CHAPTER 4

APPLICATIONS IN FRACTURE MECHANICS

The great majority of the punch and crack problems solved deals with the stresses and displacements in the plane z=0 only. Some solutions of this kind have been presented in previous chapters. There are just a few *complete* solutions published (Sneddon, 1951; Elliott, 1949; Westmann, 1965), where explicit expressions are given for the field of displacements and stresses for the simplest axisymmetric problems (a circular punch and a penny-shaped crack). The explicit expressions for the field of displacements due to an elliptic crack can be found in (Kassir and Sih, 1975). Knowledge of complete solutions is indispensable for consideration of more complicated problems of crack interactions, influence of external loads on punches and cracks, etc.

We present in this chapter a *complete* solution to the problem of a penny-shaped crack in a transversely isotropic elastic space, subjected to an arbitrary normal and tangential loading. All the relevant Green's functions are given explicitly in terms of elementary functions. An approximate analytical solution is given for a flat crack of arbitrary shape. The solution's accuracy is high, which is mainly due to the fact that it becomes exact in the case of an elliptical crack. The derivation of non-singular governing integral equations enables us to consider a very close interaction of coplanar cracks. Some of the material presented in this Chapter is still unpublished. The rest follows the papers (Fabrikant, 1987a, 1987b, 1987f, 1987g, 1988b, 1989).

4.1 Flat crack under arbitrary normal loading

A general solution to some mixed problems in terms of three harmonic functions was given in Chapter 2. We show here that in the case of a flat crack under normal loading, the general solution can be expressed through just *one* such function. Consider a transversely isotropic elastic space weakened by a flat crack S in the plane z=0, with arbitrary pressure p applied to the crack faces. Due to symmetry, the problem can be formulated as follows: find the solution to the set of differential equations (2.1.3) for a half-space $z \geq 0$, subject to the mixed boundary conditions on the plane z=0:

$$\sigma_z = -p(x,y), \text{ for } (x,y) \in S; \qquad w = 0, \text{ for } (x,y) \notin S;$$

$$\tau_z = 0 \text{ , for } -\infty < (x,y) < \infty. \tag{4.1.1}$$

These conditions can be satisfied by a representation in terms of one harmonic function. Let us put, according to (2.1.13),

$$F_1(z) = c_1 F(z_1), \qquad F_2(z) = c_2 F(z_2), \qquad F_3(z) = 0 \text{ .} \tag{4.1.2}$$

Expressions of the type $F_1(z)$ and $F(z_1)$, *etc.*, everywhere in the book should be understood as $F_1(x,y,z)$ and $F(x,y,z_1)$ respectively. The substitution of (4.1.2) and the last of expressions (2.1.12) in the third condition (4.1.1) yields:

$$c_1 = -c_2 \gamma_1 / m_1 \gamma_2 \tag{4.1.3}$$

We can represent the function F as the potential of a simple layer, i.e.

$$F(\rho,\phi,z) \equiv F(z) = \int\!\!\int_S \frac{\omega(N)\ \mathrm{d}S}{R(M,N)} \text{ ,} \tag{4.1.4}$$

where ω stands for the crack face displacement $w(x,y,0)$, $R(M,N)$ is the distance between the points $M(\rho,\phi,z)$ and $N(r,\psi,0)$, and the integration is taken over the crack domain S. Expression (4.1.4) satisfies the second condition (4.1.1) identically, due to the well known property of the potential of a simple layer. Inside the crack the same property gives:

$$\frac{\partial F}{\partial z}\bigg|_{z=0} = -2\pi\omega = -2\pi w(x,y,0) \tag{4.1.5}$$

Now expressions (4.1.2), (4.1.4), (4.1.5), and (2.1.6) give the second equation for c_1 and c_2:

$$-m_1 c_1/\gamma_1 - m_2 c_2/\gamma_2 = 1/2\pi \tag{4.1.6}$$

The constants c_1 and c_2 are determined from (4.1.3) and (4.1.6) as

$$c_1 = -\frac{\gamma_1}{2\pi(m_1 - 1)}, \qquad c_2 = -\frac{\gamma_2}{2\pi(m_2 - 1)}. \tag{4.1.7}$$

The potential functions will be given by

$$F_1(z) = -\frac{\gamma_1}{2\pi(m_1 - 1)} F(z_1), \qquad F_2(z) = -\frac{\gamma_2}{2\pi(m_2 - 1)} F(z_2). \tag{4.1.8}$$

The substitution of (4.1.8) and (2.1.12) in the first condition (4.1.1) leads to the governing integral equation:

$$p(N_0) = -\frac{1}{4\pi^2 H} \Delta \int\int_S \frac{\omega(N)\ \mathrm{d}S}{R(N_0,N)}, \tag{4.1.9}$$

where, as before, $R(N_0,N)$ stands for the distance between two points N_0 and N, and both N_0, $N \in S$. The following identities were used:

$$m_1 m_2 = 1, \qquad (m_1 - 1)/(m_1 + 1) = 2\pi A_{44} H(\gamma_1 - \gamma_2). \tag{4.1.10}$$

We next consider the penny-shaped crack in more detail. We shall return to the case of a general crack in section 4.8 below.

Green's functions for a penny-shaped crack. An exact solution in elementary functions is possible when the crack is circular. Let a be the radius of the crack. The governing integral equation (4.1.9) can be rewritten in polar coordinates as follows (see section 2.8)

$$p(\rho,\phi) = -\frac{1}{\pi^2 H\rho} \mathcal{L}\left(\frac{1}{\rho}\right) \frac{\mathrm{d}}{\mathrm{d}\rho} \int_0^{\rho} \frac{x\ \mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \mathcal{L}(x^2)$$

$$\times \frac{\mathrm{d}}{\mathrm{d}x} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho_0}\right) \omega(\rho_0,\phi). \tag{4.1.11}$$

The integral operator inverse to (4.1.11) is defined by (2.8.2)

$$\omega(\rho,\ \phi) = 4H\int_\rho^a \frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\rho_0 \mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) p(\rho_0,\ \phi). \tag{4.1.12}$$

Another form of solution can be obtained from (1.4.33), namely,

$$\omega = \frac{2}{\pi} H\int_0^{2\pi}\int_0^a \frac{p(\rho_0,\phi_0)}{R} \tan^{-1}\left(\frac{\eta}{R}\right) \rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0, \tag{4.1.13}$$

where

$$R=[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}, \qquad \eta = (a^2 - \rho^2)^{1/2}\ (a^2 - \rho_0^2)^{1/2}/a. \tag{4.1.14}$$

In this chapter, we do not restrict attention to the plane z=0: our purpose is obtaining a *complete* solution. We shall call $F(\rho,\phi,z)$, as defined by (4.1.4), the main potential function since both functions F_1 and F_2 become easily available, when F is found. The substitution of (4.1.13) in (4.1.4) allows us to express the main potential function as follows:

$$F(\rho,\phi,z) = \frac{2}{\pi}H\int_0^{2\pi}\int_0^a K(\rho,\phi,z;\ \rho_0,\phi_0)\ p(\rho_0,\phi_0)\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0, \tag{4.1.15}$$

where the Green's function K reads:

$$K(M;N_0) = K(\rho,\phi,z;\ \rho_0,\phi_0)$$

$$= \int_0^{2\pi}\int_0^a \frac{1}{R(N,N_0)} \tan^{-1}\left[\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right] \frac{r\mathrm{d}r\mathrm{d}\psi}{R(M,N)}. \tag{4.1.16}$$

Here $R(\cdot,\cdot)$ denotes the distance between respective points: $M(\rho,\phi,z)$, $N(r,\psi,0)$, and $N_0(\rho_0,\phi_0,0)$. Although we can not compute the integral in (4.1.16) in elementary functions, all its derivatives can be expressed in elementary functions, due to the fundamental integral established in section 1.6. Making use of (1.6.19), one can write:

$$\frac{\partial K}{\partial z} = -\frac{2\pi}{R(M,N_0)}\tan^{-1}\left[\frac{h}{R(M,N_0)}\right], \tag{4.1.17}$$

where

$$h = (a^2 - l_1^2)^{1/2}(a^2 - \rho_0^2)^{1/2}/a \;, \tag{4.1.18}$$

and the contraction l_1 everywhere in the book stands for $l_1(a)$, as defined by (0.18). Note that h tends to η, as defined by (4.1.14), for $z \to 0$ and $\rho < a$. Expressions (4.1.15) and (4.1.17) allow us to write:

$$\frac{\partial F}{\partial z} = -4H\int_0^{2\pi}\int_0^{a}\frac{1}{R(M,N_0)}\tan^{-1}\left[\frac{h}{R(M,N_0)}\right]p(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0. \tag{4.1.19}$$

The integral in (4.1.19), although looking difficult to compute even for p=const, can be expressed in elementary functions for any polynomial loading. This becomes evident, if we use the equivalent representations through the $\mathcal{L}$-operator (see 1.4.31):

$$\frac{\partial F}{\partial z} = -8\pi H\int_{l_2(0)}^{l_2}\frac{dx}{(x^2 - \rho^2)^{1/2}}\int_0^{g(x)}\frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)p(\rho_0,\phi). \tag{4.1.20}$$

Here

$$g(x) = x\left[1 + \frac{z^2}{\rho^2 - x^2}\right]^{1/2}, \tag{4.1.21}$$

and the contraction l_2 everywhere in the paper stands for $l_2(a)$, as defined by (0.14). Using the change of variables $x=l_2(t)$, $t=g(x)$, expression (4.1.20) can be rewritten as follows:

$$\frac{\partial F}{\partial z} = -8\pi H\int_0^{a}\frac{dl_2(t)}{[l_2^2(t) - \rho^2]^{1/2}}\int_0^{t}\frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right)p(\rho_0,\phi). \tag{4.1.22}$$

Since the function F vanishes at infinity, it can be determined from (4.1.22) in the form

$$F(\rho,\phi,z) = -8\pi H\int_{\infty}^{z} dz \int_0^a \frac{dl_2(t)}{[l_2^2(t) - \rho^2]^{1/2}} \int_0^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right) p(\rho_0,\phi). \tag{4.1.23}$$

By using the property

$$\frac{\partial l_2(t)}{\partial t} = -\frac{[l_2^2(t) - \rho^2]^{1/2}}{[\rho^2 - l_1^2(t)]^{1/2}} \frac{\partial l_1(t)}{\partial z},$$

which is a consequence of formulae (A4.1.28) and (A4.1.29) from Appendix A4.1, expression (4.1.23) can be modified as follows:

$$F(\rho,\phi,z) = 8\pi H\int_0^a dt \int_0^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \int_0^{l_1(t)} \frac{dy}{(\rho^2 - y^2)^{1/2}} \mathcal{L}\left(\frac{y^2\rho_0}{t^2\rho}\right) p(\rho_0,\phi). \tag{4.1.24}$$

Expression (4.1.24) is convenient for exact evaluation of the potential function F and proves that it can be expressed in elementary functions for arbitrary polynomial loading. A simple change of variables gives another formula, equivalent to (4.1.24):

$$F(\rho,\phi,z) = 8\pi H\int_0^a t\,dt \int_0^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \int_{l_2(t)}^{\infty} \frac{dx}{x(x^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) p(\rho_0,\phi). \tag{4.1.25}$$

We can proceed now with the remaining derivatives of the Green's function K, defined by (4.1.16). Differentiation of (4.1.16) yields:

$$\Lambda K(\rho,\phi,z;\rho_0,\phi_0) = -\int_0^{2\pi}\int_0^a \frac{\rho e^{i\phi} - re^{i\psi}}{R^3(M,N)} \tan^{-1}\left[\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right] \frac{r\,dr\,d\psi}{R(N,N_0)}. \tag{4.1.26}$$

This integral is computed in Appendix A4.3. By using (A4.3.11), one can write:

$$\Lambda K(\rho,\phi,z;\ \rho_0,\phi_0) = \frac{2\pi}{\bar{q}}\left[\frac{z}{R_0}\tan^{-1}\frac{h}{R_0} - \frac{(a^2 - \rho_0^2)^{1/2}}{\bar{s}}\tan^{-1}\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right], \tag{4.1.27}$$

where Λ is given by (2.1.5), h is defined by (4.1.18), and

$$\bar{q} = \rho e^{-i\phi} - \rho_0 e^{-i\phi_0} , \qquad \bar{s} = (a^2 - \rho\rho_0 e^{-i(\phi-\phi_0)})^{1/2} ,$$

$$R_0 = R(M,N_0) = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2} . \tag{4.1.28}$$

The other derivatives, which will be needed for the complete solution, are:

$$\frac{\partial^2}{\partial z^2} K(\rho,\phi,z; \rho_0,\phi_0) = 2\pi\left\{\frac{z}{R_0^3} \tan^{-1}\left(\frac{h}{R_0}\right) - \frac{h}{z[R_0^2 + h^2]}\left[\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2}\right]\right\}, \tag{4.1.29}$$

$$\frac{\partial}{\partial z}\Lambda K(\rho,\phi,z; \rho_0,\phi_0) = 2\pi\left\{\frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^3} \tan^{-1}\left(\frac{h}{R_0}\right)\right.$$

$$\left. + \frac{h}{R_0^2 + h^2}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^2}\right]\right\}, \tag{4.1.30}$$

$$\Lambda^2 K(\rho,\phi,z; \rho_0,\phi_0) = 2\pi\left\{\frac{(a^2 - \rho_0^2)^{1/2}}{\bar{q}\ \bar{s}}\left(\frac{2}{\bar{\bar{q}}} - \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right) \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)\right.$$

$$- \frac{z(3R_0^2 - z^2)}{\bar{q}^2 R_0^3} \tan^{-1}\left(\frac{h}{R_0}\right) + \frac{(a^2 - \rho_0^2)^{1/2}(l_2^2 - a^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\ \bar{s}^2[l_2^2 - \rho\rho_0 e^{-i(\phi-\phi_0)}]}$$

$$\left. - \frac{zh}{R_0^2 + h^2}\left[\frac{q}{\bar{q}R_0^2} - \frac{\rho^2 e^{2i\phi}}{(l_2^2 - l_1^2)(l_2^2 - \rho^2)}\right]\right\} . \tag{4.1.31}$$

This concludes the general solution to the problem of a penny-shaped crack subjected to an arbitrary pressure. Formulae (4.1.17) and (4.1.24–4.1.31) are the main results of this section.

Exercise 4.1

1. Prove the identity (4.1.10).

2. Establish (4.1.19).

3. Verify the derivation of (4.1.27)–(4.1.31).

4. Find the Green's functions for a semi-infinite plane crack in a transversely isotropic space, subjected to a normal loading.
Hint: consider the limiting case of (4.1.24–4.1.31), when the radius $a\to\infty$, and the coordinate origin moves from the circle centre to its boundary.

4.2 Point force loading of a penny-shaped crack

Consider a penny-shaped crack opened by two equal concentrated forces P applied in opposite directions at the point $(\rho_0,\phi_0,0^{\pm})$, $\rho_0<a$. Formulae (2.1.6), (2.1.12), (4.1.8), (4.1.17), and (4.1.24–4.1.31) give a complete solution for the field of displacements and stresses in elementary functions, namely,

$$u = \frac{2}{\pi} HP \left[\frac{\gamma_1}{m_1 - 1} f_1(z_1) + \frac{\gamma_2}{m_2 - 1} f_1(z_2) \right], \tag{4.2.1}$$

$$w = \frac{2}{\pi} HP \left[\frac{m_1}{m_1 - 1} f_2(z_1) + \frac{m_2}{m_2 - 1} f_2(z_2) \right], \tag{4.2.2}$$

$$\sigma_1 = \frac{2P}{\pi^2(\gamma_1 - \gamma_2)} \left\{ \left[\frac{\gamma_1}{(m_1 + 1)\gamma_3^2} - \frac{1}{\gamma_1} \right] f_3(z_1) - \left[\frac{\gamma_2}{(m_2 + 1)\gamma_3^2} - \frac{1}{\gamma_2} \right] f_3(z_2) \right\}, \tag{4.2.3}$$

$$\sigma_2 = \frac{4}{\pi} HA_{66}P \left[\frac{\gamma_1}{m_1 - 1} f_4(z_1) + \frac{\gamma_2}{m_2 - 1} f_4(z_2) \right], \tag{4.2.4}$$

$$\sigma_z = \frac{P}{\pi^2(\gamma_1 - \gamma_2)} \left[\gamma_1 f_3(z_1) - \gamma_2 f_3(z_2) \right], \tag{4.2.5}$$

$$\tau_z = \frac{P}{\pi^2(\gamma_1 - \gamma_2)} \left[f_5(z_1) - f_5(z_2) \right], \tag{4.2.6}$$

where

$$f_1(z) = \frac{1}{\bar{\bar{q}}} \left[\frac{(a^2 - \rho_0^2)^{1/2}}{\bar{s}} \tan^{-1}\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}} - \frac{z}{R_0} \tan^{-1}\frac{h}{R_0} \right], \tag{4.2.7}$$

$$f_2(z) = \frac{1}{R_0} \tan^{-1}\left(\frac{h}{R_0} \right) , \tag{4.2.8}$$

$$f_3(z) = \left\{ -\frac{z}{R_0^3} \tan^{-1}\left(\frac{h}{R_0} \right) + \frac{h}{z(R_0^2 + h^2)} \left[\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2} \right] \right\} , \tag{4.2.9}$$

$$f_4(z) = \frac{(a^2 - \rho_0^2)^{1/2}}{\bar{q}\ \bar{s}} \left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2} - \frac{2}{\bar{\bar{q}}} \right) \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}} \right)$$

$$+ \frac{z(3R_0^2 - z^2)}{\bar{q}^2 R_0^3} \tan^{-1}\left(\frac{h}{R_0}\right) - \frac{(a^2 - \rho_0^2)^{1/2}(l_2^2 - a^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\ \bar{s}^2[l_2^2 - \rho\rho_0 e^{-i(\phi-\phi_0)}]}$$

$$+ \frac{zh}{R_0^2 + h^2} \left[\frac{q}{\bar{q}R_0^2} - \frac{\rho^2 e^{2i\phi}}{(l_2^2 - l_1^2)(l_2^2 - \rho^2)} \right], \tag{4.2.10}$$

$$f_5(z) = -\left\{ \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^3} \tan^{-1}\left(\frac{h}{R_0}\right) + \frac{h}{R_0^2 + h^2}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^2} \right] \right\} . \tag{4.2.11}$$

It is reminded that $R_0=[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}$. The expression for σ_z (4.2.5) simplifies when $z=0$ and $\rho>a$, namely,

$$\sigma_z = \frac{P}{\pi^2} \frac{(a^2 - \rho_0^2)^{1/2}}{(\rho^2 - a^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]} . \tag{4.2.12}$$

Defining the stress intensity factor

$$k_1 = \lim_{\rho\to a}\{(\rho - a)^{1/2}\sigma_z\} ,$$

the following result may be obtained from (4.2.12):

$$k_1 = \frac{P}{\pi^2(2a)^{1/2}} \frac{(a^2 - \rho_0^2)^{1/2}}{a^2 + \rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)} . \tag{4.2.13}$$

One can write for an arbitrarily distributed pressure:

$$k_1 = \frac{1}{\pi^2(2a)^{1/2}} \int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{1/2} p(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{a^2 + \rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)} ,$$

which corresponds to the well known result (Cherepanov, 1974).

Exercise 4.2

1. Derive the solution (4.2.1)–(4.2.6) for the case of an isotropic body.

2. Verify the derivation of (4.2.7)–(4.2.11).

4.3 Concentrated load outside a circular crack

Consider a transversely isotropic space weakened by a penny-shaped crack of radius a in the plane z=0. Let a concentrated force P be applied at an arbitrary point (ρ,ϕ,z) in the Oz direction. The crack faces are stress-free. Let us find the crack opening displacement and the opening mode stress intensity factor k_1.

Consider the second system in equilibrium: two unit concentrated forces Q applied normally to the crack faces in opposite directions at the point $(\rho_0,\phi_0,0^{\pm})$. Denote the normal displacement in the space due to the forces Q by w_Q; while w_P is the crack opening displacement due to force P. Application of the reciprocal theorem to the two systems yields

$$Qw_P = Pw_Q ,$$

which gives the crack opening displacement

$$w_P(\rho_0,\phi_0) = \frac{2}{\pi} HP \left[\frac{m_1}{m_1 - 1} f_2(z_1) + \frac{m_2}{m_2 - 1} f_2(z_2) \right] , \tag{4.3.1}$$

with f_2 defined by (4.2.8). The stress intensity factor can be determined by

$$k_1(\phi_0) = \frac{1}{8\pi H} \lim_{\rho_0\to a} \frac{w_P(\rho_0,\phi_0)}{(a - \rho_0)^{1/2}}$$

$$= \frac{P}{2(2a)^{1/2}\pi^2}\left[\frac{m_1}{m_1-1}f_6(z_1) + \frac{m_2}{m_2-1}f_6(z_2)\right],$$

where

$$f_6(z) = (a^2 - l_1^2)^{1/2}/r_a^2, \qquad r_a^2 = \rho^2 + a^2 - 2\rho a\cos(\phi - \phi_0) + z^2. \tag{4.3.2}$$

The stress intensity factor vanishes as z tends to zero for $\rho \geq a$.

In the case of an isotropic body, expression (4.3.1) transforms into

$$w_P(\rho_0,\phi_0) = \frac{P}{\pi^2\mu}\left\{\frac{1-\nu}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right.$$

$$\left. - \frac{1}{2}\left[\frac{h}{R_0^2+h^2}\left(\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2}\right) - \frac{z^2}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)\right]\right\}. \tag{4.3.3}$$

Here μ is the shear modulus, and ν is Poisson's ratio. The corresponding expression for the stress intensity factor will take the form:

$$k_1(\phi_0) = \frac{P(a^2 - l_1^2)^{1/2}}{(8a)^{1/2}\pi^2 r_a^2}\left[1 + \frac{1}{1-\nu}\left(\frac{z^2}{r_a^2} - \frac{\rho^2 - l_1^2}{2(l_2^2 - l_1^2)}\right)\right]. \tag{4.3.4}$$

In the case of axial symmetry $\rho=0$, and formulae (4.3.3–4.3.4) simplify as follows:

$$w_P(\rho_0,\phi_0) = \frac{P}{\pi^2\mu}\left\{\left[\frac{1-\nu}{(\rho_0^2+z^2)^{1/2}} + \frac{z^2}{(\rho_0^2+z^2)^{3/2}}\right]\tan^{-1}\left(\frac{a^2-\rho_0^2}{\rho_0^2+z^2}\right)^{1/2}\right.$$

$$\left. + \frac{z^2(a^2-\rho_0^2)^{1/2}}{2(\rho_0^2+z^2)(a^2+z^2)}\right\},$$

$$k_1 = \frac{Pa^{1/2}}{2\pi^2\sqrt{2}(a^2+z^2)}\left[1 + \frac{1}{1-\nu}\frac{z^2}{a^2+z^2}\right],$$

which is in agreement with the results reported by Collins (1962), who considered the axisymmetric case only.

Exercise 4.3

1. Verify (4.3.2)

2. Derive a complete solution for the case of an arbitrary point force applied outside a penny shaped crack in a transversely isotropic space.

4.4 Plane crack under arbitrary shear loading

Consider a transversely isotropic elastic space weakened by a flat crack S in the plane z=0, with arbitrary shear loading applied to the crack faces antisymmetrically. The problem can be formulated as follows: find the solution to the set of differential equations (2.1.3) for a half-space $z\geq 0$, subject to the mixed boundary conditions on the plane z=0:

$$\tau_z = -\tau(x,y), \text{ for } (x,y)\in S; \qquad u = 0, \text{ for } (x,y)\notin S;$$

$$\sigma_z = 0, \text{ for } -\infty<(x,y)<\infty. \qquad (4.4.1)$$

It is no longer possible to present the solution in the form (4.1.2). A more complicated representation is necessary, namely,

$$F_1 = c_1(\Lambda\overline{\chi}_1 + \overline{\Lambda}\chi_1), \qquad F_2 = c_2(\Lambda\overline{\chi}_2 + \overline{\Lambda}\chi_2), \qquad F_3 = c_3(\Lambda\overline{\chi}_3 - \overline{\Lambda}\chi_3). \qquad (4.4.2)$$

Here c_1, c_2 and c_3 are the as yet unknown constants; χ_1, χ_2, and χ_3 are the as yet unknown complex harmonic functions. A bar indicates the complex conjugate value throughout this book. Introducing the notation $z_k=z/\gamma_k$, for k=1,2,3, we assume also that

$$\chi_1(z) = \chi(z_1), \qquad \chi_2(z) = \chi(z_2), \qquad \chi_3(z) = \chi(z_3). \qquad (4.4.3)$$

This assumption will allow us to reduce the problem to finding just *one* harmonic function which is much easier than searching for three. By substituting (4.4.3) into the third equation (2.1.12), we obtain the first equation for the constants, namely,

$$c_1 + m_2c_2 = 0. \qquad (4.4.4)$$

The third condition in (4.4.1) is thus satisfied. Substitution of (4.4.2) in (2.1.6) yields

$$u = c_1(\Lambda^2\overline{\chi}_1 + \Delta\chi_1) + c_2(\Lambda^2\overline{\chi}_2 + \Delta\chi_2) + \mathrm{i}c_3(\Lambda^2\overline{\chi}_3 - \Delta\chi_3),$$

(4.4.5)

where the differential operators Λ and Δ are defined by (2.1.5). When z=0, equation (4.4.5) transforms into

$$u = (c_1 + c_2 + \mathrm{i}c_3)\Lambda^2\bar{\chi} + (c_1 + c_2 - \mathrm{i}c_3)\Delta\chi. \tag{4.4.6}$$

It is convenient to assume

$$c_1 + c_2 + \mathrm{i}c_3 = 0. \tag{4.4.7}$$

This assumption simplifies (4.4.6) as follows

$$u = (c_1 + c_2 - \mathrm{i}c_3)\Delta\chi, \tag{4.4.8}$$

and makes it possible to represent

$$\chi(M) = \iint_S \ln[R(M,N) + z]\ u(N)\ \mathrm{d}S_N\ . \tag{4.4.9}$$

The representation (4.4.9) satisfies the second condition (4.4.1) identically, and inside the crack the following equation becomes valid

$$c_1 + c_2 - \mathrm{i}c_3 = 1/2\pi. \tag{4.4.10}$$

The solution of the set of equations (4.4.4), (4.4.7), and (4.4.10) gives

$$c_1 = -\frac{1}{4\pi(m_1 - 1)}\ , \qquad c_2 = -\frac{1}{4\pi(m_2 - 1)}\ , \qquad c_3 = \frac{\mathrm{i}}{4\pi}\ . \tag{4.4.11}$$

Substitution of (4.4.2) and (4.4.11) in the last of expressions (2.1.12) gives the following expression for the tangential stress:

$$\tau_z = -\frac{A_{44}}{4\pi}\frac{\partial}{\partial z}\left[\frac{m_1 + 1}{m_1 - 1}(\Lambda^2\bar{\chi}_1 + \Delta\chi_1)\right.$$

$$\left. + \frac{m_2 + 1}{m_2 - 1}(\Lambda^2\bar{\chi}_2 + \Delta\chi_2) + (\Lambda^2\bar{\chi}_3 - \Delta\chi_3)\right]. \tag{4.4.12}$$

Expression (4.4.12) simplifies for z=0

$$\tau_z = -\frac{A_{44}}{4\pi}\left[\left(\frac{m_1+1}{(m_1-1)\gamma_1}+\frac{1}{\gamma_3}\right)\Lambda^2\frac{\partial\bar{\chi}}{\partial z}+\left(\frac{m_2+1}{(m_2-1)\gamma_2}-\frac{1}{\gamma_3}\right)\Delta\frac{\partial\chi}{\partial z}\right]. \tag{4.4.13}$$

Finally, satisfaction of the first condition (4.4.1) yields the governing integro-differential equation:

$$\tau(N_0) = -\frac{1}{2\pi^2(G_1^2-G_2^2)}\left[G_1\Delta\int\int_S\frac{u(N)}{R(N,N_0)}\,dS_N + G_2\Lambda^2\int\int_S\frac{\bar{u}(N)}{R(N,N_0)}\,dS_N\right], \tag{4.4.14}$$

where the elastic constants G_1 and G_2 are defined by (2.1.9).

Green's functions in the case of shear loading. The integro-differential equation (4.4.14) was solved exactly for a penny-shaped crack in section 2.7. The closed form solution is (see 2.7.53)

$$u(\rho,\phi) = \frac{G_1}{\pi}\int_0^{2\pi}\int_0^a\left[\frac{1}{R}\tan^{-1}\frac{\eta}{R}-\frac{G_2^2}{G_1^2}\frac{(3-\bar{t})\,\eta}{a^2(1-\bar{t})^2}\right]\tau(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0$$

$$+\frac{G_2}{\pi}\int_0^{2\pi}\int_0^a\left[\frac{q}{R\bar{q}}\tan^{-1}\frac{\eta}{R}+\frac{\eta\,[(q/\bar{q})-te^{2i\phi_0}]}{a^2(1-t)(1-\bar{t})}\right]\bar{\tau}(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0\ , \tag{4.4.15}$$

where R and η are defined by (4.1.14), q is defined by (4.1.28), a bar indicates the complex conjugate value, and

$$t = \frac{\rho\rho_0}{a^2}\,e^{i(\phi-\phi_0)}. \tag{4.4.16}$$

The potential functions can be found by substitution of (4.4.15) and (4.4.9) in (4.4.2) and evaluation of the resulting integrals. This looks at first somewhat difficult, nevertheless, it will be shown here that all the Green's functions can be expressed in elementary functions. Note the following property:

$$\Lambda\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{(3-t)\,\eta}{a^2(1-t)^2}\right]$$

$$= -\bar{\Lambda}\left[\frac{q}{R\bar{q}}\tan^{-1}\frac{\eta}{R}+\frac{\eta\,[(q/\bar{q})-te^{2i\phi_0}]}{a^2(1-t)(1-\bar{t})}\right]. \tag{4.4.17}$$

Introduce the following notation:

$$E_1(N,N_0) = \frac{1}{R(N,N_0)} \tan^{-1}\left(\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right),$$

$$E_2(N,N_0) = \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})\ (a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{a(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2},$$

$$E_3(N,N_0) = \frac{re^{i\psi} - \rho_0 e^{i\phi_0}}{R(N,N_0)(re^{-i\psi} - \rho_0 e^{-i\phi_0})} \tan^{-1}\left(\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right)$$

$$+ \frac{a(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})(a^2 - r\rho_0 e^{-i(\psi-\phi_0)})}\left[\frac{re^{i\psi} - \rho_0 e^{i\phi_0}}{re^{-i\psi} - \rho_0 e^{-i\phi_0}} - \frac{r\rho_0}{a^2}\ e^{i(\psi+\phi_0)}\right]. \tag{4.4.18}$$

Here the points N and N_0 are characterized by the cylindrical coordinates $(r,\psi,0)$ and $(\rho_0,\phi_0,0)$ respectively. Note the following relationships of symmetry:

$$E_1(N,N_0) = E_1(N_0,N), \qquad E_2(N,N_0) = \overline{E}_2(N_0,N),$$

$$E_3(N,N_0) = E_3(N_0,N). \tag{4.4.19}$$

Let $R(M,N)$ denote the distance between the points $M(\rho,\phi,z)$ and $N(r,\psi,0)$. By using (4.4.17) one may write

$$\int\!\!\int_S \Lambda[E_1(N,N_0) - E_2(N,N_0)]\ \frac{dS_N}{R(M,N)} = -\int\!\!\int_S \overline{\Lambda}E_3(N,N_0)\ \frac{dS_N}{R(M,N)}. \tag{4.4.20}$$

Integration by parts in (4.4.20) leads to an important property:

$$\int\!\!\int_S [E_1(N,N_0) - E_2(N,N_0)]\Lambda\left(\frac{1}{R(M,N)}\right)dS_N = -\int\!\!\int_S E_3(N,N_0)\overline{\Lambda}\left(\frac{1}{R(M,N)}\right)dS_N. \tag{4.4.21}$$

Two more properties can be obtained by applying Λ and $\overline{\Lambda}$ to both sides of (4.4.21), namely,

$$\iint_S [E_1(N,N_0) - E_2(N,N_0)]\Lambda^2\left(\frac{1}{R(M,N)}\right)\mathrm{d}S_N = -\iint_S E_3(N,N_0)\Delta\left(\frac{1}{R(M,N)}\right)\mathrm{d}S_N, \tag{4.4.22}$$

$$\iint_S [E_1(N,N_0) - E_2(N,N_0)]\Delta\left(\frac{1}{R(M,N)}\right)\mathrm{d}S_N = -\iint_S E_3(N,N_0)\overline{\Lambda}^2\left(\frac{1}{R(M,N)}\right)\mathrm{d}S_N. \tag{4.4.23}$$

The properties (4.4.21–4.4.23) will allow us to substitute the evaluation of various integrals involving E_3, which look very formidable, by evaluation of integrals involving expressions E_1 and E_2, some of which have already been computed (4.1.27–4.1.31), and the remaining can be evaluated relatively easy (see Appendix A4.4).

Introduce the notation

$$X = \Lambda\overline{\chi} + \overline{\Lambda}\chi, \qquad Y = \Lambda\overline{\chi} - \overline{\Lambda}\chi. \tag{4.4.24}$$

In order to obtain the complete solution, we shall need the following expressions for various derivatives of X and Y: the tangential displacements are defined by ΛX and ΛY; the normal displacements by $\partial X/\partial z$; the field of stresses may be computed through $\partial^2 X/\partial z^2$, $\Lambda^2 X$, $\Lambda^2 Y$, $\Lambda(\partial X/\partial z)$, $\Lambda(\partial Y/\partial z)$. All the Green's functions involved can be expressed as various derivatives of two fundamental functions, namely,

$$K_1(M,N_0) = \iint_S E_1(N,N_0)\ \ln[R(M,N) + z]\ \mathrm{d}S_N\ ,$$

$$K_2(M,N_0) = \iint_S E_2(N,N_0)\ \ln[R(M,N) + z]\ \mathrm{d}S_N\ . \tag{4.4.25}$$

Rewrite formula (4.4.15) as

$$u(N) = \frac{G_1}{\pi}\iint_S \left[E_1(N,N_0) - \frac{G_2^2}{G_1^2}\,\overline{E}_2(N,N_0)\right]\ \tau(N_0)\ \mathrm{d}S_{N_0}$$
$$+ \frac{G_2}{\pi}\iint_S E_3(N,N_0)\ \overline{\tau}(N_0)\ \mathrm{d}S_{N_0}. \tag{4.4.26}$$

By substituting (4.4.9) and (4.4.26) in (4.4.24) and using the properties (4.4.21–4.4.23), we obtain the following results:

$$X = \frac{G_1 - G_2}{\pi}\left[\overline{\Lambda}\iint_S\left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda\iint_S\left(K_1 + \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right],$$

$$Y = \frac{G_1 + G_2}{\pi}\left[-\overline{\Lambda}\iint_S\left(K_1 - \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda\iint_S\left(K_1 - \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right].$$

We shall only need the following derivatives of X and Y for the complete solution:

$$\Lambda X = \frac{G_1 - G_2}{\pi}\left[-\frac{\partial^2}{\partial z^2}\iint_S\left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda^2\iint_S\left(K_1 + \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right], \tag{4.4.27}$$

$$\Lambda Y = \frac{G_1 + G_2}{\pi}\left[\frac{\partial^2}{\partial z^2}\iint_S\left(K_1 - \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda^2\iint_S\left(K_1 - \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right], \tag{4.4.28}$$

$$\frac{\partial X}{\partial z} = \frac{G_1 - G_2}{\pi}\frac{\partial}{\partial z}\left[\overline{\Lambda}\iint_S\left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda\iint_S\left(K_1 + \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right], \tag{4.4.29}$$

$$\frac{\partial Y}{\partial z} = \frac{G_1 + G_2}{\pi}\frac{\partial}{\partial z}\left[-\overline{\Lambda}\iint_S\left(K_1 - \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda\iint_S\left(K_1 - \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right], \tag{4.4.30}$$

$$\frac{\partial}{\partial z}\Lambda X = \frac{G_1 - G_2}{\pi}\frac{\partial}{\partial z}\left[-\frac{\partial^2}{\partial z^2}\iint_S\left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right)\tau\,\mathrm{d}S + \Lambda^2\iint_S\left(K_1 + \frac{G_2}{G_1}K_2\right)\overline{\tau}\,\mathrm{d}S\right], \tag{4.4.31}$$

$$\frac{\partial}{\partial z}\Lambda Y = \frac{G_1 + G_2}{\pi}\frac{\partial}{\partial z}\left[\frac{\partial^2}{\partial z^2}\iint_S\left(K_1 - \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda^2\iint_S\left(K_1 - \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right], \tag{4.4.32}$$

$$\frac{\partial^2 X}{\partial z^2} = \frac{G_1 - G_2}{\pi}\frac{\partial^2}{\partial z^2}\left[\overline{\Lambda}\iint_S\left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right)\tau\ \mathrm{d}S + \Lambda\iint_S\left(K_1 + \frac{G_2}{G_1}K_2\right)\overline{\tau}\ \mathrm{d}S\right],$$

(4.4.33)

$$\Lambda^2 X = \frac{G_1 - G_2}{\pi}\, \Lambda\left[-\frac{\partial^2}{\partial z^2}\int\!\!\int_S \left(K_1 + \frac{G_2}{G_1}\overline{K}_2\right) \tau \, dS + \Lambda^2 \int\!\!\int_S \left(K_1 + \frac{G_2}{G_1}K_2\right) \bar{\tau} \, dS\right],$$

(4.4.34)

$$\Lambda^2 Y = \frac{G_1 + G_2}{\pi}\, \Lambda\left[\frac{\partial^2}{\partial z^2}\int\!\!\int_S \left(K_1 - \frac{G_2}{G_1}\overline{K}_2\right) \tau \, dS + \Lambda^2 \int\!\!\int_S \left(K_1 - \frac{G_2}{G_1}K_2\right) \bar{\tau} \, dS\right],$$

(4.4.35)

All the integrals in (4.4.27–4.4.35) are computed in elementary functions in Appendix A4.4 and in (4.1.27–4.1.31).

The results above may be applied to solving the problem of a tangential point force loading of a penny-shaped crack. The solution will give us all the Green's functions, related to the case. Consider an infinite transversely isotropic solid weakened in the plane $z=0$ by a penny-shaped crack of radius a. Let two equal concentrated forces $T=T_x+iT_y$ be applied to the crack faces antisymmetrically at the point $N_0(\rho_0,\phi_0,0^{\pm})$. The previously obtained results give the complete solution in elementary functions:

$$u = \frac{\gamma_1\gamma_2 H}{\pi} \sum_{k=1}^{2} \frac{1}{m_k - 1}\left\{-\left[f_2(z_k) + \frac{G_2}{G_1}\bar{f}_7(z_k)\right]T + \left[f_{16}(z_k) + \frac{G_2}{G_1} f_8(z_k)\right]\overline{T}\right\}$$

$$+ \frac{\beta}{\pi}\left\{\left[f_2(z_3) - \frac{G_2}{G_1}\bar{f}_7(z_3)\right]T + \left[f_{16}(z_3) - \frac{G_2}{G_1} f_8(z_3)\right]\overline{T}\right\}, \quad (4.4.36)$$

$$w = \frac{2}{\pi} H\gamma_1\gamma_2 \, \Re\sum_{k=1}^{2} \frac{m_k}{(m_k - 1)\gamma_k}\left[\bar{f}_1(z_k) + \frac{G_2}{G_1}\bar{f}_9(z_k)\right]T, \quad (4.4.37)$$

$$\sigma_1 = \Re\left\{\frac{2\gamma_1\gamma_2}{\pi^2(\gamma_1 - \gamma_2)} \sum_{k=1}^{2}(-1)^{k+1}\left[\frac{1}{\gamma_3^2(m_k + 1)} - \frac{1}{\gamma_k^2}\right]\left[\bar{f}_5(z_k) + \frac{G_2}{G_1}\bar{f}_{10}(z_k)\right]T\right\},$$

(4.4.38)

$$\sigma_2 = -\frac{2}{\pi}A_{66}H\gamma_1\gamma_2 \sum_{k=1}^{2}\frac{1}{m_k - 1}\left\{\left[f_5(z_k) + \frac{G_2}{G_1}\bar{f}_{13}(z_k)\right]T\right.$$

$$+ \left[f_{11}(z_k) + \frac{G_2}{G_1} f_{12}(z_k) \right] \overline{T} \Bigg\} - \frac{1}{\pi^2 \gamma_3} \Bigg\{ \left[-f_5(z_3) + \frac{G_2}{G_1} \bar{f}_{13}(z_3) \right] T$$

$$+ \left[f_{11}(z_3) - \frac{G_2}{G_1} f_{12}(z_3) \right] \overline{T} \Bigg\}, \tag{4.4.39}$$

$$\sigma_z = \Re \left\{ \frac{\gamma_1 \gamma_2}{\pi^2 (\gamma_1 - \gamma_2)} \sum_{k=1}^{2} (-1)^{k+1} \left[\bar{f}_5(z_k) + \frac{G_2}{G_1} \bar{f}_{10}(z_k) \right] T \right\}, \tag{4.4.40}$$

$$\tau_z = \frac{\gamma_1 \gamma_2}{2\pi^2 (\gamma_1 - \gamma_2)} \sum_{k=1}^{2} \frac{(-1)^k}{\gamma_k} \left\{ \left[f_3(z_k) + \frac{G_2}{G_1} \bar{f}_{14}(z_k) \right] T + \left[-f_4(z_k) + \frac{G_2}{G_1} f_{15}(z_k) \right] \overline{T} \right\}$$

$$+ \frac{1}{2\pi^2} \left\{ \left[f_3(z_3) - \frac{G_2}{G_1} \bar{f}_{14}(z_3) \right] T + \left[f_4(z_3) + \frac{G_2}{G_1} f_{15}(z_3) \right] \overline{T} \right\}. \tag{4.4.41}$$

Here $\Re$ indicates the real part, the elastic coefficients are defined by (2.1.9), and the functions f with subindex less than 6 are given by formulae (4.2.7–4.2.11), and the remaining functions are computed in Appendix A4.4, namely,

$$f_7(z) = \frac{ha^2}{s^2} \left[\frac{3}{s^2} - \frac{t}{l_2^2 - a^2 t} - \frac{3(l_2^2 - a^2)^{1/2}}{s^3} \tan^{-1}\left(\frac{s}{(l_2^2 - a^2)^{1/2}} \right) \right], \tag{4.4.42}$$

$$f_8(z) = \frac{1}{\bar{q}} (a^2 - \rho_0^2)^{1/2} \left\{ \frac{(\bar{\zeta} - 1)^{1/2}}{\bar{q}} \left[\tan^{-1}\left(\frac{1}{(\bar{\zeta} - 1)^{1/2}} \right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}} \right) \right] \right.$$

$$\left. - \frac{e^{i\phi}}{\rho} \left[\frac{(a^2 - l_1^2)^{1/2}}{a} \left(1 + \frac{\rho^2}{l_2^2 - \rho\rho_0 e^{i(\phi - \phi_0)}} \right) - 1 \right] \right\}, \tag{4.4.43}$$

$$f_9(z) = -\rho e^{i\phi} \frac{(a^2 - \rho_0^2)^{1/2}}{a^3} \left\{ \frac{1}{t} \sin^{-1}\left(\frac{a}{l_2}\right) + \frac{a(l_2^2 - a^2)^{1/2}}{(1 - t)(l_2^2 - \rho\rho_0 e^{i(\phi - \phi_0)})} \right.$$

$$- \frac{1}{t(1-t)^{3/2}} \tan^{-1}\left(\frac{a(1-t)^{1/2}}{(l_2^2-a^2)^{1/2}}\right) \Bigg\}, \tag{4.4.44}$$

$$f_{10}(z) = -\frac{h\rho e^{i\phi}(3l_2^2 - a^2 t)}{(l_2^2-l_1^2)(l_2^2-a^2t)^2}, \tag{4.4.45}$$

$$f_{11}(z) = \frac{1}{\bar q}\Bigg\{ \frac{3R_0^4+6R_0^2z^2-z^4}{R_0^3\bar q^2}\tan^{-1}\left(\frac{h}{R_0}\right) - (a^2-\rho_0^2)^{1/2}\Bigg[\frac{z}{\bar s}\left(\frac{8}{\bar q^2} - \frac{4\rho_0 e^{i\phi_0}}{\bar s^2 \bar q}\right.$$

$$\left. + \frac{3\rho_0^2 e^{2i\phi_0}}{\bar s^4}\right)\tan^{-1}\left(\frac{\bar s}{(l_2^2-a^2)^{1/2}}\right) - \frac{e^{i\phi}}{\rho}\left(\frac{2e^{i\phi}}{\rho}+\frac{3}{\bar q}\right)$$

$$- \frac{3(\bar\zeta-1)^{1/2}}{\bar q^2}\left(\tan^{-1}\frac{1}{(\bar\zeta-1)^{1/2}} - \tan^{-1}\frac{(a^2-l_1^2)^{1/2}}{a(\bar\zeta-1)^{1/2}}\right)\Bigg]$$

$$+ \frac{ha^2e^{i\phi}}{\rho\bar s^2}\left[\frac{2\rho_0e^{i\phi_0}}{\bar s^2} - \frac{2e^{i\phi}}{\rho} - \frac{2}{\bar q} + \left(\frac{\rho_0 e^{i\phi_0}}{\bar s^2}-\frac{2}{\bar q}\right)\frac{(l_2^2-a^2)\bar t}{l_2^2-a^2\bar t}\right]$$

$$- \frac{h}{R_0^2+h^2}\left[\frac{\bar q\rho e^{3i\phi}}{l_2^2-l_1^2} + \frac{e^{i\phi}(l_2^2-\rho^2)}{\rho\bar q} - \frac{z^2 q}{R_0^2\bar q} + 2e^{2i\phi}\right]\Bigg\}, \tag{4.4.46}$$

$$f_{12}(z) = \frac{1}{\bar q}(a^2-\rho_0^2)^{1/2}\Bigg\{\frac{3(\bar\zeta-1)^{1/2}}{\bar q^2}\left[\tan^{-1}\left(\frac{1}{(\bar\zeta-1)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{a(\bar\zeta-1)^{1/2}}\right)\right]$$

$$- \frac{e^{2i\phi}(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}\left[\frac{l_2^2+\rho^2}{l_2^2-\rho\rho_0e^{i(\phi-\phi_0)}} + \frac{2\rho^2(l_2^2-a^2)}{(l_2^2-\rho\rho_0e^{i(\phi-\phi_0)})^2}+1\right]$$

$$+ \frac{e^{i\phi}}{\rho}\left[\frac{3}{\bar q}+\frac{2e^{i\phi}}{\rho} - \frac{(a^2-l_1^2)^{1/2}}{a}\left(\frac{l_2^2+2\rho^2}{\bar q(l_2^2-\rho\rho_0e^{i(\phi-\phi_0)})} + 2\left(\frac{1}{\bar q}+\frac{e^{i\phi}}{\rho}\right)\right)\right]\Bigg\}, \tag{4.4.47}$$

$$\bar{f}_{13}(z) = -h\left\{\frac{a^2}{\bar{s}^2}\rho_0 e^{i\phi_0}\left[\frac{15(l_2^2 - a^2)^{1/2}}{\bar{s}^5}\tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right) - \frac{15}{\bar{s}^4}\right.\right.$$

$$\left.\left. + \frac{5}{\bar{s}^2(l_2^2 - a^2\bar{t})} + \frac{2\bar{t}}{(l_2^2 - a^2\bar{t})^2}\right] + \frac{\rho e^{i\phi}(3l_2^2 - a^2\bar{t})}{(l_2^2 - l_1^2)(l_2^2 - a^2\bar{t})^2}\right\}, \tag{4.4.48}$$

$$f_{14}(z) = \frac{(a^2 - \rho_0^2)^{1/2}}{a^3(1 - t)}\left\{\frac{a(l_2^2 - a^2)^{1/2}}{(l_2^2 - l_1^2)(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})}\left[\frac{3(l_2^2 - l_1^2 t)}{1 - t}\right.\right.$$

$$\left.\left. + \frac{\rho\rho_0 e^{i(\phi-\phi_0)}(2l_2^2 + l_1^2 t - 3\rho^2)}{l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)}}\right] - \frac{3}{(1 - t)^{3/2}}\tan^{-1}\left(\frac{a(1 - t)^{1/2}}{(l_2^2 - a^2)^{1/2}}\right)\right\}, \tag{4.4.49}$$

$$f_{15}(z) = \frac{\rho^2 e^{2i\phi}(a^2 - \rho_0^2)^{1/2}(l_2^2 - a^2)^{1/2}(3l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})}{l_2^2(l_2^2 - l_1^2)(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})^2}, \tag{4.4.50}$$

$$f_{16}(z) = \frac{1}{\bar{\bar{q}}}\left\{\frac{R_0^2 + z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h}{R_0}\right) + (a^2 - \rho_0^2)^{1/2}\left[\frac{z}{\bar{\bar{s}}}\left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2} - \frac{2}{\bar{\bar{q}}}\right)\tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left. + \frac{(\bar{\zeta} - 1)^{1/2}}{\bar{q}}\left(\tan^{-1}\frac{1}{(\bar{\zeta} - 1)^{1/2}} - \tan^{-1}\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right) + \frac{e^{i\phi}}{\rho}\right] - \frac{e^{i\phi}ha^2}{\rho\bar{s}^2}\right\}. \tag{4.4.51}$$

The solution (4.4.36–4.4.51) presents, in fact, the explicit expressions for the Green's functions, and allows us to write a complete solution for the case of arbitrary tangential loading in quadratures. The general results simplify significantly for $z=0$, namely,

$$u = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\frac{\eta}{R} - \frac{G_2^2}{G_1^2}\frac{(3 - \bar{t})\eta}{a^2(1 - \bar{t})^2}\right]T$$

$$+ \frac{G_2}{\pi}\left[\frac{q}{R\bar{\bar{q}}}\tan^{-1}\frac{\eta}{R} + \frac{\eta[(q/\bar{q}) - te^{2i\phi_0}]}{a^2(1 - t)(1 - \bar{t})}\right]\bar{T}, \quad \text{for } \rho<a, \tag{4.4.52}$$

$$w = H\alpha(a^2 - \rho_0^2)^{1/2}\Re\left\{\left[\frac{1}{qs} + \frac{G_2}{G_1}\frac{e^{-i\phi_0}}{\rho_0}\left(\frac{a^2}{\bar{s}^3} - \frac{1}{a}\right)\right]T\right\} \qquad \text{for } \rho\le a,$$

$$w = \frac{2}{\pi}H\alpha(a^2 - \rho_0^2)^{1/2}\Re\left\{\left[\frac{1}{qs}\tan^{-1}\frac{s}{(\rho^2 - a^2)^{1/2}} + \frac{G_2}{G_1}\left(\frac{a^2e^{-i\phi_0}}{\rho_0\bar{s}^3}\tan^{-1}\frac{\bar{s}}{(\rho^2 - a^2)^{1/2}}\right.\right.\right.$$

$$\left.\left.\left. - \frac{(\rho^2 - a^2)^{1/2}}{\bar{s}^2q} - \frac{e^{-i\phi_0}}{a\rho_0}\sin^{-1}\left(\frac{a}{\rho}\right)\right)\right]T\right\}, \qquad \text{for } \rho>a \tag{4.4.53}$$

$$\sigma_1 = \frac{2}{\pi^2}\Re\left\{\left(2\pi HA_{66}\gamma_1\gamma_2 - \frac{\gamma_1 + \gamma_2}{\gamma_1\gamma_2}\right)\left[\frac{1}{qR}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{a^2(1 - t)(1 - \bar{t})}\left(\frac{1}{q}\right.\right.\right.$$

$$\left.\left.\left. + \frac{\rho e^{-i\phi}}{a^2 - \rho^2}\right) + \frac{G_2}{G_1}\frac{\eta\rho e^{-i\phi}(3 - \bar{t})}{(a^2 - \rho^2)a^2(1 - \bar{t})^2}\right]T\right\}, \qquad \text{for } \rho<a,$$

$$\sigma_1 = 0, \qquad \text{for } \rho>a, \tag{4.4.54}$$

$$\sigma_2 = \frac{1}{\pi^2}\left\{\left(2\pi A_{66}H\gamma_1\gamma_2 + \frac{1}{\gamma_3}\right)\left[f_5(0)T + \frac{G_2}{G_1}f_{12}(0)\bar{T}\right]\right.$$

$$\left. + \left(2\pi A_{66}H\gamma_1\gamma_2 - \frac{1}{\gamma_3}\right)\left[f_{11}(0)\bar{T} + \frac{G_2}{G_1}\bar{f}_{13}(0)T\right]\right\},$$

$$\sigma_z = 0, \tag{4.4.55}$$

$$\tau_z = \frac{(a^2 - \rho_0^2)^{1/2}}{\pi^2(\rho^2 - a^2)^{1/2}}\left[\frac{T}{R^2} + \frac{G_2}{G_1}\frac{e^{2i\phi}(3\rho - \rho_0e^{i(\phi-\phi_0)})\bar{T}}{\rho(\rho - \rho_0e^{i(\phi-\phi_0)})^2}\right], \qquad \text{for } \rho>a. \tag{4.4.56}$$

Here R and η are defined by (4.1.14). The second and the third mode stress intensity factors can be expressed through the decomposition $\tau^{(n)}=\tau_{zn}+i\tau_{tz}$, which is related to τ_z by a relationship $\tau_z=\tau^{(n)}e^{i\phi}$. Introducing the complex stress intensity factor

$$k_2 + ik_3 = \lim_{\rho\to a}\{(\rho - a)^{1/2}\tau_z e^{-i\phi}\}, \tag{4.4.57}$$

one gets from (4.4.56)

$$k_2 + ik_3 = \frac{(a^2 - \rho_0^2)^{1/2}}{\pi^2(2a)^{1/2}}\left[\frac{Te^{-i\phi}}{\rho_0^2 + a^2 - 2a\rho_0\cos(\phi-\phi_0)}\right.$$
$$\left. + \frac{G_2}{G_1}\,\frac{e^{i\phi}(3a - \rho_0 e^{i(\phi-\phi_0)})\overline{T}}{a(a - \rho_0 e^{i(\phi-\phi_0)})^2}\right]. \tag{4.4.58}$$

In the general case of arbitrarily distributed loading, the stress intensity factor takes the form

$$k_2 + ik_3 = \int_0^{2\pi}\int_0^{a}\frac{(a^2 - \rho_0^2)^{1/2}}{\pi^2(2a)^{1/2}}\left[\frac{e^{-i\phi}\tau(\rho_0,\phi_0)}{\rho_0^2 + a^2 - 2a\rho_0\cos(\phi-\phi_0)}\right.$$
$$\left. + \frac{G_2}{G_1}\,\frac{(3a - \rho_0 e^{i(\phi-\phi_0)})\, e^{i\phi}\, \overline{\tau}(\rho_0,\phi_0)}{a(a - \rho_0 e^{i(\phi-\phi_0)})^2}\right]\rho_0 d\rho_0 d\phi_0, \tag{4.4.59}$$

which is in agreement with (2.7.30).

Exercise 4.4

1. Derive the general solution in the case of isotropy.
2. Find the isotropic equivalent of (4.4.14).
3. Verify the property (4.4.17).
4. Derive the solution (4.4.36)–(4.4.51).
5. Rederive the solution (4.4.36)–(4.4.51) for the case of isotropy.

4.5 Penny-shaped crack under uniform pressure

Let a penny-shaped crack of radius a be opened by the pressure p=const. In this case one gets from (4.1.12)

$$\omega(\rho,\phi) = 4Hp(a^2 - \rho^2)^{1/2}. \tag{4.5.1}$$

The potential function F can be obtained by substitution of (4.5.1) in (4.1.4). The integral can be computed in elementary functions (A4.1.4), giving

$$F = 2\pi Hp\left[(2a^2 + 2z^2 - \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{2a^2 - 3l_1^2}{a}(l_2^2 - a^2)^{1/2}\right]. \tag{4.5.2}$$

The complete solution can be expressed through various derivatives of the potential function, as prescribed by formulae (2.1.6) and (2.1.12). All the derivatives are given in Appendix A4.1. The solution is:

$$u = 2Hp\rho e^{i\phi}\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\left[\sin^{-1}\left(\frac{a}{l_{2k}}\right) - \frac{a(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2}\right]\right\}, \tag{4.5.3}$$

$$w = 4Hp\sum_{k=1}^{2}\left\{\frac{m_k}{m_k - 1}\left[\operatorname{sign}(z)(a^2 - l_{1k}^2)^{1/2} - z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]\right\}, \tag{4.5.4}$$

$$\sigma_1 = 8HpA_{66}\sum_{k=1}^{2}\left\{\frac{\gamma_k^2 - (m_k + 1)\gamma_3^2}{\gamma_k(m_k - 1)}\left[\frac{a(l_{2k}^2 - a^2)^{1/2}}{(l_{2k}^2 - l_{1k}^2)} - \sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]\right\}, \tag{4.5.5}$$

$$\sigma_2 = -8HpA_{66}ae^{2i\phi}\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\left[\frac{l_{1k}^2(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2(l_{2k}^2 - l_{1k}^2)}\right]\right\}, \tag{4.5.6}$$

$$\sigma_z = \frac{2p}{\pi(\gamma_1 - \gamma_2)}\sum_{k=1}^{2}\left\{(-1)^{k+1}\gamma_k\left[\frac{a(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2 - l_{1k}^2} - \sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]\right\}, \tag{4.5.7}$$

$$\tau_z = \frac{2pa^2\rho e^{i\phi}}{\pi(\gamma_1 - \gamma_2)}\sum_{k=1}^{2}\left\{(-1)^k\frac{(a^2 - l_{1k}^2)^{1/2}}{l_{2k}^2(l_{2k}^2 - l_{1k}^2)}\right\}, \tag{4.5.8}$$

Here the notation was introduced

$$l_{1k} = \frac{1}{2}\{[(a + \rho)^2 + z_k^2]^{1/2} - [(a - \rho)^2 + z_k^2]^{1/2}\},$$

$$l_{2k} = \frac{1}{2}\{[(a + \rho)^2 + z_k^2]^{1/2} + [(a - \rho)^2 + z_k^2]^{1/2}\},$$

$$z_k = z/\gamma_k, \qquad \text{for } k=1,2. \tag{4.5.9}$$

The problem was first solved by Elliott (1949) by the integral transform method. Our results are essentially in agreement with those of Elliott, who expressed them in the form of integrals involving Bessel functions, namely,

$$C_n^m = \int_0^\infty x^{n-2}\cos(x)\ J_m(x\frac{\rho}{a})\ \exp(-x\frac{z}{a})\ dx,$$

$$S_n^m = \int_0^\infty x^{n-2}\sin(x)\ J_m(x\frac{\rho}{a})\ \exp(-x\frac{z}{a})\ dx.$$

These integrals can be computed in elementary functions, and the results in our notation are

$$S_1^0 = \sin^{-1}(\frac{a}{l_2}), \qquad S_2^0 = \frac{a(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}, \qquad C_2^0 = \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2},$$

$$C_3^0 = \frac{a(a^2 - l_1^2)^{1/2}[l_2^4 + a^2(\rho^2 - 2a^2 - 2z^2)]}{(l_2^2 - l_1^2)^3},$$

$$S_3^0 = \frac{a(l_2^2 - a^2)^{1/2}[a^2(2a^2 + 2z^2 - \rho^2) - l_1^4]}{(l_2^2 - l_1^2)^3},$$

$$C_1^1 = \frac{z[a - (a^2 - l_1^2)^{1/2}]}{\rho(a^2 - l_1^2)^{1/2}}, \qquad S_1^1 = \frac{a - (a^2 - l_1^2)^{1/2}}{\rho},$$

$$S_1^2 = \frac{z[a - (a^2 - l_1^2)^{1/2}]^2}{\rho^2(a^2 - l_1^2)^{1/2}}, \qquad S_2^1 = \frac{a l_1(l_2^2 - a^2)^{1/2}}{l_2(l_2^2 - l_1^2)},$$

$$C_2^1 = \frac{a - (a^2 - l_1^2)^{1/2}}{\rho} - \frac{a l_1 (a^2 - l_1^2)^{1/2}}{l_2 (l_2^2 - l_1^2)},$$

$$C_3^1 = \frac{a^2 \rho (l_2^2 - a^2)^{1/2} (l_2^2 + 3l_1^2 - 4a^2)}{(l_2^2 - l_1^2)^3},$$

$$S_3^1 = \frac{a^2 \rho (a^2 - l_1^2)^{1/2} (3l_2^2 + l_1^2 - 4a^2)}{(l_2^2 - l_1^2)^3},$$

$$C_2^2 = \frac{2a[(l_2^2 - a^2)^{1/2} - z]}{\rho^2} - \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2},$$

$$S_2^2 = \frac{2a[a - (a^2 - l_1^2)^{1/2})]}{\rho^2} - \frac{a(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2},$$

$$C_3^2 = \frac{2a}{\rho} \left[\frac{a - (a^2 - l_1^2)^{1/2}}{\rho} - \frac{a l_1 (a^2 - l_1^2)^{1/2}}{l_2 (l_2^2 - l_1^2)} \right]$$

$$- \frac{a(a^2 - l_1^2)^{1/2} [l_2^4 + a^2(\rho^2 - 2a^2 - 2z^2)]}{(l_2^2 - l_1^2)^3}.$$

There are some misprints (or errors) in Elliott's paper. For example, according to his formula (4·2·5), the tangential displacement u vanishes on the plane z=0 which cannot be correct; there are some missing terms and obvious misprints in formula (4·2·6).

In the limiting case of $\gamma_1 \to \gamma_2 \to \gamma_3 \to 1$, $m_1 \to m_2 \to 1$, $H=(1-\nu)/2\pi\mu$, $A_{44}=A_{66}=\mu$, formulae (4.5.3–4.5.8) give the complete solution for an isotropic body. By using the L'Hôpital rule, one obtains

$$u = \frac{p\rho e^{i\phi}}{2\pi\mu} \left\{ (1-2\nu) \left[\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} - \sin^{-1}\left(\frac{a}{l_2}\right) \right] + \frac{2a^2 |z| (a^2 - l_1^2)^{1/2}}{l_2^2 (l_2^2 - l_1^2)} \right\}, \tag{4.5.10}$$

$$w = \frac{p}{\pi\mu}\left\{2(1 - \nu)\left[\frac{z}{|z|}(a^2 - l_1^2)^{1/2} - z\sin^{-1}(\frac{a}{l_2})\right]\right.$$

$$\left. + z\left[\sin^{-1}(\frac{a}{l_2}) - \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}\right]\right\}, \tag{4.5.11}$$

$$\sigma_1 = \frac{2p}{\pi}\left\{(1 + 2\nu)\left[\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} - \sin^{-1}(\frac{a}{l_2})\right]\right.$$

$$\left. + \frac{az^2[l_1^4 + a^2(2a^2 + 2z^2 - 3\rho^2)]}{(l_2^2 - l_1^2)^3(l_2^2 - a^2)^{1/2}}\right\}, \tag{4.5.12}$$

$$\sigma_2 = \frac{2p}{\pi}\frac{al_1^2 e^{2i\phi}(l_2^2 - a^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)}\left\{1-2\nu + \frac{z^2[a^2(6l_2^2 - 2l_1^2 + \rho^2) - 5l_2^4]}{(l_2^2 - l_1^2)^2(l_2^2 - a^2)}\right\}, \tag{4.5.13}$$

$$\sigma_z = \frac{2p}{\pi}\left\{\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} - \sin^{-1}(\frac{a}{l_2}) - \frac{az^2[l_1^4 + a^2(2a^2 + 2z^2 - 3\rho^2)]}{(l_2^2 - l_1^2)^3(l_2^2 - a^2)^{1/2}}\right\}, \tag{4.5.14}$$

$$\tau_z = -\frac{2p}{\pi}\frac{zl_1 e^{i\phi}(l_2^2 - a^2)^{1/2}[a^2(4l_2^2 - 5\rho^2) + l_1^4]}{l_2(l_2^2 - l_1^2)^3}\ . \tag{4.5.15}$$

This problem was first solved by Sneddon (1951), using the integral transform method. He was seemingly unable to compute the potential function (4.5.2), so he resorted to differentiation under the integral sign, with a subsequent computation of various integrals involving Bessel functions. His final results are given as elementary functions of the four parameters

$$r = (1 + (z/a)^2)^{1/2}, \qquad R^2 = [(\rho/a)^2 + (z/a)^2 - 1]^2 + 4(z/a)^2,$$

$$\theta = \tan^{-1}(a/z), \qquad \phi = \tan^{-1}\left(\frac{2az}{\rho^2 + z^2 - a^2}\right).$$

This choice of parameters is not the best possible. Here is one illustration. The expression for S_1^0 in Sneddon's notation takes the form (Sneddon, 1951, p.497)

$$S_1^0 = \tan^{-1}\left(\frac{r\sin\theta + \sqrt{R}\sin(\phi/2)}{r\cos\theta + \sqrt{R}\cos(\phi/2)}\right),$$

with the limitation $\rho \neq 0$, and no indication of what the result would be if $\rho=0$. In our notation the corresponding result is $\sin^{-1}(a/l_2)$, with no limitations attached. Introduction of Sneddon's parameters r and θ seems to be unnecessary. There exist relationships between his parameters R and ϕ, and our l_1 and l_2, namely,

$$R = (l_2^2 - l_1^2)/a^2, \qquad \sin(\phi/2) = (a^2 - l_1^2)^{1/2}/(l_2^2 - l_1^2)^{1/2}.$$

These relationships may be used to compare the solutions, which are in good agreement, except for some misprints: factor ζ is missing in Sneddon's formula (139, p. 496), and the last term in his formula (145, p. 499) should read $-S_1^0$, rather than $+S_0^0$.

Exercise 4.5

1. Establish the result (4.5.3)–(4.5.8).

2. Derive (4.5.10)–(4.5.15).

3. Derive the expressions for the polar components of stresses and displacements equivalent to (4.5.10)–(4.5.15).

4. Find the complete solution in the case where the penny-shaped crack is loaded by a normal stress, whose magnitude is proportional to the x-coordinate.

4.6 Penny-shaped crack under uniform shear loading

Consider a circular crack of radius a in a transversely isotropic elastic space, subjected to a uniform shear loading τ, where τ is a complex constant. The solution of the integro-differential equation (4.4.14) in this case is

$$u = \frac{2(G_1^2 - G_2^2)}{G_1}\tau(a^2 - \rho^2)^{1/2}, \qquad \text{for } z=0 \text{ and } \rho \leq a. \tag{4.6.1}$$

Substitution of (4.6.1) in (4.4.9) leads to the integral

$$\frac{2(G_1^2 - G_2^2)}{G_1} \tau \int_0^{2\pi} \int_0^a (a^2 - \rho_0^2)^{1/2} \ln(R_0 + z) \rho_0 d\rho_0 d\phi_0 ,$$

which has been computed in Appendix A4.1, with all the necessary derivatives. The complete solution will take the form:

$$u = \frac{G_1^2 - G_2^2}{G_1} \left\{ \frac{1}{m_1 - 1}\left[f_{17}(z_1)\bar{\tau} + f_{18}(z_1)\tau \right] + \frac{1}{m_2 - 1}\left[f_{17}(z_2)\bar{\tau} + f_{18}(z_2)\tau \right] + f_{17}(z_3)\bar{\tau} - f_{18}(z_3)\tau \right\}, \tag{4.6.2}$$

$$w = \frac{G_1^2 - G_2^2}{2G_1}(\bar{\tau}\rho e^{i\phi} + \tau\rho e^{-i\phi}) \sum_{k=1}^{2} \frac{m_k}{m_k - 1}\left[\sin^{-1}\left(\frac{a}{l_{2k}}\right) - \frac{a(l_{2k}^2 - a^2)^{1/2}}{l_{2k}^2} \right] \frac{1}{\gamma_k}, \tag{4.6.3}$$

$$\sigma_1 = \frac{2(G_1^2 - G_2^2)A_{66}}{G_1} a(\bar{\tau}e^{i\phi} + \tau e^{-i\phi}) \sum_{k=1}^{2} \frac{\gamma_3^2(m_k + 1) - \gamma_k^2}{\gamma_k^2(m_k - 1)} \frac{l_{1k}(a^2 - l_{1k}^2)^{1/2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)}, \tag{4.6.4}$$

$$\sigma_2 = \frac{2(G_1^2 - G_2^2)A_{66}}{G_1} \left\{ \frac{1}{m_1 - 1}\left[f_{19}(z_1)\bar{\tau} + f_{20}(z_1)\tau \right] + \frac{1}{m_2 - 1}\left[f_{19}(z_2)\bar{\tau} + f_{20}(z_2)\tau \right] + f_{19}(z_3)\bar{\tau} - f_{20}(z_3)\tau \right\}, \tag{4.6.5}$$

$$\sigma_z = \frac{2\gamma_1\gamma_2\beta(\tau e^{-i\phi} + \bar{\tau}e^{i\phi})}{\pi G_1(\gamma_1 - \gamma_2)} \sum_{k=1}^{2} (-1)^k \frac{a l_{1k}(a^2 - l_{1k}^2)^{1/2}}{l_{2k}(l_{2k}^2 - l_{1k}^2)}, \tag{4.6.6}$$

$$\tau_z = \frac{2\gamma_1\gamma_2}{\pi G_1} \left\{ \frac{\beta}{\gamma_1 - \gamma_2} \sum_{k=1}^{2} \frac{(-1)^k}{\gamma_k}[f_{21}(z_k)\tau + f_{22}(z_k)\bar{\tau}] + H[f_{21}(z_3)\tau - f_{22}(z_3)\bar{\tau}] \right\}. \tag{4.6.7}$$

Here

$$f_{17}(z) = e^{2i\phi} \frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^2}, \tag{4.6.8}$$

$$f_{18}(z) = z\sin^{-1}(\frac{a}{l_2}) - (a^2 - l_1^2)^{1/2}, \tag{4.6.9}$$

$$f_{19}(z) = e^{3i\phi} \frac{a l_1(a^2 - l_1^2)^{1/2}}{l_2(l_2^2 - l_1^2)} - 4e^{i\phi} f_{17}(z), \tag{4.6.10}$$

$$f_{20}(z) = e^{i\phi} \frac{a l_1(a^2 - l_1^2)^{1/2}}{l_2(l_2^2 - l_1^2)}, \tag{4.6.11}$$

$$f_{21}(z) = -\sin^{-1}(\frac{a}{l_2}) + \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}, \tag{4.6.12}$$

$$f_{22}(z) = \frac{e^{2i\phi} a l_1^2 (l_2^2 - a^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)}, \tag{4.6.13}$$

A complete solution to this problem for the case of *isotropy* can be found in (Westmann, 1965).

In the case of isotropy, formulae (4.6.2–4.6.7) transform into

$$u = \frac{1}{\pi\mu(2 - \nu)} \left\{ \left[(-5 + 4\nu) z\sin^{-1}(\frac{a}{l_2}) + 4(1 - \nu)(a^2 - l_1^2)^{1/2} \right] \tau \right.$$

$$\left. + \frac{za(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} \left(\tau + \bar{\tau} e^{2i\phi} \frac{l_1^2}{l_2^2} \right) \right\}, \tag{4.6.14}$$

$$w = \frac{(\bar{\tau}e^{i\phi} + \tau e^{-i\phi})\rho}{\pi\mu(2 - \nu)} \left\{ \frac{1-2\nu}{2} \left[\sin^{-1}(\frac{a}{l_2}) - \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} \right] + \frac{za^2(a^2 - l_1^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)} \right\}, \tag{4.6.15}$$

$$\sigma_1 = \frac{2(\bar{\tau}e^{i\phi} + \tau e^{-i\phi})}{\pi(2 - \nu)} \left\{ -2(1 + \nu) \frac{a l_1(a^2 - l_1^2)^{1/2}}{l_2(l_2^2 - l_1^2)} \right.$$

$$+ \frac{zl_1(l_2^2 - a^2)^{1/2}[a^2(4l_2^2 - 5\rho^2) + l_1^4]}{l_2(l_2^2 - l_1^2)^3}\Bigg\}, \qquad (4.6.16)$$

$$\sigma_2 = -\frac{2e^{i\phi}}{\pi(2 - \nu)}\Bigg\{4(1 - \nu)\frac{al_1(a^2 - l_1^2)^{1/2}}{l_2(l_2^2 - l_1^2)}\tau$$

$$+ \frac{zl_1(l_2^2 - a^2)^{1/2}}{l_2(l_2^2 - l_1^2)}\left[\frac{4a^2}{l_2^2}\bar{\tau}e^{2i\phi} - \frac{a^2(4l_2^2 - 5\rho^2) + l_1^4}{(l_2^2 - l_1^2)^2}(\tau + \bar{\tau}e^{2i\phi})\right]\Bigg\}, \qquad (4.6.17)$$

$$\sigma_z = -\frac{2(\bar{\tau}e^{i\phi} + \tau e^{-i\phi})}{\pi(2 - \nu)}\frac{zl_1(l_2^2 - a^2)^{1/2}[a^2(4l_2^2 - 5\rho^2) + l_1^4]}{l_2(l_2^2 - l_1^2)^3}, \qquad (4.6.18)$$

$$\tau_z = \frac{2}{\pi(2 - \nu)}\Bigg\{\Bigg[(2 - \nu)\left(\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} - \sin^{-1}\left(\frac{a}{l_2}\right)\right)$$

$$+ \frac{z(a^2 - l_1^2)^{1/2}[l_1^4 + a^2(2a^2 + 2z^2 - 3\rho^2)]}{(l_2^2 - l_1^2)^3}\Bigg]\tau + \Bigg[\nu a(l_2^2 - a^2)^{1/2}$$

$$+ \frac{z(a^2 - l_1^2)^{1/2}[a^2(6l_2^2 - 2l_1^2 + \rho^2) - 5l_2^4]}{(l_2^2 - l_1^2)^2}\Bigg]\frac{l_1^2e^{2i\phi}}{l_2^2(l_2^2 - l_1^2)}\bar{\tau}\Bigg\}. \qquad (4.6.19)$$

Exercise 4.6

1. Investigate a penny-shaped crack of radius a in a transversely isotropic elastic body, subjected to the shear loading $\tau = c\rho e^{i\phi}$, where c=const.
Answer: the main potential function will be proportional to the integral

$$I = \int_0^{2\pi}\int_0^a \ln(R_0 + z)(a^2 - \rho_0^2)^{1/2}e^{i\phi_0}\rho_0^2\, d\rho_0 d\phi_0$$

$$= \frac{\pi}{8}\rho e^{i\phi}\left\{(a^2 - l_1^2)^{1/2}\left[\frac{4}{3}a^2 + 7\rho^2 - \frac{19}{3}l_1^2 - 4l_2^2\right.\right.$$

$$\left.\left. + \frac{2(8a^4 + 4a^2l_1^2 + 3l_1^4)}{15\rho^2}\right] + z(4a^2 - 3\rho^2 + 4z^2)\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{16a^5}{15\rho^2}\right\}.$$

It might be a good exercise for the reader to perform the differentiation and write a complete solution. As an example, here are the first two derivatives

$$\frac{\partial I}{\partial z} = \frac{\pi}{8}\rho e^{i\phi}\left\{a(l_2^2 - a^2)^{1/2}\left[15\frac{l_1^2}{a^2} - 12 - 2\frac{l_1^2}{l_2^2}\right] + \sin^{-1}\left(\frac{a}{l_2}\right)\left[4a^2 - 3\rho^2 + 12z^2\right]\right\},$$

$$\frac{\partial^2 I}{\partial z^2} = \pi\rho e^{i\phi}\left[(a^2 - l_1^2)^{1/2}\left(\frac{a^2}{l_2^2} - 3\right) + 3z\sin^{-1}\left(\frac{a}{l_2}\right)\right].$$

Note: $\partial I/\partial z$ is proportional to the main potential function for the case of linear *normal* loading of a penny-shaped crack.

2. Find the expressions for the energy release rate by using a procedure similar to the one employed by Kassir and Sih (1968).
Answer: $\mathcal{G}_1 = 2\pi^2 H k_1^2$, $\mathcal{G}_2 = \pi^2(G_1 - G_2)k_2^2$, $\mathcal{G}_3 = \pi^2(G_1 + G_2)k_3^2$.

3. Find the Green's functions for a semi-infinite plane crack in a transversely isotropic space, subjected to a shear loading.
Hint: consider the limiting case of (4.4.36–4.4.41), when the radius $a\to\infty$, and the coordinate origin moves from the circle centre to its boundary.

4.7 Asymptotic behavior of stresses and displacements near the crack rim

Kassir and Sih (1975) have derived expressions relating the stress intensity factors to the field of stresses and displacements in the immediate vicinity of the crack edge, by using their very complicated solution for an elliptical crack. Their derivation looks so complicated that nobody so far was capable to repeat it and verify its accuracy. There is a notion in fracture mechanics that the asymptotic behavior is defined completely by three stress intensity factors, and is invariant for any crack, with a smooth boundary. If this is so, then we can obtain the same results from the much simpler solution for a penny-shaped crack. The results, presented here, are simpler than those of Kassir and Sih, and obtained in a simple manner.

Asymptotic behavior for mode I loading. Though formulae (4.5.3–4.5.8) are valid for a penny-shaped crack subjected to a *uniform* pressure only, they can be used to obtain some general results which are valid for an arbitrary crack with a smooth boundary subjected to *arbitrary* loading, in other words, we can explore the asymptotic behavior of displacements and stresses near the rim of a general crack. Introduce the local system of spherical coordinates (r,θ,ϕ), with the coordinate origin at the crack rim. The following asymptotics are valid for the main parameters used in (4.5.3–4.5.8):

$$\rho = a + r\cos\theta, \qquad l_{1k} \approx a - \frac{1}{2}rS_k^2, \qquad l_{2k} \approx a + \frac{1}{2}rT_k^2,$$

$$z = r\sin\theta, \qquad l_{2k}^2 - l_{1k}^2 \approx 2arQ_k, \qquad (a^2 - l_{1k}^2)^{1/2} \approx (ar)^{1/2}S_k,$$

$$(l_{2k}^2 - a^2)^{1/2} \approx (ar)^{1/2}T_k, \qquad \sin^{-1}\left(\frac{a}{l_2}\right) \approx -\left(\frac{r}{a}\right)^{1/2}T_k. \tag{4.7.1}$$

Here the notation was introduced:

$$Q_k = [\cos^2\theta + (1/\gamma_k^2)\sin^2\theta]^{1/2} \qquad S_k = [Q_k - \cos\theta]^{1/2},$$

$$T_k = [Q_k + \cos\theta]^{1/2}, \qquad \text{for } k = 1,2,3. \tag{4.7.2}$$

Introducing the opening mode stress intensity factor

$$k_1 = \frac{p\sqrt{2a}}{\pi}, \tag{4.7.3}$$

the following asymptotic expressions can be derived by substitution of (4.7.1) and (4.7.3) in (4.5.3–4.5.8):

$$u = u_n = -2\pi Hk_1\sqrt{2r}\left[\frac{\gamma_1 T_1}{m_1 - 1} + \frac{\gamma_2 T_2}{m_2 - 1}\right] + 0(1), \tag{4.7.4}$$

$$w = 2\pi Hk_1\sqrt{2r}\left[\frac{m_1 S_1}{m_1 - 1} + \frac{m_2 S_2}{m_2 - 1}\right] + 0(r), \tag{4.7.5}$$

$$\sigma_1 = 2\pi A_{66}Hk_1\left(\frac{2}{r}\right)^{1/2}\sum_{k=1}^{2}\frac{\gamma_k^2 - (m_k + 1)\gamma_3^2}{\gamma_k(m_k - 1)}\frac{T_k}{Q_k} + 0(1), \tag{4.7.6}$$

$$\sigma_2 = -2\pi A_{66}Hk_1\left(\frac{2}{r}\right)^{1/2}\sum_{k=1}^{2}\frac{\gamma_k T_k}{(m_k - 1)Q_k} + 0(\sqrt{r}), \tag{4.7.7}$$

$$\sigma_z = \frac{k_1}{\sqrt{2r}(\gamma_1 - \gamma_2)}\left[\frac{\gamma_1 T_1}{Q_1} - \frac{\gamma_2 T_2}{Q_2}\right] + 0(1), \tag{4.7.8}$$

$$\tau_z = \tau_{zn} = -\frac{k_1}{\sqrt{2r}(\gamma_1 - \gamma_2)}\left[\frac{S_1}{Q_1} - \frac{S_2}{Q_2}\right] + 0(\sqrt{r}), \tag{4.7.9}$$

These results were computed for $\phi=0$. This assumption allows us to avoid a cumbersome axis transformation, without loss of generality. The parameter σ_1 in this case is interpreted as the sum $\sigma_n + \sigma_t$, and $\sigma_2 = \sigma_n - \sigma_t + 2i\tau_{nt}$. By taking the sum and the difference of (4.7.6) and (4.7.7), one can get

$$\sigma_n = \frac{k_1}{\sqrt{2r}(\gamma_1 - \gamma_2)}\left[-\frac{T_1}{\gamma_1 Q_1} + \frac{T_2}{\gamma_2 Q_2}\right],$$

$$\sigma_t = \pi A_{66}Hk_1\left(\frac{2}{r}\right)^{1/2}\left[\frac{[2\gamma_1^2 - (m_1 + 1)\gamma_3^2]T_1}{\gamma_1(m_1 - 1)Q_1} + \frac{[2\gamma_2^2 - (m_2 + 1)\gamma_3^2]T_2}{\gamma_2(m_2 - 1)Q_2}\right].$$

Formulae (4.7.4–4.7.9) are essentially in agreement with the results of Kassir and Sih (1975), except for some misprints, for example, one should read $\sqrt{n_1}$ and $\sqrt{n_2}$ instead of n_1 and n_2 in the denominator of the terms in curly brackets of their formula (8.94c). In order to compare our results with those of Kassir and Sih, one should keep in mind that their definition of the stress intensity factor is $\sqrt{2}$ times greater than ours, their notation n_k corresponds to our γ_k^2; Kassir and Sih seem to have not noticed the properties (4.1.10) and the relationship $S_k=(\sin\theta)/(\gamma_k T_k)$, which in some cases can be used to simplify their results significantly. For example, they have an expression $[A_{13}m_k - A_{11}\gamma_k^2]/[A_{44}(m_k + 1)]$ in formula (8.95a), without realizing that it is equal to -1 for $k=1,2$.

The asymptotic behavior of the displacements and stresses near the crack edge in an isotropic body can be found from either (4.7.4–4.7.9) or (4.5.10–4.5.15). The result is

$$u = u_n = \frac{k_1\sqrt{r}}{2\mu} \cos(\frac{\theta}{2}) \left[2(1 - \nu) - \cos^2(\frac{\theta}{2})\right] + 0(1), \tag{4.7.10}$$

$$w = \frac{k_1\sqrt{r}}{\mu} \sin\frac{\theta}{2} \left[2(1 - \nu) - \cos^2(\frac{\theta}{2})\right] + 0(r), \tag{4.7.11}$$

$$\sigma_1 = \frac{k_1}{\sqrt{r}} \cos\frac{\theta}{2} \left[1 + 2\nu - \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right] + 0(1), \tag{4.7.12}$$

$$\sigma_2 = \frac{k_1}{\sqrt{r}} \cos\frac{\theta}{2} \left[1 - 2\nu - \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right] + 0(\sqrt{r}), \tag{4.7.13}$$

$$\sigma_z = \frac{k_1}{\sqrt{r}} \cos\frac{\theta}{2} \left[1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right] + 0(1), \tag{4.7.14}$$

$$\tau_z = \tau_{zn} = \frac{k_1}{2\sqrt{r}} \sin\theta \cos\frac{3\theta}{2} + 0(\sqrt{r}), \tag{4.7.15}$$

which is in agreement with the results given in (Sih and Liebowitz, 1968).

Asymptotic behavior for mode II and III loading. We can derive again some results of general nature, namely, the asymptotic behavior of the field of stresses and displacements in the neighbourhood of the edge of a flat crack with a smooth boundary. We recall that at $\phi=0$ the decompositions $u=u_x+iu_y$ and $\tau_z=\tau_{zx}+i\tau_{yz}$ are equal to $u^{(n)}=u_n+iu_t$ and $\tau^{(n)}=\tau_{zn}+i\tau_{tz}$ respectively; σ_1 is understood as $\sigma_n+\sigma_t$, and $\sigma_2=\sigma_n-\sigma_t+2i\tau_{nt}$. This will allow us to avoid a cumbersome axis transformation. The complex stress intensity factor, introduced in (4.4.57), can be expressed through the prescribed shear loading τ as

$$k = k_2 + ik_3 = \frac{\sqrt{2a}}{\pi} \left[\tau + \frac{G_2}{G_1} \bar{\tau}\right], \tag{4.7.16}$$

and its inversion gives

$$\tau = \frac{\pi G_1(kG_1 - \bar{k}G_2)}{\sqrt{2a}(G_1^2 - G_2^2)}. \tag{4.7.17}$$

Substitution of (4.7.1) and (4.7.17) in (4.6.2–4.6.13) yields

$$u_n + iu_t = \frac{k_2\gamma_1\gamma_2\sqrt{2r}}{A_{44}(\gamma_1 - \gamma_2)}\left[-\frac{S_1}{m_1 + 1} + \frac{S_2}{m_2 + 1}\right] + \frac{ik_3\gamma_3\sqrt{2r}}{A_{44}} S_3, \tag{4.7.18}$$

$$w = \frac{k_2\gamma_1\gamma_2\sqrt{2r}}{A_{44}(\gamma_1 - \gamma_2)}\left[-\frac{m_1T_1}{\gamma_1(m_1 + 1)} + \frac{m_2T_2}{\gamma_2(m_2 + 1)}\right], \tag{4.7.19}$$

$$\sigma_1 = 2\pi k_2\gamma_1\gamma_2 HA_{66} \left(\frac{2}{r}\right)^{1/2} \sum_{k=1}^{2} \frac{\gamma_3^2(m_k + 1) - \gamma_k^2}{\gamma_k^2(m_k - 1)} \frac{S_k}{Q_k}, \tag{4.7.20}$$

$$\sigma_2 = 2\pi k_2\gamma_1\gamma_2 HA_{66}\left(\frac{2}{r}\right)^{1/2}\left[\frac{S_1}{(m_1 - 1)Q_1} + \frac{S_2}{(m_2 - 1)Q_2}\right] - \frac{ik_3\sqrt{2}S_3}{\gamma_3\sqrt{r}Q_3}, \tag{4.7.21}$$

$$\sigma_z = -\frac{k_2\gamma_1\gamma_2}{\sqrt{2r}(\gamma_1 - \gamma_2)} \left[\frac{S_1}{Q_1} - \frac{S_2}{Q_2}\right], \tag{4.7.22}$$

$$\tau_{zn} + i\tau_{tz} = -\frac{k_2\gamma_1\gamma_2}{\sqrt{2r}(\gamma_1 - \gamma_2)} \left[\frac{T_1}{\gamma_1 Q_1} - \frac{T_2}{\gamma_2 Q_2}\right] + \frac{ik_3T_3}{\sqrt{2r}Q_3}. \tag{4.7.23}$$

By taking the sum and the difference of (4.7.20) and (4.7.21), one gets

$$\sigma_n = \frac{k_2\gamma_1\gamma_2}{\sqrt{2r}(\gamma_1 - \gamma_2)} \left[\frac{S_1}{\gamma_1^2 Q_1} - \frac{S_2}{\gamma_2^2 Q_2}\right], \tag{4.7.24}$$

$$\sigma_t = \frac{k_2\gamma_1\gamma_2}{\sqrt{2r}}\left\{\frac{1}{(\gamma_1 - \gamma_2)}\left[\left(\frac{1}{\gamma_1^2} - \frac{1}{\gamma_3^2}\right)\frac{S_1}{Q_1} - \left(\frac{1}{\gamma_2^2} - \frac{1}{\gamma_3^2}\right)\frac{S_2}{Q_2}\right]\right.$$

$$\left. + 2\pi HA_{66}\left(\frac{S_1}{Q_1} + \frac{S_2}{Q_2}\right)\right\}, \tag{4.7.25}$$

$$\tau_{nt} = -\frac{k_3 S_3}{\sqrt{2r}\gamma_3 Q_3}. \tag{4.7.26}$$

Our formulae (4.7.18–4.7.26) are in relatively good agreement with similar results of Kassir and Sih (1975), except for formula (8.96b, p.371) for u_t which should correspond to the imaginary part of our (4.7.18). Formula by Kassir and Sih (8.96b) seems to be in error because it implies that u_t depends on k_2, γ_1, and γ_2, which is wrong: our result relates u_t to k_3 and γ_3 only, as it should be. There are several misprints in their formulae (8.96a) and (8.96c). The remaining formulae are in agreement, though the formulae by Kassir and Sih (1975) look more complicated than ours, mainly because they did not notice the properties (4.1.10) which could make some expressions much simpler.

Exercise 4.7

1. Establish (4.7.1).

2. Derive (4.7.4)–(4.7.9).

3. Derive (4.7.18)–(4.7.23).

4. Find the asymptotic behavior of the stresses and displacements near the crack rim for the case of mode II and III loading in isotropic bodies.

Answer: $u_n + iu_t = \frac{\sqrt{r}}{\mu} \sin\frac{\theta}{2} \left\{ \left[2(1 - \nu) + \cos^2(\frac{\theta}{2}) \right] k_2 + 2ik_3 \right\},$

$$w = k_2 \frac{\sqrt{r}}{\mu} \cos(\frac{\theta}{2}) \left[-(1 - 2\nu) + \sin^2(\frac{\theta}{2}) \right],$$

$$\sigma_1 = \frac{k_2}{\sqrt{r}} \sin(\frac{\theta}{2}) \left[-2(1 + \nu) - \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2}) \right],$$

$$\sigma_2 = \frac{\sin(\theta/2)}{\sqrt{r}} \left\{ \left[-2(1 - \nu) - \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2}) \right] k_2 - 2ik_3 \right\},$$

$$\sigma_z = \frac{k_2}{2\sqrt{r}} \sin\theta \cos(\frac{3\theta}{2}),$$

$$\tau_{zn} + i\tau_{tz} = \frac{\cos(\theta/2)}{\sqrt{r}} \left\{\left[1 - \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right]k_2 + ik_3\right\}.$$

Note: compare with the results presented in Sih and Liebowitz (1968).

$$\tau_{nt} = -\frac{k_3 S_3}{\sqrt{2r}\gamma_3 Q_3}. \tag{4.7.26}$$

Our formulae (4.7.18–4.7.26) are in relatively good agreement with similar results of Kassir and Sih (1975), except for formula (8.96b, p.371) for u_t which should correspond to the imaginary part of our (4.7.18). Formula by Kassir and Sih (8.96b) seems to be in error because it implies that u_t depends on k_2, γ_1, and γ_2, which is wrong: our result relates u_t to k_3 and γ_3 only, as it should be. There are several misprints in their formulae (8.96a) and (8.96c). The remaining formulae are in agreement, though the formulae by Kassir and Sih (1975) look more complicated than ours, mainly because they did not notice the properties (4.1.10) which could make some expressions much simpler.

Exercise 4.7

1. Establish (4.7.1).

2. Derive (4.7.4)–(4.7.9).

3. Derive (4.7.18)–(4.7.23).

4. Find the asymptotic behavior of the stresses and displacements near the crack rim for the case of mode II and III loading in isotropic bodies.

Answer: $$u_n + iu_t = \frac{\sqrt{r}}{\mu} \sin\frac{\theta}{2} \left\{\left[2(1 - \nu) + \cos^2(\frac{\theta}{2})\right]k_2 + 2ik_3\right\},$$

$$w = k_2\frac{\sqrt{r}}{\mu} \cos(\frac{\theta}{2}) \left[-(1 - 2\nu) + \sin^2(\frac{\theta}{2})\right],$$

$$\sigma_1 = \frac{k_2}{\sqrt{r}} \sin(\frac{\theta}{2})\left[-2(1 + \nu) - \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})\right],$$

$$\sigma_2 = \frac{\sin(\theta/2)}{\sqrt{r}} \left\{\left[-2(1 - \nu) - \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})\right]k_2 - 2ik_3\right\},$$

$$\sigma_z = \frac{k_2}{2\sqrt{r}} \sin\theta \cos(\frac{3\theta}{2}),$$

$$\tau_{zn} + i\tau_{tz} = \frac{\cos(\theta/2)}{\sqrt{r}} \left\{\left[1 - \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right]k_2 + ik_3\right\}.$$

Note: compare with the results presented in Sih and Liebowitz (1968).

4.8 Flat crack of general shape

The general method is applied here to the analysis of an elastic space weakened by a flat crack of arbitrary shape under the action of a uniform normal pressure. A simple yet accurate relationship is established between the crack face displacements and the applied pressure for an arbitrary flat crack. Specific formulae are derived for a crack in the shape of a polygon, a rectangle, a rhombus, a cross, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature, and their accuracy is confirmed. A similar approach can be used for the analysis of a crack under a general polynomial loading. The material in this section follows the paper (Fabrikant, 1987b).

Theory. Consider an elastic space weakened in the plane z=0 by a flat crack occupying the domain S whose boundary is given in polar coordinates as

$$\rho = a(\phi). \tag{4.8.1}$$

Let a uniform pressure p be applied normally to the crack faces in opposite directions. The governing integral equation in this case is given by (4.1.9). The approach is based on the integral representation of the reciprocal of the distance between two points established in (1.1.27). Substitution of (1.1.27) into (4.1.9) gives, after interchanging the order of integration

$$\sigma(\rho,\phi) = -\frac{1}{2\pi^3 H}\Delta\int_0^{\rho}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_0^{2\pi}d\phi_0\int_x^{a(\phi_0)}\frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)}{(\rho_0^2 - x^2)^{1/2}}w(\rho_0,\phi_0)\rho_0 d\rho_0. \tag{4.8.2}$$

It is noteworthy that the change of the order of integration which led to (4.8.2) is valid inside the circle $\rho\leq\min\{a(\phi)\}$ only, and this is one of the reasons why the accuracy generally deteriorates for domains with the aspect ratio very far away from unity. Nevertheless, one can obtain from (4.8.2) the *exact* solution for an ellipse and sufficiently accurate formulae for various crack shapes as will be demonstrated further on. Let the normal displacements of the crack face be

$$w = \frac{\delta}{a(\phi)}\left[a^2(\phi) - \rho^2\right]^{1/2}, \tag{4.8.3}$$

where δ is a constant to be defined. Now substituting (4.8.3) in (4.8.2), we can verify how close to a constant will be the traction σ producing the displacements (4.8.3). Integration with respect to ρ_0 gives

$$\sigma(\rho,\phi) = -\frac{\delta}{8\pi^2 H}\Delta\sum_{n=-\infty}^{\infty}\int_0^{\rho}(\frac{x}{\rho})^{|n|}\frac{x dx}{(\rho^2 - x^2)^{1/2}}\int_0^{2\pi}\frac{a^2(\phi_0) - x^2}{a^2(\phi_0)}$$

$$\times F\left(2-\frac{|n|}{2},\ \frac{1}{2};\ 2;\ 1-\frac{x^2}{a^2(\phi_0)}\right)e^{i(\phi-\phi_0)}d\phi_0. \tag{4.8.4}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the normal traction can be done separately for each value of n. The zero*th* term has the form

$$\sigma_0 = \frac{\delta B}{8\pi H}, \tag{4.8.5}$$

where the notation

$$B = \int_0^{2\pi}\frac{d\phi}{a(\phi)} \tag{4.8.6}$$

was introduced. It is clear that the value of the integral in (4.8.6) will depend not only on the domain contour but also on the location of the coordinate origin. The following argument might be useful for establishing certain rules in this regard. According to (4.8.3), the coordinate origin location corresponds to the point where the crack face displacement attains its maximum. We shall call this point *the crack centre*. In the case of a crack domain with one axis of symmetry, we may conclude from physical considerations that this point should be located at the axis. When this domain possesses two axes of symmetry the location of the crack center is at their intersection, i.e. at the center of gravity of the domain. It is noteworthy that the integral (4.8.6) attains its minimum in this case. One can extend this rule to a general crack, namely, the crack center should be identified with the point inside S where the integral (4.8.6) reaches its minimum. Direct computations for various domains indicate that this minimum is, in general, sufficiently flat, so that in many cases one may locate the crack center at the center of gravity, without any significant loss in accuracy. We shall discuss this in more detail further on when considering the domain S in the shape of a circular segment and sector.

It is important to note that the second harmonic is equal to zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. Here is the expression for the fourth harmonic

$$\sigma_4 = -\frac{4\delta}{5\pi^2 H}\rho\int_0^{2\pi}\frac{\cos 4(\phi - \phi_0)d\phi_0}{a^2(\phi_0)} \tag{4.8.7}$$

The investigation of the fourth and further harmonics shows that their amplitude decreases for general domains, and they vanish in the case of an ellipse. If we assume that $p \approx \sigma_0$ then the remaining harmonics may be called the solution error. This implies the establishment of the following relationship between the applied traction p and the maximum displacement of the crack face

$$p = \frac{\delta B}{8\pi H} \tag{4.8.8}$$

One can verify that in the case of an ellipse, the solution given by (4.8.3) and (4.8.8) is *exact*. We expect it to be reasonably accurate for a crack of general shape. This assumption will be justified in the next Section where several particular crack configurations are considered. We also expect (4.8.3) to be sufficiently accurate in the neighbourhood of the crack center, though the relative error might be quite significant close to the boundary.

The crack energy can be defined as

$$W = \iint_S \sigma w dS. \tag{4.8.9}$$

Feeding (4.8.3) and (4.8.8) in (4.8.9) yields

$$W = \frac{16\pi H p^2 A}{3B}, \tag{4.8.10}$$

where A is the crack area. Introduce the average displacement δ_{av} as

$$\delta_{av} = \frac{1}{A}\iint_S w dS.$$

Substitution of (4.8.3) in the last expression gives

$$\delta_{av} = \frac{2}{3}\,\delta.$$

Define the dimensionless parameter τ in the form

$$\tau = \frac{\delta_{av}}{2\pi H p\sqrt{A}} = \frac{W}{2\pi H p^2 A^{3/2}}. \tag{4.8.11}$$

The physical meaning of τ can be defined either as a ratio of the average displacement to a certain fixed displacement, or as a ratio of the crack energy to a certain fixed one. Both forms (4.8.11) lead to the following expression for τ:

$$\tau = \frac{8}{3\sqrt{A}B}. \tag{4.8.12}$$

One can deduce that the value of τ does not depend on the size of the domain S, and is determined by its shape only. It attains its maximum in the case of a circle, so that $0 \le \tau \le 4/(3\pi^{3/2}) = 0.2394$. Tabulation of the coefficient τ for various crack shapes might prove very useful since its knowledge allows us to find the maximum (or average) crack face displacement and the crack energy by using (4.8.11). It might seem more logical to define τ as the ratio of the given crack energy to the energy of a circular crack having the same area. In this case the value of τ would vary between zero and unity. The main reason for the definition (4.8.12) was the desire to preserve the bridge between the crack problems in mechanics and the mathematically equivalent problems in electrostatics. The value of τ defined by (4.8.11) corresponds exactly to the coefficient of electrical polarizability in the theory of wave propagation through small apertures (Bethe 1944). We did not find in the mechanics literature any report containing numerical data for nonelliptic cracks which could be compared with the theory of this paper. The situation is slightly better in electrical sciences. Cohn (1952) has measured the coefficient of electrical polarizability of several aperture configurations experimentally. Numerical solution to the same problem was given by De Meulenaere and Van Bladel (1977), and by Okon and Harrington (1981). These numerical and experimental data will be used to estimate the accuracy of the proposed theory. An empirical formula for the coefficient of electrical polarizability was proposed by Fikhmanas and Fridberg (1973). This formula in our notation reads

$$\tau = \frac{8\sqrt{A}}{3\pi L}\,, \tag{4.8.13}$$

where L stands for the perimeter of the domain S. Formula (4.8.13) is also exact for an ellipse. It is of interest to compare its performance with our (4.8.12). Several crack shapes are considered for this purpose. A high degree of accuracy of formula (4.8.12) is confirmed by comparison with available numerical solutions.

Example 1: Polygon. Consider a flat crack in the shape of a polygon with n sides, with the only limitation that the function $a(\phi)$ describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the crack center as it was defined earlier. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} intersect, is numbered k+1. It is clear that the value of the index n+1 is to be understood as 1. Denote the distance from the crack center to the kth apex as b_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. Then formulae (4.8.6) and (4.8.12) yield the following expression for the coefficient τ:

$$\tau = \frac{8}{3\sqrt{A}}\left\{\sum_{k=1}^{n}\left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_k^2}\right]^{1/2} + \left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_{k+1}^2}\right]^{1/2}\right\}^{-1}. \tag{4.8.14}$$

In the case of a regular polygon formula (4.8.14) simplifies to

$$\tau = \frac{4\sqrt{\cot(\pi/n)}}{3n^{3/2}\sin(\pi/n)}\,. \tag{4.8.15}$$

Formula (4.8.13) gives for a regular polygon

$$\tau = \frac{4}{3\pi} \left[\frac{\cot(\pi/n)}{n} \right]^{1/2}. \qquad (4.8.16)$$

It is of interest to compare the numerical results due to (4.8.15) and (4.8.16). Here the relevant computations are presented

n=	3	4	5	6	7	8	∞
formula (4.8.15) τ=	0.2251	0.2357	0.2380	0.2388	0.2391	0.2392	0.2394
formula (4.8.16) τ=	0.1862	0.2122	0.2227	0.2280	0.2312	0.2331	0.2394
discrepancy (%)	17.3	10.0	6.5	4.5	3.3	2.6	0.0

While both formulae in the limiting case $n\to\infty$ give the same exact result for a circle, their discrepancy for small n is quite significant, so it is important to establish which one is more accurate. We have not found any data for an equilateral triangle. If one takes the experimental result by Cohn for a square τ=0.2274 as exact, then our formula (4.8.15) is in error by 3.6% while formula (4.8.16) due to Fikhmanas and Fridberg is in error by 6.7%. The numerical result due to Okon and Harrington for a square is 0.2258 which also favours our formula. In the case of a regular hexagon, the result by Okon and Harrington is 0.2375, so that our result differs by 0.5% only, while the error of (4.8.16) is 4%. It is noteworthy that the value of τ does not change significantly in the whole range $3\le n<\infty$.

We can also compare the normal displacements along a central line of a hexagon perpendicular to its side, given by (4.8.3) with numerical data due to Okon and Harrington (1981). Here are the results ($w*$ stands for $w/2\pi Hp\sqrt{A}$)

ρ/a=	0.	0.1667	0.3333	0.5000	0.6667	0.8333
Okon *et al* $w*$=	0.351	0.346	0.331	0.305	0.263	0.210
formula (4.8.3) $w*$=	0.357	0.352	0.3366	0.3092	0.266	0.1973
discrepancy (%)	-1.7	-1.7	-1.4	-1.4	-1.2	6.0

As we expected, the agreement is good, except for the points very close to the boundary.

Example 2: Rectangle. Consider a rectangular crack, a and b being its semiaxes along the axes Ox and Oy respectively. Introduce the aspect ratio $\varepsilon=b/a\le1$. Formula (4.8.14) in this case reduces to

$$\tau = \frac{\sqrt{\varepsilon}}{3(1+\varepsilon^2)^{1/2}} . \qquad (4.8.17)$$

Formula (4.8.13) in this case gives

$$\tau = \frac{4\sqrt{\varepsilon}}{3\pi(1 + \varepsilon)} . \tag{4.8.18}$$

We present below the results of computations due to (4.8.17) and (4.8.18) compared with the experimental results of Cohn If one assumes the results of

ε=	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
experiment τ=	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (4.8.17) τ=	0.1049	0.1277	0.1462	0.1749	0.2108	0.2309	0.2357
discrepancy (%)	12.7	9.5	6.6	2.3	-0.7	-2.6	-3.7
formula (4.8.18) τ=	0.1220	0.1429	0.1582	0.1788	0.2001	0.2100	0.2122
discrepancy (%)	-1.5	-1.3	-1.1	0.1	4.4	6.7	6.7

Cohn to be exact then our formula performs better for $\varepsilon \geq 0.5$ while the formula by Fikhmanas and Fridberg is more accurate for $\varepsilon < 0.5$. If instead we take the numerical results received in a personal communication from De Smedt as correct then the conclusion might be different. For example, his value of τ for $\varepsilon=0.1$ is 0.1142; now our result is in error by 8% while the result by Fikhmanas and Fridberg is in error by -7%. At this moment nobody seems to know which estimate is correct. We can also compare the dimensionless displacements w^* due to (4.8.3) with the numerical results received in a personal communication from De Smedt for a rectangle with aspect ratio $\varepsilon=0.5$ (as before, w^* stands for $w/2\pi Hp\sqrt{A}$). Here are the data computed along the axis Ox for $y/b=0.025$.

x/a=	0.0250	0.2250	0.4250	0.6250	0.8250	0.9750
De Smedt w^*=	0.3161	0.3118	0.2989	0.2713	0.2107	0.0852
formula (4.8..3) w^*=	0.3158	0.3081	0.2862	0.2469	0.1787	0.0703
Discrepancy (%)	0.1	1.2	4.2	9.0	15.2	17.5

The agreement is not bad except for the zone $x/a>0.625$. Here are the data computed along the axis Oy for $x/a=0.025$. We observe here a good agreement

y/b=	0.0250	0.1250	0.2250	0.3250	0.4250	0.4750
De Smedt w^*=	0.3161	0.3067	0.2836	0.2424	0.1690	0.0976
formula (4.8.3) w^*=	0.3158	0.3062	0.2824	0.2403	0.1666	0.0987
Discrepancy (%)	0.1	0.2	0.4	0.8	1.4	-1.2

even close to the boundary which may be attributed to the fact that the crack shape in the Oy direction is very close to the two-dimensional case.

$$\tau = \frac{\sqrt{2\varepsilon}}{3(1 + \varepsilon)} . \tag{4.8.19}$$

The result due to Fikhmanas and Fridberg is

$$\tau = \frac{2\sqrt{2\varepsilon}}{3\pi(1 + \varepsilon^2)^{1/2}} . \tag{4.8.20}$$

We did not find in the mechanics literature any result related to a crack with a rhombus planform. The coefficient of electrical polarizability for a diamond with the aspect ratio ε=0.5 was found numerically by Okon and Harrington as τ=0.2082. Our result is 0.2222 (discrepancy 6.7%) while formula (4.8.20) gives 0.1898 (discrepancy 8.9%). We have received two sets of data in personal communications from De Smedt and Lee. Here are the data received as compared to formulae (4.8.19) and (4.8.20) The data received from Lee is given

ε=	0.100	0.200	0.333	0.500	0.800	1.000
De Smedt τ=	0.111	0.151	0.182	0.204	0.219	0.221
formula (4.8.19) τ=	0.136	0.176	0.204	0.222	0.234	0.236
Discrepancy (%)	-21.9	-16.4	-12.0	-9.0	-6.8	-6.6
formula (4.8.20) τ=	0.094	0.132	0.164	0.190	0.210	0.212
Discrepancy (%)	15.1	12.8	9.8	6.9	4.4	4.1

as a function of the angle $\alpha = \tan^{-1}\varepsilon$ We have presented both sets of data in

α(deg.)=	10.	15.	20.	25.	30.	40.	45.
Lee τ=	0.147	0.174	0.193	0.207	0.216	0.226	0.228
formula (4.8.19) τ=	0.168	0.192	0.209	0.220	0.227	0.235	0.236
Discrepancy (%)	-14.2	-10.6	-8.1	-6.3	-5.2	-3.8	-3.6
formula (4.8.20) τ=	0.124	0.150	0.170	0.186	0.197	0.211	0.212
Discrepancy (%)	15.8	13.7	11.8	10.1	8.5	6.9	6.7

order to underline the fact that there is no really reliable data as yet. The first set of data suggests that the formula by Fikhmanas and Fridberg is the more accurate, while the second set favours ours. It is noteworthy that formula (4.8.19) seems to give the upper bound, and formula (4.8.20) provides the lower bound, their average being very close to the numerical data.

We can also compare the normal displacements due to our (4.8.3) with a similar result due to Okon and Harrington (1981). Here are the data computed along a central line parallel to its side ($w*$ stands for $w/2\pi Hp\sqrt{A}$). The agreement is worse if the comparison is made along the major axis. This is mainly due to the assumption of a square root singularity in (4.8.3) which does

$\rho/a=$	0.	0.3333	0.6667
Okon *et al.* $w*=$	0.335	0.304	0.257
formula (4.8.3) $w*=$	0.3333	0.3142	0.2484
discrepancy (%)	0.5	-3.4	3.3

Example 4: Circular segment. Let the radius r and the angle 2α be the segment parameters. Direct numerical computations show that the crack center can be identified with the center of gravity, with an error comparable with the accuracy of the theory presented. The location of the center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)} .$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k\cos\phi + (1 - k^2\sin^2\phi)^{1/2}], \quad \text{for } 0\le\phi\le\pi-\gamma \text{ or } \pi+\gamma\le\phi<2\pi,$$

and

$$a(\phi) = r\,\frac{k - \cos\alpha}{\cos(\pi - \phi)} , \qquad \text{for } \pi-\gamma\le\phi\le\pi+\gamma, \tag{4.8.21}$$

where $\gamma = \tan^{-1}(\sin\alpha/(k - \cos\alpha))$. Feeding of (4.8.21) in (4.8.6) and (4.8.12) gives

$$\tau = \frac{4}{3(\alpha - \frac{1}{2}\sin 2\alpha)^{1/2}}\left[\frac{k\sin\gamma + E(\pi-\gamma,k)}{1 - k^2} + \frac{\sin\gamma}{k - \cos\alpha}\right]^{-1} . \tag{4.8.22}$$

where $E(\cdot,\cdot)$ stands for the incomplete elliptic integral of the second kind. The formula due to Fikhmanas and Fridberg gives

$$\tau = \frac{4(\alpha - \frac{1}{2}\sin 2\alpha)^{1/2}}{3\pi(\alpha + \sin\alpha)} . \tag{4.8.23}$$

The coefficient of electrical polarizability for a semi-circle was computed by Okon and Harrington as $\tau=0.2161$. Our result due to (4.8.22) is $\tau=0.2163$ which is practically identical to the previously mentioned one. The result due to (4.8.23) is $\tau=0.2069$ (discrepancy 4.3%). An additional confirmation of correctness of the new method can be obtained by observing the plot of the electrical polarizability density distribution for a semi-circle presented by Okon and Harrington (1981). Its maximum is located at a distance $\approx 0.47r$ from the circle's center. Our definition of the crack center requiring the minimization of the integral (4.8.6) gives its coordinate at $0.48r$ which is very close. The center of gravity of the semi-circle is located at $0.42r$.

Example 5: Circular sector. Let r and 2α be its radius and the polar angle. The crack center is assumed to be located on the axis of symmetry at a distance kr from the circle's center. Numerical computations show that the crack center may be located at the center of gravity for $0.1\pi<\alpha<0.6\pi$. In this case the value of k is defined by $k = 2\sin\alpha/(3\alpha)$. In the range $\alpha<0.1\pi$ or $\alpha>0.6\pi$, the value of k should be found from the minimum condition for the integral (4.8.6). Repetition of the procedure described in the previous paragraph leads to the following result

$$\tau = \frac{4}{3\sqrt{\alpha}}\left[\frac{k\sin\gamma + \mathrm{E}(\gamma,k)}{1 - k^2} + \frac{\cos\alpha + \cos(\alpha - \gamma)}{k\sin\gamma}\right]^{-1}. \tag{4.8.24}$$

Here, $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$. The formula due to Fikhmanas and Fridberg reads

$$\tau = \frac{4\sqrt{\alpha}}{3\pi(1 + \alpha)} . \tag{4.8.25}$$

Note that neither (4.8.24) nor (4.8.25) reduce to the exact value for a circle when $\alpha=\pi$. This is due to the fact that we do not really have the case of a penny-shaped crack when α approaches π: we have a circular crack which has its faces bonded along the radius $\phi=\pi$. This case has not been considered by other authors so we cannot say which formula is more accurate. Okon and Harrington obtained in the case of a quadrant $\tau=0.2269$, formula (4.8.24) gives $\tau=0.2308$ (discrepancy 1.7%), and formula (4.8.25) gives $\tau=0.2107$ (discrepancy 7%). It is noteworthy that the value of τ for a quadrant is greater than that for a semi-circle. The general impression is that our theory in the particular cases of a circular sector and segment provides the upper bound for τ while the formula due Fikhmanas and Fridberg gives the lower bound.

Example 6: Cross. Consider a crack configuration obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon=b/a\le 1$. The area can be expressed as

$$A = 4a^2\varepsilon(2 - \varepsilon) ,$$

The following expression can be obtained for τ:

$$\tau = \frac{\sqrt{2\varepsilon}}{6\sqrt{2 - \varepsilon}\{[2(1 + \varepsilon^2)]^{1/2} - 1\}} \tag{4.8.26}$$

The formula due to Fikhmanas and Fridberg is

$$\tau = \frac{2\sqrt{\varepsilon(2 - \varepsilon)}}{3\pi} . \tag{4.8.27}$$

Here, we present the results given by formulae (4.8.26) and (4.8.27) compared to the experimental results of Cohn and the numerical results by De Meulenaere and Van Bladel (1977), and those received in personal communication from De Smedt

ε=	0.1000	0.2000	0.3000	0.4000	0.6000	0.8000	1.0000
experimental τ=	0.0942	0.1333	0.1609	–	–	–	0.2274
De Meulenaere τ=	–	–	–	0.19	0.22	0.23	0.238
De Smedt τ=	0.0835	0.1183	–	0.1767	0.2084	0.2193	0.2212
formula (4.8.26) τ=	0.1284	0.1777	0.2078	0.2252	0.2376	0.2372	0.2357
formula (4.8.27) τ=	0.0925	0.1273	0.1515	0.1698	0.1944	0.2079	0.2122

We did not compute the discrepancy since the data disagreement is too large thus making all the data not very reliable. The general impression is that our (4.8.26) gives the upper bound for τ while the formula due to Fikhmanas and Fridberg provides the lower bound. This conclusion might be wrong if the numerical results received in the personal communication from De Smedt are correct. For example, his result for ε=0.1 is τ=0.08347 which differs from the experimental result by 11%. All this proves one point: the existing numerical methods are too crude and there is a need to develop some new and more reliable numerical methods.

It should be noted that the function defined by (4.8.26) is not monotonic: a relatively flat maximum is observed for $\varepsilon \approx 0.7$. The remaining data are monotonic. We have no rigorous proof to claim that the quantitative behavior of (4.8.26) is correct while the other data behavior is not, but we can indicate that the value of τ for a quadrant is also greater than that for a semi-circle, and this is mainly due to the fact that the shape of a quadrant is more close to the shape of a circle than that of a semi-circle. A similar statement can be made about a cross with the aspect ratio $\varepsilon \approx 0.7$ as compared to a square.

Discussion. The majority of the examples considered indicate that the exact result is sandwiched between the results given by our (4.8.12) and by the formula due to Fikhmanas and Fridberg (4.8.13). In this sense the formulae act as upper and lower bounds respectively, which leads to a conjecture: for an arbitrary contour one of the inequalities holds, namely, either $\tau_{12} \le \tau_{exact} \le \tau_{13}$, or $\tau_{12} \ge \tau_{exact} \ge \tau_{13}$. We can indicate one way to disprove the conjecture. A look at the table related to a rectangle in the previous Section indicates that our formula (4.8.17) should give the lower bound for small aspect ratio ε, and it should give the upper bound for ε close to unity, and vice-versa for the formula (4.8.18). This means that there should be a value of ε for which both formulae (4.8.17) and (4.8.18) are *exact*, and give the same result. By equating (4.8.17) and (4.8.18), one gets the value of the aspect ratio

$$\varepsilon_{1,2} = \frac{\pi^2 \pm 4(2\pi^2 - 16)^{1/2}}{16 - \pi^2},$$

which yields ε_1=0.3482 and ε_2=2.8716, with an obvious property $\varepsilon_1\varepsilon_2$=1. The corresponding value of τ is 0.18576. From the table of the previous Section, one can see that for ε=0.3 the exact value of τ should be between 0.1749 and 0.1788. If we could be sure that the experimental value 0.1789 is exact then this would disprove the conjecture, but at the moment none of the existing experimental or numerical techniques can offer such an accuracy. This can be achieved on the basis of the method for accurate evaluation of singular two-dimensional integrals presented in (Fabrikant, 1986e). A significant effort is required to prove or to disprove the conjecture, and is left to the interested reader.

The accuracy of formula (4.8.12) can be improved by taking into consideration the fourth harmonic (4.8.7) in combination with the variational approach (Noble, 1960). The following functional is stationary at the exact solution of (4.1.9)

$$I(w) = 2\int\int_S \sigma(M)w(M)\mathrm{d}S_M + \frac{1}{4\pi^2 H}\int\int_S w(M)\left[\Delta\int\int_S \frac{w(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \qquad (4.8.28)$$

Taking

$$-\frac{1}{4\pi^2 H}\,\Delta\int\int_S \frac{w(N)}{R(M,N)}\,\mathrm{d}S_N \approx \sigma_0 + \sigma_4,$$

where σ_0 and σ_4 are defined by (4.8.5) and (4.8.7) respectively, and substituting them in (4.8.28), we obtain a functional which can be considered as a function of δ. From the extremum condition

$$\frac{\partial I}{\partial \delta} = 0$$

one finally gets

$$\tau = \frac{8}{3B\sqrt{A}(1 - c)}, \tag{4.8.29}$$

where

$$c = \frac{3(F_c E_c + F_s E_s)}{5AB} ,$$

and the following geometrical characteristics were introduced

$$F_c = \int_0^{2\pi} \frac{\cos 4\phi \, d\phi}{a^2(\phi)} , \qquad F_s = \int_0^{2\pi} \frac{\sin 4\phi \, d\phi}{a^2(\phi)} ,$$

$$E_c = \int_0^{2\pi} a^3(\phi)\cos 4\phi \, d\phi, \qquad E_s = \int_0^{2\pi} a^3(\phi)\sin 4\phi \, d\phi.$$

The results of computations due to (4.8.29) for a rectangle are presented below against the experimental results of Cohn Comparison of this table with a

ε=	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
Cohn τ=	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (4.8.29) τ=	0.1054	0.1290	0.1484	0.1785	0.2125	0.2257	0.2278
discrepancy (%)	12.3	8.6	5.2	0.2	-1.5	-0.3	-0.2

corresponding one presented earlier indicates that the variational approach does improve the accuracy, though the improvement is still not sufficient for small ε. There is no proof that the variational approach will always improve the accuracy. On the contrary, one can find quite a few examples when the accuracy deteriorates. This can usually be observed for domains with a very small aspect ratio ε. It is up to the user to decide whether the more cumbersome computations are worth somewhat better accuracy.

In this section we have considered in detail only the case of a uniform crack pressure. Some considerations can be presented for a general case. It is known (see, for example, 4.1.13) that in the case of a penny-shaped crack the following relationship can be established between the displacements w and the internal pressure σ

$$w(\rho,\phi) = \frac{2}{\pi} H \int_0^{2\pi} \int_0^{a} \tan^{-1}\left[\frac{[(a^2 - \rho^2)(a^2 - \rho_0^2)]^{1/2}}{aR}\right] \frac{\sigma(\rho_0,\phi_0)}{R} \rho_0 d\rho_0 d\phi_0 ,$$

where $R^2=\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)$. The following generalization of the last formula seems to be natural

$$w(\rho,\phi) = \frac{2}{\pi}H\int_0^{2\pi}\int_0^{a(\phi_0)} \tan^{-1}\left[\frac{[(a^2(\phi) - \rho^2)(a^2(\phi_0) - \rho_0^2)]^{1/2}}{a(\phi)R}\right] \frac{\sigma(\rho_0,\phi_0)}{R} \rho_0 d\rho_0 d\phi_0. \tag{4.8.30}$$

Though a complete investigation of (4.8.30) is beyond the scope of this book, there is reason to believe that it will be sufficiently accurate for a general loading of cracks whose aspect ratio is not far away from unity. Here is an example. Let us compute the displacement w_0 at the centre of an elliptical crack (a and b are the semiaxes of the ellipse, $a>b$) under a uniform pressure p. The result due to (4.8.30) is $w_0=(8/\pi)HpbK(k)$, where $K(k)$ is the complete elliptic integral of the first kind and $k=(1-b^2/a^2)^{1/2}$. The exact result is $w_0=2\pi Hpb/E(k)$. Both results are close to each other for small k, coinciding in the case of a circle (k=0). Direct computation shows that the error of the approximate expression does not exceed 7% for an ellipse with aspect ratio $b/a\geq0.5$. In the case of a square the dimensionless displacement w^* at its centre can be computed from (4.8.30) as $(4/\pi^2)\ln(1+\sqrt{2})=0.3572$, while a similar result due to the experimental value of Cohn is $(3/2)\tau=0.3411$, and the discrepancy does not exceed 5%. Of course, these examples do not prove anything conclusively, but they make it quite clear that expression (4.8.30) is worth to investigate further. A similar statement can be made about the following expression

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2[\rho^2- a^2(\phi)]^{1/2}}\int_0^{2\pi}\int_0^{a(\phi_0)} \frac{[a^2(\phi_0) - \rho_0^2]^{1/2}\sigma(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)},$$

which gives the normal stress distribution outside a circular crack directly through its values inside the crack. The future research will show whether the last expression is useful for a general crack.

Investigation of the stress intensity factors was beyond the main scope of this section, but we can show that some simple formulae may be derived to give results which are close to (Weaver, 1977). Make use of the asymptotic relationship between the limiting values of the crack opening displacements and the stress intensity factor, namely,

$$w = 2H(2\pi r)^{1/2}k_1.$$

Consider a rectangular crack of dimensions $2a$ and $2b$ ($a>b$), subjected to a uniform pressure p. By substituting (4.8.3) in the last expression, the following formulae are obtained for the dimensionless stress intensity factor $\kappa=k_1/[p(\pi b)^{1/2}]$: along the shorter side $\kappa=3\tau$; along the longer side $\kappa=3\tau(a/b)^{1/2}$. In the case of a square our formulae give $\kappa=3\times0.2357=0.7071$. The result due to Weaver is about 0.73, with the discrepancy of about 3%. In the case of b/a=0.3, our result at the middle of the longer side is κ=0.98 which is very close to the result of Weaver. Expression (4.8.17) indicates that the limiting value of the stress intensity factor as $(b/a)\to0$ is $\kappa\to1$, as it should be in the case of an infinite strip slit. The only discrepancy with the results of Weaver (1977) is the value of the dimensionless stress intensity factor along the shorter side: according to our formula it should *decrease* with b/a, tending to zero as $(b/a)^{1/2}$; in the paper by Weaver (1977) its value *increases*. The reader is referred also to the paper (Fabrikant, 1987f) where some inconsistencies of (Kassir and Sih, 1975) in defining the stress intensity factor for elliptical cracks are pointed out.

The mathematically equivalent problem of sound penetration through an aperture of general shape in a soft screen is solved in (Fabrikant, 1988d). The same apparatus is used in the investigation of electrical polarizability of small apertures of general shape (Fabrikant, 1987c).

Exercise 4.8

1. Derive (4.8.5).

2. Establish (4.8.14).

3. Try to prove or disprove the conjecture, expressed in the section 'Discussion' above.

4. Find the domain of validity of formula (4.8.30) for an elliptical crack. (Find the ratio of ellipse semiaxes, for which the error does not exceed 5%.)

5. Solve the problem of a general flat crack subjected to normal loading, with its magnitude proportional to the x-coordinate. (Bending of an elastic space, with a flat crack of general shape).

4.9 General crack under uniform shear

Let the crack boundary be described in polar coordinates by the equation $\rho=a(\phi)$, where $a(\phi)$ is a single-valued continuous function. We can always choose the coordinate axis orientation so that the first harmonic will vanish from the Fourier expansion of $a(\phi)$. An approximate analytical solution of (4.4.14) for a general crack can be obtained by the method used in previous section. The method uses the following representation

$$\iint_S \frac{u(N)}{R(N,N_0)}\,dS_N = \int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)}{(\rho_0^2 - x^2)^{1/2}}\, u(\rho_0,\phi_0)\rho_0 d\rho_0. \tag{4.9.1}$$

Consider the case of a uniform shear loading. Let the tangential displacements have the form

$$u(\rho,\phi) = u_0[a^2(\phi) - \rho^2]^{1/2}/a(\phi)\ , \tag{4.9.2}$$

where u_0 is an as yet unknown complex constant. Substitution of (4.9.2) in (4.9.1) yields, after integration and retaining the first two harmonics only,

$$\iint_S \frac{u(N)}{R(N,N_0)}\,dS_N \approx \frac{u_0}{16\pi(G_1^2 - G_2^2)} \int_0^{2\pi} \Big\{ 2a(\phi_0) - \frac{\rho^2}{a(\phi_0)}\Big[1 + 3\cos 2(\phi-\phi_0)\Big]\Big\} d\phi_0. \tag{4.9.3}$$

By substituting (4.9.3) in (4.4.14) and performing necessary differentiation, we obtain the relationship between the shear loading and the amplitude of the tangential displacements, namely,

$$\tau = \frac{1}{4\pi(G_1^2 - G_2^2)} \Big[u_0 G_1 B_1 + 3\bar{u}_0 G_2 B_2 \Big]\ , \tag{4.9.4}$$

where

$$B_1 = \int_0^{2\pi} \frac{d\phi}{a(\phi)}\ , \qquad B_2 = \int_0^{2\pi} \frac{e^{2i\phi} d\phi}{a(\phi)}\ . \tag{4.9.5}$$

Equation (4.9.4) can be solved, to give

$$u_0 = \frac{4\pi(G_1^2 - G_2^2)}{G_1^2 B_1^2 - 9G_2^2 B_2^2}\,[G_1 B_1 \tau - 3G_2 B_2 \bar{\tau}]. \tag{4.9.6}$$

The integrals in (4.9.5) can be computed easily for various crack shapes. For example, a rectangular crack with sides $2a_1$ and $2a_2$ is characterized by the values

$$B_1 = \frac{4(a_1^2 + a_2^2)^{1/2}}{a_1 a_2}, \qquad B_2 = \frac{4(a_2^2 - a_1^2)}{3a_1 a_2 (a_1^2 + a_2^2)^{1/2}}. \tag{4.9.7}$$

It is noteworthy that despite the fact that the integral representation (4.9.1) is valid inside a circle $\rho \leq \min\{a(\phi)\}$ only, and despite the approximate nature of (4.9.3), the solution given by (4.9.4–4.9.5) is *exact* for an ellipse. We did not find in the literature any reliable data related to a non-elliptical crack under shear loading, therefore it is difficult to say how accurate the solution is for various crack shapes.

Exercise 4.9

1. Derive (4.9.3).

2. Establish (4.9.6).

3. Consider the case of semicircular crack.

4. Solve the problem for a cross-shaped crack

5. Solve the problem of a general flat crack subjected to torsion.

4.10 Close interaction of pressurized coplanar circular cracks

The general method is applied here for the stress analysis of an elastic space weakened by several arbitrarily located coplanar circular cracks under the action of an arbitrary normal pressure. The governing integral equations are derived, which have definite advantages over other methods: the equations are non-singular, the iteration procedure is rapidly convergent even for very close interactions; there is no need to solve the integral equations if one is interested only in obtaining the upper and the lower bounds for the quantities of interest. In the case of the cracks which are far apart, these bounds are so close that they provide, in fact, a sufficiently accurate solution to the problem. The method allows us also to obtain a practically exact numerical solution to the problem of very close interactions. Several illustrative examples are considered.

Theory. Consider an elastic space weakened in the plane z=0 by n arbitrarily located circular cracks. The cracks do not intersect. Let the centre of the kth crack be located at the point with Cartesian coordinates x_k and y_k, its radius be denoted by a_k. Let an arbitrary pressure σ_k be applied normally to the crack faces in opposite directions. The set of governing integral equations can be written, due to (4.1.9), in the form

$$\sigma_i(M_i) = -\frac{1}{4\pi^2 H}\Delta\sum_{k=1}^{n}\int\!\!\int_{S_k}\frac{w_k(M_k)}{R(M_i,M_k)}\,dS_k\ ,\ \text{for } i = 1,2,\ \ldots\ ,n, \tag{4.10.1}$$

where Δ is the two-dimensional Laplace operator, S_k is the kth crack domain, $R(M_i,M_k)$ stands for the distance between the points M_i and M_k, ($M_i \in S_i$ and $M_k \in S_k$); w_k denotes the normal displacements of the crack face (an unknown function), σ_i stands for the normal traction acting inside the ith crack (a known function). We can single out, without loss of generality, the crack number 1, and consider the set of cracks in the local polar system of coordinates with the origin coinciding with the centre of the first crack. By using the integral representation (4.1.11), the first equation from the set (4.10.1) can be rewritten as

$$\sigma_1(\rho,\phi) = -\frac{1}{\pi^2 H\rho}\,\mathcal{L}\!\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{(\rho^2 - x^2)^{1/2}}\,\mathcal{L}(x^2)$$

$$\times\frac{d}{dx}\int_x^{a_1}\frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}}\,\mathcal{L}\!\left(\frac{1}{\rho_0}\right)w_1(\rho_0,\phi) - \frac{1}{4\pi^2 H}\sum_{k=2}^{n}\Delta\int\!\!\int_{S_k}\frac{w_k(M_k)}{R(M_1,M_k)}\,dS_k. \tag{4.10.2}$$

Since the integrals under the summation sign in (4.10.2) are non-singular, the differentiation can be performed under the integral sign, with the result

$$\sigma_1(\rho,\phi) = -\frac{1}{\pi^2 H\rho}\,\mathcal{L}\!\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{(\rho^2 - x^2)^{1/2}}\,\mathcal{L}(x^2)$$

$$\times\frac{d}{dx}\int_x^{a_1}\frac{\rho_0\ d\rho_0}{(\rho_0^2 - x^2)^{1/2}}\,\mathcal{L}\!\left(\frac{1}{\rho_0}\right)w_1(\rho_0,\phi) - \frac{1}{4\pi^2 H}\sum_{k=2}^{n}\int\!\!\int_{S_k}\frac{w_k(M_k)}{R^3(M_1,M_k)}\,dS_k. \tag{4.10.3}$$

Formula (4.1.12) allows us to express w_1 from (4.10.3) by constructing an inverse operator, namely,

$$w_1(\rho,\phi) = 4H\int_\rho^{a_1} \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma_1(\rho_0,\phi)$$

$$+ \frac{(a_1^2 - \rho^2)^{1/2}}{\pi^2} \sum_{k=2}^{n} \iint_{S_k} \frac{w_k(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{(\rho_0^2 - a_1^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]} .$$

(4.10.4)

Here we used the following integral representation for $1/R^3$

$$\frac{1}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0))^{3/2}} = \frac{2}{\pi\rho}\mathcal{L}\left(\frac{1}{\rho}\right) \frac{d}{d\rho}\int_0^\rho \frac{\lambda\left(\frac{x^2}{\rho_0},\ \phi-\phi_0\right)x^2 dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{3/2}},$$

for $\rho_0>\rho$. (4.10.5)

The representation (4.10.5) allows us to compute various integrals involving the Abel and the $\mathcal{L}$-operators. For example, the following integral is an immediate consequence of (4.10.5)

$$\int_0^x \frac{\rho\ d\rho}{(x^2 - \rho^2)^{1/2}} \mathcal{L}(\rho)[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{-3/2}$$

$$= \frac{x}{(\rho_0^2 - x^2)^{3/2}} \lambda\left(\frac{x^2}{\rho_0},\ \phi-\phi_0\right).$$

A similar procedure can be applied to the remaining $n-1$ cracks, and the additional $n-1$ equations of the type (4.10.4) can be obtained. Note that each such equation is valid in a local system of polar coordinates related to a certain crack. The set of equations (4.10.4) can be solved numerically by iteration.

Here we show that one can obtain the upper bound, the lower bound and a reasonably accurate central estimation for all the quantities of interest without solving the integral equations (4.10.4). Since w_k does not change sign in S_k, we can apply the mean value theorem to the second integral in (4.10.4), with the result

$$w_1(\rho,\phi) = 4H\int_{\rho}^{a_1} \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma_1(\rho_0,\phi)$$

$$+ \frac{(a_1^2 - \rho^2)^{1/2}}{\pi^2}\sum_{k=2}^{n}\frac{V_k}{(r_{1k}^2 - a_1^2)^{1/2}\ [\rho^2 + r_{1k}^2 - 2\rho r_{1k}\cos(\phi-\phi_{1k})]}. \tag{4.10.6}$$

Here V_k denotes half of the volume of the opened kth crack

$$V_k = \iint_{S_k} w_k dS_k,$$

and though the exact location of the point R_{1k} with the polar coordinates (r_{1k},ϕ_{1k}) is unknown, the fact of belonging to the domain S_k limits the possible variation of the quantities of interest and allows us to obtain the upper and the lower bounds as well as a sufficiently accurate central estimation which will be discussed further. The symmetry considerations can also be used to sharpen the estimations. It will be shown further that these bounds can be so close in the case of cracks remote from each other, that they provide, in fact, a sufficiently accurate solution to the problem.

The crack volume $2V_1$ can be estimated by integration of (4.10.4) over the domain S_1. The result is

$$V_1 = 4H\int_0^{2\pi}\int_0^{a_1} \sigma(\rho,\phi)(a_1^2 - \rho^2)^{1/2}\rho d\rho d\phi$$

$$+ \frac{2}{\pi}\sum_{k=2}^{n}\iint_{S_k}\left[\frac{a_1}{(\rho^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{\rho}\right] w_k(\rho,\phi)\rho d\rho d\phi. \tag{4.10.7}$$

We can use again the mean value theorem, with the result

$$V_1 = 4H\int_0^{2\pi}\int_0^{a_1} \sigma_1(\rho,\phi)\ (a_1^2 - \rho^2)^{1/2}\rho d\rho d\phi + \frac{2}{\pi}\sum_{k=2}^{n}\left[\frac{a_1}{(\rho_{1k}^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{\rho_{1k}}\right]V_k. \tag{4.10.8}$$

Again, the point with the polar coordinate ρ_{1k} belongs to S_k thus limiting the possible variation. Integration of the remaining $n-1$ equations over the area of each crack provides finally a system of n linear algebraic equations which can be solved with respect to the unknowns V_k. Their feeding back into (4.10.6) gives the complete solution to the problem. Although the exact values of the coordinates r_{ik}, ϕ_{ik} and ρ_{ik} are not known, we can always use their maximum and minimum values in order to obtain the upper and the lower bounds for all the quantities of interest.

Defining the stress intensity factor as

$$k_k(\phi) = \lim_{\rho \to a_k} \{(\rho - a_k)^{1/2} \sigma_k(\rho,\phi)\} ,$$

one can get an equivalent expression through the crack face displacements

$$k_k(\phi) = \frac{1}{4\pi H} \lim_{\rho \to a_k} \frac{w_k(\rho,\phi)}{(a_k - \rho)^{1/2}} . \tag{4.10.9}$$

Substitution of (4.10.4) in (4.10.9) gives

$$k_1(\phi) = \frac{1}{\pi^2\sqrt{2a_1}} \int_0^{2\pi}\int_0^{a_1} \frac{(a_1^2 - \rho^2)^{1/2}\sigma_1(\rho,\phi_0)\ \rho d\rho d\phi_0}{\rho^2 + a_1^2 - 2\rho a_1\cos(\phi - \phi_0)}$$

$$+ \frac{\sqrt{a_1}}{2\sqrt{2\pi^3}H} \sum_{k=2}^{n} \iint_{S_k} \frac{w_k(\rho, \phi_0)\ \rho d\rho d\phi_0}{(\rho^2 - a_1^2)^{1/2}[\rho^2 + a_1^2 - 2\rho a_1\cos(\phi-\phi_0)]}.$$

By using again the mean value theorem, the following expression for the stress intensity factor can be obtained

$$k_1(\phi) = \frac{1}{\pi^2\sqrt{2a_1}} \int_0^{2\pi}\int_0^{a_1} \frac{(a_1^2 - \rho^2)^{1/2}\sigma_1(\rho,\phi_0)\ \rho d\rho d\phi_0}{\rho^2 + a_1^2 - 2\rho a_1\cos(\phi-\phi_0)}$$

$$+ \frac{\sqrt{a_1}}{2\sqrt{2\pi^3}H} \sum_{k=2}^{n} \frac{V_k}{(r_{1k}^2 - a_1^2)^{1/2}\ [r_{1k}^2 + a_1^2 - 2r_{1k}a_1\cos(\phi - \phi_{1k})]} . \tag{4.10.10}$$

Similar expressions can be derived for the remaining $n-1$ cracks. The first term in (4.10.10) presents the stress intensity factor of a solitary crack opened up by an arbitrary normal pressure σ_1, the remaining terms display the influence of interacting cracks. It is clear from (4.10.10) that the stress intensity factor of interacting coplanar cracks is always greater than the stress intensity factor of a solitary crack under similar pressure. This proves the results of Mastrojannis and Mura (1983) to be incorrect, since they report a decrease in the stress intensity factor along part of the boundary.

Example 1: Two cracks. Consider the case of two coplanar circular cracks with the radii a_1 and a_2 under the action of a uniform normal pressure $\sigma_1=p_1$ and $\sigma_2=p_2$ respectively. Let l be the distance between their centres. Equations (4.10.8) in this case will take the form

$$V_1 = \frac{8}{3}\pi a_1^3 H p_1 + \frac{2}{\pi}V_2\left[\frac{a_1}{(\rho_{12}^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{\rho_{12}}\right],$$

$$V_2 = \frac{8}{3}\pi a_2^3 H p_2 + \frac{2}{\pi}V_1\left[\frac{a_2}{(\rho_{21}^2 - a_2^2)^{1/2}} - \sin^{-1}\frac{a_2}{\rho_{21}}\right].$$

The solution is

$$V_1 = \frac{8}{3}\pi H\,\frac{a_1^3 p_1 + c_{12}a_2^3 p_2}{1 - c_{12}c_{21}}, \qquad V_2 = \frac{8}{3}\pi H\,\frac{c_{21}a_1^3 p_1 + a_2^3 p_2}{1 - c_{12}c_{21}}, \tag{4.10.11}$$

where

$$c_{12} = \frac{2}{\pi}\left[\frac{a_1}{(\rho_{12}^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{\rho_{12}}\right], \qquad c_{21} = \frac{2}{\pi}\left[\frac{a_2}{(\rho_{21}^2 - a_2^2)^{1/2}} - \sin^{-1}\frac{a_2}{\rho_{21}}\right]. \tag{4.10.12}$$

Here ρ_{12} varies between $l-a_2$ and $l+a_2$, and $l-a_1\leq\rho_{21}\leq l+a_1$. The upper bound for V_k corresponds to $\rho_{12}=l-a_2$ and $\rho_{21}=l-a_1$; the lower bound is achieved at $\rho_{12}=l+a_2$ and $\rho_{21}=l+a_1$. The estimations can be sharpened by using the reciprocal theorem which implies the identity $c_{12}a_2^3=c_{21}a_1^3$. This identity narrows the range of admissible variation for ρ (in the case of unequal cracks only) thus making the estimations sharper. In this vein the case of two equal cracks should be considered as the least accurate. We shall also consider *the central estimation* which corresponds to $\rho_{12}=\rho_{21}=l$. It will be shown that the central estimation gives a sufficiently accurate solution even for relatively close crack interactions.

Formulae (4.10.11–4.9.12) simplify in the case of equal cracks as $a_1=a_2=a$, and if $p_1=p_2=p$, then

$$V_1 = V_2 = V = \frac{V_0}{1 - c}, \qquad c = \frac{2}{\pi}\left[\frac{a}{(\rho^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{\rho}\right]. \tag{4.10.13}$$

Here $V_0=(8/3)\pi Ha^3p$ stands for a half of the volume of a solitary crack, and ρ varies between $l-a$ and $l+a$. Note that in the case of a uniform pressure, the crack energy W is proportional to its volume, namely, $W=pV$. It is clear from (4.10.13) that the crack interaction increases their energy. Substitution of (4.10.13) in (4.10.10) and use of the mean value theorem yield the following expression for the stress intensity factor

$$K(\phi) = K_0\left\{1 + \frac{2\varepsilon^3}{\{3\pi(1 - \varepsilon^2)^{1/2} - 6[\varepsilon - (1 - \varepsilon^2)^{1/2}\sin^{-1}\varepsilon]\}[1+\varepsilon^2-2\varepsilon\cos\phi]}\right\}, \tag{4.10.14}$$

where $K_0=p\sqrt{2a}/\pi$ corresponds to the stress intensity factor for an isolated crack under the action of a uniform normal pressure p. Here the upper bound for the stress intensity factor is given by $\varepsilon=a/(l-a)$, the lower bound corresponds to $\varepsilon=a/(l+a)$, and the central estimation is defined by $\varepsilon=a/l$. Now we need an accurate numerical solution in order to estimate the accuracy of the approximate formulae derived. Assume the crack face displacements in the form

$$w(\rho,\phi) = 4Hp(a^2 - \rho^2)^{1/2}f(\rho,\phi), \tag{4.10.15}$$

where f is as yet unknown function. It may be called *the interaction function* since it is equal to the ratio of the crack opening displacements to those of a solitary crack. The values of $f(a,\phi)$ are equal to the ratio of the stress intensity factor of interacting cracks to the stress intensity factor of a solitary crack. We shall call $f(a, \phi)$ *the interaction factor*. Substitution of (4.10.15) in (4.10.4) gives a convenient expression for the procedure of iteration

$$w(\rho, \phi) = 4Hp(a^2 - \rho^2)^{1/2}\Bigg\{ 1$$

$$+ \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a \frac{(a^2 - r_0^2)^{1/2}\; f(r_0, \psi_0)\; r_0 dr_0 d\psi_0}{(l^2 + r_0^2 + 2lr_0\cos\psi_0 - a^2)^{1/2}[r^2 + r_0^2 + 2rr_0\cos(\psi+\psi_0)]}\Bigg\}. \tag{4.10.16}$$

Here we have introduced the new variables $r=(\rho^2+l^2-2l\rho\cos\phi)^{1/2}$, $\psi=\sin^{-1}[(\rho/r)\sin\phi]$.

The integral in (4.10.16) has a logarithmic singularity for $r=l-a$, $\psi=0$, as $l\to 2a$, therefore the procedure of iteration might not be convergent for l very close to $2a$. The limiting value of l can be found by analyzing the integral operator

$$Z(f) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a \frac{(a^2 - r_0^2)^{1/2}\ f(r_0,\ \psi_0)\ r_0 dr_0 d\psi_0}{(l^2 + r_0^2 + 2lr_0\cos\psi_0 - a^2)^{1/2}[(l - a)^2 + r_0^2 + 2(l - a)r_0\cos\psi_0]}. \tag{4.10.17}$$

According to the Banach's theorem, it is sufficient to prove that the integral operator (4.10.17) is a contraction operator. We define the distance in the class of continuous functions by

$$\delta(f,f_1) = \max|f(\rho,\phi) - f_1(\rho,\phi)|.$$

We assess the value of

$$|Z(f) - Z(f_1)|$$

$$= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a \frac{(a^2 - r^2)^{1/2}\ |f(r,\ \psi) - f_1(r,\ \psi)|\ r dr d\psi}{(l^2 + r^2 + 2lr\cos\psi - a^2)^{1/2}[(l - a)^2 + r^2 + 2(l - a)r\cos\psi]}$$

$$\le \frac{\delta(f,f_1)}{\pi^2}\int_0^{2\pi}\int_0^a \frac{(a^2 - r^2)^{1/2}\ r dr d\psi}{(l^2 + r^2 + 2lr\cos\psi - a^2)^{1/2}[(l - a)^2 + r^2 + 2(l - a)r\cos\psi]}$$

$$< \frac{2\delta(f,f_1)}{\pi}\int_0^a \frac{(a^2 - r^2)^{1/2}\ r dr}{[(l - r)^2 - a^2]^{1/2}[(l - a)^2 - r^2]} < \frac{2a\ \delta(f,f_1)}{\pi[(l - a)^2 - a^2]^{1/2}} .$$

The integral operator (4.10.17) will be a contraction operator if

$$\frac{2a}{\pi[(l - a)^2 - a^2]^{1/2}} < 1 ,$$

indicating that the iteration procedure will be convergent for $l>2.18a$ which corresponds to a fairly close interaction. The estimation given above is crude. Direct computations show that the iteration procedure converges even for $l=2.0005a$ (which corresponds to the case when the shortest distance between the cracks is equal to 0.0005 of its radius), and converges rapidly: the first iteration with $f\equiv 1$ has the maximum relative error less than 2%, and the sixth iteration may be considered practically as an exact solution since the maximum relative error becomes less than 10^{-7}. The accuracy of the first iteration improves as the distance between cracks increases. For example, the first iteration for l=10 is practically exact with maximum relative error less than 10^{-7}. We could not go closer than $l=2.0005a$, not because of non-convergence, but because the standard subroutine DBLIN from IMSL library, which was used to compute the integrals, failed giving terminal errors. Though we do not have a rigorous proof, it seems probable that the iteration procedure is theoretically convergent for an arbitrarily small distance between cracks.

The values of the interaction function $f(\rho,\phi)$ are presented in (Fabrikant, 1987g) for various ratio of l/a. We limit ourselves here to just one abbreviated table, related to the closest interaction considered, with $l=2.0005a$. The reader is referred to the original paper for additional data. The first line in Table 4.10.1

Table 4.10.1. The interaction function for $l=2.0005a$

ρ ϕ	0.	30.	60.	90.	120.	150.	180.
1.00000	2.77577	1.18634	1.06686	1.03605	1.02471	1.02009	1.01881
0.91667	1.42135	1.16479	1.06729	1.03761	1.02614	1.02137	1.02003
0.83333	1.27366	1.14560	1.06728	1.03916	1.02766	1.02277	1.02139
0.75000	1.20176	1.12870	1.06682	1.04067	1.02928	1.02431	1.02289
0.66667	1.15812	1.11396	1.06593	1.04214	1.03102	1.02601	1.02455
0.58333	1.12865	1.10117	1.06464	1.04353	1.03287	1.02788	1.02642
0.50000	1.10743	1.09012	1.06299	1.04482	1.03483	1.02996	1.02850
0.41667	1.09146	1.08058	1.06103	1.04597	1.03692	1.03228	1.03086
0.33333	1.07903	1.07234	1.05883	1.04696	1.03911	1.03486	1.03353
0.25000	1.06913	1.06519	1.05646	1.04777	1.04142	1.03775	1.03657
0.16667	1.06108	1.05899	1.05396	1.04836	1.04382	1.04101	1.04007
0.08333	1.05442	1.05358	1.05141	1.04872	1.04630	1.04468	1.04411
0.00000	1.04884	1.04884	1.04884	1.04884	1.04884	1.04884	1.04884

gives the ratio of the stress intensity factor of the interacting cracks to the stress intensity factor of a solitary crack under the same uniform load. All the computations were made with the relative error not exceeding 10^{-6}. It was established that Collins' (1963) formulae are accurate within 1% for $l>2.5a$. The relative error of the central estimation corresponding to (4.10.14) does not exceed 2% for $l>2.5a$. One can also notice that the central estimation is always slightly below the exact value, thus giving in the case of two cracks a very close lower bound for the quantities of interest. The same can be said about the formulae by Collins (1963). The accuracy of the central estimation deteriorates rapidly as l decreases, for example, the maximum error in the stress intensity factor for l=2.2 (the distance between cracks is 0.2 of its radius) is about 10%. The accuracy of the central estimation of the crack energy, corresponding to (4.10.13) is much better, and is discussed in more detail in the next section. In the examples to follow we shall consider only the central estimation for all the quantities.

Example 2: Infinite row of equal cracks. Let the crack radius be a, and the distance between the adjacent crack centres be l. The cracks are opened by a uniform pressure p. The central estimation for the crack opening volume $2V$ can be defined, according to (4.10.8), by a single equation

$$V = \frac{8}{3}\pi H a^3 p + \frac{4}{\pi} V \sum_{k=1}^{\infty}\left[\frac{a}{(k^2l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{kl}\right],$$

with the result

$$V = \frac{V_0}{1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\left[\frac{a}{(k^2l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{kl}\right]}, \tag{4.10.18}$$

where $V_0=(8/3)\pi Hp$ is the corresponding result for the case of a solitary crack. The crack opening displacement will take the form, according to (4.10.6)

$$w(\rho,\phi) = (a^2 - \rho^2)^{1/2}\left[4pH + \frac{2V}{\pi^2}\sum_{k=1}^{\infty}\frac{\rho^2 + (kl)^2}{(k^2l^2 - a^2)^{1/2}[\rho^4 + (kl)^4 - 2(\rho kl)^2\cos 2\phi]}\right],$$

and, as an immediate consequence of the previous expression, the stress intensity factor will be defined by

$$K(\phi) = K_0\left[1 + \frac{V}{2\pi^2 pH}\sum_{k=1}^{\infty}\frac{a^2 + (kl)^2}{(k^2l^2 - a^2)^{1/2}[a^4 + (kl)^4 - 2(akl)^2\cos 2\phi]}\right],$$

where K_0 stands, as before, for the stress intensity factor of a solitary crack.

Example 3: Polygonal configuration. Consider a circular crack of radius b, its centre coinciding with the centre of a regular polygon, surrounded by n cracks of radius a, with their centres located at the apices of the polygon. Let l be the distance from the polygon centre to its apex. Let a uniform pressure p_c open up the central crack, and a uniform pressure p act inside the cracks located at the polygon apices. Due to the system symmetry, the crack opening volume can be defined by just two equations

$$V_c = \frac{8}{3}\pi H b^3 p_c + \frac{2}{\pi} nV \left[\frac{b}{(l^2 - b^2)^{1/2}} - \sin^{-1}\frac{b}{l} \right],$$

$$V = \frac{8}{3}\pi H a^3 p + \frac{2}{\pi} V_c \left[\frac{a}{(l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l} \right]$$

$$+ \frac{2}{\pi} V \sum_{k=2}^{n} \left[\frac{a}{(l_k^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l_k} \right],$$

where V_c and V denote half of the volume of the central crack and the apex crack respectively, and $l_k = 2l\sin(\pi k/n)$. The solution is

$$V_c = \frac{8}{3}\pi H \frac{b^3 p_c c_{22} + a^3 p c_{12}}{c_{11}c_{22} - c_{12}c_{21}}, \qquad V = \frac{8}{3}\pi H \frac{b^3 p_c c_{21} + a^3 p c_{11}}{c_{11}c_{22} - c_{12}c_{21}}, \tag{4.10.19}$$

where

$$c_{11} = 1, \qquad c_{22} = 1 - \frac{2}{\pi}\sum_{k=2}^{n}\left[\frac{a}{(l_k^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l_k} \right],$$

$$c_{21} = \frac{2}{\pi}\left[\frac{a}{(l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l} \right], \qquad c_{12} = \frac{2}{\pi} n \left[\frac{b}{(l^2 - b^2)^{1/2}} - \sin^{-1}\frac{b}{l} \right].$$

The central crack opening displacements are, according to (4.10.6),

$$w_c(\rho, \phi) = 4Hp_c(b^2 - \rho^2)^{1/2}\Bigg[1$$

$$+ \frac{V}{4\pi^2 H p_c (l^2 - b^2)^{1/2}} \sum_{k=1}^{n} \frac{1}{\rho^2 + l^2 - 2\rho l\cos(\phi - \phi_k)} \Bigg] \tag{4.10.20}$$

Assuming $\phi_k = 2\pi k/n$, the summation in (4.10.20) can be performed, with the result

$$w_c(\rho,\phi) = 4Hp_c(b^2 - \rho^2)^{1/2}\Bigg[1$$

$$+ \frac{nV}{4\pi^2 Hp_c(l^2 - b^2)^{1/2}} \frac{l^{2n} - \rho^{2n}}{(l^2 - \rho^2)\ (\rho^{2n} + l^{2n} - 2\rho^n l^n \cos n\phi)}\Bigg].$$

The stress intensity factor for the central crack takes the form

$$K_c = K_0\Bigg[1 + \frac{nV}{4\pi^2 Hp_c} \frac{l^{2n} - b^{2n}}{(l^2 - b^2)^{3/2}\ (b^{2n} + l^{2n} - 2b^n l^n \cos n\phi)}\Bigg].$$

Due to symmetry of the system, all the apex located cracks will have the same characteristics. The following expressions are valid in a local system of polar coordinates, with its origin at the crack centre and the polar axis coinciding with the line connecting the polygon centre with its apex. The crack opening displacements are

$$w(\rho,\phi) = 4Hp(a^2 - \rho^2)^{1/2}\Bigg\{1 + \frac{1}{4\pi^2 Hp}\Bigg[\frac{V_c}{(l^2 - a^2)^{1/2}\ (\rho^2 + l^2 + 2l\rho\cos\phi)}$$

$$+ \sum_{k=2}^{n} \frac{V}{(l_k^2 - a^2)^{1/2}\ [\rho^2 + l_k^2 + 2l_k\rho\cos(\phi + \psi_k)]}\Bigg]\Bigg\},$$

where V and V_c are defined by (4.10.19); $l_k=2l\sin[\pi(k-1)/n]$, and $\psi_k=2\pi(k-1)/n$. The stress intensity factor can be written as

$$K = K_0\Bigg\{1 + \frac{1}{4\pi^2 Hp}\Bigg[\frac{V_c}{(l^2 - a^2)^{1/2}\ (a^2 + l^2 + 2la\cos\phi)}$$

$$+ \sum_{k=2}^{n} \frac{V}{(l_k^2 - a^2)^{1/2}\ [a^2 + l_k^2 + 2al_k\cos(\phi + \psi_k)]}\Bigg]\Bigg\}.$$

Discussion. It is of interest to compare our results to those available in the literature. The paper by Collins (1963), though published 23 years ago, seems to be still the most reliable source. He did not give the stress intensity factor explicitly but it could be derived easily for the case of two equal cracks, and it reads in our notation

$$K = K_0\left\{ 1 + \frac{2\varepsilon^3}{3\pi} + \frac{8\varepsilon^5}{5\pi} + \frac{4\varepsilon^6}{9\pi^2} + \frac{30\varepsilon^7}{7\pi} + \frac{12\varepsilon^8}{5\pi^2}\right.$$

$$+ \frac{4\varepsilon^4}{3\pi}\left[1 + 3\varepsilon^2 + \frac{2\varepsilon^3}{3\pi} + 9\varepsilon^4\right]\cos\phi + \frac{4\varepsilon^5}{3\pi}\left[1 + \frac{18\varepsilon^2}{5} + \frac{2\varepsilon^3}{3\pi}\right]\cos 2\phi$$

$$\left. + \frac{4\varepsilon^6}{3\pi}\left[1 + \frac{21\varepsilon^2}{5}\right]\cos 3\phi + \frac{4\varepsilon^7}{3\pi}\cos 4\phi + \frac{4\varepsilon^8}{3\pi}\cos 5\phi\right\}. \tag{4.10.21}$$

Our result (4.10.14), if expanded in series, reads

$$K = K_0\left[1 + \frac{2\varepsilon^3}{3\pi}\left(1 + 2\sum_{k=1}^{\infty}\varepsilon^k\cos k\phi\right)\left(1 + \frac{3}{2}\varepsilon^2 + \frac{2\varepsilon^3}{3\pi} + \frac{7}{4}\varepsilon^4 + \ldots\right)\right]. \tag{4.10.22}$$

Comparison of (4.10.21) with (4.10.22) reveals quite a few common terms. Though Collins himself assumed that his results are valid only for the cracks whose radius is much smaller than the distance between their centres, the numerical results indicate that (4.10.21) is accurate within 1% for $l \geq 2.5a$ which corresponds to the shortest distance reported in the literature. Direct computations show that the central estimation corresponding to (4.10.14) does not differ from Collins' (4.10.21) by more than 3% in the whole range $2a<l<\infty$, and differs by less than 0.9% for $l>2.5a$. The stress intensity factor, due to Andreikiv and Panasiuk (1971), is

$$K = K_0\left(1 + \frac{2\varepsilon^3}{3\pi} + \frac{2\varepsilon^4}{3\pi}\cos\phi\right).$$

One should notice that a factor 2 is missing in the third term of their result.

Collins (1963) gave the following expression for the crack energy of two equal cracks

$$W = W_0\left[1 + \frac{2\varepsilon^3}{3\pi} + \frac{6\varepsilon^5}{5\pi} + \frac{4\varepsilon^6}{9\pi^2} + \frac{18\varepsilon^7}{7\pi} + \frac{32\varepsilon^8}{15\pi^2} + \ldots \right],$$

where $W_0=(8/3)\pi H a^3 p^2$ is the energy of a solitary crack. Our expression for the crack energy is

$$W = \frac{W_0}{1 - c}, \tag{4.10.23}$$

where c is defined according to (4.10.13) as

$$c = \frac{2}{\pi}\left[\frac{\varepsilon}{(1 - \varepsilon^2)^{1/2}} - \sin^{-1}\varepsilon \right].$$

Series expansion of (4.10.23) results in

$$W = W_0\left[1 + \frac{2\varepsilon^3}{3\pi} + \frac{3\varepsilon^5}{5\pi} + \frac{4\varepsilon^6}{9\pi^2} + \frac{15\varepsilon^7}{28\pi} + \dots \right].$$

which is very close to Collins' result, ours being slightly lower. Table 4.10.2

Table 4.10.2. Crack energy increase (two equal cracks).

l/a	upper bound	lower bound	central estimation	result of Collins	exact result	error (%)
2.001	–	1.008800	1.035366	1.046308	1.062976	2.6
2.005	–	1.008762	1.035105	1.045921	1.061694	2.5
2.010	–	1.008714	1.034783	1.045444	1.060378	2.4
2.020	–	1.008621	1.034152	1.044510	1.058066	2.3
2.050	–	1.008349	1.032352	1.041861	1.052411	1.9
2.100	2.965366	1.007921	1.029637	1.037908	1.045276	1.5
2.200	1.498350	1.007150	1.025091	1.031413	1.035399	1.0
2.500	1.117132	1.005370	1.016120	1.019160	1.020067	0.4
3.000	1.035432	1.003526	1.008809	1.009900	1.010035	0.1
5.000	1.003526	1.001009	1.001764	1.001834	1.001835	0.0
10.000	1.000294	1.000161	1.000214	1.000216	1.000216	0.0

displays the values of W/W_0. It confirms that the error of the central estimation (4.10.23) is under 3% even for a close interaction when the shortest distance between the cracks is equal to 0.001 of their radius. Collins' results are presented in order to emphasize the accuracy of our numerical solution. The energy increase due to the crack interaction is relatively small even for very close interactions; this is mainly due to the sharp localization of interaction effects (see, for example, Table 4.10.1.)

It is also of interest to compare the upper bound for the stress intensity factor defined by (4.10.14) with the upper bound derived by Ioakimidis (1982). His result for two equal cracks reads in our notation

$$K = \frac{K_0}{1 - \dfrac{2a^3}{3\pi(l - 2a)}} \tag{4.10.24}$$

If we expand (4.10.13) in power series, retaining the first term only, the result is

$$V = \frac{V_0}{1 - \dfrac{2a^3}{3\pi(l - a)}} \tag{4.10.25}$$

Comparison of (4.10.24) with (4.10.25) explains why our estimation is sharper: we have in the denominator $l-a$ while Ioakimidis has $l-2a$. Here is a numerical example. For $l=3a$ the exact result reads $K=1.0234K_0$. Our upper bound gives $K=1.127K_0$ with an error of 10%, while the result of (4.10.24) is $K=1.269K_0$, with an error of about 25%.

Collins (1963) gave the following expression for the crack energy in the case of an infinite row of equal cracks

$$W = W_0(1 + 0.5102\varepsilon^3 + 0.7921\varepsilon^5 + 0.2603\varepsilon^6 + 1.6507\varepsilon^7 + 0.8083\varepsilon^8 + ...) \tag{4.10.26}$$

Our expression due to (4.10.18) is

$$W = \frac{W_0}{1 - \dfrac{4}{\pi}\sum_{k=1}^{\infty}\left[\dfrac{\varepsilon}{(k^2 - \varepsilon^2)^{1/2}} - \sin^{-1}\dfrac{\varepsilon}{k}\right]} \tag{4.10.27}$$

The value of W/W_0, computed on the basis of (4.10.26) and (4.10.27), can be found in (Fabrikant, 1987g). Our (4.10.27) does not differ from (4.10.26) by more than 2% in the whole range of $2.001a \le l < \infty$. Of course, this does not mean that the central estimation is so accurate; it means only that our simple approximate solution is almost as accurate as a very complicated one by Collins. We are not aware of any result in the mechanics literature to compare with our results for the polygonal configuration.

Some well known results can be simplified significantly by using the method of computation of various integrals involving distances between several points. For example, here is the set of governing integral equations derived by

Panasiuk et al (1986, p. 83) for the problem of interaction of N coplanar thin spheroidal inclusions:

$$w_n(x_n,y_n) + H\Gamma w_n(x_n,y_n) - \frac{1}{4\pi^2}\sum_{\substack{k=1\\k\neq n}}^{N}\iint_{S_k} w_k(x_k,y_k)\Gamma[(x_n - x_k)^2$$

$$+ (y_n - y_k)^2]^{-3/2}\,dx_k dy_k = f_n(x_n,y_n), \qquad \text{for } n = 1,2, \ldots ,N. \tag{4.10.28}$$

Here f_n is a known function, w_n is the normal displacement at the boundary of the nth inclusion, S_n is its median crossection, $(x_n,y_n)\in S_n$ and the integral operator Γ is defined by

$$\Gamma\Phi(x_k,y_k) = \iint_{S_k} \frac{\Phi(\xi,\eta)\, d\xi d\eta}{[(x_k - \xi)^2 + (y_k - \eta)^2]^{1/2}}$$

$$- \frac{1}{\pi^2}\iint_{\bar{S}_k}\frac{d\xi d\eta}{\{[\xi^2 + \eta^2 - a_k^2][(x_k - \xi)^2 + (y_k - \eta)^2]\}^{1/2}}$$

$$\times\iint_{S_k}\frac{\Phi(\xi_1,\eta_1)(a_k^2 - \xi_1^2 - \eta_1^2)^{1/2}}{(\xi - \xi_1)^2 + (\eta - \eta_1)^2}\, d\xi_1 d\eta_1. \tag{4.10.29}$$

Here a_k is the radius of median crossection of the kth inclusion, and $\bar{S}_k$ denotes the area outside S_k. The double and quadruple integrals in (4.10.29), which is a kernel of the integral equation (4.10.28), make any numerical solution next to impossible. Panasiuk et al. (1986) have managed to give an approximate solution for the case when the inclusions are far apart, which is of little practical value, since there is almost no interaction at such distances between the inclusions. Let us show that (4.10.28) can be simplified so that its kernel be presented in elementary functions. Making use of the integral

$$\int_0^{2\pi}\int_a^{\infty}\frac{(a^2 - \rho_0^2)^{1/2}}{(r^2 - a^2)^{1/2}}\,\frac{1}{\rho_0^2 + r^2 - 2\rho_0 r\cos(\phi_0-\psi)}\,\frac{r dr d\psi}{[\rho^2 + r^2 - 2\rho r\cos(\phi-\psi)]^{1/2}}$$

$$= \frac{2\pi}{R}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{(a^2 - \rho^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR}\right)\right],$$

one can change the order of integration in the second integral of (4.10.29), perform the integration in $\bar{S}_k$, and the integral operator Γ simplifies in polar coordinates significantly, namely,

$$\Gamma\Phi(\rho,\phi) = \frac{2}{\pi}\iint_{S_k} \frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)\,\Phi(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0,$$

which is much simpler than (4.10.29). It is reminded that R and η are defined by (4.1.14). We can also compute $\Gamma[(x_n - x_k)^2 + (y_n - y_k)^2]^{-3/2}$ in elementary functions. Indeed, one may obtain from (1.6.19)

$$\int_0^{2\pi}\int_0^{a} \frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) \frac{\rho_0 d\rho_0 d\phi_0}{[\rho_0^2 + r^2 - 2r\rho_0\cos(\phi_0-\psi)]^{3/2}}$$

$$= \frac{2\pi(a^2 - \rho^2)^{1/2}}{(r^2 - a^2)^{1/2}[r^2 + \rho^2 - 2r\rho\cos(\phi-\psi)]}, \qquad \text{for } r>a.$$

The above results simplify (4.10.28) so significantly that now it can be easily solved by iteration numerically or analytically.

Exercise 4.10

1. Derive (4.10.4).

2. Rederive (4.10.4) by using an alternative approach: the results of section 4.2 combined with the reciprocity theorem.

3. Investigate the convergence of an iterative process, applied to the integral equation (4.10.16).

4. Consider the interaction of a crack with a microcrack.

5. Consider the interaction of two cracks due to the bending of an elastic space. Assume the space stretched so that cracks do not close due to the bending.

4.11 Close interaction of coplanar circular cracks under shear loading

In this section, the general method is applied to the stress analysis of an elastic space weakened by several arbitrarily located coplanar circular cracks subjected to an arbitrary shear loading. The problem is reduced to a set of Fredholm integral equations. The number of equations is equal to the number of cracks, and can be reduced in the case of a symmetrical configuration. The equations are non-singular. It is shown that the iteration procedure is convergent, and the convergence is so rapid that a practically exact numerical solution can be obtained even for very closely located cracks. One can get an approximate *analytical* solution without having solved the integral equations. It provides sufficiently accurate estimations for the quantities of interest, like the stress intensity factor, the crack energy, the crack face displacement, *etc.* The cases of two cracks and an infinite row of equal cracks are considered as illustrative examples.

Theory. Consider an elastic space weakened in the plane $z=0$ by n arbitrarily located circular cracks. The cracks do not intersect. Let the centre of the kth crack be located at the point with Cartesian coordinates x_k and y_k, and its radius be denoted by a_k. We introduce the complex tangential displacement $u=u_x+iu_y$, and the complex shear stress $\tau=\tau_{zx}+i\tau_{yz}$. Let an arbitrary skew-symmetric shear traction τ_k be applied to the k-th crack faces. We can single out, without loss of generality, the crack number 1, and consider the set of cracks in the local polar system of coordinates with the origin coinciding with the centre of the first crack. In order to be able to use the reciprocal theorem, we need to consider the second set of tractions applied to the same crack configuration. We apply two unit concentrated forces T_x to both faces of the first crack in opposite directions at the point with polar coordinates (ρ,ϕ), and parallel to the axis Ox. We also apply shear tractions q_{kx} and q_{ky} to the remaining cracks. These tractions are chosen so as to provide zero displacement discontinuity at the crack faces, so that the whole system would behave as a single crack (number one) in an infinite body. This choice will allow us to use the Green's functions for an isolated circular crack derived in section 4.4. The following integral equation can be obtained by using the reciprocal theorem:

$$u_{1x} + \sum_{k=2}^{n} \iint_{S_k} q_{kx} u_{kx} \mathrm{d}S_k + \sum_{k=2}^{n} \iint_{S_k} q_{ky} u_{ky} \mathrm{d}S_k = \iint_{S_1} (\tau_{1x} u_{xT_x} + \tau_{1y} u_{yT_x}) \mathrm{d}S_1. \tag{4.11.1}$$

Here q_{kx} and q_{ky} stand for the shear tractions in the kth crack domain due to a pair of unit concentrated forces applied at an arbitrary point of the first crack in the direction parallel to the Ox axis; u_{xT_x} and u_{yT_x} are the tangential displacements of the first crack face due to the same unit forces; and u_{1x}, u_{kx}, and u_{ky} are the as yet unknown tangential displacements of the first and the kth crack faces respectively. Similar considerations, with the unit concentrated forces

T_y applied parallel to the Oy direction, yield the second integral equation

$$u_{1y} + \sum_{k=2}^{n} \iint_{S_k} s_{kx} u_{kx} dS_k + \sum_{k=2}^{n} \iint_{S_k} s_{ky} u_{ky} dS_k = \iint_{S_1} (\tau_{1x} u_{xT_y} + \tau_{1y} u_{yT_y}) dS_1. \tag{4.11.2}$$

The meaning of the notation in (4.11.2) is similar to that in (4.11.1). All the integrals in (4.11.1) and (4.11.2) are evaluated on one side of the relevant crack. Now we need the explicit expressions for the quantities q_{kx}, q_{ky}, s_{kx}, s_{ky}, u_{xT_x}, u_{yT_x}, u_{xT_y}, u_{yT_y}. These expressions were derived in section 4.4, and are

$$q_{kx} = -\zeta - \Re Z \ , \qquad q_{ky} = -\Im Z = s_{kx} \ , \qquad s_{ky} = -\zeta + \Re Z, \tag{4.11.3}$$

$$u_{xT_x} = \zeta_1 - \Re Z_1 + \zeta_2 + \Re Z_2, \qquad u_{yT_x} = -\Im Z_1 + \Im Z_2 \ ,$$

$$u_{xT_y} = \Im Z_1 + \Im Z_2 \ , \qquad u_{yT_y} = \zeta_1 - \Re Z_1 - \zeta_2 - \Re Z_2 \ , \tag{4.11.4}$$

where $\Re$ and $\Im$ stand for the real and the imaginary part, and

$$\zeta = \frac{(a_1^2 - \rho^2)^{1/2}}{\pi^2(\rho_0^2 - a_1^2)^{1/2} R^2}, \quad Z = \frac{G_2}{G_1} \frac{(a_1^2 - \rho^2)^{1/2}}{\pi^2(\rho_0^2 - a_1^2)^{1/2}} \frac{e^{i\phi_0}(3\rho_0 e^{-i\phi_0} - \rho e^{-i\phi})}{\rho_0(\rho e^{-i\phi} - \rho_0 e^{-i\phi_0})^2} \ ,$$

$$\zeta_1 = \frac{G_1}{\pi R} \tan^{-1} \frac{\eta(a_1)}{R} \ , \qquad Z_1 = \frac{G_2^2}{\pi G_1} \frac{(3 - t_1)\ \eta(a_1)}{a_1^2(1 - t_1)^2} \ ,$$

$$\zeta_2 = \frac{G_2 \xi}{\pi R} \tan^{-1} \frac{\eta(a_1)}{R} \ , \qquad Z_2 = \frac{G_2}{\pi a_1^2} \frac{\eta(a_1)(\xi - t_1 e^{2i\phi_0})}{(1 - t_1)(1 - \bar{t}_1)} \ . \tag{4.11.5}$$

Here

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{1/2}, \qquad \eta(x) = (x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}/x, \tag{4.11.6}$$

$$t_1 = (\rho\rho_0/a_1^2) e^{i(\phi-\phi_0)} \ , \qquad \xi = (\rho e^{i\phi} - \rho_0 e^{i\phi_0})/(\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}). \tag{4.11.7}$$

We multiply equation (4.11.2) by the imaginary unit i, add the result to (4.11.1) and, after substitution of (4.11.3), (4.11.4) and (4.11.5), obtain

$$u_1(\rho,\phi) = \frac{G_1}{\pi}\int_0^{2\pi}\int_0^{a_1}\left[\frac{1}{R}\tan^{-1}\frac{\eta(a_1)}{R} - \frac{G_2^2}{G_1^2}\frac{(3-\bar{t}_1)\,\eta(a_1)}{a_1^2(1-\bar{t}_1)^2}\right]\tau_1(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0$$

$$+ \frac{G_2}{\pi}\int_0^{2\pi}\int_0^{a_1}\left[\frac{\xi}{R}\tan^{-1}\left(\frac{\eta(a_1)}{R}\right) + \frac{\eta(a_1)\,(\xi - t_1 e^{2i\phi_0})}{a_1^2(1-t_1)(1-\bar{t}_1)}\right]\bar{\tau}_1(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0$$

$$+ \frac{(a_1^2-\rho^2)^{1/2}}{\pi^2}\sum_{k=2}^{n}\int\int_{S_k}\left[\frac{u_k(\rho_0,\phi_0)}{R^2} + \frac{G_2}{G_1}\frac{\bar{u}_k(\rho_0,\phi_0)\,(3\rho_0 e^{-i\phi_0} - \rho e^{-i\phi})e^{i\phi_0}}{\rho_0(\rho e^{-i\phi} - \rho_0 e^{-i\phi_0})^2}\right]\frac{\rho_0 d\rho_0 d\phi_0}{(\rho_0^2-a_1^2)^{1/2}}.$$

(4.11.8)

The first two integrals in (4.11.8), though looking formidable, can be computed exactly and expressed in elementary functions for any polynomial loading. They give the tangential displacements of the first crack, as if it were an isolated crack, under the prescribed loading τ_1. The remaining integrals represent the influence of the other cracks. A similar procedure can be applied to the remaining $n-1$ cracks, and the additional $n-1$ equations of the type (4.11.8) can be obtained. Note that each such equation is valid in a local system of polar coordinates related to a certain crack. The equations are non-singular. They can be solved numerically by iteration. As will be shown further, the convergence is so rapid, that the first iteration has an error less than 2.5% even for a very close interaction when two cracks are separated only by 0.01 of their radius.

In the case of a uniform shear loading $\tau=\tau_0=const$, the set of equations (4.11.8) simplifies as follows

$$u_1 = 2\tau_0\frac{G_1^2 - G_2^2}{G_1}(a_1^2-\rho^2)^{1/2} + \frac{(a_1^2-\rho^2)^{1/2}}{\pi^2}\sum_{k=2}^{n}\int\int_{S_k}\left[\frac{u_k(\rho_0,\phi_0)}{R^2}\right.$$

$$\left. + \frac{G_2}{G_1}\frac{\bar{u}_k(\rho_0,\phi_0)\,(3\rho_0 e^{-i\phi_0} - \rho e^{-i\phi})\,e^{i\phi_0}}{\rho_0(\rho e^{-i\phi} - \rho_0 e^{-i\phi_0})^2}\right]\frac{\rho_0 d\rho_0 d\phi_0}{(\rho_0^2 - a_1^2)^{1/2}}\ . \qquad (4.11.9)$$

In some cases we can obtain a sufficiently accurate *analytical* solution of the set (4.11.8) by applying the mean value theorem. The result is:

$$u_1(\rho,\phi) = \frac{G_1}{\pi}\int_0^{2\pi}\int_0^{a_1}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta(a_1)}{R}\right) - \frac{G_2^2}{G_1^2}\frac{(3 - \bar{t}_1)\ \eta(a_1)}{a_1^2(1 - \bar{t}_1)^2}\right]\tau_1(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0$$

$$+ \frac{G_2}{\pi}\int_0^{2\pi}\int_0^{a_1}\left[\frac{\xi}{R}\tan^{-1}\frac{\eta(a_1)}{R} + \frac{\eta(a_1)\ (\xi - t_1 e^{2i\phi_0})}{a_1^2(1 - t_1)(1 - \bar{t}_1)}\right]\bar{\tau}_1(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0$$

$$+ \sum_{k=2}^{n}\frac{(a_1^2 - \rho^2)^{1/2}}{\pi^2(\rho_k^2 - a_1^2)^{1/2}}\left[\frac{U_k}{\rho^2 + \rho_k^2 - 2\rho\rho_k cos(\phi-\phi_k)} + \frac{G_2}{G_1}\frac{\bar{U}_k(3\rho_k e^{-i\phi_k} - \rho e^{-i\phi})e^{i\phi_k}}{\rho_k(\rho e^{-i\phi} - \rho_k e^{-i\phi_k})^2}\right]. \quad (4.11.10)$$

Here

$$U_k = \iint_{S_k} u_k dS_k\ ,$$

and ρ_k and ϕ_k are the polar coordinates of a point inside S_k. Though the exact location of the point is unknown, the fact of belonging to the domain S_k limits the possible variation of the quantities of interest and allows the construction of upper and lower bounds as well as a sufficiently accurate *central estimation* which corresponds to the assumption of ρ_k and ϕ_k being located at the *centre* of the *k*th crack. The symmetry considerations can also be used to sharpen the estimates. It will be shown further that the central estimation provides in some cases a sufficiently accurate solution to the problem.

The value of U_1 can be estimated by integration of (4.11.8) over the domain S_1. The result is

$$U_1 = 2\frac{G_1^2 - G_2^2}{G_1}\int_0^{2\pi}\int_0^{a_1}\tau(\rho,\phi)(a_1^2 - \rho^2)^{1/2}\rho d\rho d\phi + \frac{2}{\pi}\sum_{k=2}^{n}\iint_{S_k}\left\{\left[\frac{a_1}{(\rho^2 - a_1^2)^{1/2}}\right.\right.$$

$$\left.\left. - \sin^{-1}\frac{a_1}{\rho}\right] u_k(\rho,\phi) + \frac{G_2}{G_1}a_1^3\frac{\bar{u}_k(\rho,\phi)e^{2i\phi}}{\rho^2(\rho^2 - a_1^2)^{1/2}}\right\}\rho d\rho d\phi. \quad (4.11.11)$$

We can use again the mean value theorem, with the result for the central estimation

$$U_1 = 2\,\frac{G_1^2 - G_2^2}{G_1}\int_0^{2\pi}\int_0^{a_1}\tau(\rho,\phi)(a_1^2 - \rho^2)^{1/2}\rho d\rho d\phi$$

$$+\ \frac{2}{\pi}\sum_{k=2}^{n}\left[\frac{a_1}{(l_{1k}^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{l_{1k}}\right]U_k + \frac{2G_2}{\pi G_1}\,a_1^3\sum_{k=2}^{n}\frac{\overline{U}_k e^{2i\phi_{1k}}}{l_{1k}^2(l_{1k}^2 - a_1^2)^{1/2}}\ . \tag{4.11.12}$$

Here (l_{1k},ϕ_{1k}) are the polar coordinates of the kth crack centre, with respect to the system of coordinates having its origin at the centre of the first crack. Integration of the remaining n-1 equations over the area of each crack provides finally a system of n linear algebraic equations which can be solved with respect to the unknowns U_k. Their feeding back into (4.11.10) gives the complete solution to the problem.

Define the stress intensity factor at the edge of the first crack as

$$K_1(\phi) = \lim_{\rho\to a_1}\{(\rho - a_1)^{1/2}\tau_1(\rho,\phi)\}.$$

An important feature of the present method is the possibility to compute the stress intensity factor directly through the displacements (see the derivation of (2.8.46) for details):

$$K_1(\phi) = -\frac{a_1}{\pi(G_1^2 - G_2^2)\sqrt{2a_1}}\lim_{\rho\to a_1}\left[\frac{G_1 u_1(\rho,\phi) + G_2 e^{2i\phi}\,\overline{u}_1(\rho,\phi)}{(a_1^2 - \rho^2)^{1/2}}\right]. \tag{4.11.13}$$

The stress intensity factor for the remaining cracks can be defined in a similar manner. Substitution of (4.11.8) in (4.11.13) yields, after simplification,

$$K_1(\phi) = -\frac{1}{\pi^2\sqrt{2a_1}}\left\{\int_0^{2\pi}\int_0^{a_1}\frac{(a_1^2 - \rho_0^2)^{1/2}\ \tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{a_1^2 + \rho_0^2 - 2a_1\rho_0\cos(\phi - \phi_0)}\right.$$

$$\left. +\ \frac{G_2}{G_1}\,\frac{e^{i\phi}}{a_1}\int_0^{2\pi}\int_0^{a_1}\frac{3a_1 e^{-i\phi} - \rho_0 e^{-i\phi_0}}{[a_1 e^{-i\phi} - \rho_0 e^{-i\phi_0}]^2}\,(a_1^2 - \rho_0^2)^{1/2}\ \overline{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0\right\}$$

$$- \frac{a_1}{\pi^3(G_1^2 - G_2^2)\sqrt{2a_1}} \sum_{k=2}^{n} \iint_{S_k} \left\{ \left[\frac{G_1}{R_1^2} + \frac{G_2^2}{G_1} \overline{\Theta}_1 e^{2i\phi} \right] u_k(\rho_0,\phi_0) \right.$$

$$\left. + G_2 \left[\frac{e^{2i\phi}}{R_1^2} + \Theta_1 \right] \overline{u}_k(\rho_0,\phi_0) \right\} \frac{\rho_0 d\rho_0 d\phi_0}{(\rho_0^2 - a_1^2)^{1/2}} . \qquad (4.11.14)$$

Here

$$\Theta_1 = \frac{3\rho_0 e^{-i\phi_0} - a_1 e^{-i\phi}}{\rho_0 e^{-i\phi_0}(a_1 e^{-i\phi} - \rho_0 e^{-i\phi_0})^2} , \qquad R_1 = [a_1^2 + \rho_0^2 - 2\rho_0 a_1 \cos(\phi - \phi_0)]^{1/2}.$$

The first two integrals in (4.11.14) give the stress intensity factor for an isolated crack, while the remaining integrals represent the influence of the other cracks. We shall see next how these general expressions can be applied to some specific problems.

Example: Two cracks. Consider the case of two coplanar circular cracks with radii a_1 and a_2 under the action of a uniform shear loading τ_1 and τ_2 respectively. Let l be the distance between their centres. As was established in the previous Section, the problem is reduced to evaluating the integral characteristics U_1 and U_2. We consider the central estimation only. Equations (4.11.12) in this case will take the form

$$U_1 = \frac{4}{3}\pi a_1^3 \tau_1 \frac{G_1^2 - G_2^2}{G_1} + \frac{2}{\pi} U_2 \left[\frac{a_1}{(l - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{l} \right] + \frac{2G_2}{\pi G_1} \frac{a_1^3 \overline{U}_2}{l^2(l^2 - a_1^2)^{1/2}} ,$$

$$U_2 = \frac{4}{3}\pi a_2^3 \tau_2 \frac{G_1^2 - G_2^2}{G_1} + \frac{2}{\pi} U_1 \left[\frac{a_2}{(l - a_2^2)^{1/2}} - \sin^{-1}\frac{a_2}{l} \right] + \frac{2G_2}{\pi G_1} \frac{a_2^3 \overline{U}_1}{l^2(l^2 - a_2^2)^{1/2}} . \qquad (4.11.15)$$

Strictly speaking, the mean value theorem is not applicable in this case, since the imaginary part of u does change sign inside the crack. A numerical evidence will be presented later which justifies neglect of the imaginary part. Under this assumption, the solution is

$$U_1 = \frac{4}{3}\pi \frac{G_1^2 - G_2^2}{G_1} \frac{a_1^3 \tau_1 + c_{12} a_2^3 \tau_2}{1 - c_{12} c_{21}} ,$$

$$U_2 = \frac{4}{3}\pi \frac{G_1^2 - G_2^2}{G_1} \frac{c_{21}a_1^3\tau_1 + a_2^3\tau_2}{1 - c_{12}c_{21}}, \tag{4.11.16}$$

where

$$c_{12} = \frac{2}{\pi}\left[\frac{a_1}{(l^2 - a_1^2)^{1/2}} - \sin^{-1}\frac{a_1}{l} + \frac{G_2}{G_1}\frac{a_1^3}{l^2(l^2 - a_1^2)^{1/2}}\right],$$

$$c_{21} = \frac{2}{\pi}\left[\frac{a_2}{(l^2 - a_2^2)^{1/2}} - \sin^{-1}\frac{a_2}{l} + \frac{G_2}{G_1}\frac{a_2^3}{l^2(l^2 - a_2^2)^{1/2}}\right]. \tag{4.11.17}$$

It will be shown that the central estimation gives a sufficiently accurate solution even for relatively close crack interactions.

Formulae (4.11.16–4.11.17) simplify in the case of equal cracks as $a_1=a_2=a$, and if $\tau_1=\tau_2=\tau_0$ then

$$U_1 = U_2 = U = \frac{U_0}{1 - c},$$

$$c = \frac{2}{\pi}\left[\frac{a}{(l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l} + \frac{G_2}{G_1}\frac{a^3}{l^2(l^2 - a^2)^{1/2}}\right]. \tag{4.11.18}$$

Here $U_0=(4/3)\pi a^3\tau_0(G_1^2-G_2^2)/G_1$. Note that in the case of a uniform loading, the crack energy W is proportional to U, namely, $W=\tau_0 U$. It is clear from (4.11.18) that the crack interaction increases their energy when the applied loadings act in the same direction, otherwise, their energy decreases. The crack face displacements will take the form, according to (4.11.9) and (4.11.10),

$$u_1(\rho,\phi) = (a_1^2 - \rho^2)^{1/2}\left\{2\tau_1\frac{G_1^2 - G_2^2}{G_1}\right.$$

$$\left. + \frac{U_2}{\pi^2(l^2 - a_1^2)^{1/2}}\left[\frac{1}{\rho^2 + l^2 - 2\rho l\cos\phi} + \frac{G_2}{G_1}\frac{3l - \rho e^{-i\phi}}{l(l - \rho e^{-i\phi})^2}\right]\right\},$$

$$u_2(\rho,\phi) = (a_2^2 - \rho^2)^{1/2} \left\{ 2\tau_2 \frac{G_1^2 - G_2^2}{G_1} \right.$$

$$\left. + \frac{U_1}{\pi^2(l^2 - a_2^2)^{1/2}} \left[\frac{1}{\rho^2 + l^2 + 2\rho l\cos\phi} + \frac{G_2}{G_1} \frac{3l + \rho e^{-i\phi}}{l(l + \rho e^{-i\phi})^2} \right] \right\}. \tag{4.11.19}$$

We recall that each expression in (4.11.19) is valid in a *local* system of polar coordinates, with the coordinate origin located at the centre of the respective crack. Substitution of (4.11.19) into (4.11.13) yields the expression for the stress intensity factor.

Now we need an accurate numerical solution in order to estimate the accuracy of the approximate formulae derived. For the sake of simplicity, consider the case of two equal cracks. Assume the crack face displacements in the form

$$u(\rho,\phi) = 2\tau_0 \frac{G_1^2 - G_2^2}{G_1} (a^2 - \rho^2)^{1/2} f(\rho,\phi), \tag{4.11.20}$$

where f is an as yet unknown complex function. It may be called *the interaction function* since it is equal to the ratio of the interacting crack face displacements to those of an isolated crack. The values of $f(a, \phi)$ are related to the stress intensity factor of interacting cracks through

$$K(\phi) = -\frac{\sqrt{2a}}{\pi G_1} [G_1\tau_0 f(a,\phi) + G_2\bar{\tau}_0\bar{f}(a,\phi)e^{2i\phi}]. \tag{4.11.21}$$

In the case when τ_0 is a real constant, we can neglect the imaginary part of f, and the stress intensity factor can be expressed as follows:

$$K(\phi) = K_0(\phi) f(a,\phi), \tag{4.11.22}$$

where

$$K_0(\phi) = -\frac{\sqrt{2a}}{\pi G_1} \tau_0[G_1 + G_2 e^{2i\phi}]$$

is the stress intensity factor for an isolated crack. We shall call $f(a,\phi)$ *the* *interaction factor*, since its value is approximately equal to the ratio of the stress intensity factor of an interacting crack to that of an isolated crack. Substitution of (4.11.20) into (4.11.9) gives a convenient expression for the procedure of iteration

$$u(\rho,\phi) = 2\tau_0 \frac{G_1^2 - G_2^2}{G_1} (a^2 - \rho^2)^{1/2} \Bigg\{ 1$$

$$+ \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{a} \frac{(a^2 - r_0^2)^{1/2}\ r_0 \mathrm{d}r_0 \mathrm{d}\psi_0}{(l^2 + r_0^2 + 2lr_0\cos\psi_0 - a^2)^{1/2}} \Bigg[\frac{f(r_0, \psi_0)}{r^2 + r_0^2 + 2rr_0\cos(\psi - \psi_0)}$$

$$+ \frac{G_2}{G_1} \frac{\bar{f}(r_0,\psi_0)\ (2l + r_0 e^{-i\psi_0} - re^{-i\psi})(l + r_0 e^{i\psi_0})}{(r_0^2 + l^2 + 2lr_0\cos\psi_0)\ (r_0 e^{-i\psi_0} - re^{-i\psi})^2} \Bigg] \Bigg\}. \qquad (4.11.23)$$

Here we have introduced the new variables $r=(\rho^2+l^2-2l\rho\cos\phi)^{1/2}$, $\psi=\pi-\sin^{-1}[(\rho/r)\sin\phi]$. The integral in (4.11.23) has a logarithmic singularity for $r=l-a$, $\psi=0$, as $l\to 2a$, therefore the procedure of iteration might not be convergent for l very close to $2a$. Direct computations show that the iteration procedure converges for $l=2.01a$ (which corresponds to the case when the shortest distance between the cracks is equal to 0.01 of its radius), and converges rapidly: the first iteration with $f\equiv 1$ has the maximum relative error less than 2.5%, and the sixth iteration may be considered practically as an exact solution, since the error becomes less than 10^{-7}. The accuracy of the first iteration improves as the distance between cracks increases. For example, the first iteration for $l=10$ is practically exact with maximum relative error less than 10^{-7}. We could not go closer than $l=2.01a$, not because of non-convergence, but because the standard subroutine DBLIN from the IMSL library, which was used to compute the integrals, failed, giving terminal errors. Though we do not have a rigorous proof, it seems probable that the iteration procedure is theoretically convergent for arbitrarily small distance between cracks.

In the case when $G_2=0$ (for an isotropic body this condition is equivalent to the Poisson ratio $\nu=0$), the crack interaction due to a shear loading is the same as the crack interaction due to a normal loading (compare (4.1.9) with (4.4.14)). The maximum value of the ratio G_2/G_1 for an isotropic body is 1/3, and this value was taken in all the numerical computations, in order to expose the maximum possible difference between the crack interaction due to a normal loading, and the crack interaction due to a shear loading. Some values of the interaction function $f(\rho,\phi)$ are presented in Tables 4.11.1 and 4.11.2, for the closest interaction considered, corresponding to l/a=2.01. The reader is referred to the original paper (Fabrikant, 1989) for the complete data.

Table 4.11.1. The interaction function (real part) for l=2.01a

ρ ϕ	0	15	30	45	90	135	180
1.00	2.14189	1.52935	1.24944	1.14303	1.06042	1.04512	1.04195
0.75	1.32715	1.28240	1.20537	1.14660	1.07271	1.05393	1.04976
0.50	1.19220	1.18283	1.16050	1.13539	1.08451	1.06514	1.06027
0.25	1.13108	1.12916	1.12389	1.11649	1.09335	1.07930	1.07499
0.00	1.09668	1.09668	1.09668	1.09668	1.09668	1.09668	1.09668

Table 4.11.2. The interaction function (imaginary part) for l=2.01a]

ρ ϕ	0	15	30	45	90	135	180
1.00	0.00000	-0.11448	-0.09508	-0.06835	-0.02738	-0.01092	0.00000
0.75	0.00000	-0.04250	-0.05420	-0.04922	-0.02518	-0.01053	0.00000
0.50	0.00000	-0.01526	-0.02474	-0.02774	-0.01983	-0.00918	0.00000
0.25	0.00000	-0.00458	-0.00834	-0.01082	-0.01107	-0.00609	0.00000
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

All the computations were made with a relative error not exceeding 10^{-6}. The first line in Table 4.11.1 is approximately equal to the ratio of the stress intensity factor of interacting cracks to the stress intensity factor of an isolated crack under the same uniform load. The data in Table 4.11.2 justify our neglect of the imaginary part in the interaction function when deriving the approximate analytical solutions: the maximum value of the imaginary part is less than 8% of the corresponding real part, and it reduces significantly, when the distance between the cracks increases. The data in the tables are presented for $0\leq\phi\leq\pi$. The following rules should be applied if one is interested in the value of the interaction function for $\phi>\pi$: $\Re f(\rho,\phi)=\Re f(\rho,2\pi-\phi)$ and $\Im f(\rho,\phi)=-\Im f(\rho,2\pi-\phi)$. Comparison with the results given in section 4.10 shows that the crack interaction due to a shear loading (when acting on both cracks in the same direction) is stronger than the interaction due to a normal loading. For example, the maximum value of the interaction factor for the case l=2.01a and G_2/G_1=1/3 is 2.1419 (Table 4.11.1) while the corresponding value in the case of a normal loading is 1.8613.

using the computed data and compare it with the approximate values due to (4.11.18). The ratio W/W_0 (W_0 is the energy of an isolated crack) is given in Table 4.11.3. The approximate value was computed as $W/W_0=1/(1-c)$, where c

Table 4.11.3. The ratio W/W_0 due to exact and approximate solutions.

l/a	10.0	3.0	2.5	2.1	2.05	2.02	2.01
exact	1.00043	1.01940	1.03817	1.08367	1.09621	1.10603	1.11000
approx.	1.00043	1.01736	1.03165	1.05802	1.06333	1.06684	1.06808
error (%)	0.0	0.2	0.6	2.4	3.0	3.5	3.8

is defined by (4.11.18). The agreement is very good for l/a=≥2.5. Even for a very close interaction (l/a=2.01) the relative error does not exceed 4%; of course, this is mainly due to the fact that the increase in the crack interaction energy is rather small. This should be attributed to a sharp localization of the interaction effects (see Table 4.11.1).

The analytical expression for the interaction function can be written, according to (4.11.19), in the form

$$f(\rho,\phi) = 1 + \frac{G_1}{2\tau_0(G_1^2 - G_2^2)} \frac{U}{\pi^2(l^2 - a^2)^{1/2}} \left[\frac{1}{\rho^2 + l^2 - 2\rho l\cos\phi} + \frac{G_2}{G_1} \frac{3l - \rho e^{-i\phi}}{l(l - \rho e^{-i\phi})^2} \right] .$$

We have computed only the interaction factor $f(a,\phi)$ due to the last formula and compared it with the exact values in Table 4.11.4. The relative error of the central estimation of the real part of the interaction factor does not exceed 3% for $l>2.5a$. Though the relative error of the imaginary part is large, this is due to the fact that the imaginary part constitutes a small percentage of the real part; the absolute error is very small, and we can consider the analytical solution (4.11.18–4.11.19) sufficiently accurate when the distance between interacting cracks is not less than half of their radius. The accuracy of the central estimation deteriorates rapidly as l decreases. One can also notice that the central estimation is always slightly below the exact value thus giving a very close lower bound for the quantities of interest in the case of two interacting cracks.

Infinite row of equal cracks. Let the crack radius be a, and the distance between the adjacent crack centres be l. The cracks are subjected to a uniform shear loading τ. The central estimation for the integral characteristic U can be

Table 4.11.4. Comparison of exact and approximate solutions for the interaction factor.

l/a	ϕ(deg.)=	0	30	60	90	120	150	180
2.50	Real exact	1.11020	1.07479	1.04136	1.02764	1.02198	1.01959	1.01892
	approximate	1.07927	1.05793	1.03465	1.02393	1.01925	1.01722	1.01664
	error (%)	2.8	1.6	0.6	0.4	0.3	0.2	0.2
	Imag. exact	0.00000	-0.01875	-0.01485	-0.00936	-0.00547	-0.00254	0.00000
	approximate	0.00000	-0.01439	-0.01210	-0.00782	-0.00461	-0.00215	0.00000
	error (%)	0.0	23.3	18.5	16.5	15.7	15.4	0.0
2.05	Real exact	1.63484	1.21826	1.08826	1.05542	1.04397	1.03945	1.03821
	approximate	1.21014	1.12652	1.06405	1.04226	1.03379	1.03029	1.02931
	error (%)	26.0	7.5	2.2	1.2	1.0	0.9	0.9
	Imag. exact	0.00000	-0.07834	-0.04299	-0.02416	-0.01359	-0.00622	0.00000
	approximate	0.00000	-0.04268	-0.02834	-0.01667	-0.00951	-0.00437	0.00000
	error (%)	0.0	45.5	34.1	31.0	30.0	29.7	0.0

defined, from (4.11.12), by a single equation

$$U = \frac{4}{3}\pi a^3\tau\,\frac{G_1^2 - G_2^2}{G_1} + \frac{4}{\pi}\sum_{k=1}^{\infty}\left\{\left[\frac{a}{(k^2l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{kl}\right]U + \frac{G_2}{G_1}\,\frac{a^3\overline{U}}{k^2l^2(k^2l^2 - a^2)^{1/2}}\right\}.$$

If we neglect the imaginary part of U then the solution is

$$U = \frac{U_0}{1 - \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty}\left\{\dfrac{a}{(k^2l^2 - a^2)^{1/2}} - \sin^{-1}\left(\dfrac{a}{kl}\right) + \dfrac{G_2}{G_1}\,\dfrac{a^3}{k^2l^2(k^2l^2 - a^2)^{1/2}}\right\}}, \tag{4.11.24}$$

where $U_0=(4/3)\pi a^3\tau(G_1^2-G_2^2)/G_1$ corresponds to the case of an isolated crack. The crack face displacement will take the form, according to (4.11.9) and (4.11.10),

$$u(\rho,\phi) = (a^2 - \rho^2)^{1/2}\left\{2\tau\,\frac{G_1^2 - G_2^2}{G_1}\right.$$

$$+\ \frac{U}{\pi^2}\sum_{k=1}^{\infty}\left[\frac{2(\rho^2+k^2l^2)}{\rho^4+k^4l^4-2\rho^2k^2l^2\cos2\phi}+\frac{G_2}{G_1}\,\frac{3k^2l^2-\rho^2e^{-2i\phi}}{(k^2l^2-\rho^2e^{-2i\phi})^2}\right]\frac{1}{(k^2l^2-a^2)^{1/2}}\Bigg\},\tag{4.11.25}$$

and substitution of (4.11.25) in (4.11.13) will give the expression for the stress intensity factor.

Discussion. It is of interest to compare our results with those available in the literature. We have found only one paper (Fu and Keer, 1969) where the problem of two interacting coplanar circular cracks was considered by a method similar to that of Collins (1963). Only the case when the distance between the crack centres l is much greater than the crack radius a ($\varepsilon=a/l\ll1$) was considered. Fu and Keer considered in detail two equal cracks subjected to a uniform shear loading, acting on both cracks in the same direction horizontally (Case a), and acting in opposite directions (Case b). There are several points in their solution which seem to be incorrect. One of the results (Fu and Keer, 1969, p. 371) states that the absolute value of the mth harmonic (m=1,2,3, ...) of the vertical displacement u_y is *equal* to the corresponding harmonic of the horizontal displacements u_x. This cannot be true, since the vertical displacements depend on the ratio G_2/G_1 (in the isotropic case this ratio is equal to $\nu/(2-\nu)$, where ν is the Poisson coefficient), and the vertical displacements vanish when $G_2=0$ ($\nu=0$)).

It is also possible to compare the expressions for the increase in the strain energy of deformation W per crack. The expression, given by Fu and Keer (1969), reads in our notation:

$$W = W_0\left[1 \mp \frac{4}{3\pi}\varepsilon^3 \mp \left(\frac{3}{5}+\frac{2}{5\pi}\right)\varepsilon^5 + \frac{16}{9\pi^2}\varepsilon^6\right],\tag{4.11.26}$$

where $\varepsilon=a/l$; the $\mp$ sign corresponds to the cases (a) and (b) respectively, and $W_0=(4/3)\pi a^3\tau^2(G_1^2-G_2^2)/G_1$ stands for the energy of an isolated crack. Note an obvious misprint in (4.11.26): the plus sign should correspond to the case (a) and minus to the case (b). Our expression for the crack energy is

$$W = \frac{W_0}{1-c}\ ,\tag{4.11.27}$$

where c is defined according to (4.11.18) as

$$c = \frac{2}{\pi}\left[\frac{a}{(l^2 - a^2)^{1/2}} - \sin^{-1}\frac{a}{l} + \frac{G_2}{G_1}\frac{a^3}{l^2(l^2 - a^2)^{1/2}}\right]. \tag{4.11.28}$$

Series expansion of (4.11.27) results in

$$W = W_0\left[1 + \frac{2\varepsilon^3}{\pi}\left(\frac{1}{3} + \frac{G_2}{G_1}\right) + \frac{\varepsilon^5}{\pi}\left(\frac{3}{5} + \frac{G_2}{G_1}\right) + \frac{4\varepsilon^6}{\pi^2}\left(\frac{1}{3} + \frac{G_2}{G_1}\right)^2 + \frac{3\varepsilon^7}{4\pi}\left(\frac{5}{7} + \frac{G_2}{G_1}\right) + \ldots\right]. \tag{4.11.29}$$

There is a definite disagreement between (4.11.26) and (4.11.29): each term in (4.11.29) depends on the ratio G_2/G_1 while each term in (4.11.26) is not dependent of the elastic constants. In the case of an isotropic body $G_2/G_1=\nu/(2-\nu)$ (as it should), and we can observe an agreement between (4.11.26) and (4.11.29) only for $\nu=1/2$ which is just a coincidence. Collins (1963) gave the following expression for the case of two cracks subjected to a *normal* loading:

$$W = W_0\left[1 + \frac{2\varepsilon^3}{3\pi} + \frac{6\varepsilon^5}{5\pi} + \frac{4\varepsilon^6}{9\pi^2} + \frac{18\varepsilon^7}{7\pi} + \frac{32\varepsilon^8}{15\pi^2} + \ldots\right]. \tag{4.11.30}$$

As was noticed before, in the case when $G_2=0$ ($\nu=0$) the interaction of cracks subjected to a shear loading is mathematically equivalent to the interaction under a normal loading, which means that both (4.11.26) and (4.11.29) should be in agreement with (4.11.30) for $\nu=0$. One can see that this is not the case for expression (4.11.26).

Exercise 4.11

1. Derive (4.11.8).

2. Establish (4.11.11).

3. Investigate convergence of the procedure of iteration, applied to (4.11.8).

4. Consider the case of a polygonal configuration of identical cracks.

Appendix A4.1

Here the main potential function is given, together with selected partial derivatives. We define the potential function by

$$\Psi = \int_0^{2\pi}\int_0^{a} (a^2 - \rho_0^2)^{1/2} \ln(R_0 + z)\ \rho_0 d\rho_0 d\phi_0 \ ,$$

where $R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}$. The integral can be computed in elementary functions:

$$\Psi = \frac{\pi}{2}\Big[z(2a^2 - \rho^2 + \frac{2}{3}z^2)\sin^{-1}(\frac{a}{l_2}) + \frac{1}{3}(a^2 - l_1^2)^{1/2}(5\rho^2 - \frac{10}{3}a^2$$

$$- 2l_2^2 - \frac{11}{3}l_1^2) + \frac{4}{3}a^3\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]\Big] \qquad \text{(A4.1.1)}$$

The following derivatives may be computed:

$$\frac{\partial\Psi}{\partial x} = \pi x\Big[-z\sin^{-1}\frac{a}{l_2} + (a^2 - l_1^2)^{1/2}(1 - \frac{l_1^2 + 2a^2}{3\rho^2}) + \frac{2a^3}{3\rho^2}\Big] \qquad \text{(A4.1.2)}$$

$$\frac{\partial\Psi}{\partial y} = \pi y\Big[-z\sin^{-1}\frac{a}{l_2} + (a^2 - l_1^2)^{1/2}(1 - \frac{l_1^2 + 2a^2}{3\rho^2}) + \frac{2a^3}{3\rho^2}\Big] \qquad \text{(A4.1.3)}$$

$$\frac{\partial\Psi}{\partial z} = \frac{\pi}{2}\Big[(2a^2 + 2z^2 - \rho^2)\sin^{-1}\frac{a}{l_2} - \frac{2a^2 - 3l_1^2}{a}(l_2^2 - a^2)^{1/2} \Big] \qquad \text{(A4.1.4)}$$

$$\Lambda\Psi = \pi\rho e^{i\phi}\Big[-z\sin^{-1}\frac{a}{l_2} + (a^2 - l_1^2)^{1/2}(1 - \frac{l_1^2 + 2a^2}{3\rho^2}) + \frac{2a^3}{3\rho^2}\Big] \qquad \text{(A4.1.5)}$$

$$\frac{\partial^2\Psi}{\partial x^2} = \pi\Big[-z\sin^{-1}\frac{a}{l_2} + (a^2 - l_1^2)^{1/2}$$

$$+ (1 - \frac{2x^2}{\rho^2})\frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^2}\Big] \qquad \text{(A4.1.6)}$$

$$\frac{\partial^2\Psi}{\partial y^2} = \pi\Big[-z\sin^{-1}\frac{a}{l_2} + (a^2 - l_1^2)^{1/2}$$

$$+ \left(1 - \frac{2y^2}{\rho^2}\right) \frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^2} \Bigg] \tag{A4.1.7}$$

$$\frac{\partial^2 \Psi}{\partial x \partial y} = -2\pi xy \, \frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^4} \tag{A4.1.8}$$

$$\Lambda^2 \Psi = -2\pi e^{2i\phi} \, \frac{2a^3 - (l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2}}{3\rho^2} \tag{A4.1.9}$$

$$\Delta \Psi = 2\pi \left[-z\sin^{-1} \frac{a}{l_2} + (a^2 - l_1^2)^{1/2} \right] \tag{A4.1.10}$$

$$\frac{\partial^2 \Psi}{\partial z^2} = 2\pi \left[z\sin^{-1} \frac{a}{l_2} - (a^2 - l_1^2)^{1/2} \right] \tag{A4.1.11}$$

$$\frac{\partial}{\partial z} \Lambda \Psi = \pi \rho e^{i\phi} \left[-\sin^{-1} \frac{a}{l_2} + \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} \right] \tag{A4.1.12}$$

$$\frac{\partial^3 \Psi}{\partial x^2 \partial z} = \pi \left\{ -\sin^{-1} \frac{a}{l_2} + \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} \left[1 + \frac{2a^2x^2}{l_2^2(l_2^2 - l_1^2)} \right] \right\} \tag{A4.1.13}$$

$$\frac{\partial^3 \Psi}{\partial y^2 \partial z} = \pi \left\{ -\sin^{-1} \frac{a}{l_2} + \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} \left[1 + \frac{2a^2y^2}{l_2^2(l_2^2 - l_1^2)} \right] \right\} \tag{A4.1.14}$$

$$\frac{\partial^3 \Psi}{\partial z^3} = 2\pi \left\{ \sin^{-1} \frac{a}{l_2} - \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} \right\} \tag{A4.1.15}$$

$$\frac{\partial^3 \Psi}{\partial x \partial y \partial z} = 2\pi \, \frac{a^3 xy (l_2^2 - a^2)^{1/2}}{l_2^4 (l_2^2 - l_1^2)} \tag{A4.1.16}$$

$$\frac{\partial^3 \Psi}{\partial x \partial z^2} = 2\pi \, \frac{a^2 x (a^2 - l_1^2)^{1/2}}{l_2^2 (l_2^2 - l_1^2)} \tag{A4.1.17}$$

$$\frac{\partial^3\Psi}{\partial y\partial z^2} = 2\pi \frac{a^2 y(a^2 - l_1^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)} \tag{A4.1.18}$$

$$\frac{\partial}{\partial z} \Lambda^2\Psi = 2\pi\rho^2 e^{2i\phi} \frac{a^3(l_2^2 - a^2)^{1/2}}{l_2^4(l_2^2 - l_1^2)} \tag{A4.1.19}$$

$$\frac{\partial}{\partial z} \Delta\Psi = 2\pi \left[-\sin^{-1} \frac{a}{l_2} + \frac{a(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2} \right] \tag{A4.1.20}$$

$$\frac{\partial^2}{\partial z^2} \Lambda\Psi = 2\pi a^2 \rho e^{i\phi} \frac{(a^2 - l_1^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)} \tag{A4.1.21}$$

$$\Lambda\Delta\Psi = - 2\pi a^2 \rho e^{i\phi} \frac{(a^2 - l_1^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)} \tag{A4.1.22}$$

$$\Lambda^3\Psi = -2\pi e^{3i\phi} \left\{ \frac{4[(l_1^2 + 2a^2)(a^2 - l_1^2)^{1/2} - 2a^3]}{3\rho^3} + \frac{a l_1(a^2 - l_1^2)^{1/2}}{l_2(l_2^2 - l_1^2)} \right\} \tag{A4.1.23}$$

$$\frac{\partial^4\Psi}{\partial z^4} = -2\pi \frac{za[l_1^4 + a^2(2a^2 + 2z^2 - 3\rho^2)]}{(l_2^2 - l_1^2)^3(l_2^2 - a^2)^{1/2}} \tag{A4.1.24}$$

$$\frac{\partial^4\Psi}{\partial\rho\partial z^3} = -2\pi \frac{l_1(l_2^2 - a^2)^{1/2}}{l_2(l_2^2 - l_1^2)^3} [a^2(4l_2^4 - 5\rho^2) + l_1^4] \tag{A4.1.25}$$

$$\frac{\partial^2}{\partial z^2} \Lambda^2\Psi = 2\pi\rho^2 e^{2i\phi} a^3 z \frac{a^2(6l_2^2 - 2l_1^2 + \rho^2) - 5l_2^4}{l_2^4(l_2^2 - l_1^2)^3(l_2^2 - a^2)^{1/2}} \tag{A4.1.26}$$

The following identities were used in the derivation of (A4.1.1–A4.1.26):

$$l_1 l_2 = a\rho, \qquad l_1^2 + l_2^2 = a^2 + \rho^2 + z^2, \tag{A4.1.27}$$

$$(l_2^2 - \rho^2)^{1/2}(l_2^2 - a^2)^{1/2} = z l_2, \qquad (a^2 - l_1^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z l_1,$$

$$(a^2 - l_1^2)^{1/2}(l_2^2 - a^2)^{1/2} = za, \qquad (l_2^2 - \rho^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z\rho. \tag{A4.1.28}$$

$$\frac{\partial l_1}{\partial z} = -\frac{z l_1}{l_2^2 - l_1^2}, \qquad \frac{\partial l_2}{\partial z} = \frac{z l_2}{l_2^2 - l_1^2},$$

$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2 - l_1^2} = \frac{\rho(a^2 - l_1^2)}{l_1(l_2^2 - l_1^2)}, \qquad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2 - l_1^2} = \frac{\rho(l_2^2 - a^2)}{l_2(l_2^2 - l_1^2)} . \tag{A4.1.29}$$

Appendix A4.2

Here we present some indefinite integrals of expressions containing l_1 and l_2.

$$\int (l_2^2 - a^2)^{1/2} dz = (a^2 - l_1^2)^{1/2} \frac{l_2^2 - 2a^2}{2a} + \frac{\rho^2}{2} \ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \tag{A4.2.1}$$

$$\int (l_2^2 - a^2)^{1/2} l_1^2 dz = -a(a^2 - l_1^2)^{1/2} \frac{l_1^2 + 2a^2}{3} + a^2\rho^2 \ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \tag{A4.2.2}$$

$$\int (a^2 - l_1^2)^{1/2} dz = \frac{2a^2 - l_1^2}{2a} (l_2^2 - a^2)^{1/2} + \frac{\rho^2}{2} \sin^{-1}(\frac{a}{l_2}), \tag{A4.2.3}$$

$$\int (a^2 - l_1^2)^{1/2} l_1^2 dz = -\frac{l_1^2(2l_1^2 + 3\rho^2)}{8a} (l_2^2 - a^2)^{1/2} + \rho^2(\frac{3}{8}\rho^2 - a^2) \sin^{-1}(\frac{a}{l_2}), \tag{A4.2.4}$$

$$\int (l_2^2 - a^2)^{1/2} \frac{l_1^2}{l_2^2} dz = a(a^2 - l_1^2)^{1/2} \left[1 - \frac{8a^2}{15\rho^2} - \frac{4a^2 + 3l_1^2}{15 l_2^2}\right], \tag{A4.2.5}$$

$$\int \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} dz = -\sin^{-1}(\frac{a}{l_2}), \tag{A4.2.6}$$

$$\int \frac{(a^2 - l_1^2)^{1/2}}{l_2^2(l_2^2 - l_1^2)} dz = \frac{1}{2a^2} \left[\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} - \sin^{-1}(\frac{a}{l_2}) \right], \tag{A4.2.7}$$

$$\int \sin^{-1}\left(\frac{a}{l_2}\right)\,dz = z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2 - l_1^2)^{1/2} + a\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \quad (A4.2.8)$$

$$\int z\sin^{-1}\left(\frac{a}{l_2}\right)\,dz = \frac{1}{4}(2a^2 + 2z^2 + \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right) + (l_2^2 - a^2)^{1/2}\,\frac{2a^2 + l_1^2}{4a}, \quad (A4.2.9)$$

$$\int z^2\sin^{-1}\left(\frac{a}{l_2}\right)\,dz = \frac{1}{3}z^3\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{1}{18}(a^2 - l_1^2)^{1/2}(3l_2^2 + 6\rho^2 + 8a^2 - 2l_1^2)$$

$$- \frac{1}{6}a(3\rho^2 + 2a^2)\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]. \quad (A4.2.10)$$

The integration in (A4.2.1–A4.2.10) was performed by parts, with a consequent change of variables: $z=(a^2 - l_1^2)^{1/2}(\rho^2 - l_1^2)^{1/2}/l_1$ or $z=(l_2^2 - a^2)^{1/2}(l_2^2 - \rho^2)^{1/2}/l_2$.

$$\int \rho\sin^{-1}\left(\frac{a}{l_2}\right)\,d\rho = \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{z(2a^2 - l_1^2)}{2(a^2 - l_1^2)^{1/2}}, \quad (A4.2.11)$$

$$\int \rho^2\sin^{-1}\left(\frac{a}{l_2}\right)d\rho = \frac{\rho^3}{3}\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{z\rho(2a^2 - l_1^2)}{6(a^2 - l_1^2)^{1/2}}$$

$$+ \frac{1}{6}a(a^2 - 3z^2)\cosh^{-1}\frac{l_2}{(a^2 + z^2)^{1/2}} - \frac{1}{6}z(3a^2 - z^2)\sin^{-1}\frac{l_1}{(a^2 + z^2)^{1/2}}, \quad (A4.2.12)$$

$$\int a\sin^{-1}\left(\frac{a}{l_2}\right)\,da = \frac{1}{4}\Big[(2a^2 + 2z^2 - \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right)$$

$$+ l_1(\rho^2 - l_1^2)^{1/2} - 2z(a^2 - l_1^2)^{1/2}\Big]. \quad (A4.2.13)$$

The integration in (A4.2.11–A4.2.12) was performed by parts, with a consequent change of variables: $\rho=y[1+z^2/(a^2-y^2)]^{1/2}$, which corresponds to the substitution $l_2=y$. A similar remark is valid for formula (A4.2.13).

Appendix A4.3

Two important integrals are computed here. The first integral is

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\rho e^{i\phi} - r e^{i\psi}}{R^3(M,N)}\tan^{-1}\left[\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right]\frac{r\,dr\,d\psi}{R(N,N_0)}\ . \tag{A4.3.1}$$

The integral (A4.3.1) can be computed indirectly by using (40), which leads to an equivalent expression:

$$\int_\infty^z \Lambda\left[\frac{1}{R(M,N_0)}\tan^{-1}\left(\frac{h}{R(M,N_0)}\right)\right]dz \tag{A4.3.2}$$

Let us make use of the following identities:

$$\Lambda h = -\frac{h\rho e^{i\phi}}{l_2^2 - l_1^2}\ , \tag{A4.3.3}$$

$$\Lambda\left[\frac{1}{R(M,N_0)}\tan^{-1}\left(\frac{h}{R(M,N_0)}\right)\right]$$

$$= -\frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{h}{R_0^2 + h^2}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^2}\right]. \tag{A4.3.4}$$

The notation R_0 in this Appendix is used as a contraction for $R(M,N_0)$. The substitution of (A4.3.4) in (A4.3.2) yields, after integration by parts:

$$\int_\infty^z \Lambda\left[\frac{1}{R(M,N_0)}\tan^{-1}\left(\frac{h}{R(M,N_0)}\right)\right]dz$$

$$= -\frac{1}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}}\left[\frac{z}{R_0}\tan^{-1}\frac{h}{R_0} - \int_\infty^z \frac{z}{R_0}\frac{\partial}{\partial z}\left(\tan^{-1}\frac{h}{R_0}\right)dz\right]$$

$$-\int_\infty^z\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^2}\right]\frac{h\,dz}{R_0^2 + h^2}\ . \tag{A4.3.5}$$

The following identities are to be used now:

$$\frac{\partial h}{\partial z} = \frac{h(\rho^2 - l_1^2)}{z(l_2^2 - l_1^2)} \,, \tag{A4.3.6}$$

$$\frac{\partial}{\partial z} \tan^{-1}\left(\frac{h}{R_0}\right) = \frac{hR_0}{z(R_0^2 + h^2)} \left[\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2} \right] . \tag{A4.3.7}$$

The substitution of (A4.3.7) in (A4.3.5) allows us to proceed:

$$- \frac{1}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} \left\{ \frac{z}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) - \int_\infty^z \frac{h}{R_0^2 + h^2} \left[\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2} \right] dz \right\}$$

$$- \int_\infty^z \left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{R_0^2} \right] \frac{h dz}{R_0^2 + h^2} = - \frac{z}{(\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}) R_0} \tan^{-1}\left(\frac{h}{R_0}\right)$$

$$+ \int_\infty^z \frac{h \; dz}{(R_0^2 + h^2)(l_2^2 - l_1^2)} \left[\frac{\rho^2 - l_1^2}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} - \rho e^{i\phi} \right]$$

$$- \int_\infty^z \frac{h dz}{R_0^2 (R_0^2 + h^2)} \left[\frac{z^2}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} + \rho e^{i\phi} - \rho_0 e^{i\phi_0} \right]$$

$$= \frac{1}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} \left\{ - \frac{z}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) + \int_\infty^z \left[\frac{\rho\rho_0 e^{i(\phi-\phi_0)} - l_1^2}{l_2^2 - l_1^2} - 1 \right] \frac{h dz}{R_0^2 + h^2} \right\}$$

$$= \frac{1}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} \left\{ - \frac{z}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) + \int_\infty^z \left[\frac{\rho\rho_0 e^{i(\phi-\phi_0)} - l_2^2}{l_2^2 - l_1^2} \right] \frac{h dz}{R_0^2 + h^2} \right\} \tag{A4.3.8}$$

Taking into consideration the identity

$$R_0^2 + h^2 = (l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)}) \; (l_2^2 - \rho\rho_0 e^{-i(\phi-\phi_0)})/l_2^2 \;, \tag{A4.3.9}$$

the integral in (A4.3.8) can be transformed as follows:

$$\int \frac{h l_2^2 \ \mathrm{d}z}{(l_2^2 - l_1^2)(l_2^2 - \rho\rho_0 \mathrm{e}^{-i(\phi-\phi_0)})} = \frac{(a^2 - \rho_0^2)^{1/2}}{a} \int \frac{(a^2 - l_1^2)^{1/2} l_2^2 \ \mathrm{d}l_2}{z l_2 (l_2^2 - \rho\rho_0 \mathrm{e}^{-i(\phi-\phi_0)})}$$

$$= (a^2 - \rho_0^2)^{1/2} \int \frac{l_2 \mathrm{d}l_2}{(l_2^2 - a^2)^{1/2}(l_2^2 - \rho\rho_0 \mathrm{e}^{-i(\phi-\phi_0)})} = \frac{(a^2 - \rho_0^2)^{1/2}}{\bar{s}} \tan^{-1} \frac{(l_2^2 - a^2)^{1/2}}{\bar{s}} , \tag{A4.3.10}$$

where $\bar{s} = (a^2 - \rho\rho_0 \mathrm{e}^{-i(\phi-\phi_0)})^{1/2}$. Finally, formulae (A4.3.8) and (A4.3.10) allow us to compute the original integral (A4.3.1):

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{\rho \mathrm{e}^{i\phi} - r \mathrm{e}^{i\psi}}{R^3(M,N)} \tan^{-1} \left[\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{a R(N,N_0)} \right] \frac{r \mathrm{d}r \mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{1}{\rho \mathrm{e}^{-i\phi} - \rho_0 \mathrm{e}^{-i\phi_0}} \left[\frac{(a^2 - \rho_0^2)^{1/2}}{\bar{s}} \tan^{-1} \frac{\bar{s}}{(l_2^2 - a^2)^{1/2}} - \frac{z}{R(M,N_0)} \tan^{-1} \frac{h}{R(M,N_0)} \right] . \tag{A4.3.11}$$

It is reminded that h is defined by (4.1.18), and $R(M,N_0) = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2]^{1/2}$. The right-hand side in (A4.3.11) simplifies in the limiting case of $\rho_0 \to \rho$ and $\phi_0 \to \phi$, namely,

$$\frac{\rho \mathrm{e}^{i\phi}}{2(a^2 - \rho^2)^{1/2}} \left[\frac{1}{(a^2 - \rho^2)^{1/2}} \tan^{-1} \frac{(a^2 - \rho^2)^{1/2}}{(l_2^2 - a^2)^{1/2}} - \frac{(l_2^2 - a^2)^{1/2}}{l_2^2 - \rho^2} \right] . \tag{A4.3.12}$$

The second integral to be computed is:

$$I_2 = \int \frac{z}{R_0^3} \left[\frac{R_0}{h} + \tan^{-1} \left(\frac{h}{R_0} \right) \right] \mathrm{d}z . \tag{A4.3.13}$$

We proceed with integration by parts. The result is

$$I_2 = \int \frac{z \mathrm{d}z}{R_0^2 h} - \frac{1}{R_0} \tan^{-1} \left(\frac{h}{R_0} \right) + \int \frac{\mathrm{d}z}{R_0} \frac{\mathrm{d}}{\mathrm{d}z} \tan^{-1} \left(\frac{h}{R_0} \right) . \tag{A4.3.14}$$

We modify (A4.3.7) as follows:

$$\frac{\mathrm{d}}{\mathrm{dz}}\tan^{-1}\left(\frac{h}{R_0}\right) = \frac{R_0}{R_0^2 + h^2}\left[\frac{h(\rho^2 - l_1^2)}{z(l_2^2 - l_1^2)} + \frac{z}{h}\right] - \frac{z}{hR_0}$$

$$= \frac{R_0(l_2^2 - a^2)^{1/2}(l_2^4 - \rho^2\rho_0^2)}{(a^2 - \rho_0^2)^{1/2}(l_2^2 - l_1^2)l_2^2(R_0^2 + h^2)} - \frac{z}{hR_0}\ . \qquad \text{(A4.3.15)}$$

By substituting (A4.3.15) in (A4.3.14), and taking into consideration (A4.3.9) and (A4.1.29), we obtain

$$I_2 = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)$$

$$+ \frac{1}{(a^2 - \rho_0^2)^{1/2}}\int\frac{(l_2^4 - \rho^2\rho_0^2)\ \mathrm{d}l_2}{(l_2^2 - \rho^2)^{1/2}(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})(l_2^2 - \rho\rho_0 e^{-i(\phi-\phi_0)})}$$

(A4.3.16)

The integral in (A4.3.16) is elementary, i.e.

$$I_2 = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{(a^2 - \rho_0^2)^{1/2}}\left\{\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]\right.$$

$$\left. - \frac{1}{(\zeta - 1)^{1/2}}\tan^{-1}\left[\frac{a(\zeta - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right] - \frac{1}{(\bar{\zeta} - 1)^{1/2}}\tan^{-1}\left[\frac{a(\bar{\zeta} - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right]\right\},\ \zeta = \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}$$

(A4.3.17)

Since the integration was indefinite, we might have lost a function of the variables, other than z. This function can be found from the condition that the result of integration should not have a logarithmic singularity at $\rho=0$ or at $q=0$. The functions eliminating such a singularity are $\tan^{-1}[(\zeta - 1)^{1/2}]$ and $\tan^{-1}[(\bar{\zeta} - 1)^{1/2}]$. The final result can now be represented in the form

$$\int\frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]\mathrm{d}z$$

$$= -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{(a^2 - \rho_0^2)^{1/2}}\left\{\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]\right.$$

$$- 2\Re\left(\frac{1}{(\zeta - 1)^{1/2}}\left[\tan^{-1}\left(\frac{a(\zeta - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right) - \tan^{-1}(\zeta - 1)^{1/2}\right]\right)\Bigg\}, \qquad \zeta = \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}$$

The last expression proves the correctness of formula (5.1.13).

Appendix A4.4

Some integrals related to the problem of a penny-shaped crack under shear loading are presented here, without derivation. The first integral, which can be computed directly, is:

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2}\,\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(M,N)}\,r\,dr\,d\psi$$

$$= \pi\,\frac{(a^2 - \rho_0^2)^{1/2}}{a^3}\left\{\frac{\rho^2}{t}\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{(l_2^2 - a^2)^{1/2}[l_1^2(4 - t) - 3a^2]}{a(1 - t)^2}\right.$$

$$\left. + \frac{1}{(1 - t)^{3/2}}\left[\frac{3z^2}{1 - t} + a^2(3 - 2t) - \frac{\rho^2}{t}\right]\tan^{-1}\left(\frac{a(1 - t)^{1/2}}{(l_2^2 - a^2)^{1/2}}\right)\right\}. \tag{A4.4.1}$$

Here t is defined by (4.4.16). Application of the operator Λ to both sides of (A4.4.1) yields

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2}\,\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^3(M,N)}(\rho e^{i\phi} - re^{i\psi})\,r\,dr\,d\psi$$

$$= -2\pi\rho e^{i\phi}\,\frac{(a^2 - \rho_0^2)^{1/2}}{a^3}\left\{\frac{1}{t}\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{a(l_2^2 - a^2)^{1/2}}{(1 - t)(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})}\right.$$

$$\left. - \frac{1}{t(1 - t)^{3/2}}\tan^{-1}\left(\frac{a(1 - t)^{1/2}}{(l_2^2 - a^2)^{1/2}}\right)\right\}. \tag{A4.4.2}$$

Another application of Λ to both sides of (A4.4.2) gives

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^5(M,N)} 3(\rho e^{i\phi} - re^{i\psi})^2 \, r\mathrm{d}r\mathrm{d}\psi$$

$$= 2\pi \frac{\rho^2 e^{2i\phi}(a^2 - \rho_0^2)^{1/2}(l_2^2 - a^2)^{1/2}(3l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})}{l_2^2(l_2^2 - l_1^2)(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})^2} . \tag{A4.4.3}$$

Differentiation with respect to z of both sides of (A4.4.1) results in

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{z(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^3(M,N)} \, r\mathrm{d}r\mathrm{d}\psi$$

$$= 2\pi \frac{ha^2}{s^2}\left[\frac{3}{s^2} - \frac{t}{l_2^2 - a^2 t} - \frac{3(l_2^2 - a^2)^{1/2}}{s^3} \tan^{-1}\left(\frac{s}{(l_2^2 - a^2)^{1/2}}\right)\right]. \tag{A4.4.4}$$

Here h is defined by (4.1.18). Another differentiation of both sides of (A4.4.4) with respect to z yields

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^3(M,N)}\left(1 - \frac{3z^2}{R^2(M,N)}\right) r\mathrm{d}r\mathrm{d}\psi$$

$$= 2\pi \frac{(a^2 - \rho_0^2)^{1/2}}{a^3(1 - t)} \left\{ \frac{a(l_2^2 - a^2)^{1/2}}{(l_2^2 - l_1^2)(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})} \left[\frac{3(l_2^2 - l_1^2 t)}{1 - t} \right.\right.$$

$$\left.\left. + \frac{\rho\rho_0 e^{i(\phi-\phi_0)}(2l_2^2 + l_1^2 t - 3\rho^2)}{l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)}} \right] - \frac{3}{(1 - t)^{3/2}} \tan^{-1}\left(\frac{a(1 - t)^{1/2}}{(l_2^2 - a^2)^{1/2}}\right) \right\} . \tag{A4.4.5}$$

Application of the operator Λ to both sides of (A4.4.4) yields

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{z(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^5(M,N)} 3(\rho e^{i\phi} - re^{i\psi}) \, r\mathrm{d}r\mathrm{d}\psi$$

$$= 2\pi \frac{h\rho e^{i\phi}(3l_2^2 - a^2 t)}{(l_2^2 - l_1^2)(l_2^2 - a^2 t)^2} . \qquad \text{(A4.4.6)}$$

A different result is obtained if Λ is applied to a complex conjugate of expression (A4.4.4), namely,

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{-i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{-i(\psi-\phi_0)})^2} \frac{z(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR^5(M,N)} 3(\rho e^{i\phi} - re^{i\psi}) \, r dr d\psi$$

$$= 2\pi h \left\{ \frac{a^2}{\bar{s}^2} \rho_0 e^{i\phi_0} \left[\frac{15(l_2^2 - a^2)^{1/2}}{\bar{s}^5} \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}} \right) - \frac{15}{\bar{s}^4} \right.\right.$$

$$\left.\left. + \frac{5}{\bar{s}^2(l_2^2 - a^2\bar{t})} + \frac{2\bar{t}}{(l_2^2 - a^2\bar{t})^2} \right] + \frac{\rho e^{i\phi}(3l_2^2 - a^2\bar{t})}{(l_2^2 - l_1^2)(l_2^2 - a^2\bar{t})^2} \right\} . \qquad \text{(A4.4.7)}$$

Integration of both sides of (A4.4.1) with respect to z gives

$$\int_0^{2\pi}\int_0^a \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{a} \ln[R(M,N) + z] \, r dr d\psi$$

$$= \pi(a^2 - \rho_0^2)^{1/2} \left\{ 2\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] - 2 + \frac{(a^2 - l_1^2)^{1/2}}{a^3(1 - t)} \left[- \frac{z^2}{1 - t} + \rho^2 \right.\right.$$

$$\left. - \frac{1}{3}(l_1^2 + 2a^2) \right] + \frac{z\bar{\zeta}}{a} \sin^{-1}\left(\frac{a}{l_2}\right) + \frac{z}{a^3(1 - t)^{3/2}} \left[\frac{z^2}{1 - t} + a^2(3 - 2t - \bar{\zeta}) \right]$$

$$\left. \times\tan^{-1}\left(\frac{a(1 - t)^{1/2}}{(l_2^2 - a^2)^{1/2}} \right) + 2(\bar{\zeta} - 1)^{1/2} \left[\tan^{-1}\left(\frac{1}{(\bar{\zeta} - 1)^{1/2}} \right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}} \right) \right] \right\} . \qquad \text{(A4.4.8)}$$

Here ζ is defined by (A4.3.17), and the bar, as usual, indicates the complex conjugate value. A similar integration with respect to z of (A4.4.3) yields

$$\int_0^{2\pi}\int_0^{a} \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{a} \Lambda^2\{\ln[R(M,N) + z]\} \, r\mathrm{d}r\mathrm{d}\psi$$

$$= \frac{2\pi}{\bar{\bar{q}}} (a^2 - \rho_0^2)^{1/2} \left\{ -\frac{(\bar{\zeta} - 1)^{1/2}}{\bar{q}} \left[\tan^{-1}\left(\frac{1}{(\bar{\zeta} - 1)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right)\right]\right.$$

$$\left. + \frac{e^{i\phi}}{\rho}\left[\frac{(a^2 - l_1^2)^{1/2}}{a} \left(1 + \frac{\rho^2}{l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)}} \right) - 1 \right]\right\} . \qquad (A4.4.9)$$

It is reminded that q is defined by (4.1.28), and

$$\Lambda^2\ln[R(M,N) + z] = -(\rho e^{i\phi} - re^{i\Psi})^2 \frac{2R(M,N) + z}{R^3(M,N)\ [R(M,N) + z]^2} \qquad (A4.4.10)$$

Yet another application of the operator Λ to (A4.4.9) gives

$$\int_0^{2\pi}\int_0^{a} \frac{(3a^2 - r\rho_0 e^{i(\psi-\phi_0)})}{(a^2 - r\rho_0 e^{i(\psi-\phi_0)})^2} \frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{a} \Lambda^3\{\ln[R(M,N) + z]\} \, r\mathrm{d}r\mathrm{d}\psi$$

$$= \frac{2\pi}{\bar{\bar{q}}} (a^2 - \rho_0^2)^{1/2} \left\{ \frac{3(\bar{\zeta} - 1)^{1/2}}{\bar{q}^2} \left[\tan^{-1}\left(\frac{1}{(\bar{\zeta} - 1)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right)\right]\right.$$

$$- \frac{e^{2i\phi}(a^2 - l_1^2)^{1/2}}{a(l_2^2 - l_1^2)} \left[\frac{l_2^2 + \rho^2}{l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)}} + \frac{2\rho^2(l_2^2 - a^2)}{(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})^2} + 1\right]$$

$$\left. + \frac{e^{i\phi}}{\rho}\left[\frac{3}{\bar{\bar{q}}} + \frac{2e^{i\phi}}{\rho} - \frac{(a^2 - l_1^2)^{1/2}}{a}\left(\frac{l_2^2 + 2\rho^2}{\bar{q}(l_2^2 - \rho\rho_0 e^{i(\phi-\phi_0)})} + 2\left(\frac{1}{\bar{\bar{q}}} + \frac{e^{i\phi}}{\rho}\right)\right)\right]\right\} . \qquad (A4.4.11)$$

Here

$$\Lambda^3\ln[R(M,N) + z] = (\rho e^{i\phi} - re^{i\Psi})^3 \frac{8R^2(M,N) + 9R(M,N)z + 3z^2}{R^5(M,N)\ [R(M,N) + z]^3} . \qquad (A4.4.12)$$

Formula (4.1.27) can be used to obtain some additional results. Integration of

(4.1.27) with respect to z gives

$$\int_0^{2\pi}\int_0^a \frac{\rho e^{i\phi} - re^{i\psi}}{R(M,N)[R(M,N) + z]} \tan^{-1}\left(\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right) \frac{r\mathrm{d}r\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{\bar{q}}}\left\{ R_0\tan^{-1}\left(\frac{h}{R_0}\right) - (a^2 - \rho_0^2)^{1/2}\left[\frac{z}{\bar{\bar{s}}} \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left. - (\bar{\zeta} - 1)^{1/2}\left(\tan^{-1}\frac{1}{(\bar{\zeta} - 1)^{1/2}} - \tan^{-1}\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right)\right]\right\} . \tag{A4.4.13}$$

The following indefinite integrals were used here

$$\int \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right) \mathrm{d}z = z \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)$$

$$+ \bar{s}\left[\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] + (\zeta - 1)^{1/2} \tan^{-1}\left(\frac{a(\zeta - 1)^{1/2}}{(a^2 - l_1^2)^{1/2}}\right)\right] ,$$

$$\int \frac{z}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) \mathrm{d}z = R_0\tan^{-1}\left(\frac{h}{R_0}\right) - (a^2 - \rho_0^2)^{1/2}\left[-\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]\right.$$

$$\left. + (\bar{\zeta} - 1)^{1/2}\tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right) + (\zeta - 1)^{1/2}\tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{a(\zeta - 1)^{1/2}}\right)\right] . \tag{A4.4.14}$$

By applying the operator Λ to both sides of (A4.4.13), one gets

$$\int_0^{2\pi}\int_0^a \frac{(\rho e^{i\phi} - re^{i\psi})^2[2R(M,N) + z]}{R^3(M,N)\ [R(M,N) + z]^2} \tan^{-1}\left(\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right) \frac{r\mathrm{d}r\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{\bar{q}}}\left\{ \frac{R_0^2 + z^2}{R_0\bar{q}} \tan^{-1}\left(\frac{h}{R_0}\right) + (a^2 - \rho_0^2)^{1/2}\left[\frac{z}{\bar{\bar{s}}} \left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2} - \frac{2}{\bar{\bar{q}}}\right)\tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)\right.\right.$$

$$+ \frac{(\bar{\zeta} - 1)^{1/2}}{\bar{q}} \left(\tan^{-1}\frac{1}{(\bar{\zeta} - 1)^{1/2}} - \tan^{-1}\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right) + \left.\frac{e^{i\phi}}{\rho}\right] - \frac{e^{i\phi} h a^2}{\rho \bar{s}^2} \Bigg\},$$

(A4.4.15)

and yet another application of Λ to (A4.4.15) yields

$$\int_0^{2\pi}\int_0^a \frac{(\rho e^{i\phi} - re^{i\psi})^3[8R^2(M,N) + 9R(M,N)z + 3z^2]}{R^5(M,N)\ [R(M,N) + z]^3}$$

$$\times \tan^{-1}\left(\frac{(a^2 - r^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right) \frac{r\mathrm{d}r\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\Bigg\{ \frac{3R_0^4 + 6R_0^2 z^2 - z^4}{R_0^3\bar{q}^2} \tan^{-1}(\frac{h}{R_0}) + (a^2 - \rho_0^2)^{1/2} \frac{e^{i\phi}}{\rho}\left(\frac{2e^{i\phi}}{\rho} + \frac{3}{\bar{\bar{q}}}\right)$$

$$- (a^2 - \rho_0^2)^{1/2}\Bigg[\frac{z}{\bar{\bar{s}}} \left(\frac{8}{\bar{q}^2} - \frac{4\rho_0 e^{i\phi_0}}{\bar{s}^2\bar{q}} + \frac{3\rho_0^2 e^{2i\phi_0}}{\bar{s}^4}\right) \tan^{-1}\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)$$

$$- \frac{3(\bar{\zeta} - 1)^{1/2}}{\bar{q}^2} \left(\tan^{-1}\frac{1}{(\bar{\zeta} - 1)^{1/2}} - \tan^{-1}\frac{(a^2 - l_1^2)^{1/2}}{a(\bar{\zeta} - 1)^{1/2}}\right)\Bigg]$$

$$+ \frac{ha^2 e^{i\phi}}{\rho\bar{s}^2} \left[\frac{2\rho_0 e^{i\phi_0}}{\bar{s}^2} - \frac{2e^{i\phi}}{\rho} - \frac{2}{\bar{\bar{q}}} + \left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2} - \frac{2}{\bar{\bar{q}}}\right)\frac{(l_2^2 - a^2)\bar{t}}{l_2^2 - a^2\bar{t}}\right]$$

$$- \frac{h}{R_0^2 + h^2} \left[\frac{\bar{q}\rho e^{3i\phi}}{l_2^2 - l_1^2} + \frac{e^{i\phi}(l_2^2 - \rho^2)}{\rho\bar{q}} - \frac{z^2 q}{R_0^2\bar{\bar{q}}} + 2e^{2i\phi}\right]\Bigg\}.$$

(A4.4.16)