

CHAPTER 5

APPLICATION TO CONTACT PROBLEMS

The various elastic contact problems, solved in Chapters 2 and 3, deal primarily with the stresses and displacements in the plane $z=0$. We concentrate here on the *complete* solutions. The solution is called complete when explicit expressions are given for the field of stresses and displacements in the whole half-space. The complete solution, combined with the reciprocal theorem, enables us to solve more complicated problems of punch interactions, influence of external loads on punches, etc. An approximate *analytical* solution is given to the non-classical punch problem. Significant part of the material presented here is as yet unpublished. The rest follows the papers (Fabrikant, 1974a, 1986a, 1986c, 1986i, 1987h).

5.1 Contact problem for a smooth punch.

A smooth punch is pressed against a transversely isotropic elastic half-space $z \geq 0$ by a normal force P . The term 'smooth' is used to identify a punch which does not exert any shear traction at its base. Let S denote the domain of contact. The mixed boundary conditions on the plane $z=0$ are:

$$\begin{aligned} \sigma_z &= 0, \quad \text{for } (\rho, \phi) \notin S; & w &= \omega(\rho, \phi), \quad \text{for } (\rho, \phi) \in S, \\ \tau_z &= 0, \quad \text{for } -\infty < (x, y) < \infty. \end{aligned} \tag{5.1.1}$$

As in the previous chapter, we may assume again that

$$F_1(z) = c_1 F(z_1), \quad F_2(z) = c_2 F(z_2), \quad F_3(z) = 0. \tag{5.1.2}$$

Substitution of (5.1.2) and (2.1.12) in the third condition (5.1.1) yields:

$$c_1 = -c_2\gamma_1/m_1\gamma_2. \quad (5.1.3)$$

Now we need to define the main potential function F so that its second z -derivative vanishes at $z=0$ outside the domain of contact. Comparison with (4.1.4) gives

$$F(\rho, \phi, z) = F(M) = \int_S \int \ln[R(M, N) + z] \sigma(N) dS_N. \quad (5.1.4)$$

The simple layer potential property yields inside the domain of contact:

$$\frac{\partial^2 F}{\partial z^2} \Big|_{z=0} = -2\pi\sigma, \quad (5.1.5)$$

which, combined with (2.1.12), gives the second equation for the constants c_1 and c_2

$$2\pi A_{44}[(1 + m_1)c_1 + (1 + m_2)c_2] = 1. \quad (5.1.6)$$

Equations (5.1.3) and (5.1.6) determine the constants

$$c_1 = H\gamma_1/(m_1 - 1), \quad c_2 = H\gamma_2/(m_2 - 1). \quad (5.1.7)$$

The simplifications made in (5.1.7) are due to the properties (4.1.10).

Finally, substitution of (5.1.2), (5.1.4) and (5.1.7) in the second of equations (2.1.6) yields for $z=0$ the well known governing integral equation of elastic contact problem for a smooth punch:

$$\omega(N_0) = H \int_S \int \frac{\sigma(N)}{R(N, N_0)} dS_N. \quad (5.1.8)$$

An approximate analytical solution of (5.1.8) for a punch of arbitrary shape will be given later. We consider in more detail the case of a circular punch. The integral equation (5.1.8) can be rewritten for a circular domain of contact of radius a as follows (c.f. 1.4.5)

$$4H \int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} L\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0, \phi) = \omega(\rho, \phi)$$

Its exact closed form solution is (c.f. 1.4.10)

$$\begin{aligned} \sigma(\rho, \phi) = & -\frac{1}{\pi^2 H \rho} L(\rho) \frac{d}{d\rho} \int_{\rho}^a \frac{x dx}{(x^2 - \rho^2)^{1/2}} \\ & \times L\left(\frac{1}{x^2}\right) \frac{d}{dx} \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} L(\rho_0) \omega(\rho_0, \phi). \end{aligned} \quad (5.1.9)$$

The potential function F can be found in two stages. First of all, one has from (5.1.4):

$$\frac{\partial F}{\partial z} = \int_0^{2\pi} \int_0^a \frac{\sigma(r, \psi) r dr d\psi}{[\rho^2 + r^2 - 2r\rho \cos(\phi - \psi) + z^2]^{1/2}} \quad (5.1.10)$$

Substitution of (5.1.9) in (5.1.10) yields, after integration (c.f. 1.4.21)

$$\frac{\partial F}{\partial z} = \frac{1}{\pi^2 H} \int_0^{2\pi} \int_0^a \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right] \frac{z}{R_0^3} \omega(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0. \quad (5.1.11)$$

Here $R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2]^{1/2}$, and h is defined by (4.1.18). The next integration of (5.1.11) with respect to z gives the expression for the potential function directly in terms of the prescribed displacement under the punch, namely,

$$F(\rho, \phi, z) = \frac{1}{\pi^2 H} \int_0^{2\pi} \int_0^a \mathfrak{X}(\rho, \phi, z; \rho_0, \phi_0) \omega(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0, \quad (5.1.12)$$

where

$$\mathfrak{X}(\rho, \phi, z; \rho_0, \phi_0) = -\frac{1}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{(a^2 - \rho_0^2)^{1/2}} \left\{ \ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \right.$$

$$- 2\Re\left\{\frac{1}{(\zeta - 1)^{1/2}}\left[\tan^{-1}\left(\frac{a(\zeta - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right) - \tan^{-1}(\zeta - 1)^{1/2}\right]\right\}, \quad \zeta = \frac{\rho}{\rho_0} e^{i(\phi - \phi_0)}$$
(5.1.13)

The details of integration are presented in Appendix A4.3. Now all the Green's functions related to the field of displacements and stresses can be obtained in elementary functions from (5.1.13) by differentiation

$$\frac{\partial \mathcal{K}}{\partial z} = \frac{z}{R_0^3} \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right], \quad (5.1.14)$$

$$\Lambda \mathcal{K} = \frac{q}{R_0^3} \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right] - \frac{1}{hq} \left\{ 1 - \frac{(a^2 - l_1^2)^{1/2}}{a} - \frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}} \left[\tan^{-1}\left(\frac{a(\bar{\zeta} - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right) - \tan^{-1}(\bar{\zeta} - 1)^{1/2} \right] \right\}, \quad (5.1.15)$$

$$\frac{\partial^2 \mathcal{K}}{\partial z^2} = \frac{1}{R_0^3} \left(1 - \frac{3z^2}{R_0^2} \right) \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right] + \frac{1}{h(R_0^2 + h^2)} \left[\frac{z^2}{R_0^2} - \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} \right], \quad (5.1.16)$$

$$\Lambda \frac{\partial \mathcal{K}}{\partial z} = - \frac{3zq}{R_0^5} \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right] + \frac{z}{h(R_0^2 + h^2)} \left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{q}{R_0^2} \right], \quad (5.1.17)$$

$$\begin{aligned} \Lambda^2 \mathcal{K} = & - \frac{3q^2}{R_0^5} \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right) \right] + \frac{1}{h(R_0^2 + h^2)} \left[\frac{q^2}{R_0^2} - \frac{l_2^2 - \rho^2}{l_2^2 - l_1^2} e^{2i\phi} \right] \\ & + \frac{3}{q} \left\{ \frac{1}{h} - \frac{1}{(a^2 - \rho_0^2)^{1/2}} \left(1 + \frac{1}{(\bar{\zeta} - 1)^{1/2}} \left[\tan^{-1}\left(\frac{a(\bar{\zeta} - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right) - \tan^{-1}(\bar{\zeta} - 1)^{1/2} \right] \right) \right\} \\ & + \frac{e^{i\phi}}{\rho q (a^2 - \rho_0^2)^{1/2}} \left[1 - \frac{a^3 \bar{\zeta}}{(a^2 - l_1^2)^{1/2} (a^2 \bar{\zeta} - l_1^2)} \right]. \end{aligned} \quad (5.1.18)$$

The following identities were used in the simplification of (5.1.14–5.1.18):

$$(l_2^2 - \rho_0^2)(a^2 - l_1^2) + a^2 q \bar{q} = a^2 (R_0^2 + h^2),$$

$$(a^2 \zeta - l_1^2)(a^2 \bar{\zeta} - l_1^2) = (R_0^2 + h^2) l_1^2 a^2 / \rho_0^2,$$

$$(l_2^2 - \rho_0^2)(q\bar{q} + z^2) - q\bar{q}(a^2 - \rho_0^2) = (l_2^2 - a^2)(R_0^2 + h^2). \quad (5.1.19)$$

Formulae (5.1.13–5.1.18) are the main new results of this section. They give a complete solution for the contact problem under consideration when combined with formulae (2.1.6), (2.1.12), (5.1.2) and (5.1.7). When the prescribed displacement can be expressed as a sum of powers in x and y , the complete solution is elementary. Some particular examples will be considered further.

Exercise 5.1

1. Derive 5.1.13.
2. Verify (5.1.14)–(5.1.18).
3. Prove the identities (5.1.19).
4. Derive the general solution for the case of isotropy.

5.2 Flat centrally loaded circular punch.

We consider a transversely isotropic elastic half-space $z \geq 0$, indented by a rigid circular punch of radius a . The punch loading is statically equivalent to a centrally applied normal force P . Denote the punch settlement by $\omega = \text{const}$. Solution of the integral equation (5.1.8) is given by (5.1.9), and in this particular case takes the form

$$\sigma(\rho, \phi) = \frac{\omega}{\pi^2 H (a^2 - \rho^2)^{1/2}}. \quad (5.2.1)$$

Integration of the last expression over the circle $\rho \leq a$ relates the punch settlement ω to the total force P as

$$P = \frac{2\omega a}{\pi H}.$$

The substitution of (5.2.1) in (5.1.4) leads to an integral which can be evaluated by differentiation of the expression for the main potential function (A4.1.1) with respect to a . The result is

$$F = \frac{2\omega}{\pi H} \left[z \sin^{-1} \left(\frac{a}{l_2} \right) - (a^2 - l_1^2)^{1/2} + a \ln [l_2 + (l_2^2 - \rho^2)^{1/2}] \right]. \quad (5.2.2)$$

The appropriate differentiation of the potential function (5.2.2) gives the complete solution:

$$u = \frac{2\omega a e^{i\phi}}{\pi \rho} \sum_{k=1}^2 \frac{\gamma_k}{m_k - 1} \left[1 - \frac{(a^2 - l_{1k}^2)^{1/2}}{a} \right], \quad (5.2.3)$$

$$w = \frac{2\omega}{\pi} \sum_{k=1}^2 \frac{m_k}{m_k - 1} \sin^{-1} \left(\frac{a}{l_{2k}} \right), \quad (5.2.4)$$

$$\sigma_1 = - \frac{4\omega A_{66}}{\pi} \sum_{k=1}^2 \frac{\gamma_k^2 - (m_k + 1)\gamma_3^2}{\gamma_k(m_k - 1)} \frac{(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2 - l_{1k}^2}, \quad (5.2.5)$$

$$\sigma_2 = \frac{4\omega A_{66} e^{2i\phi}}{\pi} \sum_{k=1}^2 \frac{\gamma_k}{(m_k - 1)} \left\{ \frac{(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2 - l_{1k}^2} - \frac{2a}{\rho^2} \left[1 - \frac{(a^2 - l_{1k}^2)^{1/2}}{a} \right] \right\}, \quad (5.2.6)$$

$$\sigma_z = \frac{\omega}{\pi^2 H(\gamma_1 - \gamma_2)} \sum_{k=1}^2 (-1)^k \gamma_k \frac{(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2 - l_{1k}^2}, \quad (5.2.7)$$

$$\tau_z = \frac{\omega e^{i\phi}}{\pi^2 H(\gamma_1 - \gamma_2)} \sum_{k=1}^2 (-1)^k \frac{l_{1k}(l_{2k}^2 - a^2)^{1/2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)}. \quad (5.2.8)$$

This problem was first solved by Elliott (1949). His solution is essentially in agreement with ours. The fact that he uses the notation ρ as an elastic parameter and as a polar radius simultaneously, somewhat complicates the comparison.

Introduce the stress intensity factor as

$$k_1 = \lim_{\rho \rightarrow a} \{(a - \rho)^{1/2} \sigma_z\}, \text{ for } z=0. \quad (5.2.9)$$

Substitution of (5.2.7) in (5.2.9) yields

$$k_1 = -\frac{\omega}{\pi^2 H \sqrt{2a}} = -\frac{P}{2\pi} \left(\frac{a}{2}\right)^{1/2}. \quad (5.2.10)$$

The asymptotic behavior of the field of stresses and displacements near the punch edge can be derived by substitution of (4.7.1) and (5.2.10) in (5.2.3–5.2.8). The result is:

$$u = u_n = 2\pi H k_1 \sqrt{2r} \left[\frac{\gamma_1 S_1}{m_1 - 1} + \frac{\gamma_2 S_2}{m_2 - 1} \right] + 0(1), \quad (5.2.11)$$

$$w = -2\pi H k_1 \sqrt{2r} \left[\frac{m_1 T_1}{m_1 - 1} + \frac{m_2 T_2}{m_2 - 1} \right] + 0(1), \quad (5.2.12)$$

$$\sigma_1 = 2\pi A_{66} H k_1 \left(\frac{2}{r}\right)^{1/2} \sum_{k=1}^2 \frac{\gamma_k^2 - (m_k + 1)\gamma_3^2}{\gamma_k(m_k - 1)} \frac{S_k}{Q_k} + 0(1), \quad (5.2.13)$$

$$\sigma_2 = -2\pi A_{66} H k_1 \left(\frac{2}{r}\right)^{1/2} \sum_{k=1}^2 \frac{\gamma_k S_k}{(m_k - 1)Q_k} + 0(\sqrt{r}), \quad (5.2.14)$$

$$\sigma_z = \frac{k_1}{\sqrt{2r}(\gamma_1 - \gamma_2)} \left[\frac{\gamma_1 S_1}{Q_1} - \frac{\gamma_2 S_2}{Q_2} \right] + 0(1), \quad (5.2.15)$$

$$\tau_z = \tau_{zn} = -\frac{k_1}{\sqrt{2r}(\gamma_1 - \gamma_2)} \left[\frac{T_1}{Q_1} - \frac{T_2}{Q_2} \right] + 0(\sqrt{r}), \quad (5.2.16)$$

Comparison of (5.2.11–5.2.16) with (4.7.4–4.7.9) indicates that they become identical if one substitutes formally θ by $\pi - \theta$. This is easy to explain. The punch problem is mathematically equivalent to an external crack problem. There exist a notion that the asymptotic behavior of stresses and displacements near the edge of an arbitrary flat crack with a smooth boundary is completely defined by three stress intensity factors. This means that the asymptotics of an internal crack and an external one should be the same. The system of local axes was introduced in previous section so that the angle θ was measured from the direction outside the penny-shaped crack. In the case of the punch problem the axis On should be directed into the circle, this will make the expressions (5.2.11–5.2.16) identical to (4.7.4–4.7.9).

Exercise 5.2

1. Find the complete solution to the problem of a smooth flat centrally loaded circular punch in the case of an isotropic half-space.

Answer:

$$u = \frac{\omega e^{i\phi}}{\pi(1-\nu)} \left\{ -(1-2\nu) \left[\frac{a - (a^2 - l_1^2)^{1/2}}{\rho} \right] + \frac{z l_1 (l_2^2 - a^2)^{1/2}}{l_2 (l_2^2 - l_1^2)} \right\},$$

$$w = \frac{2\omega}{\pi} \left[\sin^{-1} \left(\frac{a}{l_2} \right) + \frac{z(a^2 - l_1^2)^{1/2}}{2(1-\nu)(l_2^2 - l_1^2)} \right],$$

$$\sigma_1 = - \frac{2\omega\mu}{\pi(1-\nu)} \left[(1+2\nu) \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} + \frac{z^2 [l_1^4 - a^2(2a^2 - \rho^2 + 2z^2)]}{(a^2 - l_1^2)^{1/2} (l_2^2 - l_1^2)^3} \right],$$

$$\sigma_2 = \frac{2\omega\mu e^{2i\phi}}{\pi(1-\nu)} \left\{ -(1-2\nu) \left[\frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} - \frac{2[a - (a^2 - l_1^2)^{1/2}]}{\rho^2} \right] - \frac{az(l_2^2 - a^2)^{1/2} [2l_1^4 + \rho^2(l_1^2 + 3l_2^2 - 6a^2)]}{l_2^2 (l_2^2 - l_1^2)^3} \right\},$$

$$\sigma_z = \frac{2\omega\mu}{\pi(1-\nu)} \left\{ - \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} + \frac{z^2 [l_1^4 + a^2(\rho^2 - 2a^2 - 2z^2)]}{(a^2 - l_1^2)^{1/2} (l_2^2 - l_1^2)^3} \right\},$$

$$\tau_z = - \frac{2\omega\mu}{\pi(1-\nu)} \frac{z\rho e^{i\phi} (a^2 - l_1^2)^{1/2} (3l_2^2 + l_1^2 - 4a^2)}{(l_2^2 - l_1^2)^3}.$$

Hint: use formulae (5.2.3–5.2.8).

2. Rewrite the result of the Exercise 1 in polar coordinates.

Answer:

$$u_\rho = \frac{\omega}{\pi(1-\nu)} \left\{ -(1-2\nu) \left[\frac{a - (a^2 - l_1^2)^{1/2}}{\rho} \right] + \frac{z l_1 (l_2^2 - a^2)^{1/2}}{l_2 (l_2^2 - l_1^2)} \right\},$$

$$\sigma_{\rho} = \frac{2\omega\mu}{\pi(1-\nu)} \left\{ (1-2\nu) \frac{a - (a^2 - l_1^2)^{1/2}}{\rho^2} - \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} - \frac{az(l_2^2 - a^2)^{1/2}[l_1^4 - l_2^4 + \rho^2(l_1^2 + 3l_2^2 - 4a^2)]}{l_2^2(l_2^2 - l_1^2)^3} \right\},$$

$$\sigma_{\phi} = -\frac{2\omega\mu}{\pi(1-\nu)} \left\{ (1-2\nu) \frac{a - (a^2 - l_1^2)^{1/2}}{\rho^2} + \frac{(a^2 - l_1^2)^{1/2}[a^2 - (1-2\nu)l_2^2]}{l_2^2(l_2^2 - l_1^2)} \right\},$$

$$\tau_{\rho z} = -\frac{2\omega\mu}{\pi(1-\nu)} \frac{z\rho(a^2 - l_1^2)^{1/2}(3l_2^2 + l_1^2 - 4a^2)}{(l_2^2 - l_1^2)^3}.$$

5.3 Inclined circular punch on an elastic half-space

The case of a flat circular punch pressed against a transversely isotropic elastic half-space by a non-centrally applied force P can be considered as a superposition of two problems: that of a centrally loaded punch, which was considered earlier, and a punch subjected to a tilting moment \mathcal{M} which will be considered here. Let the displacements under the punch be

$$\omega = b_x y - b_y x. \quad (5.3.1)$$

Here b_x and b_y are the tilting angles about the axes Ox and Oy respectively. Introduce the complex tilting angle

$$b = b_x + ib_y. \quad (5.3.2)$$

Expression (5.3.1) can now be rewritten as

$$\omega(\rho, \phi) = \Im\{\bar{b}\rho e^{i\phi}\}. \quad (5.3.3)$$

Here \Im indicates the imaginary part. The substitution of (5.3.3) in (5.1.9) yields

$$\sigma(\rho, \phi) = \frac{2\Im\{\bar{b}\rho e^{i\phi}\}}{\pi^2 H(a^2 - \rho^2)^{1/2}}. \quad (5.3.4)$$

Evaluation of the potential function (5.1.4) leads to the integral

$$\int_0^{2\pi} \int_0^a \frac{\rho_0 e^{i\phi_0}}{(a^2 - \rho_0^2)^{1/2}} \ln(R_0 + z) \rho_0 d\rho_0 d\phi_0. \quad (5.3.5)$$

Taking into consideration that

$$\frac{\rho e^{i\phi}}{(a^2 - \rho^2)^{1/2}} = -\Lambda(a^2 - \rho^2)^{1/2},$$

the integral in (5.3.5) can be evaluated by parts, with the result given by (A4.1.5). The potential function takes the form:

$$F = \frac{2\Im\{\bar{b}\rho e^{i\phi}\}}{\pi H} \left[z \sin^{-1} \frac{a}{l_2} - (a^2 - l_1^2)^{1/2} \left(1 - \frac{l_1^2 + 2a^2}{3\rho^2} \right) - \frac{2a^3}{3\rho^2} \right]. \quad (5.3.6)$$

All the necessary derivatives are readily available from Appendix A4.1, and we can write the complete solution:

$$u = \frac{2i}{\pi} \sum_{k=1}^2 \frac{\gamma_k}{m_k - 1} \left\{ b \left[z_k \sin^{-1} \left(\frac{a}{l_{2k}} \right) - (a^2 - l_{1k}^2)^{1/2} \right] - \bar{b} e^{2i\phi} \frac{2a^3 - (l_{1k}^2 + 2a^2)(a^2 - l_{1k}^2)^{1/2}}{3\rho^2} \right\}, \quad (5.3.7)$$

$$w = \frac{2}{\pi} (b_{xy} - b_{yx}) \sum_{k=1}^2 \frac{m_k}{m_k - 1} \left[\sin^{-1} \left(\frac{a}{l_{2k}} \right) - \frac{a(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2} \right], \quad (5.3.8)$$

$$\sigma_1 = -\frac{8A_{66}}{\pi} (b_{xy} - b_{yx}) a^2 \sum_{k=1}^2 \frac{\gamma_k^2 - (m_k + 1)\gamma_3^2}{\gamma_k(m_k - 1)} \frac{(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2(l_{2k}^2 - l_{1k}^2)}, \quad (5.3.9)$$

$$\sigma_z = \frac{4A_{66}ie^{i\phi}}{\pi} \sum_{k=1}^2 \frac{\gamma_k}{m_k - 1} \left\{ b \frac{al_{1k}(a^2 - l_{1k}^2)^{1/2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)} - \bar{b}e^{2i\phi} \left[\frac{4[(l_{1k}^2 + 2a^2)(a^2 - l_{1k}^2)^{1/2} - 2a^3]}{3\rho^3} + \frac{al_{1k}(a^2 - l_{1k}^2)^{1/2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)} \right] \right\}, \quad (5.3.10)$$

$$\sigma_z = \frac{2(b_x y - b_y x)}{\pi^2 H(\gamma_1 - \gamma_2)} \sum_{k=1}^2 (-1)^k \gamma_k \frac{a^2(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2(l_{2k}^2 - l_{1k}^2)}, \quad (5.3.11)$$

$$\tau_z = \frac{i}{\pi^2 H(\gamma_1 - \gamma_2)} \sum_{k=1}^2 (-1)^{k+1} + \bar{b}e^{2i\phi} \frac{al_{1k}^2(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2(l_{2k}^2 - l_{1k}^2)}. \quad (5.3.12)$$

Note. Some of the integrals involving special functions can now be computed simply by comparison of the solutions obtained by the integral transform method with the corresponding solution given by the present method. For example, comparison of (4.1.24) with formulae (1.42) and (1.43) of (Kassir and Sih, 1975) leads to

$$\int_0^\infty J_{n+1/2}(as) J_n(\rho s) e^{-sz} \frac{ds}{\sqrt{s}} = \left(\frac{2}{\pi a}\right)^{1/2} (a\rho)^{-n} \int_0^{l_1} \frac{x^{2n} dx}{(\rho^2 - x^2)^{1/2}}.$$

We have verified numerically that the last formula is correct for non-integer n as well. An enormous amount of material on Bessel functions exists in the literature, so it is difficult to claim that the last result is new, but there is some chance that it is new, since our notation l_1 and l_2 does not seem to have been used before. Here is another example of just how useful this notation is. Suppose that we need to compute the integral

$$\int_0^{\infty} \sin ax \, J_1(\rho x) \, e^{-zx} \frac{dx}{x^2}. \quad (5.3.13)$$

We could not find this integral in the tables, but we have found another one (Gradshtein and Ryzhik, 1963, formula 6.752.2)

$$\int_0^{\infty} \sin ax \, J_1(\rho x) \, e^{-zx} \frac{dx}{x} = \frac{a}{\rho}(1 - r), \quad (5.3.14)$$

where the parameter r is a positive root of the equation

$$a^2 = \frac{\rho^2}{1 - r^2} - \frac{z^2}{r^2}. \quad (5.3.15)$$

The original integral (5.3.13) can be computed by integration of both sides of (5.3.14) with respect to z . We need to get an explicit expression for r from the fourth order algebraic equation (5.3.15), substitute the result in (5.3.14) and integrate the result with respect to z which does not seem possible at first. The introduction of the parameters l_1 and l_2 allows us to find the positive root of (5.3.15) in a very simple form, namely, $r=(a^2 - l_1^2)^{1/2}/a$. The z -integration can be performed using (A4.2.3), and the final result is

$$\int_0^{\infty} \sin ax \, J_1(\rho x) e^{-zx} \frac{dx}{x^2} = \frac{(2a^2 - l_1^2)(l_2^2 - a^2)^{1/2} - 2a^2 z}{2a\rho} + \frac{\rho}{2} \sin^{-1}\left(\frac{a}{l_2}\right). \quad (5.3.16)$$

Exercise 5.3

1. Consider the interaction of a concentrated load P_0 , applied at an arbitrary point (ρ, ϕ, z) in the z -direction, with a flat circular punch of radius a .

Solution:

One can deduce from (5.2.4) that the normal displacement w at the point (ρ, ϕ, z) , due to a unit force applied to the punch, is

$$w = \frac{H}{a} \sum_{k=1}^2 \frac{m_k}{m_k - 1} \sin^{-1}\left(\frac{a}{l_{2k}}\right).$$

Application of the reciprocal theorem immediately gives the punch settlement ω due to the load P_0

$$\omega = \frac{H}{a} P_0 \sum_{k=1}^2 \frac{m_k}{m_k - 1} \sin^{-1}\left(\frac{a}{l_{2k}}\right).$$

The tilting angle δ of the punch can be obtained in a similar manner from (5.3.8). The result is

$$\delta = \frac{3H}{2a^3} P_0 \sum_{k=1}^2 \frac{m_k}{m_k - 1} \left[\rho \sin^{-1}\left(\frac{a}{l_{2k}}\right) - \frac{l_{1k}(l_{2k}^2 - a^2)^{1/2}}{l_{2k}} \right].$$

All the other parameters of interest can be found in a similar manner.

2. Investigate the problem of interaction between a flat circular punch and an arbitrarily located tangential force $T = T_x + iT_y$.

5.4 Flat punch of arbitrary planform under the action of a normal centrally applied force

The general method is applied here to the non-classical punch problem. A simple yet accurate relationship is established between the punch settlement and the applied force for an arbitrary flat punch. Specific formulae are derived for a punch whose planform has the shape of a polygon, a triangle, a rectangle, a rhombus, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature, and a good accuracy is confirmed.

Theory. The presentation in this section will be made in terms of the elastic contact problem but one should keep in mind that all the results will be applicable in other branches of engineering science. Here, we outline the idea of the analytical treatment of the elastic contact problems which allows us to derive simple yet accurate formulae for various punch shapes. The governing integral equation is given by (5.1.8). The analytical approach is based on the integral representation for the reciprocal distance established in (1.1.27).

Substitution of (1.1.27) into (5.1.8) gives, after changing the order of integration

$$w(\rho, \phi) = \frac{2}{\pi} H \int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right)}{(\rho_0^2 - x^2)^{1/2}} \sigma(\rho_0, \phi_0) \rho_0 d\rho_0. \quad (5.4.1)$$

Consider a flat-ended punch with a planform S whose boundary is defined in polar coordinates as

$$\rho = a(\phi). \quad (5.4.2)$$

Let the normal pressure distribution under the punch be

$$\sigma = \frac{ca(\phi)}{\left[a^2(\phi) - \rho^2\right]^{1/2}} \quad (5.4.3)$$

where c is a constant which can be determined from the condition that the integral of σ over S should give the total force P .

$$\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{ca(\phi)}{\left[a^2(\phi) - \rho^2\right]^{1/2}} \rho d\rho = c \int_0^{2\pi} a^2(\phi) d\phi = 2Ac = P, \quad (5.4.4)$$

where A is the area of S . It is noteworthy that the total force does not depend on the location of the coordinate system origin. This location can be determined from the condition that the stress distribution (5.4.3) should not produce any tilting moment about the origin, which leads to the two equations

$$\int_0^{2\pi} a^3(\phi) \cos\phi \, d\phi = 0, \quad \int_0^{2\pi} a^3(\phi) \sin\phi \, d\phi = 0.$$

The left hand sides of both equations are proportional to the x and y coordinates of the center of gravity which means that the origin of the system of polar coordinates should be located at the center of gravity of the domain of contact S . One gets immediately from (5.4.4) that

$$\sigma = \frac{P a(\phi)}{2A \left[a^2(\phi) - \rho^2 \right]^{1/2}}. \quad (5.4.5)$$

For the case of a flat punch $w = \delta = \text{const.}$ Now substituting (5.4.5) in (5.4.1), we can verify how close to a constant will be the displacements, produced by the traction distribution (5.4.5). Integration with respect to ρ_0 gives

$$w(\rho, \phi) = \frac{HP}{2A} \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x dx}{(\rho^2 - x^2)^{1/2}} \times \int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(1 - \frac{|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) d\phi_0. \quad (5.4.6)$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the normal displacements can be carried out separately for each harmonic. The zeroth harmonic has the form

$$w_0 = \frac{HP}{2A} \frac{\pi}{2} \int_0^{2\pi} a(\phi) d\phi. \quad (5.4.7)$$

It is important to note that the second harmonic is equal to zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. The expression for the fourth harmonic is

$$w_4 = \frac{HP}{2A} \frac{8}{35} \rho^3 \int_0^{2\pi} \frac{e^{4i(\phi-\phi_0)} d\phi_0}{a^2(\phi_0)}. \quad (5.4.8)$$

The investigation of further harmonics shows that their amplitude decreases.

Now consider in more detail the case of a square of side $2l$. The equation of the boundary in this case is $a(\phi) = l/\cos\phi$ for $-\pi/4 < \phi < \pi/4$, and the pattern is repeated outside this range. We can evaluate several non-zero harmonics:

$$w_0 = \frac{HP}{2A} 4\pi l \ln(1 + \sqrt{2}), \quad w_4 = \frac{HP}{2A} \frac{32\rho^3 \cos 4\phi}{105l^2}, \quad (5.4.9)$$

$$\begin{aligned}
w_8 &= -\frac{HP}{2A} \frac{64}{3465} \rho \cos 8\phi \left[\left(\frac{\rho}{l}\right)^2 + \frac{12}{13} \left(\frac{\rho}{l}\right)^4 + \frac{20}{13} \left(\frac{\rho}{l}\right)^6 \right], \\
w_{12} &= \frac{HP}{2A} \rho \cos 12\phi \left[\frac{6 \Gamma(\frac{9}{2})}{\Gamma(\frac{17}{2})} \left(\frac{\rho}{l}\right)^2 + \frac{144 \Gamma(\frac{7}{2})}{\Gamma(\frac{19}{2})} \left(\frac{\rho}{l}\right)^4 \right. \\
&\quad \left. + \frac{1200 \Gamma(\frac{5}{2}) \Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{21}{2})} \left(\frac{\rho}{l}\right)^6 + \frac{1680 \Gamma(\frac{7}{2})}{\Gamma(\frac{23}{2})} \left(\frac{\rho}{l}\right)^8 + \frac{3780 \Gamma(\frac{11}{2})}{\Gamma(\frac{25}{2})} \left(\frac{\rho}{l}\right)^{10} \right].
\end{aligned}$$

If we assume that the punch settlement $\delta \approx w_0$ then the remaining harmonics may be called the solution error. Direct computations show that the error is less than 3% inside the circle $\rho \leq l$. The error is reasonably small outside the circle reaching 20% at the apex, and decreasing very rapidly with the distance from the apex. Taking into consideration that the error sign fluctuation will result in even smaller error in the total force value, we may assume (5.4.7) being the relationship between the punch settlement and the total force which can be rewritten in the form

$$\delta \approx \frac{HP}{g\sqrt{A}}, \quad (5.4.10)$$

where A is the punch base area, and g is a dimensionless coefficient depending on the punch geometry only.

$$g = \frac{2\sqrt{A}}{\pi^2 r_a}, \quad (5.4.11)$$

where

$$r_a = \frac{1}{2\pi} \int_0^{2\pi} a(\phi) \, d\phi \quad (5.4.12)$$

can be called the *average radius* with respect to the center of gravity. If the correct pressure distribution required to yield $w = \delta$ over the punch could be found then the total force P would be related to δ by a formula

$$\delta = \frac{HP}{g_1 \sqrt{A}} .$$

Our basic assumption is that the g of equation (5.4.10), which we can calculate, is likely to be a good approximation to g_1 . The problem now is to find the value of g for various punch planforms. One can compute the coefficient g for the square from (5.4.9) as

$$g = \frac{1}{\pi \ln(1 + \sqrt{2})} = 0.3611,$$

which is very close to the approximate value 0.3607 given by Maxwell for the capacity of the square. Using the electrostatic analogy, one can easily deduce that our coefficient g is related to the capacity C of a flat lamina by

$$g = \frac{C}{\sqrt{A}} . \tag{5.4.13}$$

Of course, closeness to the result of Maxwell does not mean that ours is so accurate. The value of g which seems to be accurate was obtained in (Noble, 1960) and (Solomon, 1964a), and is 0.367, so that our result is in error by 1.6% which is not bad. Now it seems reasonable to assume that formulae (5.4.10–5.4.12) are valid for an arbitrary flat punch, and we need to verify how good they really are for each specific case.

We have found in the literature only one *general* formula of the type (5.4.10) suggested by Solomon (1964b). His result expressed through the coefficient g reads

$$g = \frac{2^{9/8}}{\pi^{11/8}} \frac{I_0^{1/8}}{A^{1/4}} . \tag{5.4.14}$$

where I_0 stands for the polar moment of inertia. One can easily verify that formula (5.4.14) is exact for a circle, so one should expect it to be sufficiently accurate for domains with the aspect ratio not far away from unity, but the error might be quite significant for oblong domains. For example, in the case of an ellipse with semi-axes a and b formula (5.4.14) gives

$$g = \frac{2^{7/8}}{\pi^{3/2}} \left(\frac{a}{b} + \frac{b}{a} \right)^{1/8} .$$

The error of this formula can be quite significant for large aspect ratio $\varepsilon=a/b$. Our formulae (5.4.10–5.4.12) in the case of an ellipse are exact.

Example 1: Polygon. Consider a flat punch, shaped as a polygon with n sides, with the only limitation that the function $a(\phi)$ describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n , a_k being the length of the k th side. The apex at which the sides a_k and a_{k+1} intersect is numbered $k+1$. It is clear that the value of index equal to $n+1$ is understood as 1. We denote the distance from the center of gravity to the k th apex as b_k . Let A_k be the area of the triangle formed by a_k , b_k and b_{k+1} , the total area A of the polygon being equal to the sum of A_k . Then formulae (5.4.11) and (5.4.12) yield the following expression for the coefficient g

$$g = \frac{2\sqrt{A}}{\pi \sum_{k=1}^n \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}}. \quad (5.4.15)$$

The formula due to Solomon (5.4.14) in this case gives

$$g = \frac{2^{9/8}}{\pi^{11/8}} \left\{ \sum_{k=1}^n \frac{2A_k^3}{A^2 a_k^2} \left[1 + \frac{a_k^4 + 3(b_{k+1}^2 - b_k^2)^2}{48A_k^2} \right] \right\}^{1/8}. \quad (5.4.16)$$

In the case of a regular polygon formulae (5.4.15) and (5.4.16) simplify to

$$g = \frac{4\sqrt{\tan(\pi/n)}}{\pi\sqrt{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}}, \quad (5.4.17)$$

and

$$g = \frac{2^{9/8}}{\pi^{11/8}} \left[\frac{2\cos^2(\pi/n) + 1}{3n\sin(2\pi/n)} \right]^{1/8} \quad (5.4.18)$$

respectively. Formulae (5.4.17) and (5.4.18), though looking different, give practically the same results in the whole range $3 \leq n < \infty$. Consider several particular values of n . For an equilateral triangle ($n=3$) formula (5.4.17) gives $g = 0.3673$. The value of g , which seems likely to be accurate, can be computed from (Solomon 1964a), and is equal to 0.3829, so that our result is in error by 4.1%. As we have seen earlier, the error of (5.4.17) for a square is 1.6%. Since formula (5.4.17) in the limiting case $n \rightarrow \infty$ gives the exact result for a circle $g=2/\pi^{3/2}=0.35917$, we should expect that the error of (5.4.17) will decrease with n . For a regular pentagon $g=0.3599$. We did not find in the literature anything to compare with this result. The value of g for a regular hexagon is

0.3595 which is within the bounds given by Pólya and Szegő (1951) $0.35917 < g < 0.3635$, and it is quite clear that the maximum possible error indeed decreases with n . It is noteworthy that the value of g does not change significantly in the whole range $3 \leq n < \infty$.

Example 2: Triangle. In the case of a triangular punch with the sides a_1 , a_2 and a_3 , formula (5.4.15) simplifies as follows:

$$g = \frac{6}{\pi\sqrt{A}} \left[\frac{1}{a_1} \ln \frac{b_1 + b_2 + a_1}{b_1 + b_2 - a_1} + \frac{1}{a_2} \ln \frac{b_2 + b_3 + a_2}{b_2 + b_3 - a_2} + \frac{1}{a_3} \ln \frac{b_3 + b_1 + a_3}{b_3 + b_1 - a_3} \right]^{-1}. \quad (5.4.19)$$

The parameters in (5.4.19) can be determined from the well known geometric formulae

$$A = [p(p - a_1)(p - a_2)(p - a_3)]^{1/2}, \quad p = (a_1 + a_2 + a_3)/2,$$

$$b_1 = \frac{1}{3}[2(a_1^2 + a_3^2) - a_2^2]^{1/2}, \quad b_2 = \frac{1}{3}[2(a_2^2 + a_1^2) - a_3^2]^{1/2},$$

$$b_3 = \frac{1}{3}[2(a_3^2 + a_2^2) - a_1^2]^{1/2}.$$

We are unaware of any report treating a triangle of general type but certain particular cases have been considered, so we can compare the results. When $a_1 = a_2 = l$, and the angle between these two sides is equal to α , the formula for the coefficient g can be rewritten in the form

$$g = \frac{6}{\pi\sqrt{\tan(\alpha/2)}} \left[2\sin\left(\frac{\alpha}{2}\right) \ln\left(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}\right) + \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right]^{-1}, \quad (5.4.22)$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

The isosceles triangle was considered by Rvachev *et al* (1977) who gave an approximate expression for the stress distribution but, for some reason, did not give the relationship between the total force and the punch settlement. They have though presented a graph depicting the location of the point of application of the force P as a function of the angle α . Their graph indicates a small variation (within 0.01 of the height of the triangle) of the coordinate about the center of gravity. We think that Rvachev *et al* did not realize that this fluctuation should be attributed to the approximate nature of their method, and

that the exact location of the point is at the center of gravity. Of course, the previous statement should be understood as a conjecture since our method is also approximate.

The case of $\alpha=\pi/2$ was considered by Pólya and Szegő (1951) who gave the following bounds for g : $0.35917 < g < 0.4282$. Okon and Harrington (1970) obtained $g = 0.3867$ as the most probable result. Our result is $g = 0.374$ which is well inside the admissible region and differs by 3.3% from the result of Okon and Harrington. The following bounds were established by Pólya and Szegő for a triangle with the sides a , $a/2$ and $a\sqrt{3}/2$: $0.35917 < g < 0.517$. Our result $g = 0.3822$ is within this interval. There seems to be no other source to compare with this result.

Example 3: Rectangle. Consider a punch with a rectangular base, a and b being its semiaxes. Introduce the aspect ratio $\varepsilon=a/b$. Formula (5.4.15) in this case reduces to

$$g = \frac{2}{\pi \left[\sqrt{\varepsilon} \sinh^{-1}\left(\frac{1}{\varepsilon}\right) + \frac{1}{\sqrt{\varepsilon}} \sinh^{-1}\varepsilon \right]} \quad (5.4.21)$$

Howe (1920) suggested an approximate formula for the capacitance of a rectangle which in terms of the coefficient g reads

$$g = \frac{1}{2\sqrt{\varepsilon} \left[\frac{1}{\varepsilon} \sinh^{-1}\varepsilon + \sinh^{-1}\left(\frac{1}{\varepsilon}\right) + \frac{\varepsilon}{3} + \frac{1}{3\varepsilon^2} - \frac{(\varepsilon^2 + 1)^{3/2}}{3\varepsilon^2} \right]} \quad (5.4.22)$$

The result due to Solomon (5.4.16) in this case takes the form

$$g = \frac{2^{9/8}}{\pi^{11/8}} \left[\frac{\varepsilon}{12} + \frac{1}{12\varepsilon} \right]^{1/8} \quad (5.4.23)$$

We have found in the literature some numerical results which seem to be more or less accurate. Borodachev and Galin (1974) have considered the case of a narrow rectangular punch, and Noble (1960) investigated an equivalent problem of the electric charge distribution on a rectangular lamina. Their data, expressed in terms of the coefficient g , are presented below and compared with our result (5.4.21) and those due to Howe (5.4.22) and Solomon (5.4.23). The following relationship can be established between Borodachev-Galin's coefficient γ and our g : $\gamma=1/2\pi g\sqrt{\varepsilon}$. Several useful conclusions can be drawn from the data presented. It seems logical to assume that the error of an approximate formula should change monotonically (or to have only one extremum) with respect to a certain

$\varepsilon =$	0.020	0.050	0.100	0.125	0.150	0.200	0.250	0.500	1.000
Borodachev and Galin	0.7375	0.5661	0.4819	–	0.4458	0.4259	–	–	–
Noble	–	–	–	0.4543	–	–	0.4047	0.3762	0.3670
Formula (5.4.21)	0.8031	0.6072	0.5037	0.4771	0.4576	0.4306	0.4128	0.3742	0.3612
Howe (5.4.22)	0.6916	0.5317	0.4481	0.4268	0.4112	0.3899	0.3759	0.3462	0.3363
Solomon (5.4.23)	0.5402	0.4819	0.4423	0.4304	0.4211	0.4071	0.3969	0.3715	0.3613
Discrepancy (%)									
Formula (5.4.21)	-8.9	-7.3	-4.5	-5.0	-2.6	-1.1	-2.0	0.5	1.6
Howe (5.4.22)	6.2	6.1	7.0	6.1	7.8	8.5	7.1	8.0	8.4
Solomon (5.4.23)	26.7	14.9	8.2	5.3	5.5	4.4	1.9	1.3	1.6

parameter. The fact that the discrepancy due to each formula jumps when moving from the data due to Noble to those by Borodachev and Galin, indicates that the results of at least one author are not exact. Our formula seems to perform better than the other two in a sufficiently wide range of aspect ratio. As expected, the formula due to Solomon performs well only when the aspect ratio is not far away from unity. There seems to be little change in the error of Howe's formula (5.4.22). If this is really so, then its accuracy can be improved dramatically just by multiplication by a constant factor, say, 1.07.

Example 4: Rhombus. Let α be the angle at one of the rhombus apices. Formula (5.4.15) in this case yields

$$g = \frac{2}{\pi \sqrt{\sin \alpha} \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1}}. \quad (5.4.24)$$

The same formula in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = a/b$ has the form

$$g = \frac{[2(\varepsilon + 1/\varepsilon)]^{1/2}}{\pi \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}}}. \quad (5.4.25)$$

We did not find in mechanics literature any result related to a punch with a rhombus planform. In electrical sciences, the related problem of the capacity of a diamond was solved numerically by Okon and Harrington (1970). Their result, expressed in terms of the coefficient g , for a diamond with the aspect ratio $a:b=0.7:1.65$ is $g=0.3855$. Formula (5.4.25) gives $g=0.3744$ which differs by 3% from the result of Okon and Harrington. They also considered a rhombus with the aspect ratio 1:2. Their result, $g=0.3705$, almost coincides with ours which is

$g=0.3698$.

Example 5: Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2 \sin^3 \alpha}{3(\alpha - \frac{1}{2} \sin 2\alpha)}. \quad (5.4.26)$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k \cos \phi + (1 - k^2 \sin^2 \phi)^{1/2}], \quad \text{for } 0 \leq \phi \leq \pi - \gamma \text{ or } \pi + \gamma \leq \phi < 2\pi;$$

and

$$a(\phi) = r \frac{k - \cos \alpha}{\cos(\pi - \phi)} \quad \text{for } \pi - \gamma \leq \phi \leq \pi + \gamma. \quad (5.4.27)$$

Substitution of (5.4.27) into (5.4.11–5.4.12) gives

$$g = \frac{2(\alpha - \frac{1}{2} \sin 2\alpha)^{1/2}}{\pi \left[2E(k) - E(\gamma, k) - k \sin \gamma + (k - \cos \alpha) \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right]}. \quad (5.4.28)$$

where $\gamma = \tan^{-1}(\sin \alpha / (k - \cos \alpha))$. We have found only one numerical example to verify the accuracy of (5.4.28): Okon and Harrington (1970) have computed the capacity of a semicircle. Their result expressed through the coefficient g is 0.3724. Formula (5.4.28) gives 0.3714 with the discrepancy of 0.3%.

Example 6: Circular sector. Repetition of the procedure, described in the previous paragraph, leads to the following result for a circular sector of subtended angle 2α :

$$g = \frac{2\sqrt{\alpha}}{\pi \left[E(\gamma, k) - k \sin \gamma + k \sin \alpha \ln[\cot(\alpha/2) \cot((\gamma - \alpha)/2)] \right]}. \quad (5.4.29)$$

Here, $k=2\sin\alpha/(3\alpha)$, and $\gamma=\tan^{-1}(\sin\alpha/(\cos\alpha-k))$. Okon and Harrington (1970) obtained $g=0.3668$ for the case of a quadrant. Formula (5.4.29) for $\alpha=\pi/4$ gives $g=0.3639$, with the discrepancy of 0.8%.

Discussion. We might enquire whether there exists any contour, other than an ellipse, for which an expression of the type (5.4.5) would be an *exact* solution to the integral equation (5.1.8). Expression (5.4.6) can provide the sufficient conditions:

$$\int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(1 - \frac{|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) d\phi_0 \quad (5.4.30)$$

should be equal to zero for $n \neq 0$. The integral (5.4.30) will vanish for all odd n if $a(\phi)$ contains even harmonics only. In the case of even n , the hypergeometric function in (5.4.30) represents a finite polynomial in $x/a(\phi)$ of degree not greater than $n-2$ which means that the integral will vanish if $(a(\phi))^{-2}$ contains harmonics not higher than the second, which corresponds to an ellipse. The question as to whether these conditions are also necessary requires an additional investigation.

Solomon's formula (5.4.14) can be considered as a particular case of a more general one, namely

$$g = \frac{2}{\pi\sqrt{A}} \left[\frac{1}{2\pi} \int_0^{2\pi} (a(\phi))^m d\phi \right]^{-1/2m} \left[\frac{1}{2\pi} \int_0^{2\pi} (a(\phi))^n d\phi \right]^{1/2n}. \quad (5.4.31)$$

for $m=2$ and $n=4$. One may ask now whether this choice of the parameters m and n is in any sense optimal. Direct computations show that for a regular polygon, $m=1$ and $n=9$ give much better accuracy than (5.4.18). In the case of a rectangle $m=1$ and $n=7$ are more accurate than (5.4.23). It is clear that the formula chosen by Solomon (5.4.14) is not the best particular case of (5.4.31). One can suggest many formulae of the type (5.4.31) but this would be just an exercise in curve-fitting, which is outside the scope of this book.

Mossakovskii (1972) considered the case of a flat punch of nearly circular cross-section. He assumed a solution for the contact tractions in the form

$$\sigma(\rho, \phi) = \frac{F_0(\rho, \phi) + \alpha F_1(\rho, \phi) + \alpha^2 F_2(\rho, \phi) + \dots}{\left[a^2(\phi) - \rho^2 \right]^{1/2}}, \quad (5.4.32)$$

where α is a small parameter. Expression (5.4.32) has two essential disadvantages as compared with (5.4.3): (i) even in the case of an ellipse expression (5.4.32) gives an infinite series rather than the closed form exact solution; (ii) the solution (5.4.32) at the point $\rho=0$ is a function of ϕ which is physically meaningless. Though Mossakovskii's method can give reasonably

accurate estimations for the relationship between the punch settlement and applied force, his claim of being able to evaluate the stress distribution exerted by the punch seems to be wrong because of incorrect assumption of a square root singularity at the punch boundary (for example, in the case of a square punch).

The mathematically similar problem of sound penetration through an aperture of general shape in a rigid flat screen was considered in (Fabrikant, 1986b). The same method was used to find the electrical capacity of flat laminae (Fabrikant, 1986k).

Exercise 5.4

1. Establish (5.4.6).
2. Derive (5.4.7).
3. Verify (5.4.15).
4. Derive (5.4.19).
5. Generalize the theory for the case of a non-homogeneous half-space, with the modulus of elasticity being proportional to a power function of the depth.

5.5 Inclined flat punch of general shape

An approximate analytical solution is given here to the contact problem of a flat inclined punch of arbitrary planform under the action of a normal non-centrally applied force. Some accurate relationships are established between the tilting moments and the angles of inclination of an arbitrary flat punch. Specific formulae are derived for a punch whose planform has the shape of a polygon, a triangle, a rectangle, a rhombus, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature, and their accuracy is confirmed.

Theory. Consider a flat-ended punch with a planform S whose boundary is given in polar coordinates as

$$\rho = a(\phi).$$

where the function $a(\phi)$ is bounded and single-valued. The punch is pressed against an elastic half-space by a normal force P applied at the point with cartesian coordinates x_0 and y_0 . This loading is statically equivalent to a centrally applied force P and two tilting moments $M_x = Py_0$ and $M_y = -Px_0$. The case of a centrally applied force was considered in the previous section. It

remains here to consider the punch under the action of the tilting moments, and to superpose the results. Repeating the procedure of the section 5.4, we come to the same governing integral equation, namely,

$$w(\rho, \phi) = \frac{2}{\pi} H \int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right)}{(\rho_0^2 - x^2)^{1/2}} \sigma(\rho_0, \phi_0) \rho_0 d\rho_0. \quad (5.5.1)$$

Let the normal displacements under the punch be

$$w = \alpha_x y - \alpha_y x, \quad (5.5.2)$$

where α_x and α_y are the tilting angles about the axes Ox and Oy respectively. It is necessary to relate these angles to the tilting moments.

Present the normal stress distribution under the punch as

$$\sigma = \frac{a(\phi)\rho(p_1 \cos\phi + p_2 \sin\phi)}{\left[a^2(\phi) - \rho^2\right]^{1/2}}, \quad (5.5.3)$$

where p_1 and p_2 are the as yet unknown constants. Make use of the condition that the integral of σ over S should be equal to zero. Since p_1 and p_2 are independent, this leads to the two equations

$$\int_0^{2\pi} (a(\phi))^3 \cos\phi \, d\phi = 0, \quad \int_0^{2\pi} (a(\phi))^3 \sin\phi \, d\phi = 0. \quad (5.5.4)$$

It follows that (5.5.4) will be satisfied if and only if the origin of coordinates is located at the center of gravity of the domain of contact. The axis orientation will be discussed later.

The relationships between the tilting moments and the parameters p_1 and p_2 can be established from the statics conditions

$$M_x = Py_0 = \iint_S \sigma y dS, \quad M_y = -Px_0 = -\iint_S \sigma x dS,$$

which leads to

$$M_x = \frac{8}{3}(p_1 I_{xy} + p_2 I_x), \quad M_y = \frac{8}{3}(p_1 I_y + p_2 I_{xy}), \quad (5.5.5)$$

where I_x , I_y and I_{xy} are the well known quantities of the moments of inertia and the product of inertia respectively. Now it is necessary to relate p_1 and p_2 to the angles α_x and α_y . This can be done by substitution of (5.5.3) into (5.5.1) which, after integration with respect to ρ_0 , yields

$$w(\rho, \phi) = H \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^2 dx}{(\rho^2 - x^2)^{1/2}} \\ \times \int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(\frac{3-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) (p_1 \cos\phi_0 + p_2 \sin\phi_0) d\phi_0. \quad (5.5.6)$$

Here F denotes the Gauss hypergeometric function. Further evaluation of the normal displacements can be carried out separately for each harmonic. Note that the zeroth and all the even harmonics of w will be zero if $a(\phi)$ contains only even harmonics. The first harmonic will take the form

$$w_1(\rho, \phi) = \frac{\pi}{2} H \rho \int_0^{2\pi} \cos(\phi - \phi_0) (p_1 \cos\phi_0 + p_2 \sin\phi_0) a(\phi_0) d\phi_0,$$

which can be simplified as

$$w_1(\rho, \phi) = \frac{\pi}{2} H \rho [(p_1 J_y + p_2 J_{xy}) \cos\phi + (p_1 J_{xy} + p_2 J_x) \sin\phi], \quad (5.5.7)$$

where the following quantities were introduced

$$J_x = \int_0^{2\pi} a(\phi) \sin^2\phi d\phi, \quad J_y = \int_0^{2\pi} a(\phi) \cos^2\phi d\phi, \\ J_{xy} = \int_0^{2\pi} a(\phi) \sin\phi \cos\phi d\phi \quad (5.5.8)$$

These quantities do not seem to have been used before in engineering practice so they do not have an accepted name. Since their tensor properties are similar to

those of the moments of inertia, we shall call J_x and J_y the linear moments of a two-dimensional domain about the axes Ox and Oy respectively, while J_{xy} will be called the linear product of a two-dimensional domain about the axes Ox and Oy

It is important to note that the third harmonic is equal to zero for an arbitrary contour. Here is the expression for the fifth harmonic

$$w_5(\rho, \phi) = \frac{128}{315} H \rho^4 \int_0^{2\pi} \frac{\cos 5(\phi - \phi_0)}{a^2(\phi_0)} (p_1 \cos \phi_0 + p_2 \sin \phi_0) d\phi_0,$$

which can be modified as

$$\begin{aligned} w_5(\rho, \phi) = & \frac{64}{315} H \rho^4 \{ [(A_{c6} + A_{c4})p_1 + (A_{s6} - A_{s4})p_2] \cos 5\phi \\ & + [(A_{s6} + A_{s4})p_1 + (A_{c4} - A_{c6})p_2] \sin 5\phi \}. \end{aligned} \quad (5.5.9)$$

Here, the following geometrical characteristics of the domain of contact were introduced

$$\begin{aligned} A_{c4} &= \int_0^{2\pi} \frac{\cos 4\phi \, d\phi}{(a(\phi))^2}, & A_{c6} &= \int_0^{2\pi} \frac{\cos 6\phi \, d\phi}{(a(\phi))^2}, \\ A_{s4} &= \int_0^{2\pi} \frac{\sin 4\phi \, d\phi}{(a(\phi))^2}, & A_{s6} &= \int_0^{2\pi} \frac{\sin 6\phi \, d\phi}{(a(\phi))^2}. \end{aligned} \quad (5.5.10)$$

Investigation of further harmonics shows that their amplitude decreases.

Now consider in more detail the case of a square of side $2l$. The equation of the boundary in this case is $a(\phi) = l/\cos\phi$ for $-\pi/4 < \phi < \pi/4$, and the pattern is repeated outside this range. We can evaluate the first two non-zero harmonics:

$$\begin{aligned} w_1 &= \pi H l \rho \ln(1 + \sqrt{2})(p_1 \cos \phi + p_2 \sin \phi), \\ w_5 &= \frac{128 H \rho^4}{945 l^2} (p_1 \cos 5\phi + p_2 \sin 5\phi). \end{aligned}$$

Since the amplitude of w_5 is significantly less than that of w_1 , it seems natural to assume $w \approx w_1$, and the remaining harmonics may be called the solution

error. Direct computations show that the error is less than 3% inside the circle $\rho \leq l$. The error is reasonably small outside the circle, reaching about 20% at the apex and decreasing very rapidly with the distance from the apex. Taking into consideration that the error sign fluctuation will result in even smaller error in the integral characteristics sought, a direct comparison of (5.5.2) and (5.5.7) leads to

$$\alpha_x = \frac{\pi}{2}H(p_1J_{xy} + p_2J_x) , \quad \alpha_y = -\frac{\pi}{2}H(p_1J_y + p_2J_{xy}). \quad (5.5.11)$$

The inversion of (5.5.11) gives

$$p_1 = -\frac{2}{\pi H} \frac{J_{xy}\alpha_x + J_x\alpha_y}{J_xJ_y - J_{xy}^2} , \quad p_2 = \frac{2}{\pi H} \frac{J_y\alpha_x + J_{xy}\alpha_y}{J_xJ_y - J_{xy}^2}. \quad (5.5.12)$$

Substitution of (5.5.12) in (5.5.5) finally gives the required relationship

$$M_x = \frac{16}{3\pi H}(m_{11}\alpha_x + m_{12}\alpha_y) , \quad M_y = \frac{16}{3\pi H}(m_{21}\alpha_x + m_{22}\alpha_y) \quad (5.5.13)$$

where

$$m_{11} = \frac{J_yI_x - J_{xy}I_{xy}}{J_xJ_y - J_{xy}^2} , \quad m_{12} = \frac{J_{xy}I_x - J_xI_{xy}}{J_xJ_y - J_{xy}^2} ,$$

$$m_{21} = \frac{J_{xy}I_y - J_yI_{xy}}{J_xJ_y - J_{xy}^2} , \quad m_{22} = \frac{J_xI_y - J_{xy}I_{xy}}{J_xJ_y - J_{xy}^2} .$$

It is clear that all these results can be rewritten in a matrix or a tensor form. One can verify that formulae (5.5.13) are invariant with respect to an arbitrary rotation of the axes. The same property holds for $m_{11}+m_{22}$ and $m_{12}-m_{21}$. Strictly speaking, according to the reciprocal theorem, m_{12} should be equal to m_{21} , so that formulae (5.5.13) generally do not satisfy this theorem. But we may state that this theorem is satisfied 'approximately'. We mean by this the following property which has been verified by several direct computations, namely, $|m_{12}-m_{21}|/m_{11} \ll 1$ and $|m_{12}-m_{21}|/m_{22} \ll 1$. This theorem will be satisfied exactly for any domain which has at least one axis of symmetry because in this case $m_{12}=m_{21}=0$, provided that the coordinate axes coincide with the central principal axes of the domain of contact. Since we have no numerical data for non-symmetrical domains which could be used to verify the accuracy of (5.5.13), we shall consider further only the case when the domain of contact has an axis of symmetry. In this case formulae (5.5.5), (5.5.11) and (5.5.13) simplify significantly, namely,

$$M_x = \frac{8}{3}I_x p_2, \quad M_y = -\frac{8}{3}I_y p_1, \quad (5.5.14)$$

$$\alpha_x = \frac{\pi}{2}HJ_x p_2, \quad \alpha_y = -\frac{\pi}{2}HJ_y p_1, \quad (5.5.15)$$

$$M_x = \frac{16}{3\pi H} \frac{I_x}{J_x} \alpha_x, \quad M_y = \frac{16}{3\pi H} \frac{I_y}{J_y} \alpha_y. \quad (5.5.16)$$

Returning back to the problem of a non-centrally loaded flat punch and using the results of section 5.4, we can write the following expression for the traction distribution under the punch in terms of the applied force P and the coordinates of its point of application x_0 and y_0

$$\sigma = \frac{Pa(\phi)}{2A \left[a^2(\phi) - \rho^2 \right]^{1/2}} \left[1 + \frac{3}{4}A \left(\frac{xx_0}{I_y} + \frac{yy_0}{I_x} \right) \right], \quad (5.5.17)$$

where A is the area of the domain S . An expression equivalent to (5.5.17) can be written in terms of the normal displacement δ and the tilting angles α_x and α_y

$$\sigma = \frac{2a(\phi)}{\pi H \left[a^2(\phi) - \rho^2 \right]^{1/2}} \left[\frac{\delta}{J_0} + \frac{\alpha_x y}{J_x} - \frac{\alpha_y x}{J_y} \right], \quad (5.5.18)$$

where

$$J_0 = \int_0^{2\pi} a(\phi) \, d\phi$$

The quantity J_0 may be called *the polar linear moment* due to the analogy with the moments of inertia and the property $J_0 = J_x + J_y$. One can note also that J_0 is proportional to the average polar radius. Expressions (5.5.17) and (5.5.18) are *exact* for an ellipse. We expect them to be reasonably accurate in the neighbourhood of the coordinate origin for an arbitrary punch planform with at least one axis of symmetry, while the error might become quite significant close to the boundary of the domain S .

Let us rewrite formulae (5.5.16) in the form

$$M_x = \frac{A^{3/2}}{2\pi H} h_x \alpha_x, \quad M_y = \frac{A^{3/2}}{2\pi H} h_y \alpha_y, \quad (5.5.19)$$

where

$$h_x = \frac{32I_x}{3A^{3/2}J_x}, \quad h_y = \frac{32I_y}{3A^{3/2}J_y}. \quad (5.5.20)$$

We introduced the coefficients h_x and h_y for two reasons: since they are dimensionless they characterize the shape of S and do not depend on its size; both h_x and h_y are equal to the corresponding coefficients of magnetic polarizability which will simplify the comparison of our results with the numerical data available. There is an advantage of formulae (5.5.19) over the equivalent (5.5.16): the factors depending on the shape of S are separated from those depending on its size. One can draw from (5.5.19) an immediate conclusion that in the case when a domain S is magnified so that its linear dimensions double, its area quadruples, and the tilting moment should be multiplied by 8 in order to produce the same tilting angle. This conclusion is not so clear from (5.5.16). The remaining part of this section will be devoted to the evaluation of the coefficients h_x and h_y for various punch planforms. Several punch planforms are considered. Each configuration is related to its central principal axes and assumed to have at least one axis of symmetry coinciding with the axis Ox . The high degree of accuracy of formulae (5.5.20) is confirmed by comparison with available numerical solutions.

Example 1: Polygon. Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n , a_k being the length of the k th side. The apex at which the sides a_k and a_{k+1} intersect is numbered $k+1$. It is clear that the value of index $n+1$ is to be understood as 1. We denote the distance from the center of gravity to the k th apex as b_k ; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k . Let A_k be the area of the triangle formed by a_k , b_k and b_{k+1} , the total area A of the polygon being equal to the sum of A_k . The following expressions can be obtained for the moments of inertia

$$I_x = \sum_{k=1}^n \frac{2A_k^3}{a_k^2} \left[\sin^2 \psi_k + \frac{b_{k+1}^2 - b_k^2}{4A_k} \sin 2\psi_k + \frac{a_k^4 + 3(b_{k+1}^2 - b_k^2)^2}{48A_k^2} \cos^2 \psi_k \right], \quad (5.5.21)$$

$$I_y = \sum_{k=1}^n \frac{2A_k^3}{a_k^2} \left[\cos^2 \psi_k - \frac{b_{k+1}^2 - b_k^2}{4A_k} \sin 2\psi_k + \frac{a_k^4 + 3(b_{k+1}^2 - b_k^2)^2}{48A_k^2} \sin^2 \psi_k \right]. \quad (5.5.22)$$

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^n \frac{A_k}{a_k^2} \left[-\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right) [a_k^2 - (b_k - b_{k+1})^2] \cos 2\psi_k \right. \\ \left. + 4A_k \left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right) \sin 2\psi_k + 2a_k \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k} \cos^2 \psi_k \right], \quad (5.5.23)$$

$$J_y = \sum_{k=1}^n \frac{A_k}{a_k^2} \left[\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right) [a_k^2 - (b_k - b_{k+1})^2] \cos 2\psi_k \right. \\ \left. - 4A_k \left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right) \sin 2\psi_k + 2a_k \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k} \sin^2 \psi_k \right]. \quad (5.5.24)$$

Substitution of (5.5.21–5.5.24) into (5.5.20) gives the coefficients h_x and h_y for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k-1)/n$, $A_k = [a^2 \cot(\pi/n)]/4 = [b^2 \sin(2\pi/n)]/2$, $A = nA_k$, and formulae (5.5.21–5.5.24) simplify to

$$I_x = I_y = \frac{na^4}{64} \cot \frac{\pi}{n} \left[\cot^2 \left(\frac{\pi}{n}\right) + \frac{1}{3} \right] = \frac{nb^4}{24} \sin \frac{2\pi}{n} \left[2 + \cos \frac{2\pi}{n} \right], \quad (5.5.25)$$

$$J_x = J_y = \frac{na}{4} \cot \frac{\pi}{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)} = \frac{nb}{2} \cos \frac{\pi}{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}. \quad (5.5.26)$$

Substituting (5.5.25) and (5.5.26) in (5.5.20) leads to

$$h_x = h_y = \frac{16(2 + \cos \frac{2\pi}{n})}{9 \left(n^3 \sin \frac{\pi}{n} \cos^3 \left(\frac{\pi}{n}\right) \right)^{1/2} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}}. \quad (5.5.27)$$

Consider several particular values of n . For an equilateral triangle ($n=3$) formula

(5.5.27) gives $h_x=h_y=3^{1/4}16/[27\ln(2+\sqrt{3})]=0.5922$. We did not find any numerical data to compare with this result. In the case of a square $n=4$, and $h_x=h_y=4/[9\ln(1+\sqrt{2})]=0.5043$ which is inside the interval from 0.4973 to 0.5162 given by Okon and Harrington (1981) and within 3% from the result of de Smedt (1979) which is 0.5193. Since formula (5.5.27) gives the exact result for a circle in the limiting case of $n\rightarrow\infty$, namely, $h_x=h_y=8/(3\pi^{3/2})=0.4789$, we should expect that the error of (5.5.27) will decrease with n . The value of the coefficients for a regular hexagon is $h_x=h_y=40\sqrt{2}/(3^{1/4}81\ln 3)=0.4830$ which differs by 1.4% from the result 0.49 due to Okon and Harrington (1981), and it is quite clear that the maximum possible error indeed decreases with n . It is noteworthy that the values of the coefficients do not change significantly in the whole range $3\leq n<\infty$.

Example 2: Isosceles triangle. In the case of a triangle with sides $a_1=a_2=l$, the angle between them being equal to α , formulae (5.5.20–5.5.24) give

$$I_x = \frac{1}{12}l^4 \sin\alpha \sin^2(\alpha/2), \quad I_y = \frac{1}{36}l^4 \sin\alpha \cos^2(\alpha/2),$$

$$J_x = \frac{2}{3}l \cos \frac{\alpha}{2} \left[\sin\alpha + \sin(\alpha + \gamma) - 2\sin\gamma \right. \\ \left. + 2\sin^3\left(\frac{\alpha}{2}\right) \ln\left(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}\right) + \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right],$$

$$J_y = \frac{2}{3}l \cos \frac{\alpha}{2} \left[-\sin\alpha - \sin(\alpha + \gamma) + 2\sin\gamma \right. \\ \left. + \sin\alpha \cos \frac{\alpha}{2} \ln\left(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}\right) \right],$$

with the result for the coefficients

$$h_x = 8(\tan(\alpha/2))^{3/2} \left\{ 3 \left[\sin\alpha + \sin(\alpha + \gamma) - 2\sin\gamma \right. \right. \\ \left. \left. + 2\sin^3\frac{\alpha}{2} \ln\left(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}\right) + \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right] \right\}^{-1},$$

(5.5.28)

$$h_y = 8\sqrt{\cot(\alpha/2)} \left\{ 9 \left[-\sin\alpha - \sin(\alpha + \gamma) + 2\sin\gamma \right. \right. \\ \left. \left. + \sin\alpha \cos\frac{\alpha}{2} \ln\left(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}\right) \right] \right\}^{-1},$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

The isosceles right triangle was considered by Okon and Harrington (1981) who gave the interval between 0.9829 and 1.021 for only one of the coefficients, which in our notation is h_x . Our result for h_x is 0.9255 which differs by less than 10% from theirs. The second formula (5.5.28) gives $h_y = 0.3995$, and we found nothing in the literature to compare with this result.

Example 3: Rectangle. Consider a punch with a rectangular base, a_1 and a_2 being its semiaxes. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (5.5.21–5.5.24) in this case reduce to

$$\begin{aligned} I_x &= \frac{4}{3} a_1 a_2^3, & I_y &= \frac{4}{3} a_1^3 a_2, \\ J_x &= 4a_1 \sinh^{-1}\varepsilon, & J_y &= 4a_2 \sinh^{-1}(1/\varepsilon). \end{aligned}$$

and formulae (5.5.20) yield

$$h_x = \frac{4\varepsilon^{3/2}}{9\sinh^{-1}\varepsilon}, \quad h_y = \frac{4\varepsilon^{-3/2}}{9\sinh^{-1}(1/\varepsilon)}. \quad (5.5.29)$$

The coefficients of magnetic polarizability were computed by de Smedt (1979) for a rectangle with various aspect ratios ε . Here, we present his results along with those given by (5.5.29). Our formula (5.5.29) seems to perform satisfactorily in a sufficiently wide range of aspect ratio. The approximate expression for the stress distribution under the punch according to (5.5.17) takes the form

$$\sigma = \frac{P a(\phi)}{8a_1 a_2 \left[a^2(\phi) - \rho^2 \right]^{1/2}} \left[1 + \frac{9}{4} \left(\frac{xx_0}{a_1^2} + \frac{yy_0}{a_2^2} \right) \right]. \quad (5.5.30)$$

Expression (5.5.30) can be used for analyzing the process of movement of the applied force P , say, along the axis Ox . This analysis can be done by

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $h_x=$	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (32) $h_x=$	0.1408	0.2001	0.2612	0.3265	0.4165	0.4341	0.5043
de Smedt $h_y=$	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (32) $h_y=$	4.6876	2.1488	1.2701	0.8708	0.6228	0.5929	0.5043
Discrepancy in h_x (%)	-9.4	-6.4	-3.2	-0.5	1.8	2.2	2.9
Discrepancy in h_y (%)	-14.1	-6.1	-0.8	2.1	3.1	3.3	2.9

requiring that the contact traction vanish at the edge. One can conclude from (5.5.17) that the boundary at which this occurs will always be a straight line. It is clear from (5.5.30) that the punch will be in contact with the half-space as long as $x_0 \leq 4a_1/9$, after which the punch will start separating from the half-space. Assuming that the new domain of contact is also a rectangle (of course, with a different aspect ratio), one can again apply the formulae of this section to analyze the process further. If we denote the width of the zone of separation by c , the following relationship holds

$$c = \frac{2}{5}(9x_0 - 4a_1) , \quad \text{for } x_0 \geq \frac{4}{9}a_1.$$

The last formula states, for example, that when the force P is applied at $x_0 = 13a_1/18$ only a half of the punch will be in contact with the half-space. Unfortunately, there is no data to verify these relationships. Further analysis reveals that the core inside which the force can be applied without causing any separation is a rhombus with semiaxes $4a_1/9$ and $4a_2/9$ respectively. As one knows, in the case of a circular punch the core is a circle of radius equal to one third of the radius of the punch. The results due to (5.5.30) can be compared with the numerical data received in a personal communication from de Smedt. In order to make the comparison possible, we must set $P=0$, $M_x=0$ in (5.5.30) and replace M_y by (5.5.19), with the result

$$\sigma H = \frac{9\sqrt{\varepsilon} a(\phi) h_y x}{4a_1 \left[a^2(\phi) - \rho^2 \right]^{1/2}}. \quad (5.5.31)$$

Computations due to (5.5.31) were made for $\varepsilon=0.5$ along the axis Ox , the value h_y was taken 0.8708 (see the preceding Table). Here are the results compared with those communicated by de Smedt. Since the numerical method of De Smedt is also approximate, we are using the word *discrepancy* rather than the word *error* in the tables throughout this and adjacent sections. We can also compare

$x/a_1=$	0.0833	0.2500	0.3333	0.5000	0.5833	0.6667	0.7500	0.8333	0.9167
de Smedt $\sigma H=$	0.1143	0.3501	0.4759	0.7523	0.9367	1.1460	1.4304	1.8303	2.8182
Formula (5.5.31) $\sigma H=$	0.1159	0.3577	0.4898	0.7999	0.9950	1.2392	1.5709	2.0886	3.1777
Discrepancy (%)	-1.3	-2.2	-2.9	-6.3	-6.2	-8.1	-9.8	-14.1	-12.8

the same values along the axis Oy . One can use a formula similar to (5.5.31) replacing all x by y and interchanging a_1 and a_2 , the value of h_x was taken to be 0.3265. The results are:

$y/a_2=$	0.1667	0.3333	0.5000	0.6667	0.8333
de Smedt $\sigma H=$	0.1756	0.3663	0.6011	0.9014	1.6413
our result $\sigma H=$	0.1756	0.3673	0.5998	0.9292	1.5662
Discrepancy (%)	0.0	-0.3	0.2	-3.1	4.6

The agreement is satisfactory.

Example 4: Rhombus. Let α be the angle at one of the rhombus apices, and l be its side. Formulae (5.5.21–5.5.24) in this case yield

$$I_x = \frac{1}{6}l^4 \sin\alpha \sin^2\left(\frac{\alpha}{2}\right), \quad I_y = \frac{1}{6}l^4 \sin\alpha \cos^2\left(\frac{\alpha}{2}\right), \quad A = l^2 \sin\alpha,$$

$$J_x = 2l \sin\alpha \left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\left(\frac{\alpha}{2}\right) \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right],$$

$$J_y = 2l \sin\alpha \left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\left(\frac{\alpha}{2}\right) \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right].$$

The coefficients are determined as

$$h_x = \frac{8 \sin^2 \frac{\alpha}{2}}{9(\sin\alpha)^{3/2} \left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\left(\frac{\alpha}{2}\right) \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right]},$$

$$h_y = \frac{8 \cos^2 \frac{\alpha}{2}}{9(\sin\alpha)^{3/2} \left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\left(\frac{\alpha}{2}\right) \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right]}.$$

(5.5.32)

The same formulae can be expressed in terms of the rhombus semiaxes a and b and the aspect ratio $\epsilon=b/a$, giving

$$\begin{aligned}
 h_x &= \frac{2\sqrt{2}\epsilon (1 + \epsilon^2)}{9 \left[1 - \epsilon + \frac{\epsilon^2}{(1 + \epsilon^2)^{1/2}} \ln \frac{1 + \epsilon + (1 + \epsilon^2)^{1/2}}{1 + \epsilon - (1 + \epsilon^2)^{1/2}} \right]}, \\
 h_y &= \frac{2\sqrt{2} (1 + \epsilon^2)}{9\epsilon^{3/2} \left[\epsilon - 1 + \frac{1}{(1 + \epsilon^2)^{1/2}} \ln \frac{1 + \epsilon + (1 + \epsilon^2)^{1/2}}{1 + \epsilon - (1 + \epsilon^2)^{1/2}} \right]}. \tag{5.5.33}
 \end{aligned}$$

We did not find in mechanics literature any result related to a punch with a rhombus planform. In the electrical sciences, the mathematically equivalent problem of the coefficients of magnetic polarizability of a diamond was solved numerically by de Smedt (1979). Here, we present his results compared with those given by formula (5.5.33)

$\epsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $h_x=$	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (5.5.33) $h_x=$	0.1078	0.1627	0.2258	0.2986	0.4026	0.4230	0.5043
de Smedt $h_y=$	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (5.5.33) $h_y=$	4.5987	2.1982	1.3254	0.9095	0.6388	0.6052	0.5043
Discrepancy of h_x (%)	8.7	5.9	3.6	2.2	1.8	2.1	2.9
Discrepancy of h_y (%)	25.6	18.8	13.0	8.6	4.7	4.3	2.9

The deterioration of the accuracy of (5.5.33) for small values of ϵ can be attributed to the erroneous assumption of a square root singularity in (5.5.3) which is grossly incorrect for domains with sharp angles.

The traction distribution under the punch can be expressed according to (5.5.17) as

$$\sigma = \frac{P a(\phi)}{2A \left[a^2(\phi) - \rho^2 \right]^{1/2}} \left[1 + \frac{9}{2} \left(\frac{xx_0}{a^2} + \frac{yy_0}{b^2} \right) \right].$$

Further analysis of the last expression reveals that the core inside which the force can be applied without causing any separation is a rectangle with semiaxes $2a/9$ and $2b/9$ respectively. In the case of $\epsilon=1$ the rhombus transforms into a square, and all the results are in agreement with those of the Example 3.

Example 5: Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where k is given by (5.4.26). The equation of the segment boundary with respect to its center of gravity takes the form (5.4.27).

Computation of the area and moments (5.5.21–5.5.24) yields

$$A = r^2\left(\alpha - \frac{1}{2}\sin 2\alpha\right), \quad I_x = \frac{1}{4}Ar^2(1 - k\cos\alpha), \quad I_y = \frac{1}{4}Ar^2(1+3k\cos\alpha-4k^2), \quad (5.5.34)$$

$$J_x = \frac{2}{3}r \left\{ -k\sin^3\gamma + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1 - k^2}{k^2}F(\pi-\gamma, k) \right. \\ \left. + \frac{2k^2 - 1}{k^2}E(\pi-\gamma, k) + 3(k - \cos\alpha) \left[-\sin\gamma + \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right] \right\},$$

$$J_y = \frac{2}{3}r \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1 - k^2\sin^2\gamma)^{1/2} \cos\gamma \right] \right. \\ \left. - \frac{1 - k^2}{k^2} F(\pi-\gamma, k) + \frac{1 + k^2}{k^2} E(\pi-\gamma, k) \right\}, \quad (5.5.35)$$

where $\gamma = \tan^{-1}(\sin\alpha/(k - \cos\alpha))$, and the functions $F(\cdot, \cdot)$ and $E(\cdot, \cdot)$ stand for the incomplete elliptic integrals of the first and the second kind respectively. Substituting in (5.5.20) leads to

$$h_x = \frac{4(1 - k\cos\alpha)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ -k\sin^3\gamma + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\pi-\gamma, k) \right. \\ \left. + \frac{2k^2-1}{k^2}E(\pi-\gamma, k) + 3(k - \cos\alpha) \left[-\sin\gamma + \ln \tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right) \right] \right\}^{-1},$$

$$h_y = \frac{4(1 + 3k\cos\alpha - 4k^2)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1 - k^2\sin^2\gamma)^{1/2} \cos\gamma \right] \right.$$

$$\left. - \frac{1 - k^2}{k^2} F(\pi - \gamma, k) + \frac{1 + k^2}{k^2} E(\pi - \gamma, k) \right\}^{-1}. \quad (5.5.36)$$

We can use this result to investigate the case of a circular punch under the action of a normal force P applied at $x_0 > r/3$. From the classical theory we know that there should be a separation between the punch and the half-space. Assuming that the domain of contact after separation is a circular segment, one can get the following relationship between the coordinate x_0 and the size of the segment characterized by the angle α

$$x_0 = r \frac{(1 - k)(1 + 4k)}{3(k - \cos\alpha)}. \quad (5.5.37)$$

The last expression is exact in two limiting cases: the complete circle $\alpha = \pi$ gives $x_0 = r/3$, and $\alpha \rightarrow 0$ results in $x_0 = r$. The problem of an inclined circular punch was considered numerically in the book by Rvachev and Protsenko (1977). Here, we compare the results

α (deg)=	158.4	108.1	102.0
Rvachev <i>et al.</i> $x_0 =$	0.3583	0.5833	0.6250
Formula (5.5.37) $x_0 =$	0.3543	0.5418	0.5750
Discrepancy (%)	1.1	7.1	8.0

The agreement should be considered as surprisingly good, especially taken into consideration that Rvachev and Protsenko considered the domain of contact not in the form of a segment but having a more complicated shape.

Example 6: Cross. Consider a punch configuration obtained by the orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments will take the form

$$A = 4a^2\varepsilon(2 - \varepsilon), \quad I_x = I_y = \frac{4}{3} a^4\varepsilon(1 + \varepsilon^2 - \varepsilon^3),$$

$$J_x = J_y = 4a \left[\ln[\varepsilon + (1 + \varepsilon^2)^{1/2}] + \varepsilon \ln \frac{1 + (1 + \varepsilon^2)^{1/2}}{(1 + \sqrt{2})\varepsilon} \right]. \quad (5.5.38)$$

The coefficients will be determined as

$$h_x = h_y = \frac{4\varepsilon(1 + \varepsilon^2 - \varepsilon^3)}{9\varepsilon(2 - \varepsilon)^{3/2}} \left[\ln[\varepsilon + (1 + \varepsilon^2)^{1/2}] + \varepsilon \ln \frac{1 + (1 + \varepsilon^2)^{1/2}}{(1 + \sqrt{2})\varepsilon} \right]^{-1}.$$

(5.5.39)

The values of $h=h_x=h_y$ were computed from (5.5.39) and compared with those given by de Smedt (1979). The results are presented below

$\varepsilon=$	0.1000	0.2000	0.3333	0.4000	0.5000	0.6000	0.7500	0.8000	1.0000
de Smedt $h=$	1.5910	0.8720	0.6255	0.5725	0.5267	0.5069	0.4985	0.4997	0.5193
Formula (5.5.39) $h=$	1.7382	0.8758	0.6006	0.5465	0.5049	0.4890	0.4893	0.4926	0.5043
Discrepancy (%)	-9.3	-0.4	4.0	4.5	4.1	3.5	1.9	1.4	2.9

Taking into consideration the shape complexity, we should consider the results agreement as surprisingly good, not only quantitatively but qualitatively as well: both sets of data display a relatively flat minimum around $\varepsilon=0.75$.

Discussion. It is noteworthy that the change of the order of integration which led to (5.5.1) is valid inside the circle $\rho \leq \min\{a(\phi)\}$ only. Nevertheless, one can obtain from (5.5.1) the *exact* solution for an ellipse and sufficiently accurate formulae for various punch planforms as it was demonstrated in the preceding Examples.

The accuracy of formulae (5.5.20) can be improved by taking into consideration the fifth harmonic (5.5.9) in combination with the variational approach (Noble, 1960). The following functional assumes its maximum value at the exact solution of (5.1.8)

$$I(\sigma) = \frac{2}{H} \int_S \int \sigma(M) w(M) dS_M - \int_S \int \sigma(M) \left[\int_S \int \frac{\sigma(N)}{R(M,N)} dS_N \right] dS_M. \quad (5.5.40)$$

Taking

$$H \int_S \int \frac{\sigma(N)}{R(M,N)} dS_N \approx w_1 + w_5, \quad (5.5.41)$$

and substituting (5.5.3), (5.5.7), (5.5.9), and (5.5.41) in (5.5.40), one gets, after integration with respect to ρ

$$I = \int_0^{2\pi} (a(\phi))^4 \left\{ (p_1 \cos\phi + p_2 \sin\phi) \left[\frac{4}{3H} (\alpha_x \sin\phi - \alpha_y \cos\phi) \right. \right. \\ \left. \left. - \frac{\pi}{3} (p_1 J_y + p_2 J_{xy}) \cos\phi - \frac{\pi}{3} (p_1 J_{xy} + p_2 J_x) \sin\phi - \frac{2\pi}{63} (a(\phi))^3 (lp_1 (A_{c6} + A_{c4})) \right] \right\} d\phi$$

$$+ p_2(A_{s6} - A_{s4})\cos 5\phi + [p_1(A_{s6} + A_{s4}) + p_2(A_{c4} - A_{c6})\sin 5\phi] \Bigg\} d\phi. \tag{5.5.42}$$

Considering now the functional I as a function of p_1 and p_2 , the extremum conditions

$$\frac{\partial I}{\partial p_1} = 0, \quad \frac{\partial I}{\partial p_2} = 0,$$

give two linear algebraic equations with respect to the unknowns p_1 and p_2 . The complete solution is rather cumbersome. Here, we present the final result for the coefficients h_x and h_y which are valid only for domains having at least one axis of symmetry, and the central principal axes taken as the coordinate axes

$$h_x = \frac{32I_x}{3A^{3/2}J_x(1 + \eta_x)}, \quad h_y = \frac{32I_y}{3A^{3/2}J_y(1 + \eta_y)}, \tag{5.5.43}$$

where the correction terms

$$\eta_x = \frac{(B_{c4} - B_{c6})(A_{c4} - A_{c6})}{84I_xJ_x}, \quad \eta_y = \frac{(B_{c4} + B_{c6})(A_{c4} + A_{c6})}{84I_yJ_y}, \tag{5.5.44}$$

the parameters A_{c4} and A_{c6} are defined by (5.5.10), and

$$B_{c6} = \int_0^{2\pi} (a(\phi))^7 \cos 6\phi \, d\phi, \quad B_{c4} = \int_0^{2\pi} (a(\phi))^7 \cos 4\phi \, d\phi.$$

Since expression (5.5.41) is approximate, there is no guarantee that (5.5.43) will be more accurate than (5.5.20). We performed the necessary computations for a rectangle. Here are the results compared with those of de Smedt (1979)

$\varepsilon =$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $h_x =$	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (5.5.43) $h_x =$	0.1405	0.1988	0.2577	0.3207	0.4165	0.4376	0.5331
de Smedt $h_y =$	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (5.5.43) $h_y =$	4.5856	2.0985	1.2479	0.8714	0.6463	0.6190	0.5331
Discrepancy in h_x (%)	-9.2	-5.7	-1.8	1.3	1.8	1.3	-2.7
Discrepancy in h_y (%)	-11.7	-3.6	1.0	2.0	-0.6	-1.0	-2.7

Comparison with similar data computed on the basis of formula (5.5.29) shows that the correction terms η_x and η_y in this particular case resulted in decreasing the value of discrepancy, positive as well as negative. We caution again that there is no guarantee that this will be valid for an arbitrary domain. For example, here are the data computed for a rhombus

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $h_x=$	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (5.5.43) $h_x=$	0.2268	0.1860	0.2351	0.3031	0.4058	0.4264	0.5091
de Smedt $h_y=$	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (5.5.43) $h_y=$	8.5600	2.5916	1.4196	0.9408	0.6490	0.6138	0.5091
Discrepancy of h_x (%)	-92.0	-7.6	-0.4	0.7	1.0	1.4	2.0
Discrepancy of h_y (%)	-38.5	4.2	6.8	5.4	3.2	2.9	2.0

Comparison with the data computed due to (5.5.33) indicates that the discrepancy decreased for $\varepsilon \geq 0.2$ while for $\varepsilon = 0.1$ it has jumped in the opposite direction to -92% . The main reason for this is a jump in the value of the coefficients η_x and η_y when ε is very small. The following rule of thumb may be suggested for the user wishing to improve the accuracy: when the value of the correction coefficients η_x and η_y does not exceed small percentage of unity, their inclusion generally results in an improvement in accuracy, otherwise one should not use formulae (5.5.43).

It is worthwhile to give the solution due to (5.5.42) for the case when the domain of contact has no axis of symmetry, and only the first harmonic of the displacements w_1 is taken into consideration. The result is

$$p_1 = \frac{\alpha_x(c_{22}I_{xy} - c_{12}I_x) + \alpha_y(c_{12}I_{xy} - c_{22}I_y)}{c_{11}c_{22} - c_{12}^2} \quad (5.5.45)$$

$$p_2 = \frac{\alpha_x(c_{11}I_x - c_{12}I_{xy}) + \alpha_y(c_{12}I_y - c_{11}I_{xy})}{c_{11}c_{22} - c_{12}^2}$$

where

$$c_{11} = \frac{\pi H}{2} (J_y I_y + J_{xy} I_{xy}), \quad c_{22} = \frac{\pi H}{2} (J_x I_x + J_{xy} I_{xy}),$$

$$c_{12} = \frac{\pi H}{4} (J_{xy}(I_x + I_y) + I_{xy}(J_x + J_y))$$

Formulae (5.5.45) are different from the equivalent set (5.5.12) derived earlier. In the absence of any numerical data for a general domain, it is impossible to

say whether formulae (5.5.45) are more accurate than (5.5.12), but they are definitely more complicated. It is noteworthy that in the case of a domain with an axis of symmetry both (5.5.45) and (5.5.12) simplify to the same equations (5.5.15).

One can notice a certain similarity between the formulae derived and those related to the Saint-Venant theory of bending. This similarity will become more evident if, for example, we rewrite equation (5.5.17) in the form

$$\sigma = \frac{a(\phi)}{2 \left[a^2(\phi) - \rho^2 \right]^{1/2}} \left[\frac{P}{A} + \frac{3}{4} \left(\frac{M_{xy}}{I_x} - \frac{M_{yx}}{I_y} \right) \right].$$

We think that this similarity is not a pure coincidence since the method used in this section can also be called semi-inverse. The method can be developed further into a complete Saint-Venant type theory of elastic contact problems which could combine the simplicity and accuracy sufficient for a practical engineer.

The mathematically identical problem of the magnetic polarizability of small apertures was solved in (Fabrikant, 1987j).

Exercise 5.5

1. Find the moments of inertia and the J -moments for a circular sector characterized by the radius r and the polar angle 2α .

Answer:

$$I_x = \frac{1}{4}r^4(\alpha - \frac{1}{2}\sin 2\alpha), \quad I_y = r^4 \frac{9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha}{36\alpha},$$

$$J_x = \frac{2}{3}r \left\{ \frac{2k^2-1}{k^2}E(\gamma,k) - k\sin^3\gamma - (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\gamma,k) \right. \\ \left. + 3k\sin\alpha \left[\cos\alpha + \cos(\alpha + \gamma) + \sin^2\alpha \ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right) \right] \right\},$$

$$J_y = \frac{2}{3}r \left\{ k\sin\gamma(\sin^2\gamma - 3) + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma - \frac{1-k^2}{k^2}F(\gamma,k) \right\}$$

$$+ \frac{1+k^2}{k^2}E(\gamma,k) + 3k\sin\alpha \left[\cos^2\alpha \ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right) - \cos\alpha - \cos(\alpha+\gamma) \right] \Bigg\}.$$

Here $k = 2\sin\alpha/(3\alpha)$, and $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$.

2. Find the shape coefficients h_x and h_y for the circular sector described in the Exercise 1.

Answer:

$$h_x = \frac{2(2\alpha - \sin 2\alpha)}{\alpha^{3/2}} \left\{ \frac{1 - k^2}{k^2}F(\gamma,k) - k\sin^3\gamma - (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma \right. \\ \left. + \frac{2k^2-1}{k^2}E(\gamma,k) + 3k\sin\alpha \left[\cos\alpha + \cos(\alpha+\gamma) + \sin^2\alpha \ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right) \right] \right\}^{-1},$$

$$h_y = \frac{4(9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha)}{9\alpha^{5/2}} \left\{ k\sin\gamma(\sin^2\gamma - 3) \right. \\ \left. + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma - \frac{1 - k^2}{k^2}F(\gamma,k) + \frac{1 + k^2}{k^2}E(\gamma,k) \right. \\ \left. + 3k\sin\alpha \left[-\cos\alpha - \cos(\alpha + \gamma) + \cos^2\alpha \ln\left(\cot\frac{\alpha}{2} \cot\frac{\gamma - \alpha}{2}\right) \right] \right\}^{-1}.$$

3. Verify (5.5.4).

4. Derive (5.5.7).

5. Establish (5.5.14)–(5.5.16)).

5.6 Curved punch of general planform

In this section, we analyze the elastic contact problem for a curved punch of non-elliptic planform under the action of a normal force. The punch base is assumed to be a quadratic surface. Some general relationships are established between the applied force and the punch settlement. Specific formulae are derived for a punch whose planform has the shape of a polygon, a rectangle, a rhombus and a cross. An example of a finite rigid cylinder lying on its generator and pressed against an elastic half-space is considered in detail. The method allows us to have singular tractions at the cylinder edges and zero tractions at the rest of the contact domain boundary. The last condition serves to determine the width of the domain of contact. All the formulae are checked against the previously published solutions, and a good accuracy is confirmed for a sufficiently wide range of values of aspect ratio.

Theory. Consider a punch with a curved base and arbitrary planform. The punch is pressed against an elastic half-space by a normal force P . Let the boundary of the domain of contact S be given in the polar coordinates as

$$\rho = a(\phi),$$

where the function $a(\phi)$ is bounded and single-valued. Here we assume that the domain of contact is prescribed. The case when the domain of contact is unknown (or partially unknown) will be discussed further. The punch base is assumed to be a quadratic surface. This limitation is not essential. The method can be applied to higher order surfaces as well. The transformed governing integral equation is given by (5.5.1).

Let the normal displacements under the punch be

$$w = g_0 + g_x y^2 + g_{xy} xy + g_y x^2, \quad (5.6.1)$$

where g_0 is the punch penetration, and g_x , g_y and g_{xy} are the known constants defined by the punch base geometry. Let the pressure distribution under the punch be

$$\sigma(\rho, \phi) = \frac{a(\phi)[\alpha_0 + \rho^2(\alpha_x \sin^2 \phi + \alpha_{xy} \sin \phi \cos \phi + \alpha_y \cos^2 \phi)]}{\left[a^2(\phi) - \rho^2 \right]^{1/2}}, \quad (5.6.2)$$

where α_0 , α_x , α_y and α_{xy} are the as yet unknown constants. We make use of the condition that the integral of σ over S should be equal to P . This leads to the expression

$$P = 2\alpha_0 A + \frac{8}{3} (\alpha_x I_x + \alpha_{xy} I_{xy} + \alpha_y I_y), \quad (5.6.3)$$

where A is the area of the domain of contact S ; and I_x , I_y and I_{xy} are the axial moments of inertia and the product of inertia respectively. The location of coordinate origin can be defined from the condition that the tilting moments produced by the tractions (5.6.2) should vanish. This condition yields the following set of equations:

$$\int_0^{2\pi} a^3(\phi) [\alpha_0 + \frac{3}{4} a^2(\phi) (\alpha_x \sin^2 \phi + \alpha_{xy} \sin \phi \cos \phi + \alpha_y \cos^2 \phi)] \sin \phi d\phi = 0,$$

$$\int_0^{2\pi} a^3(\phi) [\alpha_0 + \frac{3}{4} a^2(\phi) (\alpha_x \sin^2 \phi + \alpha_{xy} \sin \phi \cos \phi + \alpha_y \cos^2 \phi)] \cos \phi d\phi = 0. \quad (5.6.4)$$

The point whose coordinates satisfy (5.6.4) will be called *the punch centre*. When the domain of contact has one axis of symmetry the punch centre is located on this axis. In the case of two axes of symmetry, the punch centre coincides with the centre of gravity. Now it is necessary to relate α_0 , α_x , α_y , and α_{xy} to the parameters g_0 , g_x , g_y , and g_{xy} . This can be done by substitution of (5.6.2) into (5.5.1) which yields, after integration with respect to ρ ,

$$w(\rho, \phi) = H\alpha_0 \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(\frac{2-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) d\phi_0$$

$$+ H \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^3 dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(\frac{4-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) (\alpha_x \sin^2 \phi_0 + \alpha_{xy} \sin \phi_0 \cos \phi_0 + \alpha_y \cos^2 \phi_0) d\phi_0.$$

Here F denotes the Gauss hypergeometric function. Further evaluation of the normal displacements can be carried out separately for each harmonic. Note that all the odd harmonics of w will be zero if $a(\phi)$ contains only even harmonics. The zeroth harmonic will take the form

$$w_0(\rho, \phi) = \frac{\pi H}{4} \int_0^{2\pi} \left[2\alpha_0 + (a^2(\phi_0) + \frac{1}{2} \rho^2) \times (\alpha_x \sin^2 \phi_0 + \alpha_{xy} \sin \phi_0 \cos \phi_0 + \alpha_y \cos^2 \phi_0) \right] a(\phi_0) d\phi_0, \quad (5.6.5)$$

which can be simplified as

$$w_0(\rho, \phi) = \frac{\pi H}{4} \left[2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y + \frac{\rho^2}{2} (\alpha_x J_x + \alpha_{xy} J_{xy} + \alpha_y J_y) \right], \quad (5.6.6)$$

where the J -moments were defined in the previous section, and the following additional quantities were introduced:

$$B_x = \int_0^{2\pi} a^3(\phi) \sin^2 \phi \, d\phi, \quad B_y = \int_0^{2\pi} a^3(\phi) \cos^2 \phi \, d\phi, \\ B_{xy} = \int_0^{2\pi} a^3(\phi) \sin \phi \cos \phi \, d\phi. \quad (5.6.7)$$

Since the tensor properties of the B -moments are similar to those of the moments of inertia, we shall call B_x and B_y *the cubic moments of a two-dimensional domain about the axes Ox and Oy* respectively, B_{xy} will be called *the cubic product of a two-dimensional domain about the axes Ox and Oy* .

Here is the expression for the second harmonic

$$w_2(\rho, \phi) = \frac{3}{8} \pi H \rho^2 \int_0^{2\pi} (\alpha_x \sin^2 \phi_0 + \alpha_{xy} \sin \phi_0 \cos \phi_0 + \alpha_y \cos^2 \phi_0) a(\phi_0) \cos 2(\phi - \phi_0) d\phi_0,$$

which can be modified as

$$w_2(\rho, \phi) = \frac{3}{8} \pi H \rho^2 \{ -\alpha_x [(C_{xxxx} - C_{xxyy}) \cos 2\phi + 2C_{xxyy} \sin 2\phi] + \alpha_y [(C_{yyyy} - C_{xyxy}) \cos 2\phi + 2C_{xyxy} \sin 2\phi] + \alpha_{xy} [(C_{xyyy} - C_{xxyy}) \cos 2\phi + 2C_{xxyy} \sin 2\phi] \}. \quad (5.6.8)$$

Here, the following geometrical characteristics of the domain of contact have been

introduced:

$$\begin{aligned}
 C_{xxxx} &= \int_0^{2\pi} a(\phi) \sin^4 \phi d\phi, & C_{xxyy} &= \int_0^{2\pi} a(\phi) \sin^3 \phi \cos \phi d\phi, \\
 C_{xxyy} &= \int_0^{2\pi} a(\phi) \sin^2 \phi \cos^2 \phi d\phi, & C_{xyyy} &= \int_0^{2\pi} a(\phi) \sin \phi \cos^3 \phi d\phi, \\
 C_{yyyy} &= \int_0^{2\pi} a(\phi) \cos^4 \phi d\phi. & & (5.6.9)
 \end{aligned}$$

The C -moments will be called *the linear moments of the fourth order*. Their relationships with the J -moments are easy to establish, for example, $J_x = C_{xxxx} + C_{xxyy}$, $J_{xy} = C_{xyyy} + C_{xxyy}$, etc. It is important to note that the parameter α_0 did not appear in (5.6.8), and the parameters α_x , α_{xy} and α_y will not appear in the expression for the fourth harmonic. The investigation of further harmonics shows that their amplitude decreases. In the case of an ellipse they vanish thus making the solution *exact*. It seems natural to assume $w \approx w_0 + w_2$, and the remaining harmonics may be called the solution error. We should also take into consideration that the error sign fluctuation will result in even smaller error in the integral characteristics sought, like the total force P . Now we have an approximate expression for the displacements under the punch

$$\begin{aligned}
 w &= \frac{\pi}{4} H \left\{ 2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y + x^2 \left[-\alpha_x (C_{xxxx} - 2C_{xxyy}) \right. \right. \\
 &+ \left. \alpha_{xy} (2C_{xyyy} - C_{xxyy}) + \alpha_y (2C_{yyyy} - C_{xxyy}) \right] + y^2 \left[\alpha_x (2C_{xxxx} - C_{xxyy}) \right. \\
 &+ \left. \alpha_{xy} (2C_{xxyy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy}) \right] + 6xy \left[\alpha_x C_{xxyy} + \alpha_{xy} C_{xyyy} + \alpha_y C_{xyyy} \right] \left. \right\}. \quad (5.6.10)
 \end{aligned}$$

Comparison of (5.6.1) and (5.6.10) leads to the following set of equations

$$\frac{\pi}{4} H (2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y) = g_0,$$

$$\begin{aligned}
\frac{\pi}{4} H \left[\alpha_x (2C_{xxxx} - C_{xxyy}) + \alpha_{xy} (2C_{xxyy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy}) \right] &= g_x \\
\frac{\pi}{4} H \left[-\alpha_x (C_{xxxx} - 2C_{xxyy}) + \alpha_{xy} (2C_{xxyy} - C_{xxyy}) + \alpha_y (2C_{yyyy} - C_{xxyy}) \right] &= g_y, \\
\frac{3\pi}{2} H \left[\alpha_x C_{xxyy} + \alpha_{xy} C_{xxyy} + \alpha_y C_{xyyy} \right] &= g_{xy}.
\end{aligned} \tag{5.6.11}$$

The last three equations of (5.6.11) can be solved with respect to α_x , α_y , and α_{xy} . In the case of elastic contact problem when g_0 is prescribed, the value of α_0 can be found from the first equation (5.6.11), after which the total force P is defined by (5.6.3), and the solution is completed. When the total force P is given, the value of α_0 can be determined from (5.6.3) after which the punch penetration g_0 is given by the first equation (5.6.11).

A significant simplification occurs when the domain of contact S has at least one axis of symmetry. In this case $C_{xxyy} = C_{xyyy} = B_{xy} = 0$. The last equation (5.6.11) becomes decoupled from the previous three. The solutions can be then written explicitly

$$\begin{aligned}
\alpha_x &= \frac{4[g_x(2C_{yyyy} - C_{xxyy}) + g_y(C_{yyyy} - 2C_{xxyy})]}{3\pi H(C_{xxxx} C_{yyyy} - C_{xxyy}^2)}, \\
\alpha_y &= \frac{4[g_x(C_{xxxx} - 2C_{xxyy}) + g_y(2C_{xxxx} - C_{xxyy})]}{3\pi H(C_{xxxx} C_{yyyy} - C_{xxyy}^2)}, \\
\alpha_{xy} &= \frac{2g_{xy}}{3\pi H C_{xxyy}}.
\end{aligned} \tag{5.6.12}$$

Substitution of (5.6.12) into the first equation (5.6.11) gives

$$\alpha_0 = \frac{2}{\pi H J_0} \left[g_0 - \frac{g_x \beta_x + g_y \beta_y}{3(C_{xxxx} C_{yyyy} - C_{xxyy}^2)} \right], \tag{5.6.13}$$

where

$$\beta_x = B_x(2C_{yyyy} - C_{xxyy}) + B_y(C_{xxxx} - 2C_{xxyy}),$$

$$\beta_y = B_x(C_{yyyy} - 2C_{xxyy}) + B_y(2C_{xxxx} - C_{xxyy}).$$

Expressions (5.6.2), (5.6.3), (5.6.12), and (5.6.13) give a complete and *exact* solution for an ellipse. We shall prove them to perform well for an arbitrary punch planform. We expect (5.6.2) to be reasonably accurate in the neighbourhood of the coordinate origin, while the error might become quite significant close to the boundary of the domain S .

It is convenient to discuss the following particular cases: $g_0=2\pi\sqrt{A}$, $g_x=g_y=g_{xy}=0$; $g_y=2\pi/\sqrt{A}$, $g_x=g_{xy}=g_0=0$; and the case $g_x=2\pi/\sqrt{A}$, $g_y=g_{xy}=g_0=0$. In each case we compute the integral

$$p = \frac{H}{A} \int_S \sigma dS,$$

which is proportional to the average value of σ , and is dimensionless, thus characterizing the shape of S and being independent of its size. We shall denote these parameters by p_0 , p_y and p_x for each case respectively. There are two reasons for these definitions: *i*) they correspond exactly to the parameters used in the theory of sound penetration through holes, so it would be easy to compare the numerical results; *ii*) tabulation of these parameters for various shapes will simplify significantly the solution of any specific contact problem. Indeed, the relationship between the applied force P and the punch penetration g_0 can be rewritten as

$$P = \frac{A}{2\pi H} \left[\frac{p_0}{\sqrt{A}} g_0 + \sqrt{A}(p_y g_y + p_x g_x) \right].$$

The last expression indicates that the knowledge of the shape coefficients and the punch area is sufficient for establishing the relationship between the applied force and the punch penetration. Formulae (5.6.3), (5.6.12) and (5.6.13) lead to the following expressions for the parameters p_0 , p_y and p_x

$$p_0 = \frac{8\sqrt{A}}{J_0}, \quad p_y = \frac{8\{8J_0[I_x(C_{yyyy} - 2C_{xxyy}) + I_y(2C_{xxxx} - C_{xxyy})] - 3A\beta_y\}}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)},$$

$$p_x = \frac{8\{8J_0[I_x(2C_{yyyy} - C_{xxyy}) + I_y(C_{xxxx} - 2C_{xxyy})] - 3A\beta_x\}}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)}. \quad (5.6.14)$$

Some further simplifications take place when the domain S possesses such a symmetry that all the moments about the axis Ox are equal to the similar moments about the axis Oy . In this case

$$p_y = p_x = \frac{8(8I_0J_0 - 3AB_0)}{3A^{3/2}J_0^2}, \quad (5.6.15)$$

where the moments with the subindex 0 indicate corresponding polar moments. Formula (5.6.13) also simplifies as follows:

$$\alpha_0 = \frac{2}{\pi H J_0} \left[g_0 - \frac{B_0}{J_0} (g_x + g_y) \right]. \quad (5.6.16)$$

Formulae (5.6.2), (5.6.3), (5.6.12) and (5.6.13) are the main results of this section. The elastic contact problem for a wide variety of planforms can now be solved by a relatively simple computation of the geometrical characteristics (moments) of the domain of contact.

Several examples are considered further. We present only the necessary computations of the moments involved. The domain of contact is assumed to be prescribed. The more complicated case when the domain of contact is partially unknown is discussed in the context of the problem of a rigid roller on an elastic half-space.

Example 1: Polygon. Consider a polygon with n sides. The general notation of its parameters is the same as in the previous section, where the moments of inertia and the J -moments are computed. Here the remaining moments are presented.

The C -moments can be computed by the following formulae

$$\begin{aligned} C_{xxx} &= \sum_{k=1}^n -q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k + 4s_k \sin \psi_k \cos^3 \psi_k + 2t_k \cos^4 \psi_k, \\ C_{xxy} &= \sum_{k=1}^n v_k \cos 4\psi_k + u_k \sin 4\psi_k + s_k \cos^2 \psi_k (1 - 4\sin^2 \psi_k) \\ &\quad + \frac{1}{2} q_k \sin 2\psi_k - 2t_k \sin \psi_k \cos^3 \psi_k, \end{aligned} \quad (5.6.17)$$

$$C_{xxyy} = \sum_{k=1}^n u_k \cos 4\psi_k - v_k \sin 4\psi_k - \frac{1}{2} s_k \sin 4\psi_k + 2t_k \sin^2 \psi_k \cos^2 \psi_k, \quad (5.6.18)$$

$$C_{xyyy} = \sum_{k=1}^n -v_k \cos 4\psi_k - u_k \sin 4\psi_k - s_k \sin^2 \psi_k (1 - 4\cos^2 \psi_k) + \frac{1}{2} q_k \sin 2\psi_k - 2t_k \sin^3 \psi_k \cos \psi_k, \quad (5.6.19)$$

$$C_{yyyy} = \sum_{k=1}^n q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k - 4s_k \sin^3 \psi_k \cos \psi_k + 2t_k \sin^4 \psi_k, \quad (5.6.20)$$

where

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}, \quad s_k = 4 \frac{A_k^2}{a_k^2} \left(\frac{1}{b_k} - \frac{1}{b_{k+1}} \right),$$

$$q_k = \frac{A_k}{a_k^2} \left(\frac{1}{b_k} + \frac{1}{b_{k+1}} \right) [a_k^2 + (b_k - b_{k+1})^2]. \quad (5.6.21)$$

$$u_k = \frac{A_k}{12a_k^4} \left\{ \left[\frac{b_{k+1}^2 - b_k^2 + a_k^2}{b_{k+1}} \right]^3 + \left[\frac{b_k^2 - b_{k+1}^2 + a_k^2}{b_k} \right]^3 \right\},$$

$$v_k = \frac{16A_k^4}{3a_k^4} \left[\frac{1}{b_{k+1}^3} - \frac{1}{b_k^3} \right]. \quad (5.6.22)$$

The following formulae can be derived for the cubic moments

$$B_x = \sum_{k=1}^n -j_k \cos 2\psi_k + r_k \sin 2\psi_k + 2f_k \cos^2 \psi_k,$$

$$B_y = \sum_{k=1}^n j_k \cos 2\psi_k - r_k \sin 2\psi_k + 2f_k \sin^2 \psi_k,$$

$$B_{xy} = \sum_{k=1}^n (j_k - f_k) \sin 2\psi_k + r_k \cos 2\psi_k, \quad (5.6.23)$$

where

$$j_k = \left(\frac{2A_k}{a_k}\right)^3 \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}, \quad r_k = \left(\frac{2A_k}{a_k}\right)^2 (b_{k+1} - b_k),$$

$$f_k = \frac{1}{4}(b_k + b_{k+1})A_k \left[1 + \left(\frac{b_{k+1} - b_k}{a_k}\right)^2\right] + \frac{1}{4} j_k. \quad (5.6.24)$$

Substitution of (5.6.17–5.6.24) into (5.6.2, 5.6.3, 5.6.12 and 5.6.13) gives the complete solution for an arbitrary polygon. In the case of a regular polygon significant simplifications occur. The moments of inertia and the J -moments take the form (5.5.25) and (5.5.26), and the remaining moments are determined by

$$B_x = B_y = \frac{nb^3}{4} \left[\sin \frac{2\pi}{n} + \cos^3 \left(\frac{\pi}{n}\right) \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)} \right], \quad (5.6.25)$$

$$C_{xxxx} = C_{yyyy} = \frac{3nb}{8} \cos \frac{\pi}{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}$$

$$C_{xyxy} = \frac{nb}{8} \cos \frac{\pi}{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)} \quad (5.6.26)$$

Note that formulae (5.6.26) are valid for any regular polygon except the square, due to the fact that the trigonometric series summation

$$\sum_{k=1}^n \sin^4(k-1) \frac{2\pi}{n} = \sum_{k=1}^n \cos^4(k-1) \frac{2\pi}{n} = \frac{3n}{8}, \quad (5.6.27)$$

is not valid for a square. The C -moments for a square of side $2l$ can be expressed as

$$C_{xxxx} = C_{yyyy} = l \left[4 \ln(1 + \sqrt{2}) - \frac{2\sqrt{2}}{3} \right], \quad C_{xyxy} = \frac{2\sqrt{2}}{3} l. \quad (5.6.28)$$

Formulae (5.6.3, 5.6.12, and 5.6.13) simplify for a regular polygon

$$\alpha_x = \frac{4(5g_x + g_y)}{3\pi H J_0}, \quad \alpha_y = \frac{4(g_x + 5g_y)}{3\pi H J_0}, \quad \alpha_{xy} = \frac{16g_{xy}}{3\pi H J_0}. \quad (5.6.29)$$

Again, one should note that formulae (5.6.29) are not valid for a square. The formulae to follow are valid for any regular polygon including the square.

$$\alpha_0 = \frac{2}{\pi H J_0} \left[g_0 - \frac{B_0}{J_0} (g_x + g_y) \right],$$

$$P = \frac{4}{\pi H J_0} \left[A g_0 + \left(\frac{8}{3} I_0 - \frac{A B_0}{J_0} \right) (g_x + g_y) \right]. \quad (5.6.30)$$

The dimensionless coefficients p_y and p_x take the form

$$p_y = p_x = \frac{8\sqrt{2}}{(n \sin \frac{2\pi}{n})^{1/2} n \cos \frac{\pi}{n} \ln \frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}} \left[\frac{23+7\cos \frac{2\pi}{n}}{36} - \frac{\sin \frac{\pi}{n}}{\ln \frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}} \right]. \quad (5.6.31)$$

Consider several particular values of n . For an equilateral triangle ($n=3$) formula (5.6.31) gives $p_y=p_x=0.3782$. We are not aware of any numerical data to compare with this result. In the case of a square $n=4$, and $p_y=p_x=0.2697$. The result due to De Smedt is 0.2645, with the discrepancy less than 2%. Since formula (5.6.31) in the limiting case $n \rightarrow \infty$ gives the exact result for a circle $p_y=p_x=4/(3\pi^{3/2})=0.2394$, we should expect that the error of (5.6.31) will decrease with n . The value of the coefficients for a regular hexagon is 0.2443, and again, we did not find in the literature anything to compare with this result.

Example 2: Rectangle. Consider a punch with a rectangular base, a_1 and a_2 being its semiaxes along the axis Ox and Oy respectively. Introduce the aspect ratio $\varepsilon=a_2/a_1$. Formulae (5.6.17–5.6.24) in this case reduce to

$$I_x = \frac{4}{3} a_1 a_2^3, \quad I_y = \frac{4}{3} a_1^3 a_2,$$

$$J_x = 4a_1 \sinh^{-1} \varepsilon, \quad J_y = 4a_2 \sinh^{-1}(1/\varepsilon). \quad (5.6.32)$$

$$C_{xxx} = 4a_1 (\sinh^{-1} \varepsilon - \frac{\varepsilon}{3(1 + \varepsilon^2)^{1/2}}), \quad C_{xyy} = 4a_1 \frac{\varepsilon}{3(1 + \varepsilon^2)^{1/2}},$$

$$C_{yyy} = 4a_2 (\sinh^{-1} \frac{1}{\varepsilon} - \frac{1}{3(1 + \varepsilon^2)^{1/2}}). \quad (5.6.33)$$

$$\begin{aligned}
 B_x &= 2a_1^3 \left[\varepsilon(1 + \varepsilon^2)^{1/2} - \sinh^{-1} \varepsilon + 2\varepsilon^3 \sinh^{-1} \left(\frac{1}{\varepsilon} \right) \right], \\
 B_y &= 2a_1^3 \left[\varepsilon(1 + \varepsilon^2)^{1/2} - \varepsilon^3 \sinh^{-1} \left(\frac{1}{\varepsilon} \right) + 2 \sinh^{-1} \varepsilon \right].
 \end{aligned} \tag{5.6.34}$$

The coefficients p_y and p_x were computed by De Smedt (1979) for a rectangle with various aspect ratios ε . Here, we present his results along with those given by our method

$\varepsilon=$	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt $p_y=$	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
our method $p_y=$	3.2809	1.3959	0.7782	0.5100	0.3485	0.2697
Discrepancy in p_y (%)	-9.4	-1.7	2.0	2.5	0.2	-2.0
De Smedt $p_x=$	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
our method $p_x=$	0.0284	0.0577	0.0963	0.1431	0.2086	0.2697
Discrepancy in p_x (%)	24.6	9.7	1.9	-2.3	-3.2	-2.0

Our formulae seem to perform satisfactorily over a sufficiently wide range of aspect ratio. The traction distribution due to (5.6.3) can be compared with the numerical data received in a personal communication from De Smedt. Computations were made for $\varepsilon=0.5$, $g_y=2\pi/\sqrt{A}$, $H=1$, $g_0=g_x=0$. Here are the results along the axis Ox , compared with those communicated by De Smedt

$x/a_1=$	0.0000	0.2500	0.3333	0.5000	0.5833	0.6667	0.7500	0.8333	0.9167
De Smedt $\sigma=$	-0.4715	-0.3933	-0.3249	-0.1238	0.0452	0.2515	0.5456	0.9462	2.0580
our method $\sigma=$	-0.4731	-0.3953	-0.3314	-0.1290	0.0232	0.2273	0.5141	0.9602	1.8556
Discrepancy (%)	-0.3	-0.5	-2.0	-4.2	48.8	9.6	5.8	-1.5	9.8

The relevant results along the axis Oy are given below

$y/a_2=$	0.0000	0.1667	0.3333	.5000	.6667	.8333
De Smedt $\sigma=$	-0.4715	-0.4765	-0.4774	-0.4837	-0.5063	-0.5311
our method $\sigma=$	-0.4731	-0.4744	-0.4791	-0.4907	-0.5198	-0.6138
Discrepancy (%)	-0.3	0.5	-0.3	-1.4	-2.7	-15.6

As we predicted, the discrepancy becomes quite significant close to the boundary.

Example 3: Rhombus. Let a_1 and a_2 be its semiaxes along Ox and Oy respectively. We denote its side $l=(a_1^2+a_2^2)^{1/2}$, and introduce the aspect ratio $\varepsilon=a_2/a_1$. Formulae (5.6.17–5.6.24) in this case yield

$$I_x = \frac{l^4 \varepsilon^3}{3(1 + \varepsilon^2)^2}, \quad I_y = \frac{l^4 \varepsilon}{3(1 + \varepsilon^2)^2}, \quad A = \frac{2l^2 \varepsilon}{(1 + \varepsilon^2)}.$$

$$J_x = \frac{4l\varepsilon}{(1 + \varepsilon^2)} \left[\frac{1 - \varepsilon}{(1 + \varepsilon^2)^{1/2}} + \frac{\varepsilon^2}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right],$$

$$J_y = \frac{4l\varepsilon}{(1 + \varepsilon^2)} \left[-\frac{1 - \varepsilon}{(1 + \varepsilon^2)^{1/2}} + \frac{1}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right]. \quad (5.6.35)$$

$$B_x = \frac{2l^3 \varepsilon^3}{(1 + \varepsilon^2)^3} \left[\frac{\varepsilon^3 + 4\varepsilon - 3}{(1 + \varepsilon^2)^{1/2}} + \frac{2 - \varepsilon^2}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right],$$

$$B_y = \frac{2l^3 \varepsilon}{(1 + \varepsilon^2)^3} \left[\frac{1 + 4\varepsilon^2 - 3\varepsilon^3}{(1 + \varepsilon^2)^{1/2}} + \frac{\varepsilon^2(2\varepsilon^2 - 1)}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right]. \quad (5.6.36)$$

$$C_{xxx} = \frac{4l\varepsilon}{(1 + \varepsilon^2)^2} \left[\frac{2 - \varepsilon + 5\varepsilon^2 - 4\varepsilon^3}{3(1 + \varepsilon^2)^{1/2}} + \frac{\varepsilon^4}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right],$$

$$C_{yyy} = \frac{4l\varepsilon}{(1 + \varepsilon^2)^2} \left[\frac{-4 + 5\varepsilon - \varepsilon^2 + 2\varepsilon^3}{3(1 + \varepsilon^2)^{1/2}} + \frac{1}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right],$$

$$C_{xyy} = \frac{4l\varepsilon}{(1 + \varepsilon^2)^2} \left[\frac{1 - 2\varepsilon - 2\varepsilon^2 + \varepsilon^3}{3(1 + \varepsilon^2)^{1/2}} + \frac{\varepsilon^2}{(1 + \varepsilon^2)} \ln \frac{1 + \varepsilon + (1 + \varepsilon^2)^{1/2}}{1 + \varepsilon - (1 + \varepsilon^2)^{1/2}} \right]. \quad (5.6.37)$$

We did not find in the mechanics literature any result related to a punch with a rhombus planform. The mathematically equivalent problem of sound penetration through an aperture in the shape of a diamond was solved numerically by De Smedt (1979). Here, we present his results compared with those given by our method

$\varepsilon =$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y =$	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
our method $p_y =$	3.7425	1.6605	0.9192	0.5770	0.3661	0.2697
Discrepancy (%)	19.6	12.1	6.6	2.7	-0.2	-2.5
De Smedt $p_x =$	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
our method $p_x =$	0.1944	0.1435	0.1345	0.1532	0.2050	0.2697
Discrepancy (%)	-518.4	-161.3	-56.0	-20.6	-6.6	-2.5

Though our results are satisfactory for p_y , they are unacceptable for p_x when $\varepsilon \leq 0.5$. Despite the large relative errors of p_y and p_x for small ε , there is

reason to believe that the accuracy of solution of various contact problems will be quite satisfactory due to the following: different signs in discrepancy will partially compensate the total error; the term with p_0 generally dominates in real contact problems, and it will generally determine the total error of the solution. We shall see further that in the case of a rigid cylindrical roller our theory works well for the aspect ratios ε very far away from unity. An alternative approach which uses the variational principle and somewhat improves the accuracy, is discussed further on.

Example 4: Cross. Consider a punch with a configuration obtained by the orthogonal intersection of two equal rectangles with sides $2a$ and $2b$, ($a \geq b$). Introduce the aspect ratio as $\varepsilon = b/a$. The area and some of the moments are given in the previous section. The remaining moments are

$$B_x = B_y = 2a^3 \left\{ 2\varepsilon(1+\varepsilon^2)^{1/2} + \ln[\varepsilon + (1+\varepsilon^2)^{1/2}] + \varepsilon^3 \left[\ln \frac{1 + (1+\varepsilon^2)^{1/2}}{\varepsilon(1 + \sqrt{2})} - \sqrt{2} \right] \right\}. \quad (5.6.38)$$

The comparison between our results and those given by De Smedt (1979) are presented below

$\varepsilon =$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y = p_x =$	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
our method $p_y = p_x =$	1.6943	0.6765	0.3716	0.2683	0.2517	0.2697
Discrepancy(%)	-75.1	-39.4	-13.6	-0.5	0.3	-2.0

Taking into consideration the shape complexity, we should consider the results agreement as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$. The discrepancy becomes unacceptably large for $\varepsilon \leq 0.3$. It will be shown later on that the variational approach slightly improves the results.

Example 5: Rigid roller. Consider a rigid right cylinder of length $2l$ and radius r_0 lying on its generator and pressed against an elastic half-space by a normal centrally applied force P . This case corresponds to $g_x = g_{xy} = 0$ and $g_y = -1/(2r_0)$. All the previously derived formulae are valid here. Formulae (5.6.12) yield

$$\alpha_x = - \frac{2(C_{yyyy} - 2C_{xxyy})}{3\pi r_0 H(C_{xxxx} C_{yyyy} - C_{xxyy}^2)},$$

$$\alpha_y = - \frac{2(2C_{xxxx} - C_{xxyy})}{3\pi r_0 H(C_{xxxx} C_{yyyy} - C_{xxyy}^2)}, \quad \alpha_{xy} = 0. \quad (5.6.39)$$

Assuming that the contact region is close to a rectangle, we can use formulae (5.6.32–5.6.34) for all the moments involved. We denote the width of the contact region by $2b$. In the previously considered contact problems the contact region was prescribed. Here, we have the domain of contact partially unknown. Its length $2l$ is prescribed, and the tractions should be singular at $y=\pm l$, while its width $2b$ is as yet unknown and is to be found from the condition that the tractions vanish at $x=\pm b$.

We define α_0 from the condition

$$\alpha_0 = -\alpha_y b^2. \quad (5.6.40)$$

Returning back to (5.6.2), one can see that the condition (5.6.40) will cause σ

Fig. 5.6.1. The geometry pertinent to the rigid roller problem.]

to vanish along a part of an ellipse (see Fig. 5.6.1)

$$\frac{x^2}{b^2} + \frac{a(\phi)\text{pha}/_xy^2}{\alpha_y b^2} = 1, \quad (5.6.41)$$

while the stresses will remain singular at $y=\pm l$. Direct computations show that the ellipse semiaxis along Oy is at least several times greater than the other semiaxis, thus making the part of the ellipse (5.6.41) very close to a straight line and validating our assumption of a rectangular domain of contact. The ellipse semiaxis ratio is smallest for a square where it is equal to 2.4612. In this sense the case $l=b$ is the least accurate and will be considered in more detail further.

Formulae (5.6.3), (5.6.13) and (5.6.39) now give the following expressions for the total force P and the punch penetration g_0

$$P = \frac{8}{3}(\alpha_x I_x + \alpha_y I_y) - 2\alpha_y b^2 A, \quad (5.6.42)$$

$$g_0 = \frac{1}{4} \pi H[\alpha_x B_x + \alpha_y (B_y - 2J_0 b^2)], \quad (5.6.43)$$

where α_x and α_y are defined by (5.6.39) and the moments by (5.6.32–5.6.34), and this completes the solution. We have found only one report (Kalker, 1972) where some formulae were presented for the case of a very narrow domain of contact with the aspect ratio $\varepsilon=l/b \gg 1$. These formulae in our notation take the form

$$P = \frac{l^3}{2\varepsilon^2 r_0 H} \left[1 + \frac{1 - \ln 2}{2\ln(4\varepsilon) + 1} \right], \quad (5.6.44)$$

$$g_0 = \frac{l^2}{4r_0 \varepsilon^2} [2\ln(4\varepsilon) + 1]. \quad (5.6.45)$$

We have received in private communication from Kalker some data generated by his numerical method. Although his computer program does not give the value of aspect ratio, we managed to compare the results by the following procedure. Assuming equality of his punch settlement with ours (5.6.43), we can find the value of aspect ratio ε which, being substituted into (5.6.42) allows us to compare the total forces. The results of comparison are given in Table 5.6.1. The agreement over a wide range of aspect ratios should be considered as very good, taking into consideration the approximate nature of our theory, and also the fact that Kalker's program ignores the stress singularity at the roller edges. Such a good agreement allows us to claim that formulae (5.6.42–5.6.43) give a sufficiently accurate analytical solution to the problem of a rigid roller on an elastic half space. Surprisingly enough, our method seems to be the most accurate around $\varepsilon=1$, despite the fact that our assumption of a rectangular domain of contact is the least accurate in this case. We have compared the traction

Table 5.6.1. Comparison of our method with the numerical results of Kalker

aspect ratio ϵ	punch settlement g^*	total force P* (Kalker)	total force P* formula (5.6.42)	discrepancy (%)
23.31	0.2501E-03	0.5379E-04	0.6251E-04	-16.20
10.77	0.1000E-02	0.2590E-03	0.2891E-03	-11.65
6.799	0.2250E-02	0.6618E-03	0.7174E-03	-8.409
4.879	0.4000E-02	0.1298E-02	0.1377E-02	-6.118
3.757	0.6251E-02	0.2200E-02	0.2296E-02	-4.389
2.511	0.1225E-01	0.4934E-02	0.5019E-02	-1.729
1.843	0.2026E-01	0.9110E-02	0.9114E-02	-0.4892E-01
1.282	0.3603E-01	0.1853E-01	0.1835E-01	0.9399
0.9653	0.5633E-01	0.3240E-01	0.3208E-01	1.015
0.6741	0.1003	0.6748E-01	0.6719E-01	0.4326
0.4174	0.2263	0.1925	0.1952	-1.393
0.3018	0.4041	0.4122	0.4216	-2.279
0.2359	0.6351	0.7499	0.7717	-2.908
0.1933	0.9212	1.231	1.272	-3.335
0.1633	1.265	1.882	1.952	-3.703
0.1411	1.670	2.734	2.844	-4.043
0.1239	2.139	3.817	3.982	-4.323
0.1101	2.679	5.173	5.411	-4.612
0.9884E-01	3.297	6.847	7.185	-4.942
0.8939E-01	4.000	8.890	9.362	-5.306
0.6806E-01	6.771	18.40	19.30	-4.901

distribution due to (5.6.2) with the numerical results of Kalker. The typical picture is presented below for $r_0/l=20$, $\epsilon=0.9653$, $H=1$, $y/l=0.1$.

$x/l=$	0.1200	0.3600	0.6000	0.8400
Kalker $\sigma=$	0.7967E-02	0.7420E-02	0.6226E-02	0.3352E-02
our method $\sigma=$	0.7960E-02	0.7513E-02	0.6528E-02	0.4675E-02
discrepancy(%)	0.09	-1.25	-4.84	-39.49

As predicted, the relative error becomes quite significant close to the boundary of the contact region. A similar behavior is observed along a line parallel to the axis Ox :

$y/l=$	0.1000	0.3000	0.5000	0.7000	0.9000
Kalker $\sigma=$	0.7967E-02	0.8146E-02	0.8623E-02	0.9498E-02	0.1819E-01
our method $\sigma=$	0.7960E-02	0.8177E-02	0.8762E-02	0.1018E-01	0.1570E-01
discrepancy (%)	0.09	-0.38	-1.61	-7.17	13.67

It is also of interest to compare the numerical results by Kalker with his approximate formulae (5.6.44–5.6.45). This comparison is given in Table 5.6.2. We expected Kalker's formulae to be the most accurate for large ϵ with the

Table 5.6.2. Comparison of formulae due to Kalker with his numerical results

aspect ratio ϵ	punch settlement g^*	total force P* Kalker	total force P* formula (5.6.44)	discrepancy (%)
22.34	0.2501E-03	0.5379E-04	0.5163E-04	4.018
10.27	0.1000E-02	0.2590E-03	0.2459E-03	5.054
6.456	0.2250E-02	0.6618E-03	0.6243E-03	5.658
4.621	0.4000E-02	0.1298E-02	0.1223E-02	5.743
3.551	0.6251E-02	0.2200E-02	0.2079E-02	5.511
2.368	0.1225E-01	0.4934E-02	0.4706E-02	4.621
1.734	0.2026E-01	0.9110E-02	0.8838E-02	2.987
1.197	0.3603E-01	0.1853E-01	0.1873E-01	-1.103
0.8848	0.5633E-01	0.3240E-01	0.3471E-01	-7.126
0.5772	0.1003	0.6748E-01	0.8364E-01	-23.95

accuracy decreasing with ϵ . This is not the case: the error is almost constant in the interval $2 < \epsilon < 22$, it decreases to zero around $\epsilon=1$, after which the error changes sign and increases rapidly. This means either that the formulae of Kalker (5.6.44–5.6.45) are not exact asymptotically, or that his numerical procedure has an error of about 5%.

Discussion. An alternative method can be suggested by using the variational approach (Noble 1960). We can use again the functional (5.5.40), which assumes its stationary value at the exact solution of (5.1.8). We take

$$H \int_S \int \frac{\sigma(N)}{R(M,N)} dS_N \approx w_0 + w_2, \quad (5.6.46)$$

where σ is defined by (5.6.2) and w_0+w_2 is given by (5.6.10). Substitution of (5.6.1), (5.6.2), (5.6.10) and (5.6.46) into (5.5.40) makes it possible to consider the functional I as a function of α_0 , α_x , α_y and α_{xy} . The extremum conditions

$$\frac{\partial I}{\partial \alpha_0} = 0, \quad \frac{\partial I}{\partial \alpha_x} = 0, \quad \frac{\partial I}{\partial \alpha_y} = 0, \quad \frac{\partial I}{\partial \alpha_{xy}} = 0,$$

give four linear algebraic equations with respect to the unknowns α_0 , α_x , α_y , and α_{xy} . The complete solution is rather cumbersome. Here, we present the set of equations for the coefficients α_0 , α_x , and α_y which are valid only for domains having at least one axis of symmetry.

$$\begin{aligned}
c_{11}\alpha_0 + c_{12}\alpha_x + c_{13}\alpha_y &= 4[A g_0 + \frac{4}{3}(I_x g_x + I_y g_y)], \\
c_{12}\alpha_0 + c_{22}\alpha_x + c_{23}\alpha_y &= \frac{16}{3}[I_x g_0 + \frac{1}{5}(D_{xxxx} g_x + D_{xxyy} g_y)], \\
c_{13}\alpha_0 + c_{23}\alpha_x + c_{33}\alpha_y &= \frac{16}{3}[I_y g_0 + \frac{1}{5}(D_{xxyy} g_x + D_{yyyy} g_y)]. \tag{5.6.47}
\end{aligned}$$

Here,

$$c_{11} = 2\pi H J_0 A,$$

$$c_{12} = \frac{1}{2} \pi H [B_x A + \frac{4}{3} I_x (2J_0 + 2C_{xxxx} - C_{xxyy}) - \frac{4}{3} I_y (C_{xxxx} - 2C_{xxyy})],$$

$$c_{13} = \frac{1}{2} \pi H [B_y A + \frac{4}{3} I_y (2J_0 + 2C_{yyyy} - C_{xxyy}) - \frac{4}{3} I_x (C_{yyyy} - 2C_{xxyy})],$$

$$c_{22} = \frac{4}{15} \pi H [5B_x I_x + D_{xxxx} (2C_{xxxx} - C_{xxyy}) - D_{xxyy} (C_{xxxx} - 2C_{xxyy})],$$

$$\begin{aligned}
c_{23} &= \frac{2}{15} \pi H [5(B_x I_y + B_y I_x) - D_{xxxx} (C_{yyyy} - 2C_{xxyy}) \\
&\quad + 2D_{xxyy} (C_{xxxx} + C_{yyyy} - C_{xxyy}) - D_{yyyy} (C_{xxxx} - 2C_{xxyy})],
\end{aligned}$$

$$c_{33} = \frac{4}{15} \pi H [5B_y I_y + D_{yyyy} (2C_{yyyy} - C_{xxyy}) - D_{xxyy} (C_{yyyy} - 2C_{xxyy})]. \tag{5.6.48}$$

The D -moments are introduced similar to (5.6.9) as

$$\begin{aligned}
D_{xxxx} &= \int_0^{2\pi} a^6(\phi) \sin^4 \phi d\phi, & D_{xxyy} &= \int_0^{2\pi} a^6(\phi) \sin^2 \phi \cos^2 \phi d\phi, \\
D_{yyyy} &= \int_0^{2\pi} a^6(\phi) \cos^4 \phi d\phi. \tag{5.6.49}
\end{aligned}$$

It is clear that the variational approach solution is more cumbersome than the one introduced earlier. It remains to be seen whether it will be more accurate. One advantage should be noted: the matrix of (5.6.48) is symmetric (as it is required by the reciprocal theorem) while the matrix of (5.6.11) is generally not

symmetric.

Let us compare the results for several particular configurations. First of all, consider a regular polygon. In the tables hereafter the word *simple* refers to the method introduced earlier, while the word *variational* refers to the solution of the set of equations (5.6.47). In the case of a regular polygon we shall need the polar D -moment only

$$D_0 = \frac{1}{5} b^6 n \sin \frac{2\pi}{n} \left[1 + \frac{4}{3} \cos^2 \left(\frac{\pi}{n} \right) + \frac{8}{3} \cos^4 \left(\frac{\pi}{n} \right) \right].$$

Here are the results of computations for a regular polygon with n sides

$n=$	3	4	5	6	7	9	∞
simple $p_y=p_x=$	0.3782	0.2697	0.2502	0.2443	0.2420	0.2403	0.2394
variational $p_y=p_x=$	0.3409	0.2612	0.2472	0.2429	0.2412	0.2401	0.2394
Discrepancy (%)	9.9	3.2	1.2	0.6	0.3	0.1	0.0

Both methods seem to work well. If one considers the result by De Smedt for a square, 0.2645, as exact then this might be an indication that the variational approach is somewhat more accurate. In the limiting case of $n \rightarrow \infty$ both methods give the exact result for a circle.

The D -moments for the rectangle will take the form

$$D_{xxxx} = \frac{24}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{24}{5} a_1^5 a_2, \quad D_{xyyy} = \frac{8}{3} a_1^3 a_2^3.$$

Here are the numerical results computed for a rectangle

$\varepsilon=$	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt $p_y=$	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
variational $p_y=$	3.4239	1.4523	0.8023	0.5166	0.3437	0.2612
Discrepancy (%)	-14.2	-5.8	-1.0	1.2	1.5	1.2
De Smedt $p_x=$	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
variational $p_x=$	0.0316	0.0588	0.0939	0.1370	0.1997	0.2612
Discrepancy (%)	15.9	7.9	4.3	2.1	1.2	1.2

Again, the general impression is that the variational approach is more accurate but not in all cases. For example, the discrepancy in p_y for $\varepsilon=0.1$ increased as compared with the result from the simple method. It is up to the user to decide whether a somewhat better accuracy of the variational approach is worth more cumbersome computations.

The mathematically equivalent problem of the analytical determination of the quadratic term in the low-frequency expansion, related to the sound transmission

through an aperture in a rigid screen, is considered in (Fabrikant, 1986d).

Exercise 5.6

1. Find the D -moments for a rhombus, with semiaxes a_1 and a_2 .

Answer:

$$D_{xxxx} = \frac{4}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{4}{5} a_1^5 a_2, \quad D_{xyyy} = \frac{2}{15} a_1^3 a_2^3.$$

2. Find the *variational* solution for a rhombus and compare it with the data of De Smedt, and with the *simple* solution.

Answer: see the Table below

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=$	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
variational $p_y=$	-0.5952	3.3549	0.9465	0.5580	0.3534	0.2612
Discrepancy (%)	112.8	-77.6	3.8	6.0	3.3	0.7
De Smedt $p_x=$	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
variational $p_x=$	0.0090	0.1464	0.1110	0.1400	0.1971	0.2612
Discrepancy (%)	71.5	-166.7	-28.8	-10.2	-2.5	0.7

Conclusion: the discrepancy decreased for $\varepsilon > 0.33$, but the results are unacceptable for $\varepsilon < 0.33$.

3. Find the D -moments for a cross-shaped punch.

Answer:

$$D_{xxxx} = D_{yyyy} = \frac{24}{5} a^6 \varepsilon (1 + \varepsilon^4 - \varepsilon^5), \quad D_{xyyy} = \frac{8}{3} a^6 \varepsilon^3 (2 - \varepsilon^3).$$

4. Find the *variational* solution for a cross and compare it with the data of De Smedt, and with the *simple* solution.

Answer: see the Table below

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=p_x=$	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
variational $p_y=p_x=$	1.4346	0.5822	0.3397	0.2606	0.2482	0.2612
Discrepancy (%)	-48.3	-19.9	-3.9	2.4	1.6	1.2

Conclusion: Comparison of this table with the data from the simple solution leads to the same conclusion: the results become valid in a wider range of the aspect ratio ε , but the theory fails for very small ε .

5. Apply the theory above to the case of a roller, with the aspect ratio $\varepsilon=1$.

Answer:

$$\sigma = \frac{a(\phi)l \left[\ln(1 + \sqrt{2}) - \frac{1}{2\sqrt{2}} \left[1 - \frac{x^2}{l^2} - \frac{\sqrt{2}\ln(1 + \sqrt{2}) - 1}{2\sqrt{2}\ln(1 + \sqrt{2}) - 1} \frac{y^2}{l^2} \right] \right]}{\pi Hr_0 \ln(1 + \sqrt{2}) [3\ln(1 + \sqrt{2}) - \sqrt{2}] \left[a^2(\phi) - \rho^2 \right]^{1/2}}$$

$$\approx \frac{0.4869 a(\phi) l \left[1 - \frac{x^2}{l^2} - \frac{y^2}{(2.4612l)^2} \right]}{\pi Hr_0 \left[a^2(\phi) - \rho^2 \right]^{1/2}}$$

$$P = \frac{2l^3 [12\ln(1 + \sqrt{2}) - \sqrt{2}]}{9\pi Hr_0 \ln(1 + \sqrt{2}) [3\ln(1 + \sqrt{2}) - \sqrt{2}]} \approx \frac{1.878l^3}{\pi Hr_0},$$

$$g_0 = \frac{l^2 [13\ln^2(1 + \sqrt{2}) - 6\sqrt{2} \ln(1 + \sqrt{2}) + 2]}{4r_0(1 + \sqrt{2}) [3\ln(1 + \sqrt{2}) - \sqrt{2}]} \approx \frac{1.06558l^2}{r_0}$$

5.7 Flat flexible punch of general planform under shifting load

The following mixed boundary value problem for a transversely isotropic elastic half-space is considered: a uniform tangential displacement is prescribed over a finite domain of general shape, while the remaining part of the boundary is traction-free. The problem may be interpreted as an external crack problem, with a remote shear loading, or as a contact problem, with a flexible punch, subjected to a tangential displacement. We use the term *flexible* to indicate a special kind of a punch which does not exert any normal pressure. A general relationship is established between the resultant shifting force and the displacement. A favorable comparison is made with the available numerical results.

Theory. Consider a transversely isotropic elastic half-space $z \geq 0$, with the following mixed boundary condition prescribed on the plane $z=0$:

$$\begin{aligned} \sigma_z &= 0, \text{ for } -\infty < (x,y) < \infty; & \tau &= 0, \text{ for } (x,y) \notin S; \\ u &= u(x,y), \text{ for } (x,y) \in S. \end{aligned} \quad (5.7.1)$$

Here S denotes the domain of contact in the elastic contact problem, or for the cross-section of the connection between two elastic half-spaces in the external crack problem. Let the boundary of the domain S be expressed in polar coordinates as

$$\rho = a(\phi), \quad (5.7.2)$$

where $a(\phi)$ is a single-valued bounded function. The governing integral equation can be rewritten from (2.6.2) as follows:

$$\frac{G_1}{2} \int_0^{2\pi} \int_0^{a(\phi_0)} \frac{\tau(\rho_0, \phi_0)}{R} \rho_0 d\rho_0 d\phi_0 + \frac{G_2}{2} \int_0^{2\pi} \int_0^{a(\phi_0)} \frac{q \bar{\tau}(\rho_0, \phi_0)}{\bar{q} R} \rho_0 d\rho_0 d\phi_0 = u(\rho, \phi). \quad (5.7.3)$$

Here the overbar indicates the complex conjugate value, and

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}, \quad R^2 = q \bar{q}. \quad (5.7.4)$$

The approach is based on the integral representation for the reciprocal distance established in (1.1.27). We also need an integral representation for the kernel in the second term of expression (5.7.3). It was established in (2.5.6). We consider further only the case where the right hand side in (5.7.3) $u = \text{const}$. Substitution of (1.1.27) and (2.5.6) into (5.7.3) gives, after changing the order of integration and retaining the zero harmonic only

$$\begin{aligned} & \frac{G_1}{\pi} \int_0^{2\pi} d\phi_0 \int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^{a(\phi_0)} \frac{\tau(\rho_0, \phi_0) \rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \\ & + \frac{G_2}{\pi} \int_0^{2\pi} e^{2i\phi_0} d\phi_0 \int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^{a(\phi_0)} \frac{\rho_0^2 - 2x^2}{\rho_0 (\rho_0^2 - x^2)^{1/2}} \bar{\tau}(\rho_0, \phi_0) d\rho_0 = u. \end{aligned} \quad (5.7.5)$$

We assume the shear stress distribution in the form

$$\tau = \frac{ca(\phi)}{[a^2(\phi) - \rho^2]^{1/2}}, \quad (5.7.6)$$

where c is a complex constant. This loading is statically equivalent to a resultant force $T=T_x+iT_y$. Integration of (5.7.6) over S yields

$$T = 2Ac, \quad (5.7.7)$$

where A is the area of the domain S . It is of interest to note that the relationship (5.7.7) does not depend on the location of origin of the system of coordinates. This location can be determined from the condition that the shear tractions should not produce any torque. This leads to two equations

$$\int_0^{2\pi} (a(\phi))^3 \cos\phi \, d\phi = 0, \quad \int_0^{2\pi} (a(\phi))^3 \sin\phi \, d\phi = 0. \quad (5.7.8)$$

One can note that the left-hand side of each equation (5.7.8) is proportional to the x or y coordinates of the center of gravity. This means that the origin of the system of coordinates should be located at the center of gravity of the domain S . The axis orientation will be discussed later.

It is now necessary to relate the tangential force T to the displacement u . This can be done by substitution of (5.7.6-5.7.7) into (5.7.5) which yields, after integration with respect to ρ_0

$$u = \frac{\pi}{8A} \left[G_1 T \int_0^{2\pi} a(\phi_0) d\phi_0 + G_2 \bar{T} \int_0^{2\pi} e^{2i\phi_0} a(\phi_0) d\phi_0 \right]. \quad (5.7.9)$$

The complex expression (5.7.9) is equivalent to two real ones, namely,

$$\begin{aligned} u_x &= \frac{\pi}{8A} \{ T_x [G_1 J_0 - G_2 (J_x - J_y)] + 2T_y G_2 J_{xy} \}, \\ u_y &= \frac{\pi}{8A} \{ 2T_x G_2 J_{xy} + T_y [G_1 J_0 - G_2 (J_y - J_x)] \}, \end{aligned} \quad (5.7.10)$$

The J -moments were introduced in (5.5.8), and some of them were computed in paragraph 5.6. One can now derive specific formulae for a variety of punch shapes. We leave this exercise to the reader.

Note that the change of the order of integration in (5.7.5) is valid inside the circle $\rho=\min(a(\phi))$ only, nevertheless, the solution given by (5.7.9) is *exact* in the case of an ellipse. This is easy to explain. It is known (Willis, 1970) that a traction distribution over an ellipse, given by a polynomial multiplied by $a(\phi)/[a^2(\phi) - \rho^2]^{1/2}$, produces a polynomial displacement. In our case the displacement is constant. Since expression (5.7.9) gives the exact value of the integrals in (5.7.3) for $\rho=0$, the result is exact all over the ellipse. Indeed, formula (5.7.10) yields for an ellipse with semiaxes a and b ($a \geq b$)

$$u_x = \frac{1}{2a} \left[G_1 K + G_2 \left(\frac{(2 - k^2)K - 2E}{k^2} \right) \right] T_x,$$

$$u_y = \frac{1}{2a} \left[G_1 K - G_2 \left(\frac{(2 - k^2)K - 2E}{k^2} \right) \right] T_y,$$

where K and E are complete elliptic integrals of the first and second kind, with the argument $k=(1-(b/a)^2)^{1/2}$. In the isotropic case $G_1=(2-\nu)/(2\pi\mu)$, $G_2=\nu/(2\pi\mu)$, and the last result coincides with Mindlin's (1949).

We can always choose the coordinate axes orientation so as to make J_{xy} vanish. In this case formulae (5.7.9) simplify as follows

$$u_x = \frac{\pi}{8A} [G_1 J_0 - G_2 (J_x - J_y)] T_x, \quad u_y = \frac{\pi}{8A} [G_1 J_0 - G_2 (J_y - J_x)] T_y. \quad (5.7.11)$$

Since expressions (5.7.10) and (5.7.11) are *exact* for an ellipse, we may expect them to be reasonably accurate for an arbitrary domain S . This assumption was justified in previous paragraphs when compared with various numerical results available in the literature.

In order to verify the accuracy of our method, some numerical computations were performed by Kalker. Equations (5.7.11) may be rewritten as

$$T_x = C_x u_x, \quad T_y = C_y u_y,$$

with

$$C_x = \frac{8A}{\pi[G_1(J_x + J_y) - G_2(J_x - J_y)]}, \quad C_y = \frac{8A}{\pi[G_1(J_x + J_y) + G_2(J_x - J_y)]}. \quad (5.7.12)$$

The values of C_x and C_y , due to formulae (5.7.12), are presented in the table below, and compared with the numerical results of Kalker.

Computations were made for an isotropic body, with the shear modulus $\mu=0.5$, and the Poisson coefficient $\nu=0.3$. The accuracy of the numerical procedure was

$\varepsilon=$	0.1	0.2	0.3	0.5	0.7	1.0
C_x our method	5.16	6.12	6.87	8.1	9.19	10.67
C_x Kalker	4.644	5.650	6.445	7.766	8.904	10.43
Discrepancy (%)	10.	7.7	6.2	4.1	3.1	2.3
C_y our method	4.32	5.32	6.13	7.56	8.85	10.67
C_y Kalker	3.991	5.044	5.904	7.373	8.667	10.43
Discrepancy (%)	7.6	5.2	3.7	2.5	2.1	2.3

assessed by computation of C_x and C_y for an ellipse, for which the exact solution is well known. The exact solution was about 4% *above* the numerical result by Kalker. All our results are also above the numerical ones. If we assume that error pattern for a rectangle is the same as for an ellipse (this means, for example, that our 10% discrepancy translates into $10-4=6$ (%) error), then our formulae must be considered as surprisingly accurate over a wide range of aspect ratios. We expect the error of our method to increase monotonically with decreasing ε , since the assumed traction distribution (5.7.6) is less realistic for a narrow rectangle than for a square. The fact that the discrepancy of C_y with the numerical results does not change monotonically, indicates some flaws in the Kalker's numerical procedure. We have also compared the shear traction distribution due to (5.7.6), (5.7.7) and (5.7.12) for a square with $a_1=4$, along the line $y=0.5$. The tangential displacement u_x was assumed equal to unity, and $u_y=0$. The comparison is given in the table below.

Taking into consideration the approximate nature of both methods, the agreement

$x=$	0.5	1.5	2.5	3.5
τ our method	0.084	0.090	0.107	0.172
τ Kalker	0.0863	0.0923	0.105	0.222
Discrepancy (%)	-2.6	-2.5	1.7	-29.

should be considered surprisingly good, except for the point close to the boundary, where neither method may claim to be accurate.

Kalker has made computation for a cross, with $a=4$ and $\varepsilon=0.5$. The remaining parameters were taken the same as for rectangle. His result $C_x=C_y=9.033$, our result is 9.26, with the discrepancy of 2.4%. We have also compared the shear stress distribution along the line $y=0.5$. The tangential displacement u_x was assumed equal to unity, and $u_y=0$. The comparison is given in the table below.

τ our method	0.0996	0.104	0.124	0.199
τ Kalker	0.0997	0.106	0.120	0.254
Discrepancy (%)	-0.08	-1.9	2.9	-27.

The agreement is good, except for the point close to the boundary.

Discussion. Some qualitative analysis of the general solution is given here. Let us rewrite equations (5.7.11) as follows

$$u_x = \frac{\pi G_1 J_0}{8A} [1 - \kappa] T_x, \quad u_y = \frac{\pi G_1 J_0}{8A} [1 + \kappa] T_y, \quad \kappa = \frac{G_2 (J_x - J_y)}{G_1 J_0}. \quad (5.7.13)$$

First of all, consider the case when $J_x = J_y$. We have from (5.7.13) that $\kappa = 0$, and hence the elastic compliance will be the same in any direction. Due to the fact that a circle has the greatest value of J_0 of all domains with the same area A , one may come to the conclusion that a circular domain is the easiest to move. A numerical comparison with a square shows that the ratio of the linear polar moments of a square to that of a circle is equal to $(2/\sqrt{\pi}) \ln(1+\sqrt{2}) = 0.9945$ which is very close to unity. Taking into consideration that our theory is approximate, it is surprising that the difference can still be observed. Comparison of a cross with a circle shows that its compliance may become arbitrarily small as $\varepsilon \rightarrow 0$.

Now let us deform the domain S so that J_x is no longer equal to J_y while keeping its area constant. Formulae (5.7.13) show that it will become easier to move the domain in the oblong direction, and a greater force will be required for a displacement in the perpendicular direction. In the extreme case of a needle, the limiting value for κ is G_2/G_1 ; in the case of isotropy $(G_2/G_1) = \nu/(2-\nu) \leq 1/3$. It is noteworthy that for $G_2 = 0$, the elastic compliance does not depend on the direction of the shift. In the case of isotropy, this corresponds to the Poisson ratio being equal to zero.

Exercise 5.7

1. Verify the invariance of formulae (5.7.10) with respect to the rotation of axes.
2. Find the relationship between the shifting force and the translational displacement of an elliptical punch, when its semiaxis $a < b$.

$$\text{Answer: } u_x = \frac{1}{2a} \left[G_1 K(k_1) - G_2 \left(\frac{(2 - k^2) K(k_1) - 2E(k_1)}{k_1^2} \right) \right] T_x,$$

$$u_y = \frac{1}{2a} \left[G_1 K(k_1) + G_2 \left(\frac{(2 - k^2)K(k_1) - 2E(k_1)}{k_1^2} \right) \right] T_y.$$

3. Find the tangential compliance of a cross-shaped punch, with the longer side $2a$ and the aspect ratio ε .

Hint: use (5.7.12) and (5.5.38)

4. Find the tangential compliance in the direction of the axes of symmetry for a rhombus-shaped punch.

5.8 Reissner-Sagoci problem for general domains

The problem of torsion of a transversely isotropic elastic half-space by a punch of general planform is considered. An approximate analytical solution is obtained by the general method. A general relationship is established between the torque and the torsion angle. Some specific formulae are derived for punches having the planform of a polygon, a rectangle, and a cross.

Theory. Consider a transversely isotropic elastic half-space $z \geq 0$. A flexible punch of general planform S is attached to the half-space surface, and a torque M_z is applied, producing the torsion angle ω . We need to relate the torsion angle to the torque. The mathematical formulation of the problem leads to the following mixed boundary conditions on the plane $z=0$:

$$\begin{aligned} \sigma_z &= 0, \text{ for } -\infty < (x,y) < \infty; \quad \tau_{zx} = 0 \quad \text{and} \quad \tau_{yz} = 0, \text{ for } (x,y) \notin S; \\ u_x &= -\omega y \quad \text{and} \quad u_y = \omega x, \quad \text{for } (x,y) \in S. \end{aligned} \quad (5.8.1)$$

Here S denotes the domain subjected to torsion, and ω is the torsion angle.

Introduce the complex tangential displacements $u = u_x + iu_y$, and the complex shear tractions $\tau = \tau_{zx} + i\tau_{yz}$. Let the boundary of the domain S be expressed in polar coordinates as

$$\rho = a(\phi), \quad (5.8.2)$$

where $a(\phi)$ is a single-valued bounded function. The governing integral equation is given by (5.7.3). The approach is based on the integral representations established in (1.1.27) and (2.5.6). Substitution of (1.1.27) and (2.5.6) into

(5.7.3) gives, after changing the order of integration and retaining the first harmonic only

$$\begin{aligned}
 u = & \frac{2}{\pi\rho} G_1 \int_0^\rho \frac{x^2 dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \cos(\phi - \phi_0) d\phi_0 \int_x^{a(\phi_0)} \frac{\tau(\rho_0, \phi_0) d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \\
 & + \frac{1}{\pi\rho} G_2 \left\{ -e^{i\phi} \int_0^\rho \frac{x^2 dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} e^{i\phi_0} d\phi_0 \int_x^{a(\phi_0)} \frac{\tau(\rho_0, \phi_0) d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \right. \\
 & \left. + e^{-i\phi} \int_0^\rho \frac{x^2 dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} e^{3i\phi_0} d\phi_0 \int_x^{a(\phi_0)} \frac{3\rho_0^2 - 4x^2}{\rho_0^2(\rho_0^2 - x^2)^{1/2}} \tau(\rho_0, \phi_0) d\rho_0 \right\}.
 \end{aligned} \tag{5.8.3}$$

Note that the change of the order of integration in (5.8.3) is valid only inside the circle $\rho = \min[a(\phi)]$, and the fact that we have ignored all the harmonics but the first one. Nevertheless, it will be shown further that the results are *exact* for an ellipse, and seem to be reasonable for a wide variety of nonelliptical shapes.

Let the shear traction distribution under the punch be

$$\tau = \tau_{zx} + i\tau_{yz} = \frac{a(\phi) \rho(iq_y \cos\phi - q_x \sin\phi)}{[a^2(\phi) - \rho^2]^{1/2}}, \tag{5.8.4}$$

where q_y and q_x are the as yet unknown constants. Make use of the condition that in the case of pure torsion the resulting force should be equal to zero, which means that the integral of τ over S should vanish. Since q_y and q_x are independent, this leads to two equations, namely,

$$\int_0^{2\pi} (a(\phi))^3 \cos\phi d\phi = 0, \quad \int_0^{2\pi} (a(\phi))^3 \sin\phi d\phi = 0. \tag{5.8.5}$$

One can note that the left-hand side of each equation (5.8.5) is proportional to the x or y coordinates of the center of gravity. This means that the origin of the system of coordinates should be located at the center of gravity of the domain of contact. The axis orientation will be discussed later.

The relationships between the torque M_z and the parameters q_y and q_x can be established from the statics conditions

$$M_z = \iint_S \tau_{yz}x \, dS - \iint_S \tau_{zx}y \, dS,$$

which leads to

$$M_z = \frac{8}{3}(q_y I_y + q_x I_x). \quad (5.8.6)$$

where I_x and I_y are the moments of inertia.

Next, it is necessary to relate q_y and q_x to the torsion angle ω . This can be done by substitution of (5.8.4) in (5.8.3), which yields, after integration with respect to ρ_0

$$\begin{aligned} u &= \frac{\pi}{4} G_1 \rho \int_0^{2\pi} \cos(\phi - \phi_0)(iq_y \cos \phi_0 - q_x \sin \phi_0) d\phi_0 \\ &- \frac{\pi}{8} G_2 \rho e^{i\phi} \int_0^{2\pi} a(\phi_0)(-i\bar{q}_y \cos \phi_0 - \bar{q}_x \sin \phi_0) e^{i\phi_0} d\phi_0 \\ &+ \frac{3\pi}{8} G_2 \rho e^{-i\phi} \int_0^{2\pi} a(\phi_0)(-i\bar{q}_y \cos \phi_0 - \bar{q}_x \sin \phi_0) e^{3i\phi_0} d\phi_0. \end{aligned} \quad (5.8.7)$$

The overbar everywhere denotes the complex conjugate value. Expression (5.8.7) can also be rewritten as follows:

$$\begin{aligned} u &= \frac{\pi}{4} G_1 \rho [(iq_y J_y - q_x J_{xy}) \cos \phi + (iq_y J_{xy} - q_x J_x) \sin \phi] \\ &- \frac{\pi}{8} G_2 \rho e^{i\phi} [-\bar{q}_y (iJ_y - J_{xy}) - \bar{q}_x (J_{xy} + iJ_x)] \\ &+ \frac{3\pi}{8} G_2 \rho e^{-i\phi} \{-i\bar{q}_y [4C_{yyyy} - 3J_y + i(3J_{xy} - 4C_{xxx})] \\ &- \bar{q}_x [4C_{xyyy} - 3J_{xy} + i(3J_x - 4C_{xxxx})]\} \end{aligned} \quad (5.8.8)$$

The J -moments were introduced by (5.5.8), and the C -moments are defined by (5.6.9). Substitution of the last two conditions (5.8.1) in (5.8.8) yields

$$\begin{aligned}
\omega(-y + ix) &= \frac{\pi}{4} G_1[(iq_y J_y - q_x J_{xy})x + (iq_y J_{xy} - q_x J_x)y] \\
&- \frac{\pi}{8} G_2(x + iy)[-q_y(iJ_y - J_{xy}) - q_x(J_{xy} + iJ_x)] \\
&+ \frac{3\pi}{8} G_2(x - iy)\{-i\bar{q}_y[4C_{yyyy} - 3J_y + i(3J_{xy} - 4C_{xxxx})] \\
&- \bar{q}_x[4C_{xyyy} - 3J_{xy} + i(3J_x - 4C_{xxxx})]\} \tag{5.8.9}
\end{aligned}$$

Separation in (5.8.9) of the terms related to x and y will lead to two linear algebraic equations for the unknowns q_y and q_x . In the general case, the parameters q_y and q_x are complex, and one has to solve four linear algebraic equations. These equations may be simplified by the assumption that $J_{xy}=0$. One can always achieve this by appropriately choosing the coordinate axis directions. In the case of symmetry, these axes will coincide with the principal axes of inertia. For the sake of simplicity, we assume also that $C_{xyyy}=C_{xxxx}=0$, which will always be the case if domain S has at least one axis of symmetry. Under these assumptions, expression (5.8.9) yields just two equations with real coefficients, namely,

$$\begin{aligned}
\frac{\pi}{4} G_1 q_y J_y + \pi G_2 [q_y (\frac{5}{4} J_y - \frac{3}{2} C_{yyyy}) + q_x (\frac{3}{2} C_{xxxx} - J_x)] &= \omega, \\
\frac{\pi}{4} G_1 q_x J_x + \pi G_2 [q_y (\frac{3}{2} C_{xyyy} - J_y) + q_x (\frac{5}{4} J_x - \frac{3}{2} C_{xxxx})] &= \omega.
\end{aligned} \tag{5.8.10}$$

The solution of (5.8.10) is

$$\begin{aligned}
q_y &= \frac{4\omega[G_1 + 3G_2(3 - 4c_x)]}{\pi(G_1 + G_2)J_y[G_1 + 3G_2(3 - 2c_x - 2c_y)]}, \\
q_x &= \frac{4\omega[G_1 + 3G_2(3 - 4c_y)]}{\pi(G_1 + G_2)J_x[G_1 + 3G_2(3 - 2c_x - 2c_y)]}.
\end{aligned} \tag{5.8.11}$$

Here the parameters were introduced

$$c_x = C_{xxxx}/J_x, \quad c_y = C_{xyyy}/J_y. \tag{5.8.12}$$

By substituting (5.8.11) in (5.8.6), one can find the relationship between the

torque and the angle of torsion in the form

$$M_z = \frac{32\omega[(G_1 + 9G_2)(I_y/J_y + I_x/J_x) - 12G_2(c_x I_y/J_y + c_y I_x/J_x)]}{3\pi(G_1 + G_2)[G_1 + 3G_2(3 - 2c_x - 2c_y)]}. \quad (5.8.13)$$

One can verify that the solution (5.8.11) and (5.8.13) is *exact* for an ellipse. Indeed, consider an ellipse with semiaxes a and b ($a \geq b$). The necessary geometrical characteristics are

$$\begin{aligned} J_x &= 4b[E(k) - (1 - k^2)K(k)]/k^2, & J_y &= 4b[K(k) - E(k)]/k^2, \\ C_{xxxx} &= 4b[2(2k^2 - 1)E(k) + (1 - k^2)(2 - 3k^2)K(k)]/(3k^4), \\ C_{yyyy} &= 4b[(2 + k^2)K(k) - 2(1 + k^2)E(k)]/(3k^4), & k &= [1 - (b/a)^2]^{1/2}. \end{aligned} \quad (5.8.14)$$

Here $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind respectively. Substitution of (5.8.14) in (5.8.11) and (5.8.13) yields

$$\begin{aligned} q_y &= \frac{\omega\{G_1 k^2[E - (1-k^2)K] + G_2[(8-7k^2)E - (1-k^2)(8-3k^2)K]\}}{\pi b(G_1+G_2)\{G_1(K-E)[E-(1-k^2)K] + G_2[k^2K(K+E)-(K-E)(K+3E)]\}}, \\ q_x &= \frac{\omega\{(G_1 k^2 - 8G_2)[K - E] + G_2 k^2[5K - E]\}}{\pi b(G_1+G_2)\{G_1(K-E)[E-(1-k^2)K] + G_2[k^2K(K+E)-(K-E)(K+3E)]\}}, \end{aligned} \quad (5.8.15)$$

$$M_z = \frac{2\omega a^3\{G_1 k^2 E - G_2[8(1-k^2)(2-k^2)K - (k^4 - 16k^2 + 16)E]\}}{3(G_1+G_2)\{G_1(K-E)[E-(1-k^2)K] + G_2[k^2K(K+E)-(K-E)(K+3E)]\}}. \quad (5.8.16)$$

The abbreviations E and K in (5.8.15) and (5.8.16) stand for $E(k)$ and $K(k)$ respectively. The same results can be obtained by direct substitution of (5.8.4) in (5.7.3), with an exact computation of the integrals involved, by using relevant formulae from Appendix A5.1. In the case of isotropy, $G_1 = (2-\nu)/(2\pi\mu)$, $G_2 = \nu/(2\pi\mu)$, and formulae (5.8.15) and (5.8.16) simplify as follows:

$$\begin{aligned} q_y &= \frac{\mu\omega\{k^2[E - (1-k^2)K] + 2\nu(1-k^2)[2E - (2-k^2)K]\}}{b\{[K - E][E - (1-k^2)K] + \nu E[2E - (2-k^2)K]\}}, \\ q_x &= \frac{\mu\omega\{k^2[K - E] + 2\nu[2E - (2 - k^2)K]\}}{b\{[K - E][E - (1-k^2)K] + \nu E[2E - (2-k^2)K]\}}, \end{aligned} \quad (5.8.17)$$

$$M_z = \frac{2\pi\mu\omega a^3 \{k^4 E + 4\nu(1 - k^2)[2E - (2 - k^2)K]\}}{3\{[K - E][E - (1 - k^2)K] + \nu E[2E - (2 - k^2)K]\}}. \quad (5.8.18)$$

Formulae (5.8.17) and (5.8.18) are in agreement with the corresponding results of Mindlin (1949). Some misprints are noticed in the relevant formula of (Kassir and Sih, 1968). The result by Willis (1970) seems to be in error, since it indicates that the parameters q_y and q_x do not depend on elastic constants, which is incorrect.

In the case when $J_x = J_y$ and $c_x = c_y$, formulae (5.8.11) and (5.8.13) simplify significantly, namely,

$$q_y = q_x = \frac{8\omega}{\pi J_0(G_1 + G_2)}, \quad M_z = \frac{64I_0\omega}{3\pi J_0(G_1 + G_2)} \quad (5.8.19)$$

Formulae (5.8.11) and (5.8.13) are the main results of this section. They are exact for an ellipse, and there is reason to believe that they will be sufficiently accurate for a general shape. The derivation of specific formulae for any particular shape is left to the reader.

Example 1: Regular polygon. Consider a punch shaped as a regular polygon with n sides. The necessary moments were computed in paragraph 5.6. We can use expressions (5.8.19), with the result

$$q_y = q_x = \frac{8\omega}{\pi n b(G_1 + G_2) \cos \frac{\pi}{n} \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}},$$

$$M_z = \frac{32\omega b^3 \sin(\pi/n) [2 + \cos(2\pi/n)]}{9\pi(G_1 + G_2) \ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}}. \quad (5.8.20)$$

In the limiting case of $n \rightarrow \infty$, formulae (5.8.20) give the results for a circular punch

$$q_y = q_x = \frac{4\omega}{\pi^2 b(G_1 + G_2)}, \quad M_z = \frac{16\omega b^3}{3\pi(G_1 + G_2)} \quad (5.8.21)$$

Introduce the stiffness parameter

$$D = M_z(G_1 + G_2)/\omega = 64I_0/(3\pi J_0), \quad (5.8.22)$$

which is a geometric characteristic of domain S . If we take the ratio of the stiffness of a regular polygon D_p to the stiffness D_c of a circle, having the same area, the result is:

$$\frac{D_p}{D_c} = \frac{2\sin(\pi/n)[2 + \cos(2\pi/n)]}{3\ln \frac{1 + \sin(\pi/n)}{1 - \sin(\pi/n)}}. \quad (5.8.23)$$

Elementary analysis of (5.8.23) shows that an equilateral triangle has stiffness 1.24 times greater than that of a circle having an equal area. The corresponding ratio for a square is 1.05, and further increase in n makes the polygon stiffness practically indistinguishable from that of a circle. One may prove a theorem stating that of all simply connected domains, having the same area, the circle has the lowest stiffness. This is opposite to the relevant theorem in the Saint-Venant theory of torsion of rods, where a circle has the greatest stiffness. This is easy to explain since in the torsion of a half-space, one has to twist not only an imaginary rod, but its surroundings as well.

Example 2: Rectangle. Consider a punch with a rectangular base, a_1 and a_2 being its semiaxes along the axis Ox and Oy respectively. Introduce the aspect ratio $\varepsilon = a_2/a_1$. The area and the necessary moments were computed in paragraph 5.6. Formulae (5.8.11) and (5.8.13) for a rectangle take the form

$$q_y = \frac{\omega[(G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}\sinh^{-1}\varepsilon + 4G_2\varepsilon]}{\pi a_1 \varepsilon (G_1 + G_2) \{ (G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}\sinh^{-1}\varepsilon \sinh^{-1}(1/\varepsilon) + 2G_2[\varepsilon \sinh^{-1}(1/\varepsilon) + \sinh^{-1}\varepsilon] \}},$$

$$q_x = \frac{\omega[(G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}\sinh^{-1}(1/\varepsilon) + 4G_2]}{\pi a_1 (G_1 + G_2) \{ (G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}\sinh^{-1}\varepsilon \sinh^{-1}(1/\varepsilon) + 2G_2[\varepsilon \sinh^{-1}(1/\varepsilon) + \sinh^{-1}\varepsilon] \}}, \quad (5.8.24)$$

$$M_z = \frac{32\omega a_1^3 \{ (G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}[\varepsilon^3 \sinh^{-1}(1/\varepsilon) + \sinh^{-1}\varepsilon] + 4G_2\varepsilon(1 + \varepsilon^2) \}}{9(G_1 + G_2) \{ (G_1 - 3G_2)(1 + \varepsilon^2)^{1/2}\sinh^{-1}\varepsilon \sinh^{-1}(1/\varepsilon) + 2G_2[\varepsilon \sinh^{-1}(1/\varepsilon) + \sinh^{-1}\varepsilon] \}}. \quad (5.8.25)$$

In order to verify the accuracy of our theory, some computations were made by Kalker for a rectangle with $a_1=4$ and various aspect ratios. The half-space was assumed to be isotropic, with the shear modulus $\mu=0.5$, and the Poisson coefficient $\nu=0.3$. The quantity $C_z=M_z/\omega$ was computed by using the universal software developed by Kalker. Our results due to (5.8.25), compared with numerical results by Kalker, are presented in the table below.

$\varepsilon=$	0.1	0.2	0.3	0.5	0.7	1.0
C_z our method	48.3	62.9	77.9	113.7	160.6	258.
C_z Kalker	36.74	51.29	66.47	102.4	148.4	241.
Discrepancy (%)	23.9	17.5	14.7	9.9	7.6	6.6

The accuracy of the numerical procedure was assessed by computation of C_z for an ellipse, where the exact solution is well known. The exact solution was about 7% *above* the numerical result by Kalker. All our results are also above the numerical ones. If we assume that error pattern for a rectangle is the same, as for an ellipse, then our formulae must be considered as surprisingly accurate over a wide range of aspect ratio.

We have also compared the shear stress distribution due to (5.8.4), and (5.8.24) for a rectangle, with $a_1=4$ and $\varepsilon=0.3$, along the line $x=0.5$. The torsion angle was assumed equal to unity. The shear traction is presented in polar form. The comparison for the modulus $|\tau|$ and its argument $arg(\tau)$ in degrees is given in the table below

$y=$	0.15	0.45	0.75	1.05
$ \tau $ our method	0.139	0.219	0.375	0.806
$ \tau $ Kalker	0.145	0.228	0.366	1.11
Discrepancy (%)	-1.9	-4.2	2.3	-37.7
$arg(\tau)$ our method	112.57	141.27	154.3	161.03
$arg(\tau)$ Kalker	112.14	140.72	152.94	161.15

The agreement of the argument of τ is very close. The agreement in modulus is satisfactory, except for the point close to the boundary where neither method may claim to be accurate. The fact that the discrepancy does not change monotonically suggests some flaws in the Kalker's numerical procedure.

Example 3: Cross. Consider a punch with a configuration obtained by the orthogonal intersection of two equal rectangles with sides $2a$ and $2b$ ($a \geq b$). Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments are given in paragraph 5.5.

Formulae (5.8.11) and (5.8.13) in this case yield

$$q_y = q_x = \frac{\omega}{\pi a(G_1 + G_2) \left[\ln(\varepsilon + (1 + \varepsilon^2)^{1/2}) + \varepsilon \ln \frac{1 + (1 + \varepsilon^2)^{1/2}}{(1 + \sqrt{2})\varepsilon} \right]},$$

$$M_z = \frac{64\omega a^3 \varepsilon (1 + \varepsilon^2 - \varepsilon^3)}{9\pi(G_1 + G_2) \left[\ln[\varepsilon + (1+\varepsilon^2)^{1/2}] + \varepsilon \ln \frac{1 + (1+\varepsilon^2)^{1/2}}{(1 + \sqrt{2})\varepsilon} \right]}.$$

Kalker has made computation for a cross, with $a=4$ and $\varepsilon=0.5$. The remaining numerical data was taken the same as for rectangle. His result $C_z=154.7$, our result is 167.9, with the discrepancy of 7.9%. Taking into consideration the assumed error of Kalker's software being 7%, our result might be considered very accurate.

We have also compared the shear traction distribution along the line $y=0.5$. The torsion angle ω was assumed equal to unity. The comparison is given in the table below.

$x=$	0.5	1.5	2.5	3.5
$ \tau $ our method	0.120	0.280	0.535	1.12
$ \tau $ Kalker	0.118	0.287	0.512	1.47
Discrepancy (%)	1.5	-2.5	4.4	-23.
$arg(\tau)$ our method	135.	108.4	101.3	98.13
$arg(\tau)$ Kalker	135.	108.0	101.1	98.72

The agreement is good, except for the point close to the boundary.

Exercise 5.8

1. Establish (5.8.3).
2. Derive (5.8.8).
3. Verify (5.8.11) and (5.8.13).
4. Consider the torsion of a half-space by a rhomboidal punch.
5. Compare the torsional rigidity of a rectangular punch with the torsional rigidity of a rhomboidal one, having the same area and the same aspect ratio.

5.9 Interaction between punches subjected to normal pressure

A general theorem is established which relates the resultant forces, acting on a set of arbitrary punches, with their generalized displacements through a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located elliptical punches. Several specific examples are considered.

Theory. Consider a set of N arbitrary punches penetrating an elastic half-space $z \geq 0$. Let S_n be the domain of contact for the n th punch, and P_n be the normal force acting on the n th punch. The friction forces between the punches and the half-space are neglected. The problem is to find the relationships between the generalized displacements of the punches and the acting forces. The boundary conditions for the problem are

$$\begin{aligned} w &= w_n(M) && \text{for } M \in S_n, \\ \sigma_z(M) &= 0 && \text{for } M \notin S_n, \quad n=1,2,\dots,N, \end{aligned} \quad (5.9.1)$$

where w denotes the normal displacement of a point at the boundary $z=0$, and σ stands for the pressure distribution. The prescribed function w_n is defined by the punch face. By using the known solution of the Boussinesq problem and the principle of superposition, we can write

$$w(Q) = H \sum_{n=1}^N \int_{S_n} \frac{\sigma_n(T_n)}{R(T_n, Q)} dS_n. \quad (5.9.2)$$

Substitution of the boundary conditions (5.9.1) in (5.9.2) leads to a set of N integral equations. The exact solution of these equations is not known at the present time, even for the case of several circles. Here, we are to show that we do not really need to know these solutions if we are interested in the integral characteristics only. We can single out, without loss of generality, the first punch, and consider the related integral equation

$$w_1(Q_1) = H \int_{S_1} \frac{\sigma_1(T_1)}{R(T_1, Q_1)} dS_1 + H \sum_{n=2}^N \int_{S_n} \frac{\sigma_n(T_n)}{R(T_n, Q_1)} dS_n. \quad (5.9.3)$$

Suppose that the functions σ_0 , σ_x and σ_y are known, satisfying respectively the following integral equations inside S_1

$$\int_{S_1} \int \frac{\sigma_0(Q_1)}{R(T_1, Q_1)} dS_1 = 1, \quad (5.9.4)$$

$$\int_{S_1} \int \frac{\sigma_x(Q_1)}{R(T_1, Q_1)} dS_1 = x, \quad (5.9.5)$$

$$\int_{S_1} \int \frac{\sigma_y(Q_1)}{R(T_1, Q_1)} dS_1 = y. \quad (5.9.6)$$

Multiplication of both sides of (5.9.3) by $\sigma_0(Q_1)$ and integration over the area S_1 yields

$$\begin{aligned} \int_{S_1} \int \sigma_0(Q_1) w_1(Q_1) dS_1 &= H \int_{S_1} \int \sigma_0(Q_1) dS_1 \int_{S_1} \int \frac{\sigma_1(T_1)}{R(T_1, Q_1)} dS_1 + \\ &+ H \sum_{n=2}^N \int_{S_1} \int \sigma_0(Q_1) dS_1 \int_{S_n} \int \frac{\sigma_n(T_n)}{R(T_n, Q_1)} dS_n. \end{aligned} \quad (5.9.7)$$

By interchanging the order of integration in (5.9.7) and taking into consideration the fact that σ_0 satisfies (5.9.4), the following result can be obtained:

$$\int_{S_1} \int \sigma_0(Q_1) w_1(Q_1) dS_1 = H \left[P_1 + \sum_{n=2}^N \int_{S_n} \int w_{1n}(T_n) \sigma_n(T_n) dS_n \right], \quad (5.9.8)$$

where P_1 is the total force acting on the first punch, and

$$w_{1n}(T_n) = \int_{S_1} \int \frac{\sigma_0(Q_1)}{R(T_n, Q_1)} dS_1, \quad (5.9.9)$$

which is proportional to the normal displacement in the domain S_n due to a flat punch in S_1 under the action of a unit force. By invoking the mean value theorem, which is valid as long as σ_n does not change sign, we obtain the linear algebraic equation

$$\int_{S_1} \int \sigma_0(Q_1) w_1(Q_1) dS_1 = H \left[P_1 + \sum_{n=2}^N w_{1n}(C_n) P_n \right] \quad (5.9.10)$$

The exact location of the point C_n is not known but the fact that $C_n \in S_n$ allows only a limited variation within S_n , and in many cases provides sufficiently close upper and lower bounds for the parameters sought. By using the same procedure, $N-1$ additional linear algebraic equations can be derived for the remaining punches. This set of equations provides the necessary relationships between the normal displacements of the punches and the applied forces.

Let us derive similar relationships for the angular displacements. Multiplication of both sides of (5.9.3) by $\sigma_x(Q_1)$ and integration over the area S_1 yields

$$\begin{aligned} \int_{S_1} \int \sigma_x(Q_1) w_1(Q_1) dS_1 &= H \int_{S_1} \int \sigma_x(Q_1) dS_1 \int_{S_1} \int \frac{\sigma_1(T_1)}{R(T_1, Q_1)} dS_1 + \\ &+ H \sum_{n=2}^N \int_{S_1} \int \sigma_x(Q_1) dS_1 \int_{S_n} \int \frac{\sigma_n(T_n)}{R(T_n, Q_1)} dS_n. \end{aligned} \quad (5.9.11)$$

By changing the order of integration in (5.9.11) and taking into consideration that σ_x satisfies (5.9.5), the following result can be obtained

$$\int_{S_1} \int \sigma_x(Q_1) w_1(Q_1) dS_1 = H \left[-M_{1y} + \sum_{n=2}^N \int_{S_n} \int x \alpha_{1n}(T_n) \sigma_n(T_n) dS_n \right], \quad (5.9.12)$$

where M_{1y} is the tilting moment acting on the first punch about the axis Oy , and

$$\alpha_{1n}(T_n) = \frac{1}{x} \int_{S_1} \int \frac{\sigma_x(Q_1)}{R(T_n, Q_1)} dS_1. \quad (5.9.13)$$

If σ_n does not change sign we can use again the mean value theorem to yield

$$\int_{S_1} \int \sigma_x(Q_1) w_1(Q_1) dS_1 = H \left[-M_{1y} + \sum_{n=2}^N \alpha_{1n}(A_n) x_n P_n \right] \quad (5.9.14)$$

where $A_n(x_n, y_n) \in S_n$. The additional $N-1$ equations relating the punch rotations about the axis Oy with the corresponding tilting moments can be obtained in a similar way.

Multiply both sides of (5.9.3) by $\sigma_y(Q_1)$ and integrate over S_1 . The procedure leads to the equation

$$\int_{S_1} \int \sigma_y(Q_1) w_1(Q_1) dS_1 = H \left[M_{1x} + \sum_{n=2}^N \int_{S_n} y \beta_{1n}(T_n) \sigma_n(T_n) dS_n \right], \quad (5.9.15)$$

where M_{1x} is the tilting moment about the axis Ox acting on the first punch, and

$$\beta_{1n}(T_n) = \frac{1}{y} \int_{S_1} \int \frac{\sigma_y(Q_1)}{R(T_n, Q_1)} dS_1. \quad (5.9.16)$$

As before, application of the mean value theorem gives

$$\int_{S_1} \int \sigma_y(Q_1) w_1(Q_1) dS_1 = H \left[M_{1x} + \sum_{n=2}^N \beta_{1n}(B_n) y_n P_n \right]. \quad (5.9.17)$$

Three sets of linear algebraic equations of the type (5.9.10), (5.9.14) and (5.9.17) are the main results of this paragraph. It is clear that each equation can be interpreted in terms of the reciprocal work. In order to use them, one need to know the normal displacements outside every punch in the system due to three types of loading which, at the moment, is available for the elliptical punches only. This particular case is considered below.

Application to elliptical punches. Consider the interaction of a set of N flat elliptical punches arbitrarily located on a transversely isotropic elastic half-space. Let a_n and b_n be the major and the minor semiaxes of the n th ellipse; X_n and Y_n define its center, and θ_n be the angle between the axis Ox and the major semiaxis a_n ; and P_n be the normal force acting upon the n th punch.

The functions σ_0 , σ_x and σ_y have the form (Lur'e, 1955):

$$\sigma_0 = \frac{1}{2\pi b_1 K(k_1)} \left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2}$$

$$\sigma_x = \frac{x}{2\pi b_1 D(k_1)} \left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2}$$

$$\sigma_y = \frac{y}{2\pi b_1 B(k_1)} \left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2}$$

The boundary conditions (5.9.1) in this case will take the form

$$w_n = \delta_n - \alpha_n x + \beta_n y, \quad \text{for } n=1,2,\dots,N. \quad (5.9.18)$$

Substitution of (5.9.18) into (5.9.10), (5.9.14) and (5.9.17) yields respectively

$$\frac{a_1}{K(k_1)} \delta_1 = H \left[P_1 + \sum_{n=2}^N \frac{F(\phi_{1n}, k_1)}{K(k_1)} P_n \right], \quad (5.9.19)$$

$$\frac{a_1^3}{3D(k_1)} \alpha_1 = H \left[M_{1y} - \sum_{n=2}^N \alpha_{1n} x_n P_n \right], \quad (5.9.20)$$

$$\frac{a_1 b_1^2}{3B(k_1)} \beta_1 = H \left[M_{1x} + \sum_{n=2}^N \beta_{1n} y_n P_n \right], \quad (5.9.21)$$

where

$$\alpha_{1n} = \frac{F(\phi_{1n}, k_1) - E(\phi_{1n}, k_1)}{K(k_1) - E(k_1)}, \quad (5.9.22)$$

$$\beta_{1n} = \frac{E(\phi_{1n}, k_1) - (1 - k_1^2)F(\phi_{1n}, k_1) - k_1^2(\rho_{1n}^2 - 1)^{1/2}/\rho_{1n}(\rho_{1n}^2 - k_1^2)^{1/2}}{E(k_1) - (1 - k_1^2)K(k_1)}, \quad (5.9.23)$$

$$B(k_1) = \frac{E(k_1) - (1 - k_1^2)K(k_1)}{k_1^2}, \quad (5.9.24)$$

$$D(k_1) = \frac{K(k_1) - E(k_1)}{k_1^2}, \quad (5.9.25)$$

$K(k_1)$, $F(\phi_{1n}, k_1)$, $E(k_1)$, $E(\phi_{1n}, k_1)$ stand for the complete and incomplete elliptic integrals of the first and second kind respectively; and ρ_{1n} , ϕ_{1n} are defined by

$$\phi_{1n} = \sin^{-1}\left(\frac{1}{\rho_{1n}}\right), \quad \rho_{1n} = [L + (L^2 - k_1^2 x_n^2 / a_1^2)^{1/2}]^{1/2},$$

$$L = \frac{1}{2}[k_1^2 + (x_n^2 + y_n^2)/a_1^2],$$

where x_n and y_n are coordinates of a certain point inside S_n , and k_1 is the eccentricity of the first ellipse

$$k_1 = [1 - b_1^2/a_1^2]^{1/2}.$$

Each of the equations (5.9.19), (5.9.20) and (5.9.21) represent the first of a set of N equations. When the acting forces are known, the three sets of equations define the normal and the angular displacements of the punches. In the case where the displacements are known, the three sets of linear algebraic equations have to be solved for P_n , M_{nx} and M_{ny} . It is also important to notice that each equation in the set is valid in the system of coordinates located at the center of the ellipse.

Example 1: Two equal elliptical punches. Consider the case where $N=2$, $a_1=a_2=a$, $b_1=b_2=b$, $X_1=Y_1=0$, $X_2=l$, $Y_2=0$, $\theta_1=\theta_2=0$. If we also have $P_1=P_2=P$ then, due to the symmetry of the system, the set of equations, equivalent to (5.9.19), reduces to just one equation, namely,

$$\frac{a}{K(k)} \delta = H \left[P + \frac{F(\phi, k)}{K(k)} P \right], \quad (5.9.26)$$

with the immediate result

$$P = \frac{P_0}{1 + \frac{F(\phi, k)}{K(k)}}, \quad (5.9.27)$$

where $P_0 = \delta a / HK(k)$ denotes the force which produces a punch settlement equal to δ when acting on a solitary punch. Equation (5.9.27) shows that the interaction between the punches decreases the value of the force necessary to produce the required settlement. The upper and lower bounds for P can be obtained from (5.9.27) by taking

$$\phi = \sin^{-1}\left[\frac{a}{l-a}\right], \quad \text{and} \quad \phi = \sin^{-1}\left[\frac{a}{l+a}\right]. \quad (5.9.28)$$

respectively. We shall also consider the central estimation for P defined by

$$\phi = \sin^{-1}\left(\frac{a}{l}\right). \quad (5.9.29)$$

Figure 5.9.1 plots the ratio P/P_0 versus l/a for $a=2$, $b=1$. The solid line gives the upper bound, the dashed line gives the lower one and the circles represent the central estimation. As one can see, the maximum possible error of the

Fig. 5.9.1. Interaction between two elliptical punches.

central estimation is less than 9% for $l/a > 3.5$, it is less than 5% for $l/a > 5$, it is less than 2% for $l/a > 8$, and it is less than 1% for $l/a > 12$. Since there is no accurate solution available for this case, it is difficult to say how great is the real error of the central estimation, but there is a reason to believe that it is much less than that indicated above. This belief is founded in a comparison of the central estimation for two equal *circular* punches with the numerical solution of Kobayashi (1939). The values of the ratio P/P_0 for various $d=l/a$ are given in Table 5.9.1.

If one takes Kobayashi's solution as exact then the maximum error of the central estimation does not exceed 0.4% in the whole range of $2 \leq d < \infty$. Even if one assumes the accuracy in the case of two elliptic punches to be ten times

Table 5.9.1 Comparison of two solutions

d	upper bound	lower bound	central estimation	result of Kobayashi	error (%)
2.0	0.82213	0.50000	0.75000	0.75272	0.36
2.2	0.83172	0.61457	0.76900	0.77014	0.15
2.4	0.84030	0.66379	0.78517	0.78545	0.04
2.6	0.84804	0.69940	0.79915	0.79898	-0.02
2.8	0.85505	0.72728	0.81136	0.81096	-0.05
3.0	0.86143	0.75000	0.82213	0.82162	-0.06
3.5	0.87515	0.79241	0.84427	0.84370	-0.07
4.0	0.88638	0.82213	0.86143	0.86093	-0.06
5.0	0.90367	0.86143	0.88638	0.88602	-0.04
7.0	0.92611	0.90367	0.91637	0.91619	-0.02
10.0	0.94522	0.93381	0.94005	0.93999	-0.007
∞	1.	1.	1.	1.	0.0

worse than the accuracy of the central estimation for two circular punches, this would still give the maximum error of 4% which is not bad. Bearing this in mind, we shall evaluate the central estimation only in the examples to follow.

If the normal forces are applied centrally then the angles of inclination of the punches will be defined by (5.9.20) and (5.9.21) as

$$\alpha_1 = -\alpha_2 = -\frac{3D(k)}{a_1^3} H\alpha_{12}lP, \quad \beta_1 = \beta_2 = 0, \quad (5.9.30)$$

where

$$\alpha_{12} = \frac{F(\phi, k) - E(\phi, k)}{K(k) - E(k)}, \quad k = (1 - b^2/a^2)^{1/2}, \quad (5.9.31)$$

and ϕ is defined by (5.9.29).

In the case $\alpha_1 = \alpha_2 = 0$ there should be tilting moments applied to the punches whose value can be determined from (5.9.20) as

$$M_{1y} = -M_{2y} = \alpha_{12}lP. \quad (5.9.32)$$

Example 2: Four equal elliptical punches. Consider the configuration shown in Fig. 5.9.2. Let equal vertical forces P be applied to each punch. The punches are numbered in the clockwise direction starting from the one at the coordinate system origin. Due to the symmetry of the system, it is sufficient to consider just one equation of each of the sets (5.9.19–5.9.21). The result is

Fig. 5.9.2. Geometry of four punches interaction

$$\frac{a}{\mathbf{K}(k)} \delta = HP \left[1 + \frac{F(\phi_{12}, k)}{\mathbf{K}(k)} + \frac{F(\phi_{13}, k)}{\mathbf{K}(k)} + \frac{F(\phi_{14}, k)}{\mathbf{K}(k)} \right], \quad (5.9.33)$$

$$\frac{a^3}{3\mathbf{D}(k)} \alpha_1 = H \left[M_{1y} - Pl(\alpha_{13} + \alpha_{14}) \right], \quad (5.9.34)$$

$$\frac{ab^2}{3\mathbf{B}(k)} \beta_1 = H \left[M_{1x} + Pc(\beta_{12} + \beta_{13}) \right], \quad (5.9.35)$$

where α_{1n} and β_{1n} are defined by (5.9.22) and (5.9.23) respectively, and

$$\phi_{1n} = \sin^{-1} \left(\frac{1}{\rho_{1n}} \right), \quad \text{for } n=2,3,4; \quad (5.9.36)$$

$$\rho_{12} = \left[1 + (c^2 - b^2)/a^2 \right]^{1/2}, \quad \rho_{14} = \frac{l}{a}$$

$$\rho_{13} = \left[L + (L^2 - k^2 l^2 / a^2)^{1/2} \right]^{1/2}, \quad L = \frac{1}{2} \left[k^2 + (l^2 + c^2) / a^2 \right], \quad (5.9.37)$$

When the forces and the tilting moments are known, equations (5.9.33–5.9.35) define the normal and the angular displacements of the punches.

The mathematically equivalent problem of diffusion through perforated membranes was solved in (Fabrikant, 1985a, 1987k).

Exercise 5.9

1. Consider the interaction between the four elliptical punches depicted in Fig. 5.9.2. Find the tilting moments if the punches are not allowed to tilt.

$$\text{Answer: } M_{1x} = -Pc(\alpha_{12} + \alpha_{13}), \quad M_{1y} = Pl(\alpha_{13} + \alpha_{14}).$$

2. Find the limiting case of formulae (5.9.22–5.9.25) for the case of circular punches.

$$\text{Answer: } \alpha_{1n} \rightarrow \beta_{1n} \rightarrow \frac{2}{\pi} \left\{ \sin^{-1} \left[\frac{a_1}{r_{1n}} \right] - \frac{a_1}{r_{1n}} \left[1 - \frac{a_1^2}{r_{1n}^2} \right]^{1/2} \right\},$$

$$B(k_1) \rightarrow D(k_1) \rightarrow \pi/4, \quad \frac{F(\phi_{1n}, k_1)}{K(k_1)} \rightarrow \frac{2}{\pi} \sin^{-1} \left[\frac{a_1}{r_{1n}} \right],$$

and the equations (5.9.19–5.9.21) will take the form

$$\frac{2}{\pi} a_1 \delta_1 = H \left[P_1 + \frac{2}{\pi} \sum_{n=2}^N P_n \sin^{-1} \left(\frac{a_1}{r_{1n}} \right) \right],$$

$$\frac{4a_1^3}{3\pi} \alpha_1 = H \left[M_{1y} - \sum_{n=2}^N \alpha_{1n} x_n P_n \right],$$

$$\frac{4a_1^3}{3\pi} \beta_1 = H \left[M_{1x} + \sum_{n=2}^N \beta_{1n} y_n P_n \right],$$

where a_1 is the radius of punch one and r_{1n} is the distance between the first punch centre and a point inside S_n .

3. Find the generalized displacements of an elliptical punch due to the action of several concentrated loads, applied normally outside the punch.

Hint: Consider the procedure of shrinking of the areas S_n , $n=2,3,\dots,N$. The accuracy of equations (5.9.19–5.9.21) will increase. In the limiting case $S_n \rightarrow 0$ formulae (5.9.19–5.9.21) give an *exact* solution to the problem of several concentrated forces acting outside an elliptical punch.

4. Consider the interaction of several punches on a non-homogeneous elastic

half-space, with the modulus of elasticity being proportional to a power function of the depth.

5.10 Interaction between flexible punches under shifting loading

The mixed boundary value problem for a transversely isotropic elastic half-space is considered for the case where uniform tangential displacements are prescribed over several domains of arbitrary shape and the rest of the half-space boundary is stress free. The problem can be interpreted either as an interaction between two elastic half-spaces interconnected through several areas and subjected to remote shear loading, or as a contact problem of several flexible punches, connected to the half-space, with different tangential displacements prescribed. A general theorem is established which relates the resulting tangential forces, acting on each domain, with their generalized displacements through a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located elliptical domains subjected to uniform tangential displacements. Several specific examples are considered.

Theory. Consider a transversely isotropic elastic half-space $z \geq 0$. Let the tangential displacements be prescribed over several simply connected domains S_n while the rest of the half-space boundary is stress free. The mathematical formulation of the boundary conditions on the plane $z=0$ is

$$\begin{aligned} u_x &= u_x(x,y), \quad u_y = u_y(x,y), \quad \text{for } (x,y) \in S_n, \\ \sigma_z &= \tau_{zx} = \tau_{yz} = 0, \quad \text{for } (x,y) \notin S_n, \quad n=1,2, \dots, N. \end{aligned} \quad (5.10.1)$$

The boundary value problem (5.10.1) can be interpreted either as an interaction between several flexible punches S_n bonded to the half-space and subjected to tangential displacements, or as the case of two elastic half-spaces connected in S_n and subjected to a remote shear loading, the first interpretation being more general since the tangential displacements each punch can be treated as independent. Hereafter we shall call S_n the domain of contact for the n th punch. The punches are assumed to be flexible, so that they do not exert any normal tractions. Let T_{nx} and T_{ny} be the components of the resulting tangential force applied to the n -th punch. The problem is to find the relationships between the generalized displacements of the punches and the acting forces.

Introduce the complex tangential displacements $u = u_x + iu_y$, and the complex tangential tractions $\tau = \tau_{zx} + i\tau_{yz}$. In view of the principle of superposition, the governing integral equation can be rewritten from (5.7.5) as follows:

$$\sum_{n=1}^N \left\{ \frac{G_1}{2} \iint_{S_n} \frac{\tau(\xi, \eta)}{R} d\xi d\eta + \frac{G_2}{2} \iint_{S_n} \frac{q \bar{\tau}(\xi, \eta)}{\bar{q} R} d\xi d\eta \right\} = u(x, y). \quad (5.10.2)$$

Here

$$q = x - \xi + i(y - \eta), \quad \bar{q} = x - \xi - i(y - \eta), \quad R^2 = q\bar{q}. \quad (5.10.3)$$

Substitution of the boundary conditions (5.10.1) in (5.10.2) leads to a set of N integral equations. The exact solution of these equations is not known at present, even for the case of several circles. Here it will be shown that we do not really need to know these solutions if we are interested only in the relationship between the applied forces and the punch displacements. We can single out, without loss of generality, the first punch, and consider the related integral equation

$$\begin{aligned} u_1(x_1, y_1) &= \frac{G_1}{2} \iint_{S_1} \frac{\tau_1(\xi, \eta)}{R_{11}} d\xi d\eta + \frac{G_2}{2} \iint_{S_1} \frac{q_{11} \bar{\tau}_1(\xi, \eta)}{\bar{q}_{11} R_{11}} d\xi d\eta \\ &+ \sum_{n=2}^N \left\{ \frac{G_1}{2} \iint_{S_n} \frac{\tau_n(\xi, \eta)}{R_{1n}} d\xi d\eta + \frac{G_2}{2} \iint_{S_n} \frac{q_{1n} \bar{\tau}_n(\xi, \eta)}{\bar{q}_{1n} R_{1n}} d\xi d\eta \right\}. \end{aligned} \quad (5.10.4)$$

Here u_n stands for tangential displacement of the n -th punch, τ_n is the tangential tractions exerted by the punch, the point $(x_n, y_n) \in S_n$, and

$$\begin{aligned} q_{kn} &= x_k - \xi_n + i(y_k - \eta_n), & \bar{q}_{kn} &= x_k - \xi_n - i(y_k - \eta_n), \\ R_{kn} &= (q_{kn} \bar{q}_{kn})^{1/2}, & (\xi_n, \eta_n) &\in S_n. \end{aligned} \quad (5.10.5)$$

We have dropped for simplicity the subscripts of ξ_n and η_n in (5.10.4) and thereafter. It should not produce any confusion for the reader.

Equation (5.10.4) can be rewritten as

$$u_1(x_1, y_1) = \frac{G_1}{2} \Delta \iint_{S_1} R_{11} \tau_1(\xi, \eta) d\xi d\eta - \frac{G_2}{2} \Lambda^2 \iint_{S_1} R_{11} \bar{\tau}_1(\xi, \eta) d\xi d\eta$$

$$+ \sum_{n=2}^N \left\{ \frac{G_1}{2} \Delta \int \int_{S_n} R_{1n} \tau_n(\xi, \eta) \, d\xi d\eta - \frac{G_2}{2} \Lambda^2 \int \int_{S_n} R_{1n} \bar{\tau}_n(\xi, \eta) \, d\xi d\eta \right\}. \quad (5.10.6)$$

Suppose that the function τ_0 , is known, satisfying the following integral equation inside S_1

$$\int \int_{S_1} \tau_0(\xi, \eta) R \, d\xi d\eta = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6, \quad (5.10.7)$$

where a_1, \dots, a_6 are complex constants. One can verify that

$$\begin{aligned} \Delta \int \int_{S_1} \tau_0(\xi, \eta) R_{11} \, d\xi d\eta &= 2a_1 + 2a_2 = d_1 = \text{const}, \\ \Lambda^2 \int \int_{S_1} \tau_0(\xi, \eta) R_{11} \, d\xi d\eta &= 2a_1 - 2a_2 + 2ia_3 = -c_1 = \text{const}, \end{aligned} \quad (5.10.8)$$

which means that the function τ_0 renders constant the first two integrals in expression (5.10.4). This important property will be used in the derivation to follow. Multiplication of both sides of (5.10.4) by τ_0 and integration over the area S_1 yields

$$\begin{aligned} \int \int_{S_1} u_1(x_1, y_1) \tau_0(x_1, y_1) \, dx_1 dy_1 &= \frac{G_1}{2} \int \int_{S_1} \tau_0(x_1, y_1) \, dx_1 dy_1 \int \int_{S_1} \frac{\tau_1(\xi, \eta)}{R_{11}} \, d\xi d\eta \\ &+ \frac{G_2}{2} \int \int_{S_1} \tau_0(x_1, y_1) \, dx_1 dy_1 \int \int_{S_1} \frac{q_{11} \bar{\tau}_1(\xi, \eta)}{\bar{q}_{11} R_{11}} \, d\xi d\eta \\ &+ \sum_{n=2}^N \left\{ \frac{G_1}{2} \int \int_{S_1} \tau_0(x_1, y_1) \, dx_1 dy_1 \int \int_{S_n} \frac{\tau_n(\xi, \eta)}{R_{1n}} \, d\xi d\eta \right. \end{aligned}$$

$$+ \frac{G_2}{2} \int \int_{S_1} \tau_0(x_1, y_1) dx_1 dy_1 \int \int_{S_n} \frac{q_{1n} \bar{\tau}_n(\xi, \eta)}{\bar{q}_{1n} R_{1n}} d\xi d\eta \Bigg\}. \quad (5.10.9)$$

By changing the order of integration in (5.10.9) and taking into consideration the property (5.10.8), the following result can be obtained

$$\begin{aligned} \int \int_{S_1} u_1(x_1, y_1) \tau_0(x_1, y_1) dx_1 dy_1 &= \frac{G_1}{2} d_1 T_1 + \frac{G_2}{2} c_1 \bar{T}_1 \\ &+ \sum_{n=2}^N \left\{ \frac{G_1}{2} \int \int_{S_n} \tau_n(\xi, \eta) d\xi d\eta \int \int_{S_1} \frac{\tau_0(x_1, y_1)}{R_{1n}} dx_1 dy_1 \right. \\ &\left. + \frac{G_2}{2} \int \int_{S_n} \bar{\tau}_n(\xi, \eta) d\xi d\eta \int \int_{S_1} \frac{q_{1n} \tau_0(x_1, y_1)}{\bar{q}_{1n} R_{1n}} dx_1 dy_1 \right\}. \end{aligned} \quad (5.10.10)$$

where T_1 is the complex representation for the total tangential force acting on the first punch. Introduce the notation

$$\Phi(\xi, \eta) = \int \int_{S_1} \frac{\tau_0(x_1, y_1)}{R_{1n}} dx_1 dy_1, \quad \Psi(\xi, \eta) = \int \int_{S_1} \frac{q_{1n} \tau_0(x_1, y_1)}{\bar{q}_{1n} R_{1n}} dx_1 dy_1. \quad (5.10.11)$$

Now expression (5.10.10) can be rewritten as

$$\begin{aligned} \int \int_{S_1} u_1(x_1, y_1) \tau_0(x_1, y_1) dx_1 dy_1 &= \frac{G_1}{2} d_1 T_1 + \frac{G_2}{2} c_1 \bar{T}_1 \\ &+ \sum_{n=2}^N \left\{ \frac{G_1}{2} \int \int_{S_n} \Phi(\xi, \eta) \tau_n(\xi, \eta) d\xi d\eta + \frac{G_2}{2} \int \int_{S_n} \Psi(\xi, \eta) \tau_n(\xi, \eta) d\xi d\eta \right\}. \end{aligned} \quad (5.10.12)$$

By evoking the mean value theorem, we come to the linear algebraic equation, for the forces T_n applied to each punch.

$$\int \int_{S_1} u_1(x_1, y_1) \tau_0(x_1, y_1) dx_1 dy_1 = \frac{G_1}{2} d_1 T_1 + \frac{G_2}{2} c_1 \bar{T}_1 + \sum_{n=2}^N \left\{ \frac{G_1}{2} \Phi(x_n, y_n) T_n + \frac{G_2}{2} \Psi(\xi_n, \eta_n) \bar{T}_n \right\}. \quad (5.10.13)$$

The exact location of the points (x_n, y_n) and (ξ_n, η_n) is not known but the fact that they belong to S_n allows only a limited variation, and in many cases provides sufficiently close upper and lower bounds for the parameters sought. By a similar argument, $N-1$ additional linear algebraic equations can be derived for the remaining punches. This set of equations provides the necessary relationships between the tangential displacements of the punches and the applied forces, and represents the main result of this paragraph. It is clear that each equation can be interpreted in terms of the reciprocal work. In order to use equations (5.10.13), one needs to know the explicit expression for τ_0 which, at present, is available for the elliptical punch only.

Application to elliptical punches. Consider the interaction of a set of N flexible elliptical punches arbitrarily located on an elastic half-space, with uniform tangential displacements $u_n = \text{const}$ prescribed in S_n . Let a_n and b_n be the major and the minor semiaxes of the n th ellipse; X_n and Y_n define its center, θ_n be the angle between the axis Ox and the major semiaxis a_n , and T_n be the tangential force acting upon the n th punch. The function τ_0 for the first punch is

$$\tau_0(x, y) = \left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{1/2}. \quad (5.10.14)$$

Substitution of (5.10.14) in (5.10.10) yields, after computation of the integrals involved (see Appendix A5.1 for details)

$$2a_1 u_1 = G_1 K(k_1) T_1 + G_2 [(2 - k_1^2) K(k_1) - 2 E(k_1)] \bar{T}_1 / k_1^2 + \frac{1}{2} \sum_{n=2}^N \left\{ G_1 \int \int_{S_n} D_{1n} \tau_n dS_n + (G_2 / k_1^2) \int \int_{S_n} L_{1n} \bar{\tau}_n dS_n - i(G_2 / k_1^2) \int \int_{S_n} M_{1n} \bar{\tau}_n dS_n \right\}, \quad (5.10.15)$$

where

$$\begin{aligned}
D_{1n} &= [F(\phi_{1n}, k_1) - F(\psi_{1n}, k_1)], \\
L_{1n} &= \{(2 - k_1^2)[F(\phi_{1n}, k_1) - F(\psi_{1n}, k_1)] - 2[E(\phi_{1n}, k_1) - E(\psi_{1n}, k_1)]\}, \\
M_{1n} &= [(1 - k_1^2 \sin^2 \phi_{1n})^{1/2} - (1 - k_1^2 \sin^2 \psi_{1n})^{1/2}].
\end{aligned} \tag{5.10.16}$$

Here $k_1 = (1 - b_1^2/a_1^2)^{1/2}$, $K(k_1)$, $F(\phi_{1n}, k_1)$, $E(k_1)$, $E(\phi_{1n}, k_1)$ denote the complete and incomplete elliptic integrals of the first and second kind respectively; ψ_{1n} and ϕ_{1n} are defined according to formulae (A5.1.20–A5.1.21) from Appendix A5.1, namely,

$$\begin{aligned}
\phi_{1n} &= \tan^{-1} \frac{x_n y_n + (a_1^2 y_n^2 + b_1^2 x_n^2 - a_1^2 b_1^2)^{1/2}}{y_n^2 - b_1^2}, \\
\psi_{1n} &= \tan^{-1} \frac{x_n y_n - (a_1^2 y_n^2 + b_1^2 x_n^2 - a_1^2 b_1^2)^{1/2}}{y_n^2 - b_1^2}.
\end{aligned} \tag{5.10.17}$$

In the case when $\phi_{1n} < \psi_{1n}$, it should be replaced by $\pi + \phi_{1n}$. We recall that the point $(x_n, y_n) \in S_n$. If the mean value theorem is applicable, equation (5.10.15) transforms into

$$\begin{aligned}
2a_1 u_1 &= G_1 K(k_1) T_1 + G_2 [(2 - k_1^2) K(k_1) - 2E(k_1)] \bar{T}_1 / k_1^2 \\
&+ \frac{1}{2} \sum_{n=2}^N \left\{ G_1 [F(\phi_{1n}, k_1) - F(\psi_{1n}, k_1)] T_n + (G_2 / k_1^2) [(2 - k_1^2) (F(\phi_{1n}, k_1) \right. \\
&- F(\psi_{1n}, k_1)) - 2(E(\phi_{1n}, k_1) - E(\psi_{1n}, k_1))] \bar{T}_n \\
&\left. - i(G_2 / k_1^2) [(1 - k_1^2 \sin^2 \phi_{1n})^{1/2} - (1 - k_1^2 \sin^2 \psi_{1n})^{1/2}] \bar{T}_n \right\}.
\end{aligned} \tag{5.10.18}$$

It is important to note that in the case of symmetric configuration, some of the terms in (5.10.18) vanish; the parameters ϕ_{1n} and ψ_{1n} do not necessarily correspond to the same point (x_n, y_n) ; they should be interpreted as fuzzy quantities which can assume any value corresponding to $(x_n, y_n) \in S_n$. Equation (5.10.18) represents the first of the set of N linear algebraic equations with fuzzy coefficients, which can be obtained in a similar manner. The solution can also

be interpreted as a fuzzy domain. Analysis of this domain will allow us to find the upper and lower bounds for the quantities of interest. As we shall see further, in some cases the variation between the upper and the lower bounds becomes so small that allows us to obtain a reasonably accurate solution to the problem.

When the acting forces are known, the set of equations define the tangential displacements of the punches. In the case when the displacements are known, the set of linear algebraic equations has to be solved for the forces T_n . It is also important to notice that each equation in the set is valid in a system of coordinates located at the center of the ellipse.

Example: Two equal elliptical punches. Consider the case where $N=2$, $a_1=a_2=a$, $b_1=b_2=b$, $X_1=Y_1=0$, $X_2=l$, $Y_2=0$, $\theta_1=\theta_2=0$. Let the tangential displacements be prescribed as $u_1=u_2=u$ where u is a real quantity. Due to the symmetry of the system, we may also assume $T_1=T_2=T$, and $\Im[T]=0$ (i.e. the resulting force is directed along the axis Ox). Taking into consideration the properties

$$\begin{aligned} D_{1n}(x,y) &= D_{1n}(-x,y) = D_{1n}(x,-y) = D_{1n}(-x,-y), \\ L_{1n}(x,y) &= L_{1n}(-x,y) = L_{1n}(x,-y) = L_{1n}(-x,-y), \\ M_{1n}(x,y) &= -M_{1n}(-x,y) = -M_{1n}(x,-y) = M_{1n}(-x,-y), \end{aligned} \quad (5.10.19)$$

we have

$$\int \int_{S_n} M_{1n} \tau dS_n = 0, \quad (5.10.20)$$

and the set of equations, equivalent to (5.10.15), reduces to just one equation, namely,

$$\begin{aligned} 2au &= 2\{G_1 K(k)T + (G_2/k^2)[(2 - k^2)K(k) - 2E(k)]\bar{T}\} \\ &- \{G_1 F(k,\phi)T + (G_2/k^2)[(2 - k^2)F(k,\phi) - 2E(k,\phi)]\bar{T}\}, \end{aligned} \quad (5.10.21)$$

where, due to symmetry, ϕ is defined as

$$\phi = \cos^{-1} \left[\frac{b}{(x^2 - a^2 + b^2)^{1/2}} \right] \quad (5.10.22)$$

Introduce the notation

$$T_0 = \frac{2au}{G_1\mathbf{K}(k) + (G_2/k^2)[(2 - k^2)\mathbf{K}(k) - 2\mathbf{E}(k)]} \quad (5.10.23)$$

where T_0 can be interpreted as the force needed to produce the displacement u when acting on an isolated punch. Equation (5.10.21) can be rewritten as follows:

$$T = \frac{T_0}{2 - B}, \quad B = \frac{G_1\mathbf{F}(k,\phi) + (G_2/k^2)[(2 - k^2)\mathbf{F}(k,\phi) - 2\mathbf{E}(k,\phi)]}{G_1\mathbf{K}(k) + (G_2/k^2)[(2 - k^2)\mathbf{K}(k) - 2\mathbf{E}(k)]} \quad (5.10.24)$$

Since $B \geq 0$, one may conclude that the punch interaction reduces the force needed to produce the displacement u , as compared to an isolated punch. The upper and lower bounds for T can be obtained from (5.10.24) by taking

$$\phi = \cos^{-1} \left\{ \frac{b}{[(l + a)^2 - a^2 + b^2]^{1/2}} \right\},$$

and

$$\phi = \cos^{-1} \left\{ \frac{b}{[(l - a)^2 - a^2 + b^2]^{1/2}} \right\}. \quad (5.10.25)$$

respectively. We shall also consider the central estimation for T defined by

$$\phi = \cos^{-1} \left\{ \frac{b}{(l^2 - a^2 + b^2)^{1/2}} \right\}.$$

We can consider in a similar manner the case when the displacement is prescribed in the Oy direction by the formal substitution of u by iu , and T by iT in expression (5.10.21). The result is

$$T = \frac{T_0}{2 - C}, \quad C = \frac{G_1\mathbf{F}(k,\phi) - (G_2/k^2)[(2 - k^2)\mathbf{F}(k,\phi) - 2\mathbf{E}(k,\phi)]}{G_1\mathbf{K}(k) - (G_2/k^2)[(2 - k^2)\mathbf{K}(k) - 2\mathbf{E}(k)]}. \quad (5.10.26)$$

Figure 5.10.1 plots the ratio T/T_0 versus l/a for $a=2$, $b=1$, $(G_2/G_1)=1/3$, with the

tangential displacements being directed along the axis Ox . The upper and lower bounds are given by circles, while the central estimation is given by a solid line.

Fig. 5.10.1. Interaction between two elliptical punches,
with shifting load in the Ox direction

As one can see, the maximum possible error of the central estimation is less than 15% for $l/a \geq 2.5$, it is less than 10% for $l/a \geq 3$, it is less than 7% for $l/a > 3.5$, and it is less than 5% for $l/a \geq 4$. Since there is no accurate solution available for this case, it is difficult to say how great is the real error of the central estimation, but there is a reason to believe that it is much less than indicated above. This belief is supported by the same argument as in section 5.9.

Figure 5.10.2 gives the same plot for the case when the tangential displacements are prescribed in the Oy direction. The respective errors are 12%, 7%, 5%, and 3.5%. One can see that the accuracy of the central estimation is significantly better than in the first case. This should be expected since the interaction in the case of displacements prescribed along the axis Ox is stronger than in the second case.

Discussion. It is appropriate to consider certain limiting cases. In the case of a circular punch the eccentricity $k_1 \rightarrow 0$, and formula (5.10.15) will simplify

Fig. 5.10.2. Interaction between two elliptical punches, with shifting load in the Oy direction

$$\begin{aligned}
 2a_1 u_1 = & \frac{\pi}{2} G_1 T_1 + \sum_{n=2}^N \left\{ G_1 \int_{S_n} \int \sin^{-1} \left[\frac{a_1}{(x^2 + y^2)^{1/2}} \right] \tau_n(x, y) \, dx dy \right. \\
 & \left. + G_2 \int_{S_n} \int \frac{a_1 (x^2 + y^2 - a_1^2)^{1/2}}{(x - iy)^2} \bar{\tau}_n(x, y) \, dx dy \right\}. \quad (5.10.27)
 \end{aligned}$$

Here we used the integrals (A5.1.22–A5.1.23). The interaction of two equal circular punches subjected to loading T in the Ox direction leads to the following expression for the central estimation

$$2au = \frac{\pi}{2} G_1 T + \left[G_1 T \sin^{-1} \left(\frac{a}{l} \right) + G_2 \frac{a(l^2 - a^2)^{1/2}}{l^2} \bar{T} \right]. \quad (5.10.28)$$

Introducing again the notation

$$T_0 = \frac{4au}{\pi G_1},$$

the following result can be obtained from (5.10.28)

$$T = \frac{T_0}{1 + \frac{2}{\pi} \left[\sin^{-1} \left(\frac{a}{l} \right) + \frac{G_2}{G_1} \frac{a(l^2 - a^2)^{1/2}}{l^2} \right]} \quad (5.10.29)$$

If the tangential displacements are prescribed in the Oy direction, we obtain

$$T = \frac{T_0}{1 + \frac{2}{\pi} \left[\sin^{-1} \left(\frac{a}{l} \right) - \frac{G_2}{G_1} \frac{a(l^2 - a^2)^{1/2}}{l^2} \right]} \quad (5.10.30)$$

All the results are valid for the case of an isotropic body, provided that $G_1 = (2 - \nu)/(2\pi\mu)$ and $G_2 = \nu/(2\pi\mu)$, where μ is the shear modulus and ν is the Poisson coefficient.

Exercise 5.10

1. Verify (5.10.6).
2. Derive (5.10.12).
3. Consider the interaction of two equal elliptical punches, when their major semi-axes are orthogonal to each other.
4. Find the generalized displacements of an elliptical punch due to the action of several concentrated loads, applied tangentially outside the punch.
Hint: Consider the procedure of shrinking the areas S_n , $n=2,3,\dots,N$, to a point. The accuracy of equation (5.10.18) will increase. In the limiting case $S_n \rightarrow 0$ formula (5.10.18) gives an *exact* solution to the problem of several concentrated forces acting outside an elliptical punch.

5.11 Contact problem for a rough punch

The term *rough* punch is used here as the opposite to the term *smooth* punch, indicating a punch which does produce shear traction at its base. Consider a circular punch, with general base, penetrating transversely isotropic elastic half-space. The punch is subjected to the action of an axial force P and a tangential force $T=kP$, acting in the Ox direction, with k being the friction

coefficient. Let the domain of contact be a circle of radius a . The shear traction τ_{zx} is assumed to be proportional to the pressure σ , namely, $\tau_{zx} = k\sigma$. The governing integral equation will take the form (due to (2.2.13)):

$$\int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0}{R} = \frac{w(\rho, \phi)}{H} - \alpha k \mathfrak{K} \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}. \quad (5.11.1)$$

Here $w(\rho, \phi)$ is the punch settlement. Let us rewrite (5.11.1) in the operator form

$$\mathcal{N}\sigma = \frac{w}{H} - \alpha k \mathcal{M}\sigma, \quad (5.11.2)$$

where \mathcal{N} and \mathcal{M} are the integral operators in the left- and the right-hand sides of (5.11.1) respectively.

The values of α and k for real bodies are less than unity, and hence their product is convenient to use as a small parameter. Let the following expansion hold

$$\sigma(\rho, \phi) = \sum_{n=0}^{\infty} (\alpha k)^n \sigma_n(\rho, \phi). \quad (5.11.3)$$

By substitution of (5.11.3) in (5.11.2) and equating the terms with equal powers of the small parameter, the following infinite system of integral equations can be obtained:

$$\mathcal{N}\sigma_0 = \frac{w}{H}, \quad \mathcal{N}\sigma_1 = -\mathcal{M}\sigma_0, \quad \dots, \quad \mathcal{N}\sigma_n = -\mathcal{M}\sigma_{n-1}, \quad \dots \quad (5.11.4)$$

The operator \mathcal{N}^{-1} , inverse to \mathcal{N} is known from (1.4.10), so we can write the formal solution to the problem as follows:

$$\sigma = \sum_{n=0}^{\infty} (\alpha k)^n (-\mathcal{N}^{-1} \mathcal{M})^n \mathcal{N}^{-1} \left(\frac{w}{H} \right) \quad (5.11.5)$$

Example. Consider the case of a flat circular punch. Let ω be the settlement of its centre, and δ be its tilting angle about the Oy axis. Then

$$w(\rho, \phi) = \omega + \delta \rho \cos \phi. \quad (5.11.6)$$

Substitution of (5.11.6) in the first equation of (5.11.4) yields

$$\sigma_0(\rho, \phi) = \frac{1}{\pi^2 H} \frac{\omega + 2\delta\rho\cos\phi}{(a^2 - \rho^2)^{1/2}}, \quad (5.11.7)$$

which is, in fact, the solution for an inclined *smooth* punch. Substitution of (5.11.7) in the second equation of (5.11.4) yields the second term of the expansion

$$\sigma_1(\rho, \phi) = \frac{2}{\pi^3 H} \left\{ \frac{\omega a \ln[(a^2 - \rho^2)^{1/2}/a]}{\rho(a^2 - \rho^2)^{1/2}} \cos\phi + \delta \left[\frac{1}{\rho} \frac{d}{d\rho} \int_{\rho}^a \frac{x^2 dx}{(x^2 - \rho^2)^{1/2}} \ln\left(\frac{a+x}{a-x}\right)^{1/2} + \frac{a}{(a^2 - \rho^2)^{1/2}} + \left(\frac{a^3 \ln[(a^2 - \rho^2)^{1/2}/a]}{\rho^2(a^2 - \rho^2)^{1/2}} + \frac{a}{2(a^2 - \rho^2)^{1/2}} \right) \cos 2\phi \right] \right\}. \quad (5.11.8)$$

The procedure may be continued, and further terms of the expansion may be obtained. It is left to the interested reader. The resultant force P and the tilting moment M can be obtained by integration of the sum of (5.11.7) and (5.11.8) as follows:

$$P = \frac{2a}{\pi H} \left(\omega + \frac{\alpha k}{\pi} a \delta \right), \quad M = \frac{2a^2}{\pi H} \left(\frac{2}{3} a \delta - \frac{\alpha k}{\pi} \omega \right). \quad (5.11.9)$$

If the point, where the normal load P is applied, is shifted along the axis Ox by the distance b from the punch centre, then $M=Pb$, and the following relationship may be established between the punch settlement ω and the tilting angle δ :

$$\delta = \frac{b + (\alpha k/\pi)a}{(2/3)a - (\alpha k/\pi)b} \frac{\omega}{a}. \quad (5.11.10)$$

Formula (5.11.10) shows that the punch tilts even in the case of a central loading. A similar effect can be observed in the case of a bonded punch (Chapter 3) and in two-dimensional contact problems (Muskhelishvili, 1946). In order to eliminate tilting, the normal load should be applied at the point $x=-\alpha ka/\pi$.

Exercise 5.11

1. Verify (5.11.5).
2. Derive (5.11.7)–(5.11.8).
3. Establish (5.11.9).

4. Find the corrective term in the expression for P (5.11.9) due to σ_2 .

$$\text{Answer: } P = \frac{2a}{\pi H} \left[\omega \left(1 - \frac{1}{6} \alpha^2 k^2 \right) + \frac{\alpha k}{\pi} a \delta \right].$$

5. Find the normal displacement w outside the punch, taking into consideration the first two terms in the solution expansion.

$$\begin{aligned} \text{Answer: } w = & \frac{2}{\pi} \left[\omega \sin^{-1} \left(\frac{a}{\rho} \right) + \delta \left(\rho \sin^{-1} \left(\frac{a}{\rho} \right) - \frac{a}{\rho} (\rho^2 - a^2)^{1/2} \right) \cos \phi \right] \\ & + \frac{4\alpha k}{\pi^2} \left\{ \omega \left[\frac{a}{\rho} \cos^{-1} \left(\frac{a}{\rho} \right) + \frac{(\rho^2 - a^2)^{1/2}}{\rho} \ln \frac{\rho}{(\rho^2 - a^2)^{1/2}} \right] \cos \phi \right. \\ & + \delta (\rho^2 - a^2)^{1/2} \ln \frac{\rho}{(\rho^2 - a^2)^{1/2}} + \frac{\delta}{\rho^2} \left[\frac{2}{3} a^3 \cos^{-1} \left(\frac{a}{\rho} \right) \right. \\ & \left. \left. + (\rho^2 - a^2)^{1/2} \left(\frac{\rho^2 + 2a^2}{3} \ln \frac{\rho}{(\rho^2 - a^2)^{1/2}} - \frac{a^2}{6} \right) \right] \cos 2\phi \right\}. \end{aligned}$$

6. Try to find an exact closed form solution to the problem.

Appendix A5.1

Here the method of computation of several integrals over an elliptical domain is given. We introduce the notation

$$R = [(x - x_0)^2 + (y - y_0)^2]^{1/2}$$

$$Z_0 = \left[1 - \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \right]^{1/2}$$

The following integral is to be computed

$$I = \iint_S \frac{f(x_0, y_0) dx_0 dy_0}{R Z_0}, \quad (\text{A5.1.1})$$

where S is an ellipse with semiaxes a and b ($a \geq b$), and f is a general function. We use the method of Rostovtsev (1961), slightly modified in order to simplify

the computations. Introduce a new system of polar coordinates (r, θ) , with the origin at the point (x, y) , namely,

$$x_0 = x + r \cos \theta, \quad y_0 = y + r \sin \theta. \quad (\text{A5.1.2})$$

First of all, consider the case when $(x, y) \in S$. Substitution of (A5.1.2) in (A5.1.1) yields

$$I = \int_0^{2\pi} \frac{abd\theta}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)^{1/2}} \int_0^{r_1} \frac{f(x + r \cos \theta, y + r \sin \theta) dr}{[(r_1 - r)(r + r_2)]^{1/2}}, \quad (\text{A5.1.3})$$

where r_1 and $-r_2$ are the roots of algebraic equation

$$\frac{(x + r \cos \theta)^2}{a^2} + \frac{(y + r \sin \theta)^2}{b^2} - 1 = 0,$$

namely,

$$r_{1,2} = \frac{(Q^2 + PZ)^{1/2} \mp Q}{P}, \quad Q = \frac{x \cos \theta}{a^2} + \frac{y \sin \theta}{b^2},$$

$$P = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}, \quad Z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \quad (\text{A5.1.4})$$

Now, we may split the integral (A5.1.3) in two, according to the scheme

$$\int_0^{2\pi} d\theta \int_0^{r_1} dr = \int_0^{\pi} d\theta \int_0^{r_1} dr + \int_{\pi}^{2\pi} d\theta \int_0^{r_1} dr. \quad (\text{A5.1.5})$$

Make a formal replacement of θ by $\pi + \theta$ in the second integral of (A5.1.5). Taking into consideration that $r_1(\theta) = r_2(\pi + \theta)$, transformation of (A5.1.5) may proceed as follows

$$\int_0^{2\pi} d\theta \int_0^{r_1} dr = \int_0^{\pi} d\theta \int_0^{r_1} dr + \int_0^{\pi} d\theta \int_0^{r_2} dr = \int_0^{\pi} d\theta \int_0^{r_1} dr + \int_0^{\pi} d\theta \int_{-r_2}^0 dr. \quad (\text{A5.1.6})$$

In the last term of (A5.1.6) we have made a formal replacement of r by $-r$.

Taking into consideration that the expression $f(x+r\cos\theta, y+r\sin\theta)$ remains unchanged with the replacements of θ by $\pi+\theta$ and r by $-r$, the two integrals in (A5.1.6) can be combined, and the final result will read

$$I = \int_0^\pi \frac{abd\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} \int_{-r_2}^{r_1} \frac{f(x+r\cos\theta, y+r\sin\theta)dr}{[(r_1 - r)(r + r_2)]^{1/2}}, \quad (\text{A5.1.7})$$

We recall the integral

$$\int_{-r_2}^{r_1} \frac{r^n dr}{[(r_1 - r)(r + r_2)]^{1/2}} = r_1^n \sum_{k=0}^n \frac{\Gamma(n + \frac{1}{2} - k) \Gamma(\frac{1}{2} + k)}{\Gamma(n + 1 - k) \Gamma(k + 1)} \left[-\frac{r_2}{r_1}\right]^k. \quad (\text{A5.1.8})$$

Now various integrals can be computed fairly easily. Here are the results which are used in the main text of this book:

$$\iint_S \frac{dx_0 dy_0}{RZ_0} = \pi \int_0^\pi \frac{abd\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} = 2\pi bK(k), \quad (\text{A5.1.9})$$

$$\iint_S \frac{q dx_0 dy_0}{\bar{q}RZ_0} = \pi \int_0^\pi \frac{abe^{2i\theta} d\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} = 2\pi b[(2 - k^2)K(k) - 2E(k)]/k^2. \quad (\text{A5.1.10})$$

Now consider the case when the point $(x,y) \notin S$. In this case substitution of (A5.1.2) in (A5.1.1) yields

$$I = \int_{\alpha_1}^{\alpha_2} \frac{abd\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} \int_{-r_2}^{r_1} \frac{f(x + r\cos\theta, y + r\sin\theta) dr}{[(r_1 - r)(r + r_2)]^{1/2}}, \quad (\text{A5.1.11})$$

where α_1 and α_2 are the angles between the axis Ox and the two lines passing through the point (x,y) tangentially to the ellipse. We can find these angles by using classical formulae of differential geometry. The parametric equation of the ellipse can be written

$$X = acost, \quad Y = bsint. \quad (\text{A5.1.12})$$

The equation of the lines passing through the point (x,y) tangentially to the ellipse may be written as

$$\frac{y - bsint}{bcost} = \frac{x - acost}{-asint}. \quad (\text{A5.1.13})$$

Equation (A5.1.13) can be solved for $sint$, and the two solutions are

$$(\text{sint})_{1,2} = \frac{b[a^2y \mp x(x^2b^2 + y^2a^2 - a^2b^2)^{1/2}]}{x^2b^2 + y^2a^2}. \quad (\text{A5.1.14})$$

This gives the following two expressions for $cost$

$$(\text{cost})_{1,2} = \frac{a[b^2x \pm y(x^2b^2 + y^2a^2 - a^2b^2)^{1/2}]}{x^2b^2 + y^2a^2}. \quad (\text{A5.1.15})$$

Now we can write

$$\tan\alpha_{1,2} = \frac{y - bsint}{x - acost}. \quad (\text{A5.1.16})$$

Substitution of (A5.1.14–A5.1.15) in (A5.1.16) yields

$$\tan\alpha_{1,2} = \frac{xy \mp (a^2y^2 + b^2x^2 - a^2b^2)^{1/2}}{x^2 - a^2}. \quad (\text{A5.1.17})$$

We can now compute the integrals

$$\iint_S \frac{dx_0 dy_0}{RZ_0} = \pi \int_{\alpha_1}^{\alpha_2} \frac{abd\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} = \pi b[F(\phi, k) - F(\psi, k)], \quad (\text{A5.1.18})$$

$$\begin{aligned} \iint_S \frac{q \, dx_0 dy_0}{\bar{q}RZ_0} &= \pi \int_{\alpha_1}^{\alpha_2} \frac{abe^{2i\theta} \, d\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{1/2}} \\ &= \pi b\{(2 - k^2)[F(\phi, k) - F(\psi, k)] - 2[E(\phi, k) - E(\psi, k)]\} \end{aligned}$$

$$- 2i[(1 - k^2 \sin^2 \phi)^{1/2} - (1 - k^2 \sin^2 \psi)^{1/2}]/k^2. \quad (\text{A5.1.19})$$

Here

$$\tan \phi = \cot \alpha_1 = \frac{xy + (a^2 y^2 + b^2 x^2 - a^2 b^2)^{1/2}}{y^2 - b^2}. \quad (\text{A5.1.20})$$

$$\tan \psi = \cot \alpha_2 = \frac{xy - (a^2 y^2 + b^2 x^2 - a^2 b^2)^{1/2}}{y^2 - b^2}. \quad (\text{A5.1.21})$$

In the limiting case of a circle, $b \rightarrow a$, $k \rightarrow 0$, and formulae (A5.1.18–A5.1.19) will take the form

$$\int_S \int \frac{dx_0 dy_0}{RZ_0} = 2\pi a \sin^{-1} \frac{a}{(x^2 + y^2)^{1/2}}, \quad (\text{A5.1.22})$$

$$\int_S \int \frac{q dx_0 dy_0}{qRZ_0} = 2\pi a^2 \frac{(x^2 + y^2 - a^2)^{1/2}}{(x - iy)^2}. \quad (\text{A5.1.23})$$

The more complicated integrals can be computed in the same manner.

Here are some additional integrals, where x and y are inside the domain S .

$$\int_S \int \frac{x_0^2 dx_0 dy_0}{RZ_0} = \pi b^3 \frac{1}{k^2} \left[\frac{1}{1-k^2} \mathbf{E} - \mathbf{K} \right] + \pi b x^2 \frac{1}{k^4} \left[2\mathbf{K} - (2+k^2)\mathbf{E} \right] + \pi b y^2 \frac{1}{k^4} \left[2\mathbf{E} - (2-k^2)\mathbf{K} \right]$$

$$\int_S \int \frac{x_0 y_0 dx_0 dy_0}{RZ_0} = 2\pi b xy \frac{1-k^2}{k^4} \left[\frac{2-k^2}{1-k^2} \mathbf{E} - 2\mathbf{K} \right]$$

$$\int_S \int \frac{y_0^2 dx_0 dy_0}{RZ_0} = \pi b^3 \frac{1}{k^2} \left[\mathbf{K} - \mathbf{E} \right] + \pi b x^2 \frac{1-k^2}{k^4} \left[2\mathbf{E} - (2-k^2)\mathbf{K} \right]$$

$$+ \pi b y^2 \frac{1}{k^4} \left[2(1-k^2)^2 \mathbf{K} - (2-3k^2)\mathbf{E} \right]$$

$$\int_S \int \frac{x_0^3 dx_0 dy_0}{RZ_0} = \pi b^3 x \frac{1}{k^4} \left[\frac{2-k^2}{1-k^2} \mathbf{E} - 2\mathbf{K} \right] + \pi b xy^2 \frac{1}{k^6} \left[(-8+5k^2)\mathbf{K} + (8-k^2)\mathbf{E} \right]$$

$$+\pi b x^3 \frac{1}{3k^6} \left[(8-3k^2+k^4)\mathbf{K} - (8+k^2+2k^4)\mathbf{E} \right]$$

$$\int_S \int \frac{x_0^2 y_0 dx_0 dy_0}{RZ_0} = \pi b^3 y \frac{1}{k^4} \left[(2-k^2)\mathbf{K} - 2\mathbf{E} \right]$$

$$+\pi b x^2 y \frac{1}{k^6} \left[(-8+9k^2-k^4)\mathbf{K} + (8-5k^2-k^4)\mathbf{E} \right]$$

$$+\pi b y^3 \frac{1}{3k^6} \left[(8-11k^2+3k^4)\mathbf{K} - (8-7k^2)\mathbf{E} \right]$$

$$\int_S \int \frac{x_0 y_0^2 dx_0 dy_0}{RZ_0} = \pi b^3 x \frac{1}{k^4} \left[(2-k^2)\mathbf{K} - 2\mathbf{E} \right]$$

$$+\pi b x y^2 \frac{1}{k^6} \left[(8-15k^2+7k^4)\mathbf{K} + (-8+11k^2-2k^4)\mathbf{E} \right]$$

$$+\pi b x^3 \frac{1}{3k^6} \left[(8-9k^2+k^4)\mathbf{E} - (8-13k^2+5k^4)\mathbf{K} \right]$$

$$\int_S \int \frac{y_0^3 dx_0 dy_0}{RZ_0} = \pi b^3 y \frac{1}{k^4} \left[(2-k^2)\mathbf{E} - 2(1-k^2)\mathbf{K} \right]$$

$$+\pi b y^3 \frac{1}{3k^6} \left[(8-17k^2+11k^4)\mathbf{E} + (-8+21k^2-19k^4+6k^6)\mathbf{K} \right]$$

$$+\pi b x^2 y \frac{1}{k^6} \left[(8-19k^2+14k^4-3k^6)\mathbf{K} - (8-15k^2+7k^4)\mathbf{E} \right]$$

$$\int_S \int \frac{x_0^4 dx_0 dy_0}{RZ_0} = \pi b x^4 \frac{1}{12k^8} \left[(-48+8k^2-4k^4-6k^6)\mathbf{E} + (48-32k^2+5k^4+3k^6)\mathbf{K} \right]$$

$$+\pi b x^2 y^2 \frac{1}{2k^8} \left[(48-16k^2-k^4)\mathbf{E} - (48-40k^2+4k^4)\mathbf{K} \right]$$

$$\begin{aligned}
& +\pi b y^4 \frac{1}{4k^8} \left[(16-16k^2+3k^4)\mathbf{K} - (16-8k^2)\mathbf{E} \right] \\
& +\pi b^3 x^2 \frac{1}{2k^6} \left[\frac{8-5k^2-k^4}{1-k^2} \mathbf{E} - (8-k^2)\mathbf{K} \right] +\pi b^3 y^2 \frac{1}{2k^6} \left[(8-3k^2)\mathbf{K} - \frac{8-7k^2}{1-k^2} \mathbf{E} \right] \\
& +\pi b^5 \frac{1}{4k^4(1-k^2)} \left[\frac{-2+4k^2}{1-k^2} \mathbf{E} + (2-3k^2)\mathbf{K} \right] \\
\int_S \int \frac{x_0^3 y_0 dx_0 dy_0}{RZ_0} & = \pi b x^3 y \frac{1}{3k^8} \left[(48-40k^2-2k^6)\mathbf{E} - (1-k^2)(48-16k^2+k^4)\mathbf{K} \right] \\
& +\pi b x y^3 \frac{1}{k^8} \left[(1-k^2)(16-8k^2)\mathbf{K} - (16-16k^2+k^4)\mathbf{E} \right] \\
& +\pi b^3 x y \frac{1}{k^6} \left[(8-5k^2)\mathbf{K} - (8-k^2)\mathbf{E} \right] \\
\int_S \int \frac{x_0^2 y_0^2 dx_0 dy_0}{RZ_0} & = \pi b x^4 \frac{(1-k^2)}{12k^8} \left[(48-16k^2-k^4)\mathbf{E} - (48-40k^2+4k^4)\mathbf{K} \right] \\
& +\pi b x^2 y^2 \frac{1}{2k^8} \left[(-48+72k^2-20k^4-2k^6)\mathbf{E} + (1-k^2)(48-48k^2+5k^4)\mathbf{K} \right] \\
& +\pi b y^4 \frac{1}{12k^8} \left[(48-80k^2+31k^4)\mathbf{E} - (1-k^2)(48-56k^2+12k^4)\mathbf{K} \right] \\
& +\pi b^3 x^2 \frac{1}{2k^6} \left[(8-7k^2+k^4)\mathbf{K} - (8-3k^2)\mathbf{E} \right] \\
& +\pi b^3 y^2 \frac{1}{2k^6} \left[(8-5k^2)\mathbf{E} - (8-9k^2+2k^4)\mathbf{K} \right] +\pi b^5 \frac{1}{4k^4} \left[\frac{2-k^2}{1-k^2} \mathbf{E} - 2\mathbf{K} \right] \\
\int_S \int \frac{x_0 y_0^3 dx_0 dy_0}{RZ_0} & = \pi b x^3 y \frac{(1-k^2)}{k^8} \left[(1-k^2)(16-8k^2)\mathbf{K} - (16-16k^2+k^4)\mathbf{E} \right]
\end{aligned}$$

$$\begin{aligned}
& + \pi b x y^3 \frac{1}{3k^8} \left[(48 - 104k^2 + 64k^4 - 6k^6)E - (1 - k^2)(48 - 80k^2 + 33k^4)K \right] \\
& + \pi b^3 x y \frac{1}{k^6} \left[(8 - 7k^2)E - (1 - k^2)(8 - 3k^2)K \right] \\
\int_S \int \frac{y_0^4 dx_0 dy_0}{RZ_0} & = \pi b x^4 \frac{(1 - 2k^2 + k^4)}{4k^8} \left[(16 - 16k^2 + 3k^4)K - (16 - 8k^2)E \right] \\
& + \pi b x^2 y^2 \frac{(1 - k^2)}{2k^8} \left[(48 - 80k^2 + 31k^4)E - (1 - k^2)(48 - 56k^2 + 12k^4)K \right] \\
& + \pi b y^4 \frac{1}{12k} \left[(-48 + 136k^2 - 132k^4 + 50k^6)E + (1 - k^2)(48 - 112k^2 + 85k^4 - 24k^6)K \right] \\
& + \pi b^3 x^2 \frac{(1 - k^2)}{2k^6} \left[(8 - k^2)E - (8 - 5k^2)K \right] \\
& + \pi b^3 y^2 \frac{1}{2k} \left[(-8 + 11k^2 - 2k^4)E + (1 - k^2)(8 - 7k^2)K \right] + \pi b^5 \frac{1}{4k^4} \left[(2 + k^2)K - (2 + 2k^2)E \right]
\end{aligned}$$

The case where x and y are outside the domain S :

$$\begin{aligned}
\int_S \int \frac{x_0 dx_0 dy_0}{RZ_0} & = \pi b x \frac{1}{k^2} \left[[F(\phi) - F(\psi)] - [E(\phi) - E(\psi)] \right] \\
& + \pi b x \left[\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right] + \pi b y \frac{1}{k^2} \left[\frac{1}{\Delta(\psi)} - \frac{1}{\Delta(\phi)} \right] \\
\int_S \int \frac{y_0 dx_0 dy_0}{RZ_0} & = \pi b y \frac{1}{k^2} \left[[E(\phi) - E(\psi)] - (1 - k^2)[F(\phi) - F(\psi)] \right] \\
& - \pi b y \left[\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right] + \pi b x \frac{1 - k^2}{k^2} \left[\frac{1}{\Delta(\psi)} - \frac{1}{\Delta(\phi)} \right]
\end{aligned}$$

$$\begin{aligned}
\int_S \int \frac{x_0^2 dx_0 dy_0}{RZ_0} &= \pi b x^2 \frac{1}{2k^4} \left[-(2+k^2)[E(\phi) - E(\psi)] + 2[F(\phi) - F(\psi)] \right] \\
&+ \pi b x^2 \frac{1}{2k^2} \left[(2+k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - (1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right] \\
&+ \pi b x y \frac{1-k^2}{k^4} \left[\left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) - \frac{3-k^2}{1-k^2} \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) \right] \\
&+ \pi b y^2 \frac{1}{2k^4} \left[2[E(\phi) - E(\psi)] - (2-k^2)[F(\phi) - F(\psi)] \right] \\
&+ \pi b y^2 \frac{1}{2k^2} \left[-2 \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + \frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right] \\
&- \pi b^3 \frac{1}{2(1-k^2)} \left[\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right] \\
\int_S \int \frac{x_0 y_0 dx_0 dy_0}{RZ_0} &= \pi b x^2 \frac{1-k^2}{2k^4} \left[-3 \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (1-k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) \right] \\
&+ \pi b x y \frac{1-k^2}{k^4} \left[\left(2 + \frac{k^2}{1-k^2} \right) [E(\phi) - E(\psi)] - 2[F(\phi) - F(\psi)] \right] \\
&+ \pi b x y \frac{1-k^2}{k^2} \left[\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} - \frac{2-k^2}{1-k^2} \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) \right] \\
&+ \pi b y^2 \frac{1-k^2}{2k^4} \left[3 \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - \frac{1}{\Delta^3(\phi)} + \frac{1}{\Delta^3(\psi)} \right] + \pi b^3 \frac{1}{2k^2} \left[\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right] \\
\int_S \int \frac{y_0^2 dx_0 dy_0}{RZ_0} &= \pi b x^2 \frac{1-k^2}{2k^4} \left[2[E(\phi) - E(\psi)] - (2-k^2)[F(\phi) - F(\psi)] \right] \\
&+ \pi b x^2 \frac{1-k^2}{2k^2} \left[-2 \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + (1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \pi bxy \frac{1-k^2}{k^4} \left[(k^2-1) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) + (3-2k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) \right] \\
& + \pi by^2 \frac{1}{2k^4} \left[(3k^2-2)[E(\phi) - E(\psi)] + 2(1-k^2)^2[F(\phi) - F(\psi)] \right] \\
& + \pi by^2 \frac{1}{2k^2} \left[(2-3k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - (1-k^2) \frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right] \\
& - \pi b^3 \frac{1}{2} \left[\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right] \\
\int_S \int \frac{x_0^3 dx_0 dy_0}{RZ_0} & = \pi bx^3 \frac{1}{6k^6} \left[(8-3k^2+k^4)[F(\phi) - F(\psi)] - (8+k^2+2k^4)[E(\phi) - E(\psi)] \right] + \\
& + \pi bx^3 \frac{1}{6k^4} \left[(8+k^2+2k^4) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + \right. \\
& \quad \left. + (-7+6k^2+k^4) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) + 3(1-k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi bx^2 y \frac{1}{2k^6} \left[(9k^2-15) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + \right. \\
& \quad \left. + (1-k^2)(10-3k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) - 3(1-k^2)^2 \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi bxy^2 \frac{1}{2k^6} \left[(-8+5k^2)[F(\phi) - F(\psi)] - (8-k^2)[E(\phi) - E(\psi)] \right] \\
& + \pi bxy^2 \frac{1}{2k^4(1-k^2)} \left[(-8+3k^2+2k^4) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + \right. \\
& \quad \left. + (7-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) - 3(1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] + \\
& + \pi by^3 \frac{1}{6k^6} \left[3(5-3k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (10-8k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) + 3(1-k^2) \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \pi x b^3 \frac{1}{2k^4(1-k^2)} \left[-2[F(\phi) - F(\psi)] + (2-k^2)[E(\phi) - E(\psi)] \right] \\
& + \pi x b^3 \frac{1}{2k^2(1-k^2)} \left[(-2+k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + (1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right] \\
& + \pi y b^3 \frac{1}{2k^4} \left[3 \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) \right] \\
\int_S \int \frac{x_0^2 y_0 dx_0 dy_0}{RZ_0} & = \pi b x^3 \frac{1}{6k^6} \left[-3(1-k^2)(5-k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + \right. \\
& + (1-k^2)^2(10-k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) - 3(1-k^2)^3 \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \left. \right] \\
& + \pi b x^2 y \frac{1}{2k^6} \left[-(8-k^2)(1-k^2)[F(\phi) - F(\psi)] + (8-5k^2-k^4)[E(\phi) - E(\psi)] \right] \\
& + \pi b x^2 y \frac{1}{2k^4} \left[(8-9k^2+3k^4) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + \right. \\
& \quad \left. + (7-k^2)(1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) - 3(1-k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi b x y^2 \frac{1-k^2}{2k^6} \left[(15-3k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - (10-6k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) + \right. \\
& \quad \left. + 3(1-k^2) \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi b y^3 \frac{1}{6k^6} \left[(8-3k^2)(1-k^2)[F(\phi) - F(\psi)] - (8-7k^2)[E(\phi) - E(\psi)] \right] \\
& + \pi b y^3 \frac{1}{6k^4} \left[(8-7k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - 7(1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) + \right. \\
& \quad \left. + 3(1-k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \pi x b^3 \frac{1}{2k^4} \left[(3 - k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - (1 - k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) \right] \\
& + \pi y b^3 \frac{1}{2k^4} \left[(2 - k^2) [F(\phi) - F(\psi)] - 2[E(\phi) - E(\psi)] \right] \\
& + \pi y b^3 \frac{1}{2k^2} \left[2 \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right] \\
\int_S \int \frac{x_0 y_0^2 dx_0 dy_0}{RZ_0} & = \pi b x^3 \frac{1 - k^2}{6k^6} \left[-(8 - 5k^2) [F(\phi) - F(\psi)] + (8 - k^2) [E(\phi) - E(\psi)] \right] \\
& + \pi b x^3 \frac{1 - k^2}{6k^4} \left[-(8 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + 7(1 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \right. \right. \\
& \quad \left. \left. - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) - 3(1 - k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi b x^2 y \frac{1 - k^2}{2k^6} \left[(15 - 12k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - (1 - k^2)(10 - 4k^2) \left(\frac{1}{\Delta^3(\phi)} - \right. \right. \\
& \quad \left. \left. - \frac{1}{\Delta^3(\psi)} \right) + 3(1 - k^2)^2 \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi b x y^2 \frac{1}{2k^6} \left[(8 - 7k^2)(1 - k^2) [F(\phi) - F(\psi)] + (-8 + 11k^2 - 2k^4) [E(\phi) - E(\psi)] \right] \\
& + \pi b x y^2 \frac{1}{2k^4} \left[(8 - 11k^2 + 2k^4) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - \right. \\
& \quad \left. - (7 - 5k^2)(1 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) + 3(1 - k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi b y^3 \frac{1 - k^2}{6k^6} \left[-3(5 - 4k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (10 - 9k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) - 3(1 - k^2) \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi x b^3 \frac{1}{2k^4} \left[(2 - k^2) [F(\phi) - F(\psi)] - 2[E(\phi) - E(\psi)] \right]
\end{aligned}$$

$$\begin{aligned}
& + \pi x b^3 \frac{1}{2k^2} \left[2 \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - (1 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right] \\
& + \pi y b^3 \frac{1}{2k^4} \left[-(3 - k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (1 - k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) \right] \\
\int_S \int \frac{y_0^3 dx_0 dy_0}{RZ_0} = & \pi b x^3 \frac{(1 - k^2)^2}{6k^6} \left[3(5 - 2k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) - (1 - k^2)(10 - 2k^2) \left(\frac{1}{\Delta^3(\phi)} - \right. \right. \\
& \left. \left. - \frac{1}{\Delta^3(\psi)} \right) + 3(1 - k^2)^2 \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi b x^2 y \frac{1 - k^2}{2k^6} \left[(8 - k^2)(1 - k^2)[F(\phi) - F(\psi)] - (8 - 7k^2)[E(\phi) - E(\psi)] \right] \\
& + \pi b x^2 y \frac{1 - k^2}{2k^4} \left[(8 - 7k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) - (7 - 3k^2)(1 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right. \\
& \left. + 3(1 - k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi b x y^2 \frac{(1 - k^2)^2}{2k^6} \left[-(15 - 6k^2) \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (10 - 7k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) - 3(1 - k^2) \left(\frac{1}{\Delta^5(\phi)} - \frac{1}{\Delta^5(\psi)} \right) \right] \\
& + \pi b y^3 \frac{1}{6k^6} \left[-(8 - 13k^2 + 6k^4)(1 - k^2)[F(\phi) - F(\psi)] + (8 - 17k^2 + 11k^4)[E(\phi) - E(\psi)] \right] \\
& + \pi b y^3 \frac{1}{6k^4} \left[(-8 + 17k^2 - 11k^4) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + (1 - k^2)(7 - 8k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right. \\
& \left. - 3(1 - k^2)^2 \left(\frac{\sin \phi \cos \phi}{\Delta^5(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^5(\psi)} \right) \right] \\
& + \pi x b^3 \frac{1 - k^2}{2k^4} \left[-3 \left(\frac{1}{\Delta(\phi)} - \frac{1}{\Delta(\psi)} \right) + (1 - k^2) \left(\frac{1}{\Delta^3(\phi)} - \frac{1}{\Delta^3(\psi)} \right) \right] \\
& + \pi y b^3 \frac{1}{2k^4} \left[-2(1 - k^2)[F(\phi) - F(\psi)] + (2 - k^2)[E(\phi) - E(\psi)] \right]
\end{aligned}$$

$$+ \pi y b^3 \frac{1}{2k^2} \left[- (2 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta(\phi)} - \frac{\sin \psi \cos \psi}{\Delta(\psi)} \right) + (1 - k^2) \left(\frac{\sin \phi \cos \phi}{\Delta^3(\phi)} - \frac{\sin \psi \cos \psi}{\Delta^3(\psi)} \right) \right]$$

In the integrals above, the abbreviations $F(\phi)$, $E(\phi)$, $F(\psi)$, $E(\psi)$ were used to denote $F(\phi, k)$, $E(\phi, k)$, $F(\psi, k)$, $E(\psi, k)$ respectively. The notation $\Delta(\phi)$ stands for

$$(1 - k^2 \sin^2 \phi)^{1/2}$$

The case of a circle of radius a and x, y inside the domain S :

$$\int_S \int \frac{x_0^2 dx_0 dy_0}{RZ_0} = \pi^2 a \frac{1}{16} [4a^2 + 5x^2 - y^2]$$

$$\int_S \int \frac{x_0 y_0 dx_0 dy_0}{RZ_0} = \pi^2 a \frac{3}{8} xy$$

$$\int_S \int \frac{y_0^2 dx_0 dy_0}{RZ_0} = \pi^2 a \frac{1}{16} [4a^2 - x^2 + 5y^2]$$

$$\int_S \int \frac{x_0^3 dx_0 dy_0}{RZ_0} = \pi^2 ax \frac{1}{32} [6a^2 + 7x^2 - 3y^2]$$

$$\int_S \int \frac{x_0^2 y_0 dx_0 dy_0}{RZ_0} = \pi^2 ay \frac{1}{32} [2a^2 + 9x^2 - y^2]$$

$$\int_S \int \frac{x_0 y_0^2 dx_0 dy_0}{RZ_0} = \pi^2 ax \frac{1}{32} [2a^2 - x^2 + 9y^2]$$

$$\int_S \int \frac{y_0^3 dx_0 dy_0}{RZ_0} = \pi^2 ay \frac{1}{32} [6a^2 - 3x^2 + 7y^2]$$