) - K(k)\,E(u) + \frac{K(k)\,\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\right], \quad \Re(a) > |\Re(b)|\,, \tag{1.11.84}$$

where

$$\mathrm{cn}^2 u = 2c^2\{[(a^2+b^2+c^2)^2 - 4a^2b^2]^{1/2} - a^2 + b^2 + c^2\}^{-1}\,, \tag{1.11.85}$$

$$k^2 = \{1 - (a^2-b^2-c^2)[(a^2+b^2+c^2)^2 - 4a^2b^2]^{-1/2}\}/2\,. \tag{1.11.86}$$

Introduction of the new formalism shows that (86) is identical with the way $k$ is defined in (79) and allows to rewrite (85) as

$$\mathrm{cn}^2 u = \frac{c^2}{l_2^2 - a^2}\,, \tag{1.11.87}$$

which, in turn, gives

$$\mathrm{sn}^2 u = \frac{b^2}{l_2^2}\,, \qquad \mathrm{dn}^2 u = \frac{l_2^2 - b^2}{l_2^2 - l_1^2} \tag{1.11.88}$$

and the combination

$$\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u} = \frac{b}{\sqrt{l_2^2 - l_1^2}} \tag{1.11.89}$$

after substitution back into (84) makes the last term in (84) equal to the first term in (82). Since $\mathrm{sn}(u)$ in (88) is equal to $\sin\theta$ in (83), then $u$ in the first term of (84) is equal to $F(\theta,k)$ in the second term of (82), so that formulae (82) and (84) may be declared equivalent, provided that we can interpret $E(u)$ in the second term of (84) as $E(\theta,k)$. The same formula entered Gradshteyn and Ryzhik (1994, formula 6.662.2).

**Conclusion.** An elegant formalism is introduced, which seems to fit naturally for evaluation of integrals involving Bessel functions. Several new integral representations are obtained for infinite integrals involving product of 3 Bessel functions of first, second and third kind. In certain cases these integrals are computed in terms of elementary functions. Significant number of computed results is new and not available in existing tables. The remaining results are compared with existing ones and some are found to be in error (or misprint).

The particular case of integrands involving exponential, trigonometric and hyperbolic functions is considered by taking Bessel functions of the order 1/2.

**Appendix.** We present below examples of application of the above results. All of the integrals were checked numerically. They were submitted to and accepted for inclusion in the Sixth Edition of Gradshteyn and Ryzhik tables. Hereafter $l_1$ and $l_2$ are understood as $l_1(a,b,c)$ and $l_2(a,b,c)$. The integrals are valid for $\Re(c) > |\Im(a \pm b)|$.

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{1/2}(bx) \frac{dx}{x^{3/2}} = \frac{\sqrt{2}}{\sqrt{\pi b}\, a}\left[\frac{l_1}{2}\sqrt{a^2-l_1^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{l_1}{a}\right) + c\left(\sqrt{b^2-l_1^2}-b\right)\right],$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{1/2}(bx) \frac{dx}{x^{1/2}} = \frac{\sqrt{2}}{\sqrt{\pi b}\, a}\left(b-\sqrt{b^2-l_1^2}\right),$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{1/2}(bx)\, x^{1/2}\, dx = \frac{\sqrt{2}}{\sqrt{\pi b}\, a}\, \frac{l_1\sqrt{a^2-l_1^2}}{l_2^2-l_1^2},$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{3/2}(bx)\, x^{1/2}\, dx = \frac{2l_1^2\sqrt{b^2-l_1^2}}{\sqrt{2\pi}\, ab^{3/2}\,(l_2^2-l_1^2)},$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{3/2}(bx) \frac{dx}{x^{1/2}} = \frac{1}{\sqrt{2\pi}\, ab^{3/2}}\left[a^2\sin^{-1}\left(\frac{l_1}{a}\right) - l_1\sqrt{a^2-l_1^2}\right],$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{5/2}(bx) \frac{dx}{x^{1/2}} = \frac{c}{\sqrt{2\pi}\, ab^{5/2}}\left[\frac{l_1(3a^2-l_1^2)}{\sqrt{a^2-l_1^2}} - 3a^2\sin^{-1}\left(\frac{l_1}{a}\right)\right],$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{5/2}(bx) \frac{dx}{x^{3/2}} = \frac{1}{\sqrt{2\pi}\, ab^{5/2}}\left[\frac{l_1}{\sqrt{a^2-l_1^2}}\left(\frac{7}{8}a^4 - a^2c^2 - \frac{l_1^4}{4} - \frac{5}{8}a^2 l_1^2\right)\right.$$

$$\left. -\frac{1}{2}(l_1^2+l_2^2)\, l_1\sqrt{a^2-l_1^2} + \sin^{-1}\left(\frac{l_1}{a}\right)\left(\frac{3}{2}a^2c^2 + \frac{1}{2}a^2b^2 - \frac{3}{8}a^4\right)\right],$$

$$\int_0^\infty e^{-cx} J_1(ax)\, J_{5/2}(bx)\, \frac{dx}{x^{5/2}} = \frac{1}{\sqrt{2\pi}\, ab^{5/2}} \left[ \frac{2}{15}\, [b^{5/2} - (b^2 - l_1^2)^{5/2}] + ca^2 \sin^{-1}\!\left(\frac{l_1}{a}\right)\!\left(\frac{3}{8}\, a^2 - \frac{b^2}{2} - \frac{c^2}{2}\right) \right.$$

$$\left. + cl_1 \sqrt{a^2 - l_1^2} \left(\frac{b^2}{2} - \frac{3}{8}\, a^2 + \frac{c^2}{6} - \frac{l_1^2}{4}\right) + \frac{c^3 a^2 l_1}{3\sqrt{a^2 - l_1^2}} \right],$$

$$\int_0^\infty e^{-cx} J_2(ax)\, J_{3/2}(bx)\, x^{1/2}\, dx = \frac{2a^2 b^{3/2} \sqrt{l_2^2 - b^2}}{\sqrt{2\pi}\, (l_2^2 - l_1^2) l_2^4},$$

$$\int_0^\infty e^{-cx} J_2(ax)\, J_{3/2}(bx)\, \frac{dx}{x^{1/2}} = \frac{2b^{3/2}}{\sqrt{2\pi}\, a^2} \left( \frac{2}{3} - \frac{\sqrt{b^2 - l_1^2}}{b} + \frac{(b^2 - l_1^2)^{3/2}}{3b^3} \right),$$

$$\int_0^\infty e^{-cx} J_3(ax)\, J_{1/2}(bx)\, \frac{dx}{x^{1/2}} = \frac{\sqrt{2}}{\sqrt{\pi b}\, 3a^3} \Big[ b(3a^2 - 4b^2 + 12c^2) - \sqrt{b^2 - l_1^2}\, \big(12 l_2^2 - 16b^2$$

$$+ 4l_1^2 - 3a^2\big) \Big],$$

$$\int_0^\infty e^{-cx} J_3(ax)\, J_{3/2}(bx)\, x^{1/2}\, dx = \frac{\sqrt{2}\, b^{3/2}}{\sqrt{\pi}} \left[ \frac{4}{a^3} \left( \frac{2}{3} - \frac{\sqrt{b^2 - l_1^2}}{b} + \frac{(b^2 - l_1^2)^{3/2}}{3b^3} \right) - \frac{a\sqrt{l_2^2 - a^2}}{l_2^3 (l_2^2 - l_1^2)} \right],$$

$$\int_0^\infty e^{-cx} J_3(ax)\, J_{3/2}(bx)\, \frac{dx}{x^{1/2}} = \frac{\sqrt{2}\, b^{3/2}}{\sqrt{\pi}\, 3a^3} \left[ \sqrt{l_2^2 - b^2}\, \frac{4b^2(2b^2 - l_1^2) - l_1^4}{b^4} - 8c \right],$$

$$\int_0^\infty e^{-cx} J_3(ax)\, J_{3/2}(bx)\, \frac{dx}{x^{3/2}} = \frac{\sqrt{2}\, b^{3/2}}{\sqrt{\pi}\, 3a^3} \left[ a^2 - \frac{4}{5}\, b^2 + 4c^2 - \sqrt{b^2 - l_1^2} \left( \frac{4l_2^2}{b} - \frac{24b}{5} + \frac{8l_1^2}{5b} - \frac{a^2}{b} + \frac{l_1^4}{5b^3} \right) \right],$$

$$\int_0^\infty e^{-cx} J_3(ax)\, J_{3/2}(bx)\, \frac{dx}{x^{5/2}} = -\frac{\sqrt{2}\, b^{3/2}}{\sqrt{\pi}\, 3a^3} \Big[ \big(a^2 - \frac{4}{5}\, b^2\big) c + \frac{4}{3}\, c^3 + \sqrt{l_2^2 - b^2} \Big( a^2$$

$$+\frac{32}{15}b^2-\frac{12}{5}l_1^2-\frac{4}{3}l_2^2+\frac{2l_1^4}{5b^2}+\frac{a^4 l_1^2}{16b^4}+\frac{a^2 l_1^4}{24b^4}+\frac{l_1^6}{30b^4}\Bigg)-\frac{a^6}{16b^3}\sin^{-1}\left(\frac{b}{l_2}\right)\Bigg].$$

The 3 integrals below are also new and represent further development of (Gradshteyn and Ryzhik, 1994, formulae 6.578.4, 6.711.3 and 6.711.4) respectively for $c > a+b$.

$$\int_0^\infty t^{\rho-\mu-\nu-3} J_\mu(at)\, J_\nu(bt)\, J_\rho(ct)\, \mathrm{d}t = \frac{2^{\rho-\mu-\nu-3} a^\mu b^\nu \Gamma(\rho-1)}{c^{\rho-2}\Gamma(\mu+1)\,\Gamma(\nu+1)}\left(1-\frac{\rho-1}{\mu+1}\frac{a^2}{c^2}-\frac{\rho-1}{\nu+1}\frac{b^2}{c^2}\right),$$

$$\int_0^\infty x^{\nu-\mu-4}\sin(cx)\, J_\mu(ax)\, J_\nu(bx)\, \mathrm{d}x = \frac{a^\mu b^{-\nu} c\, \Gamma(\nu)}{2^{\mu-\nu+3}\Gamma(\mu+1)}\left(\frac{b^2}{\nu-1}-\frac{a^2}{\mu+1}-\frac{2c^2}{3}\right),$$

$$\int_0^\infty x^{\nu-\mu-3}\cos(cx)\, J_\mu(ax)\, J_\nu(bx)\, \mathrm{d}x = \frac{a^\mu b^{-\nu} \Gamma(\nu)}{2^{\mu-\nu+3}\Gamma(\mu+1)}\left(\frac{b^2}{\nu-1}-\frac{a^2}{\mu+1}-2c^2\right).$$