CHAPTER 3

MIXED-MIXED BOUNDARY VALUE PROBLEMS

The mixed-mixed problems in elasticity theory are among the most complicated due to the coupling between the normal and tangential parameters. We should mention the works of Mossakovskii (1954) and Ufliand (1956) among the first published exact solutions for the *isotropic* half-space, obtained by using various integral transforms. A more compact solution has been reported by Kapshivyi and Masliuk (1967), who used a special apparatus of *p*-analytical functions. The first *elementary* exact solution for a *transversely isotropic* elastic half-space was published in (Fabrikant, 1971a). We present here a general formulation of the internal and external mixed-mixed problems in terms of two-dimensional integral equations. Four kinds of exact solution to the internal axisymmetric problem are given, and yet another kind of solution is presented for the external axisymmetric problem. The action of a general loading on a flat circular bonded punch is considered in detail. A general solution to the non-axisymmetric internal and external problems is presented as a Fourier series expansion. The material in this chapter follows the results published in the papers (Fabrikant, 1971d, 1972, 1974b, 1975, 1976, 1986j).

3.1 General formulation of the problem

Consider a transversely isotropic elastic half-space $z\geq 0$. Let the following boundary conditions be prescribed on the plane $z=0$:

$$u = u(\rho,\phi), \quad \text{for} \quad 0\leq\rho\leq a, \quad 0\leq\phi<2\pi;$$

$$w = w(\rho,\phi), \quad \text{for} \quad 0\leq\rho\leq a, \quad 0\leq\phi<2\pi;$$

$$\sigma = \sigma(\rho,\phi), \quad \text{for} \quad a\leq\rho\leq\infty, \quad 0\leq\phi<2\pi;$$

$$\tau = \tau(\rho,\phi), \qquad \text{for} \quad a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi. \tag{3.1.1}$$

The problem, so defined, will be called *internal mixed-mixed*. The system of governing integral equations is formulated due to (2.2.12) and (2.2.13), and is

$$H\alpha\Re\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}} + H\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} = \omega_1(\rho,\phi), \tag{3.1.2}$$

$$\frac{1}{2}G_1\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} + \frac{1}{2}G_2\int_0^{2\pi}\int_0^a \frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R}$$

$$- H\alpha\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} = \omega_2(\rho,\phi). \tag{3.1.3}$$

It is reminded that the notations q and R are defined by (2.2.5) and (2.2.14) respectively. Functions ω_1 and ω_2 are known from the boundary conditions (3.1.1):

$$\omega_1(\rho,\phi) = w(\rho,\phi) - H\alpha\Re\int_0^{2\pi}\int_a^\infty \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}$$

$$- H\int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R}\ , \tag{3.1.4}$$

$$\omega_2(\rho,\phi) = u(\rho,\phi) + H\alpha\int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}}$$

$$- \frac{1}{2}G_1\int_0^{2\pi}\int_a^\infty \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} - \frac{1}{2}\ G_2\int_0^{2\pi}\int_a^\infty \frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R}\ . \tag{3.1.5}$$

The *external* mixed-mixed boundary value problem for a transversely isotropic elastic half-space can be formulated in a similar manner. The boundary conditions are:

$$
\begin{aligned}
&u = u(\rho,\phi), && \text{for} \quad a\le\rho<\infty, && 0\le\phi<2\pi;\\
&w = w(\rho,\phi), && \text{for} \quad a\le\rho<\infty, && 0\le\phi<2\pi;\\
&\sigma = \sigma(\rho,\phi), && \text{for} \quad 0\le\rho\le a, && 0\le\phi<2\pi;\\
&\tau = \tau(\rho,\phi), && \text{for} \quad 0\le\rho\le a, && 0\le\phi<2\pi.
\end{aligned}
\tag{3.1.6}
$$

The system of governing integral equations in this case takes the form

$$H\alpha\Re\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}} + H\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} = \omega_1(\rho,\phi), \tag{3.1.7}$$

$$\frac{1}{2}G_1\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} + \frac{1}{2}G_2\int_0^{2\pi}\int_a^{\infty}\frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R}$$

$$- H\alpha\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} = \omega_2(\rho,\phi). \tag{3.1.8}$$

Functions ω_1 and ω_2 are known from the boundary conditions, and are

$$\omega_1(\rho,\phi) = w(\rho,\phi) - H\alpha\Re\int_0^{2\pi}\int_0^{a}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}$$

$$- H\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R}\ , \tag{3.1.9}$$

$$\omega_2(\rho,\phi) = u(\rho,\phi) + H\alpha\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}}$$

$$-\frac{1}{2}G_1\int_0^{2\pi}\int_0^{a}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} - \frac{1}{2}\,G_2\int_0^{2\pi}\int_0^{a}\frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R}\ . \tag{3.1.10}$$

The exact closed form solution of these equations is not known at the moment, though we strongly believe that the new method is capable of furnishing such a solution. In the sections to follow we consider separately the axisymmetric case and the general one. The exact solution to both internal and external problems is obtained for the n-th harmonic, with the assumption that all the functions involved can be represented as Fourier expansions.

Exercise 3.1

1. Verify the derivation of (3.1.2–3.1.5)

2. Verify the derivation of (3.1.7–3.1.10)

3. Derive the governing integral equations for the internal mixed-mixed problem in the case of an isotropic half-space.

3.2 Internal axisymmetric mixed-mixed problem

In order to demonstrate versatility of our method, we present here four kinds of exact solution. The first kind is more convenient for the stress evaluation, while the second one has certain advantages for calculating the displacements outside the circle $\rho=a$. Since it is important to have one kind of solution easily transformed into another, the necessary relationships are established. Application of this technique to a bonded flat-ended circular punch under the action of a normal force and expanding in the radial direction is considered. The influence of an arbitrary axisymmetric normal and tangential tractions field, applied outside the punch, is investigated.

The boundary conditions in the case of axial symmetry are

$$\begin{array}{llll} u = u(\rho), & \text{for} \quad 0\le\rho\le a, & & 0\le\phi<2\pi; \\ w = w(\rho), & \text{for} \quad 0\le\rho\le a, & & 0\le\phi<2\pi; \\ \sigma = \sigma(\rho), & \text{for} \quad a\le\rho\le\infty, & & 0\le\phi<2\pi; \\ \tau = \tau(\rho), & \text{for} \quad a\le\rho\le\infty, & & 0\le\phi<2\pi. \end{array} \tag{3.2.1}$$

The set of governing integral equations will take the form

$$2H\left\{-\pi\alpha\int_{\rho}^{a}\tau(\rho_0)\mathrm{d}\rho_0 + 2\int_{0}^{\rho}\frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_{x}^{a}\frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}}\right\} = \omega_1(\rho), \tag{3.2.2}$$

$$\frac{2H}{\rho}\left\{2\gamma_1\gamma_2\int_{0}^{\rho}\frac{x^2\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_{x}^{a}\frac{\tau(\rho_0)\ \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} - \pi\alpha\int_{0}^{\rho}\sigma(\rho_0)\rho_0\mathrm{d}\rho_0\right\} = \omega_2(\rho). \tag{3.2.3}$$

The functions ω_1 and ω_2 are known from the boundary conditions, and are defined by

$$\omega_1(\rho) = w(\rho) + 2H\left\{\pi\alpha\int_{a}^{\infty}\tau(\rho_0)\mathrm{d}\rho_0 - 2\int_{a}^{\infty}\frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}}\int_{a}^{x}\frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}\right\}, \tag{3.2.4}$$

$$\omega_2(\rho) = u(\rho) - 4H\gamma_1\gamma_2\rho\int_{a}^{\infty}\frac{\mathrm{d}x}{x^2(x^2 - \rho^2)^{1/2}}\int_{a}^{x}\frac{\tau(\rho_0)\ \rho_0^2\mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}. \tag{3.2.5}$$

Multiply both sides of equation (3.2.2) by $\rho(r^2 - \rho^2)^{-1/2}\mathrm{d}\rho$, integrate with respect to ρ from zero to r and differentiate with respect to r. The result is

$$\frac{2}{\pi}\left[-\alpha\int_{0}^{a}\tau(\rho_0)\mathrm{d}\rho_0 + \alpha\ r\int_{0}^{r}\frac{\tau(\rho_0)\mathrm{d}\rho_0}{(r^2 - \rho_0^2)^{1/2}} + \int_{r}^{a}\frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - r^2)^{1/2}}\right] = \chi_1(r). \tag{3.2.6}$$

Here

$$\chi_1(r) = \frac{1}{\pi^2 H}\frac{\mathrm{d}}{\mathrm{d}r}\int_{0}^{r}\frac{\omega_1(\rho)\rho\ \mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}}. \tag{3.2.7}$$

A similar transformation can be applied to equation (3.2.3), with the result

$$\frac{2}{\pi}\left[\sqrt{\gamma_1\gamma_2}\,r\int_{r}^{a}\frac{\tau(\rho_0)\mathrm{d}\rho_0}{(\rho_0^2 - r^2)^{1/2}} - \frac{\alpha}{\sqrt{\gamma_2\gamma_2}}\int_{0}^{r}\frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(r^2 - \rho_0^2)^{1/2}}\right] = \chi_2(r), \tag{3.2.8}$$

where

$$\chi_2(r) = \frac{1}{\pi^2 H\sqrt{\gamma_1\gamma_2}}\,\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\omega_2(\rho)\rho^2\mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}}. \tag{3.2.9}$$

The set of modified integral equations (3.2.6) and (3.2.8) is solved below by four different methods. It is shown that all the solutions are consistent with each other, and give effectively the same solution.

Solution of the first kind. Assume the solution to the set of equations (3.2.2) and (3.2.3) to be defined by

$$\sigma(\rho) = \frac{1}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{f_1(t)t\ \mathrm{d}t}{(\rho^2 - t^2)^{1/2}}, \qquad \tau(\rho) = (\gamma_1\gamma_2)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{f_2(t)\ \mathrm{d}t}{(\rho^2 - t^2)^{1/2}}. \tag{3.2.10}$$

Here f_1 and f_2 are the as yet unknown stress functions. Substitution of (3.2.10) in (3.2.6) yields, after integration,

$$\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\,f_2(r) + \frac{2}{\pi}(a^2 - r^2)^{1/2}\int_0^a \frac{f_1(t)t\ \mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} = \chi_1(r) + b. \tag{3.2.11}$$

Here the notation was introduced

$$b = \frac{2\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{f_2(t)\ \mathrm{d}t}{(a^2 - t^2)^{1/2}}. \tag{3.2.12}$$

It will be shown later that we may assume b=0, without loss of generality. Hereafter the following identities are used

$$\int_0^r \frac{\mathrm{d}x}{(r^2 - x^2)^{1/2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a \frac{f(t)\ \mathrm{d}t}{(t^2 - x^2)^{1/2}} = r\int_0^a \frac{f(t)\ \mathrm{d}t}{t(t^2 - r^2)},$$

$$\int_r^a \frac{\mathrm{d}x}{(x^2 - r^2)^{1/2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{f(t)\ \mathrm{d}t}{(x^2 - t^2)^{1/2}} = (a^2 - r^2)^{1/2}\int_0^a \frac{f(t)\ \mathrm{d}t}{(t^2 - \rho^2)(a^2 - t^2)^{1/2}},$$

$$\int_r^a \frac{\mathrm{d}x}{(x^2 - r^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^a \frac{f(t)\ \mathrm{d}t}{(t^2 - x^2)^{1/2}} = -\frac{\pi}{2}\frac{f(r)}{r},$$

$$\int_0^r \frac{\mathrm{d}x}{(r^2 - x^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{f(t)\ \mathrm{d}t}{(x^2 - t^2)^{1/2}} = \frac{\pi}{2}\left[\frac{f(r) - f(0)}{r}\right]. \tag{3.2.13}$$

The identities (3.2.13) can be verified by using (1.3.2) and (1.3.9). Now substitution of (3.2.10) in (3.2.8) gives, after simplification,

$$-\frac{\alpha}{\sqrt{\gamma_1\gamma_2}} f_1(r) + \frac{2}{\pi} r(a^2 - r^2)^{1/2} \int_0^a \frac{f_2(t)\ \mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} = \chi_2(r). \tag{3.2.14}$$

Let

$$f_1(t) = -f_1(-t), \qquad f_2(t) = f_2(-t). \tag{3.2.15}$$

These assumptions allow us to rewrite equations (3.2.11) and (3.2.14) in the form

$$\frac{\alpha}{\sqrt{\gamma_1\gamma_2}} f_2(r) + \frac{1}{\pi}(a^2 - r^2)^{1/2} \int_{-a}^a \frac{f_1(t)\ \mathrm{d}t}{(t - r)(a^2 - t^2)^{1/2}} = \chi_1(r) + b,$$

$$-\frac{\alpha}{\sqrt{\gamma_1\gamma_2}} f_1(r) + \frac{1}{\pi}(a^2 - r^2)^{1/2} \int_{-a}^a \frac{f_2(t)\ \mathrm{d}t}{(t - r)(a^2 - t^2)^{1/2}} = \chi_2(r). \tag{3.2.16}$$

By introducing the complex functions $f = f_1 + if_2$ and $\chi = \chi_1 + i\chi_2$, the system depicted in (3.2.16) can be reduced to one singular integral equation, namely,

$$-i\frac{\alpha}{\sqrt{\gamma_1\gamma_2}} f(r) + \frac{1}{\pi}(a^2 - r^2)^{1/2} \int_{-a}^a \frac{f(t)\ \mathrm{d}t}{(t - r)(a^2 - t^2)^{1/2}} = \chi(r). \tag{3.2.17}$$

Multiply both sides of equation (3.2.17) by $(a+r)^{-i\theta}(a-r)^{i\theta}(r-y)^{-1}\mathrm{d}r$, where θ is an as yet unknown constant, and integrate with respect to r from $-a$ to a. The result is

$$-i \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{f(r)\ \mathrm{d}r}{r - y} - \pi f(y)\left(\frac{a + y}{a - y}\right)^{-i\theta}$$

$$- i(a^2 - y^2)^{1/2}\left(\frac{a + y}{a - y}\right)^{-i\theta} \tanh(\pi\theta) \int_{-a}^{a} \frac{f(t)\ \mathrm{d}t}{(t - y)(a^2 - t^2)^{1/2}}$$

$$+ i\tanh(\pi\theta) \int_{-a}^{a} \left(\frac{a + t}{a - t}\right)^{i\theta} \frac{f(t)\ \mathrm{d}t}{t - y} = \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\ \mathrm{d}r}{r - y} . \tag{3.2.18}$$

Defining

$$\tanh(\pi\theta) = \alpha/\sqrt{\gamma_1\gamma_2}, \tag{3.2.19}$$

equation (3.2.18) may be simplified as follows:

$$-\pi\left(\frac{a + y}{a - y}\right)^{-i\theta} \left[f(y) + \frac{i}{\pi} \tanh(\pi\theta)(a^2 - y^2)^{1/2} \int_{-a}^{a} \frac{f(t)\ \mathrm{d}t}{(t - y)(a^2 - t^2)^{1/2}}\right]$$

$$= \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\ \mathrm{d}r}{r - y} . \tag{3.2.20}$$

The singular integral may be eliminated from (3.2.20) by using (3.2.17), and the exact solution becomes available in the form

$$f(y) = -\cosh^2(\pi\theta)\left[i\tanh(\pi\theta)\chi(y) + \frac{1}{\pi}\left(\frac{a + y}{a - y}\right)^{i\theta} \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\ \mathrm{d}r}{r - y}\right]. \tag{3.2.21}$$

The following rule of interchanging the order of integration in singular integrals was employed (Muskhelishvili, 1946)

$$\int_{-a}^{a} \frac{\mathrm{d}r}{r - y} \int_{-a}^{a} \frac{f(r,t)\mathrm{d}t}{t - r} = -\pi^2 f(y,y) + \int_{-a}^{a} \mathrm{d}t \int_{-a}^{a} \frac{f(r,t)\mathrm{d}r}{(r - y)(t - r)} . \tag{3.2.22}$$

The other integrals used here can be found in Appendix A3.1. Strictly speaking, the complete solution of (3.2.17) is given by (3.2.21) plus the term $c(a+y)^{i\theta}(a-y)^{-i\theta}$, which represents the homogeneous solution of (3.2.17), with c being an arbitrary constant. The value of c is to be chosen to satisfy the condition b=0, where b is defined by (3.2.12). The appropriate integration of expression (3.2.21) shows that the condition b=0 is satisfied when c=0, and this is the reason why the final solution is written in the form (3.2.21).

The general solution is now completed, and we can consider in more detail the case of a bonded axisymmetric punch, with no tractions applied outside the punch. The total force P may be obtained by integration

$$P = 2\pi\int_0^a \sigma(\rho)\rho\mathrm{d}\rho = 2\pi\int_0^a \frac{f_1(t)t\mathrm{d}t}{(a^2 - t^2)^{1/2}} . \tag{3.2.23}$$

We recall that f_1 is an odd function, and that $f_1=\Re f$. Taking this into consideration, we obtain, after substitution of (3.2.21) in (3.2.23)

$$P = \pi\cosh(\pi\theta)\Re \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{i\theta} \chi(r)\mathrm{d}r. \tag{3.2.24}$$

The field of displacements outside the punch can be obtained by repeating the derivation of (3.2.2) and (3.2.3) for $\rho>a$, which results in

$$w(\rho) = 4H \int_0^a \frac{\mathrm{d}y}{(\rho^2 - y^2)^{1/2}} \int_y^a \frac{\sigma(x)\mathrm{d}x}{(x^2 - y^2)^{1/2}} , \qquad \text{for } \rho>a;$$

$$u(\rho) = \frac{2H}{\rho}\left[2\gamma_1\gamma_2\int_0^a \frac{y^2\mathrm{d}y}{(\rho^2 - y^2)^{1/2}} \int_y^a \frac{\tau(x)\mathrm{d}x}{(x^2 - y^2)^{1/2}} - \frac{\alpha}{2}P\right], \qquad \text{for } \rho>a.$$

(3.2.25)

Substitution of (3.2.10) in (3.2.25) and integration with respect to x leads to

$$w(\rho) = 4H \int_0^a \frac{(a^2 - y^2)^{1/2}}{(\rho^2 - y^2)^{1/2}}\, \mathrm{d}y \int_0^a \frac{f_1(t)t\mathrm{d}t}{(t^2 - y^2)(a^2 - t^2)^{1/2}}\ ,$$

$$u(\rho) = \frac{2H}{\rho}\left[2\sqrt{\gamma_1\gamma_2}\int_0^a \frac{(a^2 - y^2)^{1/2}y^2\mathrm{d}y}{(\rho^2 - y^2)^{1/2}} \int_0^a \frac{f_2(t)\mathrm{d}t}{(t^2 - y^2)(a^2 - t^2)^{1/2}} - \frac{\alpha}{2}P\right].$$

(3.2.26)

The singular integrals in (3.2.26) can be evaluated from (3.2.17) and (3.2.21), and the final result may be written as

$$w(\rho) = 2H \int_0^a \frac{\Re\{Z(y)\}}{(\rho^2 - y^2)^{1/2}}\, \mathrm{d}y,$$

$$u(\rho) = \frac{H}{\rho}\sqrt{\gamma_1\gamma_2}\left[2\int_0^a \frac{y\Im\{Z(y)\}}{(\rho^2 - y^2)^{1/2}}\, \mathrm{d}y - P\tanh(\pi\theta)\right], \text{ for } \rho > a.$$

(3.2.27)

Here

$$Z(y) = \pi\cosh^2(\pi\theta)\left[\chi(y) - \frac{i}{\pi}\tanh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta} \int_{-a}^{a} \left(\frac{a+r}{a-r}\right)^{-i\theta} \frac{\chi(r)\mathrm{d}r}{r-y}\right].$$

(3.2.28)

Formulae (3.2.10) and (3.2.21) are the main results of this section.

Solution of the second kind. Assume solution to the problem in the form

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{F_1(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}, \qquad \tau(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{F_2(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}. \tag{3.2.29}$$

Again, F_1 and F_2 are the as yet unknown stress functions. Substitution of (3.2.29) in (3.2.6) and (3.2.8) results in

$$\frac{2}{\pi}\left[\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{F_2(t)\mathrm{d}t}{t} + \frac{\alpha\ r^2}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{F_2(t)\mathrm{d}t}{t(t^2 - r^2)} - \frac{\pi}{2}F_1(r)\right] = \chi_1(r),$$

$$\frac{2}{\pi}\left[-\frac{\pi}{2}F_2(r) - \frac{\alpha\ r}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{F_1(t)\mathrm{d}t}{t^2 - r^2}\right] = \chi_2(r). \qquad (3.2.30)$$

Assuming that

$$F_1(t) = F_1(-t), \qquad F_2(t) = -F_2(-t), \qquad (3.2.31)$$

equations (3.2.30) can be rewritten as

$$-F_1(r) + \frac{\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_{-a}^{a} \frac{F_2(t)\mathrm{d}t}{t - r} = \chi_1(r),$$

$$-F_2(r) - \frac{\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_{-a}^{a} \frac{F_1(t)\mathrm{d}t}{t - r} = \chi_2(r). \qquad (3.2.32)$$

Introducing the complex functions

$$F(r) = F_1(r) + iF_2(r), \qquad \chi(r) = \chi_1(r) + i\chi_2(r), \qquad (3.2.33)$$

the system (3.2.32) can be reduced to a single equation, namely,

$$F(r) + \frac{i\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_{-a}^{a} \frac{F(t)\mathrm{d}t}{t - r} = -\ \chi(r). \qquad (3.2.34)$$

Multiplication of (3.2.34) by $(a+r)^{-i\theta}(a-r)^{i\theta}(r-y)^{-1}$ and integration with respect to r from $-a$ to a leads to

$$\int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{F(r)\mathrm{d}r}{r - y} + \frac{i\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\left[-\pi^2\left(\frac{a + y}{a - y}\right)^{-i\theta}F(y) + \left(\frac{a + y}{a - y}\right)^{-i\theta}\frac{\pi}{i}\coth(\pi\theta)\right.$$

$$\times \int_{-a}^{a} \frac{F(t)\mathrm{d}t}{t - y} - \frac{\pi}{i}\coth(\pi\theta) \int_{-a}^{a} \left(\frac{a + t}{a - t}\right)^{-i\theta} \frac{F(t)\mathrm{d}t}{t - y}\Bigg] = - \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\mathrm{d}r}{r - y}. \tag{3.2.35}$$

Here formula (3.2.22) and the integrals from Appendix 3.1 were used. Taking, as before, $\tanh(\pi\theta)=\alpha/\sqrt{\gamma_1\gamma_2}$, equation (3.2.35) can be simplified significantly, namely,

$$F(y) + \frac{i}{\pi}\coth(\pi\theta) \int_{-a}^{a} \frac{F(t)\mathrm{d}t}{t - y} = - \frac{i}{\pi}\coth(\pi\theta)\left(\frac{a + y}{a - y}\right)^{i\theta} \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\mathrm{d}r}{r - y}. \tag{3.2.36}$$

The singular integral can be eliminated from (3.2.36) by using (3.2.34), and the final result is

$$F(y) = \cosh^2(\pi\theta)\left[-\chi(y) + \frac{i}{\pi}\tanh(\pi\theta) \left(\frac{a + y}{a - y}\right)^{i\theta} \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \frac{\chi(r)\mathrm{d}r}{r - y}\right]. \tag{3.2.37}$$

The general solution may be considered completed. Some additional results are presented for the case of a bonded punch. The total force, exerted by the punch is obtained by integration of σ:

$$P = -2\pi\int_{0}^{a} F_1(t)\mathrm{d}t = -\pi \int_{-a}^{a} F_1(t)\mathrm{d}t. \tag{3.2.38}$$

Substitution of (3.2.37) in (3.2.38) yields

$$P = \pi\cosh(\pi\theta)\Re \int_{-a}^{a} \left(\frac{a + r}{a - r}\right)^{-i\theta} \chi(r)\mathrm{d}r.$$

As expected, this result is identical to (3.2.24). The displacements outside the punch can be found by substitution of (3.2.37) and (3.2.29) in (3.2.25), with the result

$$w(\rho) = -2\pi H \int_0^a \frac{\Re\{F(y)\}}{(\rho^2 - y^2)^{1/2}}\, \mathrm{d}y,$$

$$u(\rho) = -\frac{H}{\rho}\sqrt{\gamma_1\gamma_2}\left[2\pi\int_0^a \frac{y\Im\{F(y)\}}{(\rho^2 - y^2)^{1/2}}\,\mathrm{d}y + P\tanh(\pi\theta)\right], \text{ for } \rho>a. \tag{3.2.39}$$

Comparison of (3.2.28) and (3.2.37) shows that $Z(y) = -\pi F(y)$, and this means that formulae (3.2.39) actually repeat (3.2.27).

Solution of the third kind. Let the stresses in question be expressed through two new and as yet unknown functions q_1 and q_2 as follows

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{q_1(t)t\,\mathrm{d}t}{(\rho^2 - t^2)^{1/2}}, \qquad \tau(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{q_2(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}. \tag{3.2.40}$$

Substitution of (3.2.40) in (3.2.6) and (3.2.8) leads to the system

$$\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{q_2(t)t\mathrm{d}t}{t^2 - r^2} + (a^2 - r^2)^{1/2}\int_0^a \frac{q_1(t)t\mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} = \frac{\pi}{2}\chi_1(r),$$

$$-q_2(r) - \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}q_1(r) = \chi_2(r). \tag{3.2.41}$$

Function q_2 can be expressed from the second equation (3.2.41) as

$$q_2(r) = -\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}q_1(r) - \chi_2(r). \tag{3.2.42}$$

Substitution of (3.2.42) into the first equation (3.2.41) gives

$$(a^2 - r^2)^{1/2}\int_0^a \frac{q_1(t)t\mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} - \frac{\alpha^2}{\gamma_1\gamma_2}\int_0^a \frac{q_1(t)t\mathrm{d}t}{t^2 - r^2} = \psi(r). \tag{3.2.43}$$

Here

$$\psi(r) = \frac{\pi}{2}\,\chi_1(r) + \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{\chi_2(t)t\mathrm{d}t}{t^2 - r^2}. \tag{3.2.44}$$

An exact solution of the integral equation (3.2.43) can be obtained in the following manner. Introduce the notation

$$Y_\mathrm{s}(r) = \sin\left[\theta\ln\left|\frac{a + r}{a - r}\right|\right], \qquad Y_\mathrm{c}(r) = \cos\left[\theta\ln\left|\frac{a + r}{a - r}\right|\right]. \tag{3.2.45}$$

Multiply both sides of (3.2.43) by $Y_\mathrm{c}(r)/(r^2 - x^2)$ and integrate with respect to r from 0 to a. The result is

$$-\frac{\pi^2}{4x^2}\left(1 - \frac{\alpha^2}{\gamma_1\gamma_2}\right)q_1(x)Y_\mathrm{c}(x) + \frac{\pi}{2}\tanh(\pi\theta)\int_0^a\left[\frac{Y_\mathrm{s}(t)}{t} - \left(\frac{a^2 - x^2}{a^2 - t^2}\right)^{1/2}\frac{Y_\mathrm{s}(x)}{x}\right]\frac{q_1(t)\mathrm{d}t}{(x^2 - t^2)}$$

$$-\frac{\pi\alpha^2}{2\gamma_1\gamma_2}\coth(\pi\theta)\int_0^a \frac{q_1(t)}{x^2 - t^2}\left[\frac{Y_\mathrm{s}(t)}{t} - \frac{Y_\mathrm{s}(x)}{x}\right]\mathrm{d}t = \mathcal{X}_\mathrm{c}\psi(x). \tag{3.2.46}$$

Hereafter the following integral operators are introduced

$$\mathcal{X}_\mathrm{c}\psi(x) = \int_0^a \frac{Y_\mathrm{c}(r)\psi(r)\mathrm{d}r}{r^2 - x^2}, \qquad \mathcal{X}_\mathrm{s}\psi(x) = \int_0^a \frac{Y_\mathrm{s}(r)\psi(r)r\mathrm{d}r}{r^2 - x^2}. \tag{3.2.47}$$

Again, assuming $\tanh(\pi\theta)=\alpha/\sqrt{\gamma_1\gamma_2}$, equation (3.2.46) can be simplified as

$$-\frac{\pi^2 q_1(x)Y_\mathrm{c}(x)}{4x\cosh^2(\pi\theta)} + \frac{\pi}{2}\tanh(\pi\theta)Y_\mathrm{s}(x)\int_0^a\left[1 - \left(\frac{a^2 - x^2}{a^2 - t^2}\right)^{1/2}\right]\frac{q_1(t)\mathrm{d}t}{x^2 - t^2} = x\mathcal{X}_\mathrm{c}\psi(x). \tag{3.2.48}$$

A similar procedure of multiplication of (3.2.43) by $rY_\mathrm{s}(r)/(r^2 - x^2)$ leads, after simplification, to

$$-\frac{\pi^2 q_1(x)Y_s(x)}{4x\cosh^2(\pi\theta)} - \frac{\pi}{2}\tanh(\pi\theta)Y_c(x)\int_0^a\left[1-\left(\frac{a^2-x^2}{a^2-t^2}\right)^{1/2}\right]\frac{q_1(t)\,\mathrm{d}t}{x^2-t^2} = \mathcal{X}_s\psi(x). \tag{3.2.49}$$

Equations (3.2.48) and (3.2.49) finally give the solution

$$q_1(x) = -\frac{4}{\pi^2}\cosh^2(\pi\theta)\Big[xY_c(x)\mathcal{X}_c\psi(x) + Y_s(x)\mathcal{X}_s\psi(x)\Big] + b_0\frac{Y_c(x)}{x}. \tag{3.2.50}$$

The last term in (3.2.50) represents the homogeneous solution, and b_0 is an arbitrary constant. If the stresses defined by (3.2.40) are to be nonsingular at $\rho=0$, then b_0 should be equal to zero.

Substitution of (3.2.44) in (3.2.50) allows us to express the solution in terms of χ_1 and χ_2:

$$\begin{aligned} q_1(x) = -\cosh^2(\pi\theta)\Big[&\frac{2}{\pi}xY_c(x)\mathcal{X}_c\chi_1(x) + \frac{2}{\pi}Y_s(x)\mathcal{X}_s\chi_1(x) - \tanh(\pi\theta)\chi_2(x) \\ &+ \frac{2}{\pi}xY_c(x)\mathcal{X}_s\{\chi_2(x)/x\} - \frac{2}{\pi}Y_s(x)\mathcal{X}_c\{x\chi_2(x)\}\Big]. \end{aligned} \tag{3.2.51}$$

Here a modified version of formula (3.2.22) was employed:

$$\int_0^a\frac{\mathrm{d}t}{t^2-x^2}\int_0^a\frac{f(t,r)\,\mathrm{d}r}{r^2-t^2} = -\frac{\pi^2 f(x,x)}{4x^2} + \int_0^a\mathrm{d}r\int_0^a\frac{f(t,r)\,\mathrm{d}t}{(t^2-x^2)(r^2-t^2)}. \tag{3.2.52}$$

The second stress function q_2 may be obtained from (3.2.42) and (3.2.51) in the form

$$\begin{aligned} q_2(x) = \cosh^2(\pi\theta)\Big\{&\frac{2}{\pi}\tanh(\pi\theta)[xY_c(x)\mathcal{X}_c\chi_1(x) + Y_s(x)\mathcal{X}_s\chi_1(x) \\ &+ xY_c(x)\mathcal{X}_s\{\chi_2(x)/x\} - Y_s(x)\mathcal{X}_c\{x\chi_2(x)\}] - \chi_2(x)\Big\}. \end{aligned} \tag{3.2.53}$$

The general solution is completed, and we can derive some additional results for the case of a bonded punch. The total force P may be obtained by integration of the first expression in (3.2.40):

$$P = 2\pi \int_0^a \frac{q_1(t)t\mathrm{d}t}{(a^2 - t^2)^{1/2}} . \tag{3.2.54}$$

Substitution of (3.2.51) in (3.2.54) gives, after integration,

$$P = 2\pi\cosh(\pi\theta)\int_0^a [\chi_1(r)Y_c(r) + \chi_2(r)Y_s(r)]\mathrm{d}r. \tag{3.2.55}$$

The result (3.2.55) coincides with (3.2.24) when χ_1 is an even function and χ_2 is odd. The displacements outside the punch may be found by substitution of (3.2.40) in (3.2.25), and are

$$w(\rho) = 4H \int_0^a \left(\frac{a^2 - y^2}{\rho^2 - y^2}\right)^{1/2} \mathrm{d}y \int_0^a \frac{q_1(t)t\mathrm{d}t}{(t^2 - y^2)(a^2 - t^2)^{1/2}} ,$$

$$u(\rho) = -H\sqrt{\gamma_1\gamma_2}\, \frac{1}{\rho} \left[2\pi\int_0^a \frac{q_2(y)y\mathrm{d}y}{(\rho^2 - y^2)^{1/2}} + P\tanh(\pi\theta)\right], \text{ for } \rho>a. \tag{3.2.56}$$

One can prove that expressions (3.2.56) are in agreement with (3.2.28).

Solution of the fourth kind. Let the solution be

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} \int_\rho^a \frac{Q_1(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} , \qquad \tau(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho} \int_0^\rho \frac{Q_2(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} . \tag{3.2.57}$$

By using the same methods as before, the following set of equations can be obtained

$$\frac{\alpha}{\sqrt{\gamma_1\gamma_2}} Q_2(r) - Q_1(r) = \chi_1(r) + b,$$

$$(a^2 - r^2)^{1/2}\int_0^a \frac{Q_2(t)\mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} - \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{Q_1(t)\mathrm{d}t}{t^2 - r^2} = \frac{\pi}{2r}\,\chi_2(r), \tag{3.2.58}$$

where

$$b = \frac{2\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{Q_2(t)\mathrm{d}t}{(a^2 - t^2)^{1/2}}. \tag{3.2.59}$$

It will be shown later, that Q_2 can always be chosen in such a way that b=0. Now Q_1 can be expressed from the first equation (3.2.58) and substituted in the second one, with the result

$$(a^2 - r^2)^{1/2}\int_0^a \frac{Q_2(t)\mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} - \frac{\alpha^2}{\gamma_1\gamma_2}\int_0^a \frac{Q_2(t)\mathrm{d}t}{t^2 - r^2} = \Psi(r). \tag{3.2.60}$$

Here

$$\Psi(r) = \frac{\pi}{2r}\,\chi_2(r) - \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a \frac{\chi_1(r) + b}{t^2 - r^2}\,\mathrm{d}t. \tag{3.2.61}$$

Equation (3.2.60) is similar to (3.2.43), so its solution can be written down as

$$Q_2(x) = -\frac{4}{\pi^2}\cosh^2(\pi\theta)\Big[x^2 Y_c(x)\mathcal{X}_c\Psi(x) + xY_s(x)\mathcal{X}_s\Psi(x)\Big] + b_2 Y_c(x). \tag{3.2.62}$$

The last term in (3.2.62) represents the homogeneous solution, and b_2 is an arbitrary constant to be chosen from the condition b=0. Appropriate integration of (3.2.62), using the integrals from Appendix 3.1, allows us to define b_2 as follows

$$b_2 = -\frac{4}{\pi^2}\cosh^2(\pi\theta)\int_0^a \Psi(x)Y_c(x)\mathrm{d}x. \tag{3.2.63}$$

Substitution of (3.2.63) in (3.2.62) yields

$$Q_2(x) = -\frac{4}{\pi^2}\cosh^2(\pi\theta)\left[Y_c(x)\mathcal{X}_c\{x^2\Psi(x)\} + xY_s(x)\mathcal{X}_s\Psi(x)\right]. \tag{3.2.64}$$

Function Q_2 can be expressed in terms of χ_1 and χ_2 by using (3.2.61) and (3.2.59) as follows:

$$Q_2(x) = -\cosh^2(\pi\theta)\{\tanh(\pi\theta)\chi_1(x) + \frac{2}{\pi}[xY_s(x)\mathcal{X}_c\chi_1(x)$$

$$- Y_c(x)\mathcal{X}_s\chi_1(x) + Y_c(x)\mathcal{X}_c(x\chi_2(x)) + xY_s(x)\mathcal{X}_s(\chi_2(x)/x)]\}. \tag{3.2.65}$$

Consequently, from the first expression of (3.2.58), we have

$$Q_1(x) = -\cosh^2(\pi\theta)\{\chi_1(x) + \frac{2}{\pi}\tanh(\pi\theta)[\,xY_s(x)\mathcal{X}_c\chi_1(x)$$

$$- Y_c(x)\mathcal{X}_s\chi_1(x) + Y_c(x)\mathcal{X}_c(x\chi_2(x)) + xY_s(x)\mathcal{X}_s(\chi_2(x)/x)]\}. \tag{3.2.66}$$

The total force exerted by a bonded punch is defined by

$$P = -2\pi\int_0^a Q_1(t)\mathrm{d}t.$$

One can show that this result coincides with (3.2.55). The displacements outside the punch are

$$w(\rho) = -2\pi H\int_0^a \frac{Q_1(y)\mathrm{d}y}{(\rho^2 - y^2)^{1/2}},$$

$$u(\rho) = \frac{2H}{\rho}\left[2\sqrt{\gamma_1\gamma_2}\int_0^a \left(\frac{a^2 - y^2}{\rho^2 - y^2}\right)^{1/2} y^2\mathrm{d}y\int_0^a \frac{Q_2(t)\mathrm{d}t}{(t^2 - y^2)(a^2 - t^2)^{1/2}} - \frac{\alpha}{2}P\right]. \tag{3.2.67}$$

Expressions (3.2.67) are in agreement with (3.2.27).

Example 1. Consider a circular flat-ended punch of radius a, bonded to a transversely isotropic half-space, and acted upon by an axial force P. The boundary conditions (3.2.1) in this particular case will take the form

$$u(\rho) = 0, \qquad w(\rho) = w_0 = \text{const.}, \qquad \text{for } \rho<a;$$

$$\sigma(\rho) = 0, \qquad \tau(\rho) = 0, \qquad \text{for } \rho>a.$$

The stress functions are

$$f(y) = i\,\frac{w_0}{\pi^2 H}\coth(\pi\theta)\left[1 - \cosh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta}\right],$$

$$F(y) = -\frac{w_0}{\pi^2 H}\cosh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta}, \qquad Q_1(y) = -\frac{w_0}{\pi^2 H}\cosh(\pi\theta)Y_c(y),$$

$$Q_2(y) = \frac{w_0}{\pi^2 H}\coth(\pi\theta)[1- \cosh(\pi\theta)Y_c(y)].$$

$$q_1(y) = \frac{w_0\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}Y_s(y), \qquad q_2(y) = -\frac{w_0}{\pi^2 H}\cosh(\pi\theta)Y_s(y),$$

One can notice that $q_1=\Re f$, $q_2=\Im F$, $Q_1=\Re F$, and $Q_2=\Im f$, and this is why we have only two different representations for the surface tractions, namely,

$$\sigma(\rho) = \frac{w_0\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho \frac{Y_s(t)t\,dt}{(\rho^2 - t^2)^{1/2}}$$

$$= -\frac{w_0}{\pi^2 H}\cosh(\pi\theta)\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{Y_c(t)t\,dt}{(t^2 - \rho^2)^{1/2}},$$

$$\tau(\rho) = -\frac{w_0\cosh(\pi\theta)}{\pi^2 H\alpha}\frac{d}{d\rho}\int_0^\rho \frac{Y_c(t)\,dt}{(\rho^2 - t^2)^{1/2}} = -\frac{w_0\cosh(\pi\theta)}{\pi^2 H\sqrt{\gamma_1\gamma_2}}\frac{d}{d\rho}\int_\rho^a \frac{Y_s(t)\,dt}{(t^2 - \rho^2)^{1/2}}.$$

The equivalence of these representations can be verified by using the identities

$$\int_0^\rho \frac{Y_c(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} = \tanh(\pi\theta)\int_\rho^a \frac{Y_s(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + \frac{\pi}{2\cosh(\pi\theta)}, \tag{3.2.68}$$

$$\int_0^\rho \frac{Y_s(t)t\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} = \frac{\pi\theta a}{\cosh(\pi\theta)} - \tanh(\pi\theta)\int_\rho^a \frac{Y_c(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}. \tag{3.2.69}$$

These and other similar identities can be established by using the general relationship:

$$\int_\rho^a \frac{f(x)\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} = \frac{2}{\pi}\int_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_0^a \frac{f(t)t\mathrm{d}t}{t^2 - x^2} + \frac{\pi}{2\rho}\lim_{t\to 0}[tf(t)]. \tag{3.2.70}$$

The relationship between the total force P and the punch settlement w_0 is

$$P = \frac{2w_0 a\theta}{H\tanh(\pi\theta)}. \tag{3.2.71}$$

The surface displacements outside the punch are given by

$$w(\rho) = \frac{2}{\pi}\, w_0\cosh(\pi\theta)\int_0^a \frac{Y_c(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}},$$

$$u(\rho) = \frac{2}{\pi}\, w_0\sqrt{\gamma_1\gamma_2}\,\cosh(\pi\theta)\left[\int_0^a \frac{Y_s(x)x\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} - \frac{\pi a\theta}{\cosh(\pi\theta)}\right]\frac{1}{\rho}. \tag{3.2.72}$$

The integrals in (3.2.72) can be evaluated (Gradshtein and Ryzhik, 1963)

$$I_c(\rho) = \int_0^a \frac{Y_c(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{\pi\theta}{\sinh(\pi\theta)}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho^2/a^2 - 1)^{-k-1/2}}{(2k + 1)(k!)^2}\prod_{m=1}^{k}(\theta^2 + m^2), \tag{3.2.73}$$

$$I_s(\rho) = \int_0^a \frac{Y_s(x)x\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{\pi a\theta^2}{\sinh(\pi\theta)} \sum_{k=0}^{\infty} \frac{(-1)^k(\rho^2/a^2 - 1)^{-k-1/2}}{(2k + 1)k!(k + 1)!} \prod_{m=1}^{k}(\theta^2 + m^2). \tag{3.2.74}$$

For real materials, the physical constant $\theta<1$. For example, for an isotropic material

$$\theta = \frac{1}{2\pi}\ln(3 - 4\nu). \tag{3.2.75}$$

Since the Poisson coefficient $\nu\le 0.5$, this means that $0\le\theta<0.2$ for isotropic materials. By using the approximation

$$(k!)^{-2}\prod_{m=1}^{k}(\theta^2 + m^2) = 1 + O(\theta^2),$$

the summation in (3.2.73) and (3.2.74) can be performed, and the results are

$$I_c(\rho) \approx \frac{\pi\theta}{\sinh(\pi\theta)} \sin^{-1}\left(\frac{a}{\rho}\right), \tag{3.2.76}$$

$$I_s(\rho) \approx \frac{\pi\theta^2}{\sinh(\pi\theta)}\left[(\rho^2 - a^2)^{1/2}\ln\left(1 - \frac{a^2}{\rho^2}\right) + 2a\sin^{-1}\left(\frac{a}{\rho}\right)\right]. \tag{3.2.77}$$

Direct numerical computations show that the relative error of (3.2.76) is less than 6% when $(\rho/a)>1.01$, and $\theta\le 0.2$; the relative error of (3.2.77) is less than 4%. The same accuracy can be obtained for $(\rho/a)>1.1$, and $\theta\le 0.3$. The relative error of both (3.2.76) and (3.2.77) rapidly decreases when ρ increases, for example, the relative error is less than 4% and 2% respectively, for $\theta<0.9$ and $(\rho/a)>3$.

Example 2. Consider the case where no forces act upon the punch bonded to a half-space, but the punch itself expands, so that the radial displacement is proportional to the radius, with k as the coefficient, namely, inside the circle $\rho=a$ the boundary conditions are

$$w(\rho) = w_0, \qquad u(\rho) = k\rho.$$

Here w_0 is the as yet unknown punch settlement. The stress functions are

$$f(y) = \frac{\coth(\pi\theta)}{\pi^2 H}\left\{ i w_0 - \frac{2ky}{\sqrt{\gamma_1\gamma_2}} - i\cosh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta}\left[\frac{2k}{\sqrt{\gamma_1\gamma_2}}(2a\theta + iy) + w_0\right]\right\},$$

$$F(y) = -\frac{\cosh(\pi\theta)}{\pi^2 H}\left[w_0 + \frac{2k}{\sqrt{\gamma_1\gamma_2}}(2a\theta + iy)\right]\left(\frac{a+y}{a-y}\right)^{i\theta},$$

$$q_1(y) = \frac{\coth(\pi\theta)}{\pi^2 H}\left\{ w_0\cosh(\pi\theta)Y_s(y) + \frac{2k}{\sqrt{\gamma_1\gamma_2}}[\cosh(\pi\theta)(2a\theta Y_s(y)+yY_c(y)) - y]\right\},$$

$$q_2(y) = -\frac{\cosh(\pi\theta)}{\pi^2 H}\left\{ w_0 Y_s(y) + \frac{2k}{\sqrt{\gamma_1\gamma_2}}[2a\theta Y_s(y) + yY_c(y)]\right\},$$

$$Q_1(y) = -\frac{\cosh(\pi\theta)}{\pi^2 H}\left\{ w_0 Y_c(y) + \frac{2k}{\sqrt{\gamma_1\gamma_2}}[2a\theta Y_c(y) - yY_s(y)]\right\},$$

$$Q_2(y) = \frac{\coth(\pi\theta)}{\pi^2 H}\left\{ w_0[1-\cosh(\pi\theta)Y_c(y)] - \cosh(\pi\theta)\frac{2k}{\sqrt{\gamma_1\gamma_2}}[2a\theta Y_c(y) - yY_s(y)]\right\}.$$

The stresses can be defined two ways

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{q_1(x)x\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{Q_1(x)x\mathrm{d}x}{(x^2 - \rho^2)^{1/2}},$$

$$\tau(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{Q_2(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{q_2(x)\mathrm{d}x}{(x^2 - \rho^2)^{1/2}}.$$

The total force P is defined by

$$P = \frac{2a\theta}{H}\coth(\pi\theta)\left[w_0 + \frac{2ka\theta}{\sqrt{\gamma_1\gamma_2}}\right]. \tag{3.2.78}$$

When no force acts upon the punch, its normal displacement is equal to

$$w_0 = -2ka\theta/\sqrt{\gamma_1\gamma_2}.$$

The force, needed to provide a zero normal displacement is

$$P = \frac{4ka^2\theta^2}{H\alpha}.$$

The displacements outside the punch are

$$w(\rho) = -2\pi H\int_0^a \frac{Q_1(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}},$$

$$u(\rho) = -H\sqrt{\gamma_1\gamma_2}\left[2\pi\int_0^a \frac{q_2(x)x\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} + P\tanh(\pi\theta)\right]\frac{1}{\rho}.$$

One can calculate the stress at the point $\rho=0$ in elementary functions

$$\sigma(0) = \frac{\coth(\pi\theta)}{\pi H}\left\{\frac{\theta}{a}w_0\cosh(\pi\theta) + \frac{k}{\sqrt{\gamma_1\gamma_2}}[(1+4\theta^2)\cosh(\pi\theta) - 1]\right\}, \qquad \tau(0) = 0.$$

Discussion. One could notice that the four kinds of solution considered represent all combinations of the Abel type integrals with limits from 0 to ρ and from ρ to a. Solutions of the first and the second kind are more compact than the others, but they are only convenient to use when $w(\rho)$ is an even function and $u(\rho)$ is an odd one. The solution of the second kind is preferable when one is interested mainly in the displacements outside the punch, while the solution of the first kind has definite advantages when one is interested in the stress distributions. Formulae (3.2.10) are more convenient for numerical integration than (3.2.29), especially in the domain close to either $\rho=0$ or $\rho=a$: (i) the differentiation of the integrand can be performed in (3.2.10), thus avoiding numerical differentiation, which is less accurate; the differentiation in (3.2.29) is rather difficult because $F(a)$ usually is not defined; (ii) formulae (3.2.10) allow us to determine easily the stress at $\rho=0$, directly through the stress function, namely,

$$\sigma(0) = \frac{\pi}{2}f_1'(0), \qquad \tau(0) = f_2'(0)/\sqrt{\gamma_1\gamma_2},$$

while the result of (3.2.29) is rather difficult to use, for example,

$$\sigma(0) = \lim_{\rho\to 0}\left\{\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{a}\frac{F_1(x)x\mathrm{d}x}{(x^2-\rho^2)^{1/2}}\right\} = \int_0^a \frac{F_1(x)-F_1(0)}{x^2}\,\mathrm{d}x - \frac{F_1(0)}{a}. \tag{3.2.79}$$

The solutions of the third and the fourth kind are more general because they do not require either $w(\rho)$ to be an even function or $u(\rho)$ to be an odd one. The same logic of preference is applicable here: the integral representations with limits from 0 to ρ are more convenient for evaluation of the stresses while the representations with limits from ρ to a are preferable for the displacement evaluation.

It is of interest to establish relationships between the various kinds of solution. Some are obvious, due to the uniqueness of the solution, namely,

$$f_1 = q_1,\quad f_2 = Q_2,\quad F_1 = Q_1,\quad F_2 = q_2. \tag{3.2.80}$$

The other relationships may be found from (3.2.70) and the following identity

$$\int_0^{\rho}\frac{f(x)\mathrm{d}x}{(\rho^2-x^2)^{1/2}} = -\frac{2}{\pi}\int_{\rho}^{a}\frac{x\mathrm{d}x}{(x^2-\rho^2)^{1/2}}\int_0^a\frac{(a^2-y^2)^{1/2}f(y)\mathrm{d}y}{(a^2-x^2)^{1/2}(y^2-x^2)}. \tag{3.2.81}$$

Comparison of (3.2.57) and (3.2.40) with (3.2.70) and (3.2.81) yields

$$Q_1(t) = -\frac{2}{\pi}\int_0^a\frac{(a^2-y^2)^{1/2}yq_1(y)}{(a^2-t^2)^{1/2}(y^2-t^2)}\,\mathrm{d}y,$$

$$q_2(t) = -\frac{2}{\pi}\,t\int_0^a\frac{(a^2-y^2)^{1/2}Q_2(y)}{(a^2-t^2)^{1/2}(y^2-t^2)}\,\mathrm{d}y, \tag{3.2.82}$$

$$q_1(t) = \frac{2}{\pi t}\int_0^a\frac{y^2Q_1(y)}{(y^2-t^2)}\,\mathrm{d}y,\qquad Q_2(t) = \frac{2}{\pi}\int_0^a\frac{yq_2(y)}{(y^2-t^2)}\,\mathrm{d}y. \tag{3.2.83}$$

Comparison of (3.2.56) and (3.2.67) leads to slightly different expressions:

$$Q_1(t) = -\frac{2}{\pi}\int_0^a \frac{(a^2 - t^2)^{1/2} y q_1(y)\mathrm{d}y}{(a^2 - y^2)^{1/2}(y^2 - t^2)},$$

$$q_2(t) = -\frac{2}{\pi}\, t\int_0^a \frac{(a^2 - t^2)^{1/2} Q_2(y)\mathrm{d}y}{(a^2 - y^2)^{1/2}(y^2 - t^2)}. \tag{3.2.84}$$

Expressions (3.2.84) differ from (3.2.82) by the term const/$(a^2-t^2)^{1/2}$ and t·const/$(a^2-t^2)^{1/2}$ respectively. One may notice from (3.2.57) and (3.2.40) that addition of these terms to Q_1 and q_2 respectively does not affect the stresses and therefore, expressions (3.2.82) and (3.2.84) are equivalent. The same argument is applicable to q_1. Since the addition to q_1 of a term const/t does not affect the solution, an alternative to the first expression of (3.2.83) may be suggested

$$q_1(t) = \frac{2}{\pi}\, t\int_0^a \frac{Q_1(y)\mathrm{d}y}{y^2 - t^2}. \tag{3.2.85}$$

Comparison of (3.2.39) and (3.2.26) allows us to construct the relationship between the complex stress functions

$$F(t) = -\frac{1}{\pi}\int_{-a}^a \frac{(a^2 - t^2)^{1/2} f(y)\mathrm{d}y}{(a^2 - y^2)^{1/2}(y - t)}. \tag{3.2.86}$$

The inverse relationship takes the form

$$f(t) = \frac{1}{\pi}\int_{-a}^a \frac{F(y)\mathrm{d}y}{y - t}. \tag{3.2.87}$$

One can also deduce from (3.2.82) the following expression which is equivalent to (3.2.86):

$$F(t) = -\frac{1}{\pi}\int_{-a}^a \frac{(a^2 - y^2)^{1/2} f(y)\mathrm{d}y}{(a^2 - t^2)^{1/2}(y - t)}. \tag{3.2.88}$$

Considerable simplifications occur when $\theta=0$. In the case of an isotropic body this condition corresponds to the Poisson coefficient $\nu=1/2$. The stress functions will be defined by

$$f(y) = -\frac{1}{\pi}\int_{-a}^{a}\frac{\chi(r)\mathrm{d}r}{r-y}, \qquad F(y) = -\chi(y),$$

$$q_1(y) = -\frac{2}{\pi}\,y\int_{0}^{a}\frac{\chi_1(r)\mathrm{d}r}{r^2-y^2} \qquad q_2(y) = -\chi_2(y),$$

$$Q_2(y) = -\frac{2}{\pi}\int_{0}^{a}\frac{\chi_2(r)r\mathrm{d}r}{r^2-y^2} \qquad Q_1(y) = -\chi_1(y).$$

The stress distributions are

$$\sigma(\rho) = -\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{a}\frac{\chi_1(x)x\mathrm{d}x}{(x^2-\rho^2)^{1/2}}, \qquad \tau(\rho) = -\frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{a}\frac{\chi_2(x)\mathrm{d}x}{(x^2-\rho^2)^{1/2}}.$$

The displacements outside the punch simplify as follows:

$$u(\rho) = 2\pi H\sqrt{\gamma_1\gamma_2}\,\frac{1}{\rho}\int_{0}^{a}\frac{\chi_2(x)x\mathrm{d}x}{(\rho^2-x^2)^{1/2}}, \qquad w(\rho) = 2\pi H\int_{0}^{a}\frac{\chi_1(x)\mathrm{d}x}{(\rho^2-x^2)^{1/2}}. \tag{3.2.89}$$

Notice that here the normal parameters are decoupled from the tangential ones, namely, the normal displacements affect the normal pressure, and the tangential displacements produce the shear tractions only. Substitution of (3.2.7) and (3.2.9) in (3.2.89) furnishes a direct relationship between the displacements inside and outside the circle $\rho=a$

$$u(\rho) = \frac{2}{\pi}\frac{(\rho^2-a^2)^{1/2}}{\rho}\int_{0}^{a}\frac{u(x)x^2\mathrm{d}x}{(a^2-x^2)^{1/2}(\rho^2-x^2)}, \qquad \text{for } \rho>a;$$

$$w(\rho) = \frac{2}{\pi}(\rho^2 - a^2)^{1/2}\int_0^a \frac{w(x)x\mathrm{d}x}{(a^2 - x^2)^{1/2}(\rho^2 - x^2)}, \qquad \text{for } \rho>a.$$

The last expressions demonstrate a certain mathematical similarity between the normal and tangential displacements.

Exercise 3.2

1. Prove that the settlement of a smooth flat circular punch is greater then or equal to that of a bonded punch.
Hint: prove that $(\pi\theta)\geq\tanh(\pi\theta)$.
Note: a more general statement can be proven from the energy consideration.

2. An axisymmetric pressure $\sigma=\sigma(\rho)$ is applied to the annulus $b\leq\rho\leq c$ outside a flat bonded punch of radius a. Investigate its influence on the punch settlement w_0 and the traction distributions under the punch.
Solution: The right hand side in the integral equations (3.2.2) and (3.2.3) will take the form

$$\omega_1(\rho) = w_0 - 4H\int_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_b^c \frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}}, \qquad \omega_2 = 0.$$

Now, from (3.2.7) and (3.2.9)

$$\chi_1(r) = \frac{w_0}{\pi^2 H} - \frac{2}{\pi}\int_b^c \frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - r^2)^{1/2}}, \qquad \chi_2 = 0.$$

Substitution of the last expressions in the chosen kind of solution gives the stresses and the displacements. The total force can be defined as

$$P = 2\cosh(\pi\theta)\left[\frac{w_0 a\theta}{H\sinh(\pi\theta)} - 2\int_b^c I_c(\rho)\ \sigma(\rho)\rho\mathrm{d}\rho\right],$$

where I_c is defined by (3.2.73). When the punch is not subjected to a direct loading, its settlement will be

$$w_0 = \frac{2H}{a\theta}\sinh(\pi\theta)\int_b^c I_c(\rho)\sigma(\rho)\rho d\rho.$$

If no displacements are allowed for the punch, then the total force should be

$$P = -4\cosh(\pi\theta)\int_b^c I_c(\rho)\ \sigma(\rho)\rho d\rho.$$

By using the approximation (3.2.76), the results can in many cases be expressed in elementary functions. For example, in the case when $\sigma=\sigma_0=\text{const}$,

$$\int_b^c I_c(\rho)\sigma(\rho)\rho d\rho \approx \frac{\pi\theta\sigma_0}{2\sinh(\pi\theta)}\left\{c^2\sin^{-1}\left(\frac{a}{c}\right) - b^2\sin^{-1}\left(\frac{a}{b}\right)\right.$$

$$\left. + a[(c^2 - a^2)^{1/2} - (b^2 - a^2)^{1/2}]\right\}.$$

3. Investigate the influence of radial tangential tractions $\tau=\tau(\rho)$, applied at the surface of an annulus $b\le\rho\le c$, outside a bonded punch of radius a.
Solution: We have from (3.2.2) and (3.2.3)

$$\omega_1(\rho) = w_0 + 2\pi H\alpha\int_b^c \tau(\rho)d\rho,$$

$$\omega_2(\rho) = -4H\gamma_1\gamma_2\frac{1}{\rho}\int_0^\rho \frac{x^2 dx}{(\rho^2 - x^2)^{1/2}}\int_b^c \frac{\tau(\rho_0)d\rho_0}{(\rho_0^2 - x^2)^{1/2}}.$$

Substitution in (3.2.7) and (3.2.9) yields

$$\chi_1(r) = \frac{w_0}{\pi^2 H} + \frac{2}{\pi}\alpha\int_b^c \tau(\rho)d\rho, \qquad \chi_2(r) = -\frac{2}{\pi}\sqrt{\gamma_1\gamma_2}\ r\int_b^c \frac{\tau(x)dx}{(x^2 - r^2)^{1/2}}.$$

Since χ_1 is an even function, and χ_2 is an odd one, we can use any of the four kinds of solution considered above. The total force is given by (3.2.55):

$$P = 2\cosh(\pi\theta)\left\{\frac{a\theta}{H\sinh(\pi\theta)}\left[w_0 + 2\pi H\alpha\int_b^c \tau(\rho)d\rho\right] - 2\sqrt{\gamma_1\gamma_2}\int_b^c I_s(\rho)\tau(\rho)d\rho\right\}.$$

Here I_s is defined by (3.2.74). When the punch is not loaded, its settlement will be

$$w_0 = 2H\left[-\pi\alpha\int_b^c \tau(\rho)d\rho + \frac{\sinh(\pi\theta)}{a\theta}\sqrt{\gamma_1\gamma_2}\int_b^c I_s(\rho)\tau(\rho)d\rho\right].$$

The value of the axial force to provide a zero punch displacement is

$$P = 4\cosh(\pi\theta)\left\{\frac{\pi a\theta\alpha}{\sinh(\pi\theta)}\int_b^c \tau(\rho)d\rho - \sqrt{\gamma_1\gamma_2}\int_b^c I_s(\rho)\tau(\rho)d\rho\right\}.$$

4. A normal concentrated load P is applied at the point $(0,0,z)$ underneath a circular punch of radius a, bonded to a transversely isotropic elastic half-space. Find the traction distribution at the punch base and its settlement w.
Answer: as an illustration, the solution is given according to (Fabrikant, 1971a). The stresses are defined by

$$\sigma(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\left\{\int_0^\rho \frac{f_1(t)t\,dt}{(\rho^2 - t^2)^{1/2}} + \int_\rho^a \frac{f_2(t)t\,dt}{(t^2 - \rho^2)^{1/2}}\right\},$$

$$\tau(\rho) = \frac{d}{d\rho}\left\{-\frac{\alpha}{\gamma_1\gamma_2}\int_\rho^a \frac{f_1(t)dt}{(t^2 - \rho^2)^{1/2}} + \frac{1}{\alpha}\int_0^\rho \frac{f_2(t)dt}{(\rho^2 - t^2)^{1/2}}\right\}.$$

Here

$$f_1(t) = \frac{\cosh^2\pi\theta}{\pi^2}\sum_{k=1}^{2}\frac{m_k}{(m_k - 1)\sinh\pi\theta}\left[\frac{tY_c(t)\sinh\xi_k + z_k Y_s(t)\cosh\xi_k}{z_k^2 + t^2}\right],$$

$$f_2(t) = \frac{\cosh\pi\theta}{\pi^2\sqrt{\gamma_1\gamma_2}}\sum_{k=1}^{2}\frac{\gamma_k}{m_k - 1}\left[\frac{z_k Y_c(t)\sinh\xi_k + tY_s(t)\cosh\xi_k}{z_k^2 + t^2}\right],\quad \xi_k = 2\theta\tan^{-1}\left(\frac{a}{z_k}\right).$$

The resultant of the stresses is

$$N = -P\coth\pi\theta\sum_{k=1}^{2}\left[\frac{m_k\sinh\xi_k}{m_k - 1} + \frac{\gamma_k(\cosh\xi_k - 1)}{\sqrt{\gamma_1\gamma_2}(m_k - 1)}\right].$$

When no force is applied to the punch directly, its settlement due to the load P is

$$w = \frac{HP}{2a\theta}\sum_{k=1}^{2}\left[\frac{m_k\sinh\xi_k}{m_k - 1} + \frac{\gamma_k(\cosh\xi_k - 1)}{\sqrt{\gamma_1\gamma_2}(m_k - 1)}\right].$$

Note: verify that in the case $z\to 0$ $N=-P$.

3.3 External axisymmetric mixed-mixed problem

We choose this problem to demonstrate yet another kind of solution, which features two stress functions, introduced in such a way that they decouple the integral equations, so that each equation can be solved independently.

The boundary conditions in the case of axial symmetry are

$$\begin{aligned}
u &= u(\rho), && \text{for } a\le\rho\le\infty, && 0\le\phi<2\pi;\\
w &= w(\rho), && \text{for } a\le\rho\le\infty, && 0\le\phi<2\pi;\\
\sigma &= \sigma(\rho), && \text{for } 0\le\rho\le a, && 0\le\phi<2\pi;\\
\tau &= \tau(\rho), && \text{for } 0\le\rho\le a, && 0\le\phi<2\pi.
\end{aligned} \qquad (3.3.1)$$

The set of governing integral equations will take the form

$$2H\left\{-\pi\alpha\int_\rho^\infty \tau(\rho_0)d\rho_0 + 2\int_\rho^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\sigma(\rho_0)\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}}\right\} = \omega_1(\rho),$$

(3.3.2)

$$\frac{2H}{\rho}\left\{2\gamma_1\gamma_2\rho^2\int_\rho^\infty \frac{\mathrm{d}x}{x^2(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\tau(\rho_0)\rho_0^2\mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}} - \pi\alpha\int_a^\rho \sigma(\rho_0)\rho_0\mathrm{d}\rho_0\right\} = \omega_2(\rho).$$

(3.3.3)

The functions ω_1 and ω_2 are known from the boundary conditions, and are defined by

$$\omega_1(\rho) = w(\rho) - 4H\int_0^a \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}},$$

(3.3.4)

$$\omega_2(\rho) = u(\rho) - 4\gamma_1\gamma_2\frac{H}{\rho}\int_0^a \frac{x^2\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\tau(\rho_0)\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} + 2\pi\frac{H\alpha}{\rho}\int_0^a \sigma(\rho_0)\rho_0\mathrm{d}\rho_0.$$

(3.3.5)

We shall seek the solution of the system (3.3.2) and (3.3.3) in the form

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_\rho^\infty \frac{f_1(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + \int_a^\rho \frac{f_2(t)t\mathrm{d}t}{(\rho^2 - t^2)^{1/2}}\right],$$

$$\tau(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\left[C_1\int_a^\rho \frac{f_1(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} + C_2\int_\rho^\infty \frac{f_2(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}\right] + \frac{C_1 Da}{\rho(\rho^2 - a^2)^{1/2}}.$$

(3.3.6)

Here f_1 and f_2 are the as yet unknown functions, and C_1, C_2, and D are the constants to be determined. Substitution of (3.3.6) in (3.3.2) and (3.3.3) leads to two uncoupled equations which can be solved independently, provided that the constants are defined by

$$C_1 = \alpha/\gamma_1\gamma_2, \qquad C_2 = -1/\alpha. \tag{3.3.7}$$

The equations in question are

$$2H\left\{-\frac{\pi\alpha^2}{\gamma_1\gamma_2}\left[D\sin^{-1}\left(\frac{a}{\rho}\right) - \int_a^\rho \frac{f_1(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}}\right]\right.$$

$$+\ 2\int_{\rho}^{\infty}\frac{(x^2-a^2)^{1/2}}{(x^2-\rho^2)^{1/2}}\,\mathrm{d}x\int_{a}^{\infty}\frac{f_1(t)t\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-x^2)}\Bigg\}\ =\ \omega_1(\rho).$$

$$\frac{2\pi H\alpha}{\rho}\Bigg\{\frac{2\gamma_1\gamma_2}{\pi\alpha^2}\int_{\rho}^{\infty}\frac{\mathrm{d}x}{(x^2-\rho^2)^{1/2}(x^2-a^2)^{1/2}}\int_{a}^{\infty}\frac{\rho^2a^2(t^2-x^2)-x^4(t^2-a^2)}{x^2(t^2-a^2)^{1/2}(t^2-x^2)}\,f_2(t)\mathrm{d}t$$

$$+\int_{a}^{\infty}\left[\frac{t}{(t^2-a^2)^{1/2}}-1\right]f_1(t)\mathrm{d}t\ +\ aD\ -\int_{a}^{\rho}\frac{f_2(t)t\mathrm{d}t}{(\rho^2-t^2)^{1/2}}\Bigg\}\ =\ \omega_2(\rho).$$

(3.3.8)

Let us solve the first equation (3.3.8). Divide both sides by $\rho(\rho^2-r^2)^{1/2}$, integrate with respect to ρ from r to ∞, multiply the result by r, and differentiate with respect to r. The result is

$$\frac{\alpha^2}{\gamma_1\gamma_2}\int_{a}^{\infty}\frac{f_1(t)\mathrm{d}t}{t^2-r^2}\ -\int_{a}^{\infty}\frac{(r^2-a^2)^{1/2}f_1(t)t\mathrm{d}t}{r(t^2-a^2)^{1/2}(t^2-r^2)}\ =\ \psi_1(r).$$

(3.3.9)

Here

$$\psi_1(r)\ =\ \frac{1}{2\pi H}\,\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_{r}^{\infty}\frac{\omega_1(\rho)\mathrm{d}\rho}{\rho(\rho^2-r^2)^{1/2}}\right]\ -\ \frac{\alpha^2}{\gamma_1\gamma_2}\,\frac{D}{2r}\,\ln\!\left(\frac{r+a}{r-a}\right). \qquad (3.3.10)$$

Equation (3.3.9) can be solved in a manner similar to that of (3.2.43). Multiply both sides of (3.3.9) by $Y_{\mathrm{c}}(r)/(r^2-x^2)$ and integrate with respect to r from a to ∞. We use in this section the notation Y_{c} and Y_{s}, as it was defined in (3.2.45). The result of integration is

$$\frac{\pi^2}{4x^2}\left(1-\frac{\alpha^2}{\gamma_1\gamma_2}\right)Y_{\mathrm{c}}(x)f_1(x)\ -\ \frac{\pi a}{2x^2\cosh(\pi\theta)}\int_{a}^{\infty}\frac{f_1(t)\mathrm{d}t}{t(t^2-a^2)^{1/2}}$$

$$- \tanh(\pi\theta)\int_a^\infty \left[\frac{t(x^2 - a^2)^{1/2}Y_s(x)}{x^2(t^2 - a^2)^{1/2}} - \frac{Y_s(t)}{t}\right]\frac{f_1(t)\mathrm{d}t}{t^2 - x^2}$$

$$+ \frac{\pi}{2}\,\frac{\alpha^2}{\gamma_1\gamma_2}\coth(\pi\theta)\int_a^\infty \left[\frac{Y_s(x)}{x} - \frac{Y_s(t)}{t}\right]\frac{f_1(t)\mathrm{d}t}{t^2 - x^2} = \int_a^\infty \frac{\psi_1(r)Y_c(r)\mathrm{d}r}{r^2 - x^2}. \tag{3.3.11}$$

Again, expression (3.3.11) can be simplified significantly, assuming $\tanh(\pi\theta)=\alpha/\sqrt{\gamma_1\gamma_2}$,

$$\frac{\pi^2 Y_c(x)}{4x^2\cosh^2(\pi\theta)}\,f_1(x) + \frac{\pi}{2}\,\tanh(\pi\theta)\frac{Y_s(x)}{x}\int_a^\infty \left[1 - \frac{t(x^2 - a^2)^{1/2}}{x(t^2 - a^2)^{1/2}}\right]\frac{f_1(t)\mathrm{d}t}{t^2 - x^2}$$

$$- \frac{\pi a}{2x^2\cosh(\pi\theta)}\int_a^\infty \frac{f_1(t)\mathrm{d}t}{t(t^2 - a^2)^{1/2}} = \int_a^\infty \frac{\psi_1(r)Y_c(r)\mathrm{d}r}{r^2 - x^2}. \tag{3.3.12}$$

Multiplication of (3.3.9) by $rY_s(r)/(r^2{-}x^2)$ and transformations similar to those above lead to

$$\frac{\pi^2 Y_s(x)}{4x\cosh^2(\pi\theta)}\,f_1(x) - \frac{\pi}{2}\,\tanh(\pi\theta)Y_c(x)\int_a^\infty \left[1 - \frac{t(x^2 - a^2)^{1/2}}{x(t^2 - a^2)^{1/2}}\right]\frac{f_1(t)\mathrm{d}t}{t^2 - x^2}$$

$$= \int_a^\infty \frac{\psi_1(r)Y_s(r)r\mathrm{d}r}{r^2 - x^2}. \tag{3.3.13}$$

Equations (3.3.12) and (3.3.13) give the final solution

$$f_1(x) = \frac{4}{\pi^2}\cosh^2(\pi\theta)x\left[\,xY_c(x)\int_a^\infty \frac{\psi_1(r)Y_c(r)\mathrm{d}r}{r^2 - x^2} + Y_s(x)\int_a^\infty \frac{\psi_1(r)Y_s(r)r\mathrm{d}r}{r^2 - x^2}\right]. \tag{3.3.14}$$

Strictly speaking, we should have added a term $BY_c(x)$, representing the homogeneous solution, where B is an arbitrary constant. It will be shown further, that we may assume B=0, since the constant D, introduced earlier, actually plays the same role.

The second equation in (3.3.8) can be solved in a similar manner. At the first stage, it is reduced to

$$\frac{\gamma_1\gamma_2}{\alpha^2}\left[\int_a^\infty \frac{r(r^2 - a^2)^{1/2}f_2(t)\mathrm{d}t}{(t^2 - a^2)^{1/2}(t^2 - r^2)} + \int_a^\infty \frac{f_2(t)\mathrm{d}t}{(t^2 - a^2)^{1/2}}\right] - \int_a^\infty \frac{f_2(t)t\mathrm{d}t}{t^2 - r^2} = \psi_2(r), \tag{3.3.15}$$

with

$$\psi_2(r) = \frac{1}{2\pi H\alpha}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_r^\infty \frac{\omega_2(\rho)\mathrm{d}\rho}{(\rho^2 - r^2)^{1/2}}\right]. \tag{3.3.16}$$

Its solution is

$$f_2(x) = -\frac{4}{\pi^2}\sinh^2(\pi\theta)\left[xY_c(x\int_a^\infty \frac{\psi_2(r)Y_c(r)\mathrm{d}r}{r^2 - x^2} + Y_s(x)\int_a^\infty \frac{\psi_2(r)Y_s(r)r\mathrm{d}r}{r^2 - x^2}\right]. \tag{3.3.17}$$

One could notice that terms of the form const./ρ were lost during transformation of the second equation (3.3.8), due to differentiation. This means that the solution (3.3.17) satisfies the second equation (3.3.8), except for the abovementioned terms. And here the role of the constant D becomes clear: it has to be chosen so that the equation be satisfied. We show below how this is done.

Example. Let the exterior of a circle $\rho=a$ be clamped, so that $w=u=0$, for $\rho>a$. A uniform pressure σ_0 is applied inside the circle. The stress distribution outside the circle and the displacements inside are to be determined. The general solution described above yields for this particular case

$$\omega_1(\rho) = -4H\sigma_0\int_0^a \frac{(a^2 - x^2)^{1/2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}, \qquad \omega_2(\rho) = \pi H\alpha\sigma_0\frac{a^2}{\rho};$$

$$\psi_1(r) = 2\pi H\sigma_0\left[1 - \frac{(r^2 - a^2)^{1/2}}{r}\right] - \frac{\alpha^2}{\gamma_1\gamma_2}\,\frac{D}{2r}\,\ln\left(\frac{r + a}{r - a}\right), \qquad \psi_2(r) = 0;$$

$$f_1(t) = \frac{2}{\pi}\coth(\pi\theta)\sigma_0[tY_s(t) - 2a\theta Y_c(t)] - D[1 - Y_c(t)], \qquad f_2(t) = 0. \tag{3.3.18}$$

Substitution of (3.3.18) in the second equation (3.3.8) defines the constant D:

$$D = \frac{2}{\pi}a\theta\sigma_0\coth(\pi\theta). \tag{3.3.19}$$

Formulae (3.3.6), (3.3.18) and (3.3.19) give the complete solution for the traction distribution. The displacements inside the circle are given by

$$u(\rho) = 2\pi H\alpha\sigma_0\left\{\frac{2}{\pi}\,\frac{\coth(\pi\theta)}{\rho}\left[\int_a^\infty \frac{tY_s(t) - a\theta[1 + Y_c(t)]}{(t^2 - \rho^2)^{1/2}}\,t\,\mathrm{d}t\right.\right.$$

$$\left.\left. - a\theta(a^2 - \rho^2)^{1/2}\right] - \frac{\rho}{2}\right\},$$

$$w(\rho) = 4H\sigma_0\left\{\int_a^\infty \frac{(t^2 - a^2)^{1/2} - tY_c(t) - a\theta Y_s(t)}{(t^2 - \rho^2)^{1/2}}\,\mathrm{d}t\right.$$

$$\left. + \int_0^\rho \frac{(a^2 - t^2)^{1/2}}{(\rho^2 - t^2)^{1/2}}\,\mathrm{d}t - \frac{\pi}{2}\,a\theta\tanh(\pi\theta)\right\}.$$

The integrals above may be computed in elementary functions at the centre $\rho=0$, namely,

$$u(0) = 0, \qquad w(0) = 4\pi Ha\theta\sigma_0\frac{1 + \cosh(\pi\theta)}{\sinh(2\pi\theta)}.$$

The reader who is interested in numerical results is referred to (Fabrikant, 1972), where the field of tractions and displacements was computed for steel, concrete and sandstone.

The method of solution presented in this section is neither the only method available, nor the simplest one. The reader is encouraged to try various modifications of the approach presented in section 3.2.

Exercise 3.3

1. Consider a transversely isotropic elastic half-space $z\geq 0$. Let the exterior of a circle $\rho=a$ be clamped, so that $u=w=0$, for $\rho>a$. An axisymmetric pressure $\sigma(\rho)$ is applied in the annulus $b\leq\rho\leq c$, with $c\leq a$. Find the traction distribution outside the circle and the displacements inside.
Hint: use solutions presented in Exercise 3.2 as an example.

2. Subject to the conditions of the exercise above, prove that the stresses in the plane $z=0$ are in equilibrium.
Note: this is generally not true in internal problems.

3. Consider a transversely isotropic elastic half-space $z\geq 0$. Let the exterior of a circle $\rho=a$ be clamped, so that $u=w=0$, for $\rho>a$. An axisymmetric tangential traction $\tau(\rho)$ is applied in the annulus $b\leq\rho\leq c$, with $c\leq a$. Find the traction distribution outside the circle and the displacements inside.

4. Consider a transversely isotropic elastic half-space $z\geq 0$. Let the exterior of a circle $\rho=a$ be clamped, so that $u=w=0$, for $\rho>a$. Find the traction distribution outside the circle and the displacements inside due to a concentrated force P applied in the positive Oz direction at the point with cartesian coordinates $x=0$, $y=0$, $z=b$.

3.4 Generalization for a non-homogeneous half-space

Popov (1973) has considered an internal mixed-mixed problem for the case of a non-homogeneous *isotropic* half-space, with elastic modulus $E_\kappa=E_0 z^\kappa$, E_0=const, and $0\leq\kappa<1$. He reduced the problem to a generalized Abel integral equation, which he solved in terms of Jacobi polynomial expansion. The interested reader is referred to the original paper for details. We present here only the *closed form solution* of the generalized Abel integral equation, as it was given in (Fabrikant, 1976).

Consider the integral equation

$$\int_b^x \frac{F(t)\mathrm{d}t}{(x-t)^\kappa} + A\int_x^a \frac{F(t)\mathrm{d}t}{(t-x)^\kappa} = f(x), \qquad \text{for } 0<\kappa<1,\ b\leq x\leq a. \tag{3.4.1}$$

Here A is a known constant, f is a known function, and function F is to be determined. Make use of the following integral representations (Gradshtein and Ryzhik, 1963)

$$\frac{1}{(x-t)^{\kappa}} = \frac{(x-b)^{\delta}(t-b)^{1-\delta-\kappa}}{\mathrm{B}(\kappa,\delta)} \int_b^t \frac{(y-b)^{\kappa-1}(t-y)^{\delta-1}}{(x-y)^{\delta+\kappa}}\,\mathrm{d}y, \tag{3.4.2}$$

$$\frac{1}{(t-x)^{\kappa}} = \frac{(x-b)^{\delta}(t-b)^{1-\delta-\kappa}}{\mathrm{B}(\kappa,1-\delta-\kappa)} \int_b^x \frac{(y-b)^{\kappa-1}(t-y)^{\delta-1}}{(x-y)^{\delta+\kappa}}\,\mathrm{d}y. \tag{3.4.3}$$

Let us define the value of δ from the condition

$$\frac{\mathrm{B}(\kappa,1-\delta-\kappa)}{\mathrm{B}(\kappa,\delta)} = A. \tag{3.4.4}$$

By using the properties of Beta-functions, expression (3.4.4) can be simplified

$$\frac{\sin(\pi\delta)}{\sin[\pi(\delta+\kappa)]} = A.$$

The value of δ can be found from the last expression as

$$\delta = \frac{1}{\pi}\tan^{-1}\left(\frac{A\sin(\pi\kappa)}{1-A\cos(\pi\kappa)}\right) \tag{3.4.5}$$

Substitute (3.4.2) and (3.4.3) in (3.4.1) and interchange the order of integration. The result is

$$\frac{(x-b)^{\delta}}{\mathrm{B}(\kappa,\delta)} \int_b^x \frac{(y-b)^{\kappa-1}\mathrm{d}y}{(x-y)^{\kappa+\delta}} \int_y^a \frac{F(t)\mathrm{d}t}{(t-b)^{\delta+\kappa-1}(t-y)^{1-\delta}} = f(x). \tag{3.4.6}$$

The generalized Abel integral equation is now represented as a sequence of two Abel type operators, and each can be inverted. The solution is

$$F(t) = -\mathrm{B}(\kappa,\delta)\,\frac{\sin(\pi\delta)\,\sin[\pi(\kappa+\delta)]}{\pi^2(t-b)^{1-\delta-\kappa}}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^a \frac{(r-b)^{1-\kappa}\mathrm{d}r}{(r-t)^{\delta}}$$

$$\times\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{f(x)\mathrm{d}x}{(x-b)^{\delta}(r-x)^{1-\delta-\kappa}}\,.$$

(3.4.7)

The form of solution given by (3.4.7) is not the only one which can be derived. Indeed, one can use the integral representations:

$$\frac{1}{(x-t)^{\kappa}} = \frac{(a-t)^{\delta}(a-x)^{1-\delta-\kappa}}{\mathrm{B}(\delta,\kappa)} \int_x^a \frac{(a-y)^{\kappa-1}(y-x)^{\delta-1}}{(y-t)^{\delta+\kappa}}\,\mathrm{d}y, \tag{3.4.8}$$

$$\frac{1}{(t-x)^{\kappa}} = \frac{(a-t)^{\delta}(a-x)^{1-\delta-\kappa}}{\mathrm{B}(1-\delta-\kappa,\kappa)} \int_t^a \frac{(a-y)^{\kappa-1}(y-x)^{\delta-1}}{(y-t)^{\delta+\kappa}}\,\mathrm{d}y. \tag{3.4.9}$$

Substitution of (3.4.8) and (3.4.9) in (3.4.1) leads to

$$\frac{(a-x)^{1-\delta-\kappa}}{\mathrm{B}(\delta,\kappa)} \int_x^a \frac{(a-y)^{\kappa-1}\mathrm{d}y}{(y-x)^{1-\delta}} \int_b^y \frac{(a-t)^{\delta}F(t)\mathrm{d}t}{(y-t)^{\delta+\kappa}} = f(x). \tag{3.4.10}$$

The solution will now take the form

$$F(t) = -\mathrm{B}(\kappa,\delta)\,\frac{\sin(\pi\delta)\ \sin[\pi(\delta+\kappa)]}{\pi^2(a-t)^{\delta}}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t \frac{(a-r)^{1-\kappa}\mathrm{d}r}{(t-r)^{1-\kappa-\delta}}$$

$$\times\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{f(x)\mathrm{d}x}{(a-x)^{1-\delta-\kappa}(x-r)^{\delta}}\ . \tag{3.4.11}$$

We leave it to the reader to establish the equivalence of the solutions (3.4.7) and (3.4.11).

If one compares integral equations (3.2.30) and (3.4.1), the first impression is that they are so different that there is no way to relate them. This is not so. Consider a set of equations

$$-\int_0^a \left[\frac{1}{(r+t)^{\kappa}} + \frac{\mathrm{sign}(r-t)}{|r-t|^{\kappa}}\right] F_1(t)\mathrm{d}t + \cot\!\left(\frac{\pi\kappa}{2}\right) \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \int_0^a \left[\frac{1}{|r-t|^{\kappa}}\right.$$

$$-\frac{1}{(r+t)^{\kappa}}\Bigg]F_2(t)\mathrm{d}t=\frac{2\cos(\pi\kappa/2)\Gamma(1-\kappa)}{\pi^2 H}\int_0^r\frac{\omega_1(\rho)\rho\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}, \tag{3.4.12}$$

$$-\cot\left(\frac{\pi\kappa}{2}\right)\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\int_0^a\left[\frac{1}{|r-t|^{\kappa}}+\frac{1}{(r+t)^{\kappa}}-\frac{2}{t^{\kappa}}\right]F_1(t)\mathrm{d}t$$

$$+\int_0^a\left[\frac{1}{(r+t)^{\kappa}}-\frac{\operatorname{sign}(r-t)}{|r-t|^{\kappa}}-\frac{1}{t^{\kappa}}-\frac{\operatorname{sign}(t)}{t^{\kappa}}\right]F_2(t)\mathrm{d}t$$

$$=\frac{2\cos(\pi\kappa/2)\Gamma(1-\kappa)}{\pi^2 H\sqrt{\gamma_1\gamma_2}}\,r\int_0^r\frac{\omega_2(\rho)\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}\,. \tag{3.4.13}$$

In the limiting case of $\kappa\to 0$, equations (3.4.12) and (3.4.13) transform into

$$-\int_0^r F_1(t)\mathrm{d}t+\frac{\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_0^a F_2(t)\,\ln\left|\frac{r+t}{r-t}\right|\,\mathrm{d}t=\frac{1}{\pi^2 H}\int_0^r\frac{\omega_1(\rho)\rho\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}, \tag{3.4.14}$$

$$-\frac{\alpha}{\pi\sqrt{\gamma_1\gamma_2}}\int_0^a F_1(t)\,\ln\frac{t^2}{|r^2-t^2|}\,\mathrm{d}t-\int_0^r F_2(t)\mathrm{d}t=\frac{1}{\pi^2 H\sqrt{\gamma_1\gamma_2}}\,r\int_0^r\frac{\omega_2(\rho)\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}. \tag{3.4.15}$$

One can easily verify that differentiation of (3.4.14) and (3.4.15) with respect to r leads to (3.2.30). Thus, the connection is established. Introducing the complex function $F=F_1+iF_2$, equations (3.4.12) and (3.4.13) may be unified as follows:

$$-\left(1+\frac{i\alpha}{\sqrt{\gamma_1\gamma_2}}\cot\left(\frac{\pi\kappa}{2}\right)\right)\int_{-a}^{r}\frac{F(t)\mathrm{d}t}{(r-t)^{\kappa}}+\left(1-\frac{i\alpha}{\sqrt{\gamma_1\gamma_2}}\cot\left(\frac{\pi\kappa}{2}\right)\right)\int_r^a\frac{F(t)\mathrm{d}t}{(t-r)^{\kappa}}$$

$$= \frac{2\cos(\pi\kappa/2)\ \Gamma(1-\kappa)}{\pi^2 H\sqrt{\gamma_1\gamma_2}} \int_0^r \frac{\sqrt{\gamma_1\gamma_2}\rho\omega_1(\rho) + ir\omega_2(\rho)}{(r^2-\rho^2)^{1/2}}\,\mathrm{d}\rho$$

$$+ 2i\left[\int_0^a \frac{F_2(t)\mathrm{d}t}{t^\kappa} - \frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\cot(\frac{\pi\kappa}{2})\int_0^a \frac{F_1(t)\mathrm{d}t}{t^\kappa}\right]. \qquad (3.4.16)$$

We obtained the generalized Abel integral equation of the type (3.4.1), with $b=-a$, the solution of which is available in the forms (3.4.7) and (3.4.11), with the parameter δ, defined as

$$\delta = -\frac{\kappa}{2} + i\theta, \qquad (3.4.17)$$

and θ given by (3.2.19). The stress function for a flat circular bonded punch is

$$F(t) = -\frac{\sin(\pi\kappa)\ \cosh(\pi\theta)\ \Gamma(1-\kappa)}{\pi^3\kappa H}\,w_0(a^2-t^2)^{\kappa/2}\left(\frac{a+t}{a-t}\right)^{i\theta}.$$

The last result is in agreement with those of paragraph 3.2. The reader can derive several new modifications of the governing integral equations by using the integral representations

$$\int_0^{min(\rho_0,\rho)} \frac{x^{\kappa+1}\mathrm{d}x}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(\kappa+1)/2}}$$

$$= \frac{\Gamma(\kappa/2)\ \Gamma[(1-\kappa)/2]}{4\sqrt{\pi}}\left[\frac{1}{|\rho-\rho_0|^\kappa} - \frac{1}{(\rho+\rho_0)^\kappa}\right],$$

$$\int_0^{min(\rho_0,\rho)} \frac{x^{\kappa-1}\mathrm{d}x}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(\kappa+1)/2}}$$

$$= \frac{\Gamma(\kappa/2)\ \Gamma[(1-\kappa)/2]}{4\sqrt{\pi}\rho\rho_0}\left[\frac{1}{|\rho-\rho_0|^\kappa} + \frac{1}{(\rho+\rho_0)^\kappa}\right].$$

Some additional representations can be obtained by simple addition, subtraction, integration or differentiation of those above.

Exercise 3.4

1. Find a solution of the generalized Abel equation (3.4.1), with $f(x)=C=\text{const}$.

Answer: $F(t) = \dfrac{C\sin[\pi(\delta+\kappa)]}{\pi(t-b)^{1-\delta-\kappa}(a-t)^{\delta}}$

2. Find a solution of the generalized Abel equation (3.4.1), with $f(x)=Cx$, with $C=\text{const}$.

Answer: $F(t) = \dfrac{C\sin[\pi(\delta+\kappa)][t-b-\delta(a-b)+\kappa b]}{\pi\kappa(t-b)^{1-\delta-\kappa}(a-t)^{\delta}}$

3. Find a solution of the generalized Abel equation (3.4.1), with $f(x)=Cx^2$, with $C=\text{const}$.

Answer: $F(t) = \dfrac{C\sin[\pi(\delta+\kappa)][2(t-b)(t+D)+D^2-\delta(a^2-b^2)+\kappa b^2]}{\pi\kappa(1+\kappa)(t-b)^{1-\delta-\kappa}(a-t)^{\delta}}$.

Here $D=\kappa b-\delta(a-b)$.

3.5 Effect of a shearing force and a tilting moment on a bonded circular punch.

Consider a circular flat punch of radius a, bonded to a transversely isotropic elastic half-space $z\geq 0$. The punch is subjected to a shearing force T, acting in the Ox direction, and a tilting moment M. We may assume, without loss of generality, that the moment is oriented along the Oy axis. We need to find the traction distribution under the punch, and to relate the translational (u_0) and angular (δ) displacements of the punch to the applied loading parameters. The problem is considered in a separate section due to its practical importance.

The problem is characterized by the following boundary conditions on the plane $z=0$:

$$u = u_0, \qquad \text{for } 0\leq\rho\leq a, \qquad 0\leq\phi<2\pi;$$

$$w = -\delta\rho\cos\phi, \qquad \text{for } 0\leq\rho\leq a, \qquad 0\leq\phi<2\pi;$$

$$\sigma = \tau = 0, \qquad \text{for } a\leq\rho<\infty \qquad 0\leq\phi<2\pi. \tag{3.5.1}$$

The governing integral equations, due to (2.5.6), (3.1.2) and (3.1.3), take the form

$$\frac{2G_1}{\rho^2}\int\limits_0^\rho \frac{x^4\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int\limits_x^a \frac{\tau_2(\rho_0)\mathrm{d}\rho_0}{\rho_0(\rho_0^2 - x^2)^{1/2}} - \frac{2\pi H\alpha}{\rho^2}\int\limits_0^\rho \sigma_1(\rho_0)\rho_0^2\mathrm{d}\rho_0$$

$$+ \frac{2G_2}{\rho^2}\int\limits_0^\rho \frac{\rho^2 - 2x^2}{(\rho^2 - x^2)^{1/2}}\,\mathrm{d}x \int\limits_x^a \frac{\bar{\tau}_0(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} = 0,$$

$$2G_2\int\limits_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int\limits_x^a \frac{(\rho_0^2 - 2x^2)\ \bar{\tau}_2(\rho_0)\mathrm{d}\rho_0}{\rho_0(\rho_0^2 - x^2)^{1/2}} + 2\pi H\alpha\int\limits_\rho^a \sigma_{-1}(\rho_0)\mathrm{d}\rho_0$$

$$+ 2G_1\int\limits_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int\limits_x^a \frac{\tau_0(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} = u_0,$$

$$2\pi H\alpha\ \Re\left\{\frac{\mathrm{e}^{-i\phi}}{\rho}\int\limits_0^\rho \tau_0(\rho_0)\rho_0\mathrm{d}\rho_0 - \rho\mathrm{e}^{i\phi}\int\limits_\rho^a \tau_2(\rho_0)\frac{\mathrm{d}\rho_0}{\rho_0}\right.$$

$$+ \frac{4H}{\rho}\int\limits_0^\rho \frac{x^2\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int\limits_x^a \frac{\sigma_1(\rho_0)\mathrm{e}^{i\phi} + \sigma_{-1}(\rho_0)\mathrm{e}^{-i\phi}}{(\rho_0^2 - x^2)^{1/2}}\,\mathrm{d}\rho_0 = -\delta\rho\cos\phi.$$

(3.5.2)

The structure of equations (3.5.2) is such that we may assume that $\sigma_1 = \bar{\sigma}_{-1}$. The solution may be represented in the form

$$\sigma_1(\rho) = \bar{\sigma}_{-1}(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\int\limits_0^\rho \frac{\bar{f}(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}},$$

$$\tau_0(\rho) = -\frac{C}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int\limits_\rho^a \frac{f(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + \frac{D}{(a^2 - \rho^2)^{1/2}},$$

$$\tau_2(\rho) = -C\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\frac{1}{\rho^2}\int_\rho^a \frac{\bar{f}(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}\right\} - \overline{D}\,\frac{2a^2 - \rho^2}{\rho^2(a^2 - \rho^2)^{1/2}}. \tag{3.5.3}$$

Here f is the as yet unknown stress function, and C and D are the constants to be determined. Substitution of (3.5.3) in the first two equations of (3.5.2) make them identities, provided that the following conditions are satisfied

$$C = \frac{\alpha}{\gamma_1\gamma_2}, \qquad D = \frac{C}{a}\int_0^a f(t)\mathrm{d}t, \tag{3.5.4}$$

$$\frac{\pi^2 D}{2}(G_1 + G_2) + 2\pi H\alpha\int_0^a \frac{f(t)\mathrm{d}t}{(a^2 - t^2)^{1/2}} = u_0. \tag{3.5.5}$$

Expressions (3.5.4) and (3.5.5) look contradictory: we have only two constants to satisfy three equations. This will be clarified further. An additional constant, representing the homogeneous solution, will appear in the expression for f.

Substitution of (3.5.3) in the third equation (3.5.2) leads to

$$\frac{2\pi H\alpha}{\rho}\left\{-2C\int_\rho^a \frac{f(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + C\int_0^a f(t)\mathrm{d}t + aD\right\}$$

$$+ \frac{8H}{\rho}\int_0^\rho \frac{x^2(a^2 - x^2)^{1/2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_0^a \frac{f(t)\mathrm{d}t}{(t^2 - x^2)(a^2 - t^2)^{1/2}} = -\delta\rho.$$

Multiply the last expression by $\rho^2(r^2-\rho^2)^{-1/2}$, and integrate with respect to ρ from 0 to r. The result is

$$(a^2 - r^2)^{1/2}\int_0^a \frac{f(t)\mathrm{d}t}{(a^2 - t^2)^{1/2}(t^2 - r^2)} - \frac{\alpha^2}{\gamma_1\gamma_2}\int_0^a \frac{f(t)\mathrm{d}t}{t^2 - r^2} = -\frac{\delta}{\pi H}. \tag{3.5.6}$$

Equation (3.5.6) is similar to (3.2.60), with the general solution given by

(3.2.62). The solution in this particular case is

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\Big[tY_s(t) - \theta a Y_c(t)\Big] + AY_c(t). \tag{3.5.7}$$

The last term in (3.5.7) represents the homogeneous solution, with A being an arbitrary constant. Substitution of (3.5.7) in (3.5.4) and (3.5.5) allows us to define all the constants, namely,

$$D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}\,A,$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1 + G_2)}{\tanh(\pi\theta)(G_1 - G_2)}\right)\right]^{-1}. \tag{3.5.8}$$

Formulae (3.5.7), (3.5.8) and (3.5.3) determine completely the traction distribution under the punch. Now we need to relate the applied loading to the punch displacements. Make use of the equilibrium conditions:

$$T = 2\pi\int_0^a \tau_0(\rho)\rho d\rho, \qquad M = -\int_0^{2\pi}\int_0^a [\sigma_1(\rho)e^{i\phi} + \sigma_{-1}(\rho)e^{-i\phi}]\rho^2\cos\phi\ d\rho d\phi.$$

After carrying out the calculations, we obtain

$$T = 4\pi^2 A\,\frac{a\theta}{\sinh(\pi\theta)}\,\frac{\alpha}{\gamma_1\gamma_2}, \qquad M = \frac{4\delta a^3\theta(1 + \theta^2)}{3H\tanh(\pi\theta)} + \frac{4\pi^2 a^2\theta^2}{\cosh(\pi\theta)}\,A. \tag{3.5.9}$$

Expressions (3.5.8) and (3.5.9) enable us to determine the displacements of the punch

$$u_0 = \frac{1}{8a}\left[\pi(G_1 + G_2) + \frac{(1 + 4\theta^2)\tanh(\pi\theta)}{\theta(1 + \theta^2)}(G_1 - G_2)\right]T - \frac{3H\alpha}{4a^2(1 + \theta^2)}\,M,$$

$$\delta = \frac{3H\alpha}{4a^2(1 + \theta^2)}\left[-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right]. \tag{3.5.10}$$

In the case of isotropy, formulae (3.5.10) are in agreement (except for some signs) with the results of Ufliand (1967), who seems to have used a different sign convention. It is noteworthy that the tilting moment produces translational displacement of the punch, even in the absence of the shearing force. The shearing force, in turn, tilts the punch, even when no tilting moment is applied. We shall see further (section 5.11) that a similar situation takes place in the case of a finite friction between the punch and the elastic half-space.

Exercise 3.5

1. A flat circular punch of radius a is bonded to a transversely isotropic elastic half-space $z \geq 0$. A shearing force T is applied in the Oy direction. Find the tilting angle δ.

Answer: $\delta = \dfrac{3H\alpha}{4a^2(1 + \theta^2)}T$, with tilting about Ox axis. Note that the angle is positive.

2. Subject to the conditions above, find the tilting moment needed to prevent tilting.

Answer: $M = -a\theta\sqrt{\gamma_1\gamma_2}\,T$.

3. A tilting moment M about the axis Ox is applied to a flat circular punch, bonded to a transversely isotropic elastic half-space. Find the translational displacement u and its direction.

Answer: $u = \dfrac{3H\alpha}{4a^2(1 + \theta^2)}M$, in the Oy direction.

4. A flat circular bonded punch is subjected to a shearing force T, acting in the Ox direction, and a tilting moment M about the Oy axis. Find the normal displacements outside the punch.

Answer:
$$w(\rho,\phi) = \left\{ -\frac{4\delta\cosh(\pi\theta)}{\pi\rho}\int_0^a \frac{x^2 Y_c(x) + \theta a x Y_s(x)}{(\rho^2 - x^2)^{1/2}}\,dx \right.$$
$$\left. + 4\pi H A\tanh(\pi\theta)\frac{1}{\rho}\left[\frac{\pi\theta a}{\cosh(\pi\theta)} - \int_0^a \frac{xY_s(x)\,dx}{(\rho^2 - x^2)^{1/2}}\right]\right\}\cos\phi.$$

5. Express the answer above in terms of the shearing force T and tilting moment M.

Answer:
$$w(\rho,\phi) = \left\{ -\frac{3H\alpha\cosh(\pi\theta)}{\pi\rho a^2(1 + \theta^2)}\left[\frac{M}{\theta a\sqrt{\gamma_1\gamma_2}} - T\right]\int_0^a \frac{x^2 Y_c(x) + \theta a x Y_s(x)}{(\rho^2 - x^2)^{1/2}}\,dx \right.$$

$$+\ \frac{H\alpha}{\rho}T\left[1-\frac{\cosh(\pi\theta)}{\pi\theta a}\int_0^a\frac{xY_s(x)\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\right]\Bigg\}\cos\phi.$$

6. Express the normal displacement w outside the punch in terms of the stress function f.

Answer: $$w(\rho,\phi)=\left\{\frac{8H}{\rho}\int_0^a\frac{x^2\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_0^a\frac{(a^2-x^2)^{1/2}f(t)\mathrm{d}t}{(a^2-t^2)^{1/2}(t^2-x^2)}\right.$$

$$\left.+\ 4\pi H\alpha D\frac{a}{\rho}\right\}\cos\phi.$$

7. Express the tangential displacement u outside the punch in terms of the stress function f.

Answer: $u(\rho,\phi)=u_0(\rho)+u_2(\rho)\mathrm{e}^{2i\phi}$,

where

$$u_0(\rho)=\pi(G_1-G_2)\frac{\alpha}{\gamma_1\gamma_2}\int_0^a\frac{f(x)\mathrm{d}x}{(\rho^2-x^2)^{1/2}}+\pi(G_1+G_2)D\sin^{-1}\left(\frac{a}{\rho}\right),$$

$$u_2(\rho)=\frac{1}{\rho^2}\left\{2\pi H\alpha\int_0^a f(x)\mathrm{d}\{x[(a^2-x^2)^{1/2}-(\rho^2-x^2)^{1/2}]\}\right.$$

$$\left.+\ \pi(G_1+G_2)Da(\rho^2-a^2)^{1/2}\right\}.$$

8. Investigate the interaction of an arbitrary concentrated force, applied at some point inside the transversely isotropic half-space, and a flat bonded circular punch.

3.6 Non-axisymmetric internal mixed-mixed problem

The general formulation of this problem is given in section 3.1, with the boundary conditions (3.1.1), and the governing integral equations (3.1.2–3.1.5). The *closed form* exact solution is not known at the moment. We assume that all the parameters involved can be expanded in a Fourier series. An exact solution for the n-th harmonic is presented below. The governing integral equations for the n-th harmonic are

$$\frac{2}{\rho^{n+1}}\int_0^{\rho}\frac{x^{2n}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_x^a\frac{G_1x^2\tau_{n+1}(\rho_0)+G_2[2n\rho^2-(2n+1)x^2]\bar{\tau}_{-n+1}(\rho_0)}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0$$

$$-\frac{2\pi H\alpha}{\rho^{n+1}}\int_0^{\rho}\sigma_n(\rho_0)\rho_0^{n+1}\mathrm{d}\rho_0 = F_{n+1}(\rho),\ \text{ for }\ n\geq 0.$$

$$\frac{2}{\rho^{n-1}}\int_0^{\rho}\frac{x^{2n-2}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_x^a\frac{G_1\rho_0^2\tau_{-n+1}(\rho_0)+G_2[(2n-1)\rho_0^2-2nx^2]\bar{\tau}_{n+1}(\rho_0)}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0$$

$$+\ 2\pi H\alpha\rho^{n-1}\int_{\rho}^a\frac{\sigma_{-n}(\rho_0)\mathrm{d}\rho_0}{\rho_0^{n-1}} = F_{-n+1}(\rho),\qquad \text{for }\ n\geq 1;$$

$$2\pi H\alpha\ \Re\left\{\frac{\mathrm{e}^{-in\phi}}{\rho^n}\int_0^{\rho}\tau_{-n+1}(\rho_0)\rho_0^n\mathrm{d}\rho_0 - \rho^n\mathrm{e}^{in\phi}\int_{\rho}^a\frac{\tau_{n+1}(\rho_0)\mathrm{d}\rho_0}{\rho_0^n}\right\}$$

$$+\ \frac{4H}{\rho^n}\int_0^{\rho}\frac{x^{2n}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_x^a\frac{\sigma_{-n}(\rho_0)\mathrm{e}^{-in\phi}+\sigma_n(\rho_0)\mathrm{e}^{in\phi}}{\rho_0^{n-1}(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0 = \Re\{\mathrm{e}^{in\phi}\Phi_n(\rho)\},$$

$$\text{for }\ n\geq 0. \tag{3.6.1}$$

Here the right hand sides are known from the boundary conditions, and are

$$F_{n+1}(\rho) = u_{n+1}(\rho) - 2\rho^{n+1}\int_a^{\infty}\frac{\mathrm{d}x}{x^{2n+2}(x^2-\rho^2)^{1/2}}$$

$$\times \int_a^x \frac{G_1\rho_0^2\tau_{n+1}(\rho_0) + G_2[2nx^2 - (2n+1)\rho_0^2]\bar{\tau}_{-n+1}(\rho_0)}{(x^2 - \rho_0^2)^{1/2}}\,\rho_0^n d\rho_0,$$

$$F_{-n+1}(\rho) = u_{-n+1}(\rho) - 2\pi H\alpha\rho^{n-1}\int_a^\infty \frac{\sigma_{-n}(\rho_0)d\rho_0}{\rho_0^{n-1}}$$

$$-\ 2\rho^{n-1}\int_a^\infty \frac{dx}{x^{2n}(x^2 - \rho^2)^{1/2}}\int_a^x \frac{G_1x^2\tau_{-n+1}(\rho_0) + G_2[(2n-1)x^2-2n\rho^2]\bar{\tau}_{n+1}(\rho_0)}{(x^2 - \rho_0^2)^{1/2}}\,\rho_0^n d\rho_0,$$

$$\Re\{\Phi_n(\rho)e^{in\phi}\} = w_n(\rho)e^{in\phi} + w_{-n}(\rho)e^{-in\phi} + 2\pi H\alpha\ \Re\left\{e^{in\phi}\rho^n\int_a^\infty \frac{\tau_{n+1}(\rho_0)d\rho_0}{\rho_0^n}\right\}$$

$$-\ 4H\rho^n\int_a^\infty \frac{dx}{x^{2n}(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\sigma_n(\rho_0)e^{in\phi} + \sigma_{-n}(\rho_0)e^{-in\phi}}{(x^2 - \rho_0^2)^{1/2}}\,\rho_0^{n+1}d\rho_0. \tag{3.6.2}$$

The case of axial symmetry (n=0) was considered in detail in paragraph 3.2, and is not discussed here. The solution is sought for $n\geq1$. We may assume, without loss of generality, that the first two equations (3.6.1) are homogeneous. This can be achieved by addition of some special solutions to the parameters τ_{-n+1} and τ_{n+1}. These special solutions, satisfying the right hand sides of the first two equations (3.6.1), can be obtained from the results of section 2.6. Of course, this procedure will make the right hand side of the third equation (3.6.1) more complicated.

Assume the solution of the set (3.6.1), with the first two equations transformed into homogeneous, in the form

$$\sigma_n(\rho) = \bar{\sigma}_{-n}(\rho) = \frac{1}{\rho^n}\int_0^\rho \frac{t^{2n-1}df_n(t)}{(\rho^2 - t^2)^{1/2}}$$

$$= -\frac{1}{\rho^{n+1}}\frac{d}{d\rho}\int_0^\rho \frac{t^{2n-2}[(2n-1)\rho^2 - 2nt^2]}{(\rho^2 - t^2)^{1/2}} f_n(t)dt;$$

$$\tau_{-n+1}(\rho) = -C\rho^{n-2}\frac{d}{d\rho}\int_\rho^a \frac{\bar{f}_n(t)tdt}{(t^2-\rho^2)^{1/2}} + \frac{\overline{D}_n\rho^{n-1}}{(a^2-\rho^2)^{1/2}};$$

$$\tau_{n+1}(\rho) = C\rho^n\frac{d}{d\rho}\left\{\frac{1}{\rho^{2n}}\int_\rho^a y^{2n-2}dy\frac{d}{dy}\int_y^a \frac{f_n(t)tdt}{(t^2-y^2)^{1/2}}\right\}$$

$$+ D_n\rho^n\frac{d}{d\rho}\left\{\frac{1}{\rho^{2n}}\int_\rho^a \frac{t^{2n-1}dt}{(a^2-t^2)^{1/2}}\right\}$$

$$= -C\rho^n\frac{d}{d\rho}\left\{\int_\rho^a \frac{f_n(t)dt}{t(t^2-\rho^2)^{1/2}} + (2n-1)\int_\rho^a \frac{dy}{y^{2n}(y^2-\rho^2)^{1/2}}\int_y^a f_n(t)t^{2n-2}dt\right\}$$

$$- D_n\left[\frac{\rho^{n-1}}{(a^2-\rho^2)^{1/2}} + \frac{2n}{\rho^{n+1}}\int_\rho^a \frac{t^{2n-1}dt}{(a^2-t^2)^{1/2}}\right]. \tag{3.6.3}$$

Here f_n is the as yet unknown complex stress function, and C, D_n are the constants to be determined. In the following derivations we give in some cases two equivalent expressions of the same parameter, for the sake of convenience in the procedure of substitution into the governing equations (3.6.1). We present first some intermediate results related to the substitution of (3.6.3) in the first equation (3.6.1):

$$\int_x^a \frac{\bar{\tau}_{-n+1}(\rho_0)d\rho_0}{\rho_0^n(\rho_0^2 - x^2)^{1/2}} = \frac{\pi C}{2x}\left[\frac{f_n(x)}{x} - \int_x^a \frac{f_n(t)dt}{t^2}\right] + \frac{\pi}{2ax}D_n, \tag{3.6.4}$$

$$\int_x^a \frac{\tau_{n+1}(\rho_0)d\rho_0}{\rho_0^n(\rho_0^2 - x^2)^{1/2}} = \frac{\pi C}{2x}\left[\frac{f_n(x)}{x} + \frac{2n - 1}{x^{2n}}\int_x^a f_n(t)t^{2n-2}dt\right] - \frac{\pi a^{2n-1}}{2x^{2n+1}} D_n. \tag{3.6.5}$$

Substitution of (3.6.4) and (3.6.5) in the first equation (3.6.1) yields

$$(G_1-G_2)\frac{\pi C}{\rho^{n+1}}\int_0^\rho \frac{x^{2n-2}[2nx^2 - (2n-1)\rho^2]}{(\rho^2 - x^2)^{1/2}} f_n(x)dx - \frac{2\pi H\alpha}{\rho^{n+1}}\int_0^\rho \sigma_n(\rho_0)\rho_0^{n+1}d\rho_0 = 0, \tag{3.6.6}$$

provided that the following condition holds

$$D_n = \frac{(2n - 1)C}{a^{2n-1}}\int_0^a f_n(t)t^{2n-2}dt. \tag{3.6.7}$$

It is now easy to verify that substitution of the first expression (3.6.3) in (3.6.6) makes it an identity if

$$C = \frac{\alpha}{\gamma_1\gamma_2}\ . \tag{3.6.8}$$

Here are some intermediary results related to the procedure of substitution of (3.6.3) in the second equation (3.6.1):

$$\int_x^a \frac{\tau_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}} = \frac{\pi}{2}\left[C\bar{f}_n(x) + \bar{D}_n\right], \tag{3.6.9}$$

$$\int_x^a \frac{(2n - 1)\rho_0^2 - 2nx^2}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}\,\bar{\tau}_{n+1}(\rho_0)d\rho_0 = \frac{\pi}{2}\left[-C\bar{f}_n(x) + \bar{D}_n\right]. \tag{3.6.10}$$

We used in transformations some general formulae from Appendix A3.3. Substitution of (3.6.9) and (3.6.10) in the second equation (3.6.1) reduces it to

$$\frac{\pi}{\rho^{n-1}}\int_0^\rho \frac{[C(G_1-G_2)\bar{f}_n(x) + (G_1+G_2)\bar{D}_n]x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} + 2\pi H\alpha\rho^{n-1}\int_\rho^a \frac{\sigma_{-n}(\rho_0)d\rho_0}{\rho_0^{n-1}} = 0.$$

(3.6.11)

Substitution of the first expression (3.6.3) in (3.6.11) makes it an identity, subject to (3.6.8), and an additional condition

$$(G_1 + G_2)\frac{\sqrt{\pi}\,\Gamma(n - 1/2)}{2\Gamma(n)}\,D_n + \frac{2H\alpha}{a^{2n-2}}\int_0^a f_n(x)\,\frac{x^{2n-2}\mathrm{d}x}{(a^2 - x^2)^{1/2}} = 0. \tag{3.6.12}$$

The conditions (3.6.7) and (3.6.12) might look contradictory. It will be shown further (see 3.6.14) that this is not so, because an additional constant will appear in the expression for f_n, corresponding to a homogeneous solution of the singular integral equation (3.6.14).

By now we have satisfied the first two equations (3.6.1). Substitution of (3.6.3) in the third equation (3.6.1) requires the following transformation:

$$\int_x^a \frac{\mathrm{d}\rho_0}{\rho_0^{2n-1}(\rho_0^2 - x^2)^{1/2}}\int_0^{\rho_0}\frac{t^{2n-1}\mathrm{d}f_n(t)}{(\rho_0^2 - t^2)^{1/2}} = \int_0^a \Bigg[(a^2 - x^2)^{1/2}(a^2 - t^2)^{1/2}Q_n(x,t)$$

$$+ \psi_n(x,t)\,\ln\frac{a|t^2 - x^2|^{1/2}}{|t(a^2 - x^2)^{1/2} - x(a^2 - t^2)^{1/2}|}\Bigg]\,t^{2n-1}\mathrm{d}f_n(t). \tag{3.6.13}$$

Here $Q_n(x,t)$ is a polynomial in even negative powers of x and t. Although we cannot write the explicit expression for $Q_n(x,t)$, it can be computed in elementary manner for any particular n. The explicit expression for ψ_n is

$$\psi_n(x,t) = \frac{1}{\pi t x^{2n-1}}\sum_{k=0}^{n-1}\frac{\Gamma(n-k-1/2)\;\Gamma(k+1/2)}{\Gamma(n-k)\;\Gamma(k+1)}\left(\frac{x}{t}\right)^{2k}.$$

Multiply both sides of the third equation (3.6.1) by ρ^n, differentiate with respect to ρ, divide the result by $\rho^{2n-2}(r^2-\rho^2)^{1/2}$ and integrate with respect to ρ from 0 to r. This procedure allows us to split the kernel of the integral equation into two parts: a singular one and a degenerate one. The result takes the form

$$(a^2 - r^2)^{1/2}\int_0^a \frac{f_n(t)\mathrm{d}t}{(t^2 - r^2)(a^2 - t^2)^{1/2}} - \frac{\alpha^2}{\gamma_1\gamma_2}\int_0^a \frac{f_n(t)\mathrm{d}t}{t^2 - \rho^2} = \chi_n(r). \tag{3.6.14}$$

Here

$$\chi_n(r) = \frac{1}{4\pi H r}\int_0^r \frac{d\rho}{\rho^{2n-2}(r^2-\rho^2)^{1/2}}\frac{d}{d\rho}[\rho^n\Phi_n(\rho)]$$

$$-\frac{2}{\pi r}\int_0^r \frac{d\rho}{\rho^{2n-2}(r^2-\rho^2)^{1/2}}\frac{d}{d\rho}\int_0^\rho \frac{x^{2n}(a^2-x^2)^{1/2}dx}{(\rho^2-x^2)^{1/2}}\int_0^a (a^2-t^2)^{1/2}Q_n(x,t)t^{2n-1}df_n(t)$$

$$-\frac{2}{\pi^{3/2}r}\int_0^r \frac{d\rho}{\rho^{2n-2}(r^2-\rho^2)^{1/2}}\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}(a^2-x^2)^{1/2}}\int_0^a q_n(\rho,x,t)(a^2-t^2)^{1/2}t^{2n-2}df_n(t),$$

(3.6.15)

with

$$q_n(\rho,x,t) = \sum_{k=0}^{n-2}\frac{\Gamma(n-k-1/2)}{\Gamma(n-k)}\left(\frac{\rho}{t}\right)^{2k}F(2-n+k,\ \frac{1}{2};\ \frac{3}{2}-n+k;\ \frac{x^2}{t^2}). \qquad (3.6.16)$$

Note that the hypergeometric function in (3.6.16) is, in fact, a polynomial, and that all the integrals with respect to x and ρ in the degenerate part of the kernel (3.6.15) are computable in elementary function for any n. The integral equation (3.6.14) was solved in paragraph 3.2, and the solution is

$$f_n(t) = -\frac{4}{\pi^2}\cosh^2(\pi\theta)t\left[tY_c(t)\int_0^a \frac{\chi_n(r)Y_c(r)dr}{r^2-t^2}\right.$$

$$\left.+\ Y_s(t)\int_0^a \frac{\chi_n(r)Y_s(r)rdr}{r^2-t^2}\right] + A_nY_c(t). \qquad (3.6.17)$$

The last term in (3.6.17) represents the homogeneous solution, with A_n being an arbitrary constant. Its value, along with the constant D_n and others which appear due to the degenerate part of the kernel, can be found from the appropriate system of linear algebraic equations and the conditions (3.6.7) and (3.6.12). The general solution may be considered completed. The main handicap of the solution is the necessity for solving a set of linear algebraic equations whose order increases with n, thus making the exact solution for higher harmonics very cumbersome. We are not aware of any other solution for a *transversely* *isotropic* solid. The corresponding problem for an *isotropic* body was solved by Ufliand (1967) who used the method of Mehler-Fok integral transform. It has the same hindrance: one needs to solve a set of linear algebraic equations, whose order increases with n.

Example. Consider the action of a normal concentrated load P applied outside a flat circular punch of radius a, bonded to a transversely isotropic elastic half-space $z\geq 0$. We may assume, without loss of generality, that the force is applied at the point $\rho=b$, $\phi=0$ ($b>a$). Thus, the boundary conditions are

$$u = w = 0, \qquad \text{for } \rho\leq a;$$

$$\sigma = P\delta(\rho-b)\delta(\phi-0)/\rho, \qquad \tau = 0, \qquad \text{for } \rho>a.$$

The boundary conditions yield

$$F_{n+1}(\rho) = 0, \quad \text{for } n\geq 0; \qquad F_{-n+1} = -PH\alpha\,\frac{\rho^{n-1}}{b^{n}}, \qquad \text{for } n\geq 1;$$

$$\Phi_n(\rho) = -\frac{4PH}{\pi(\rho b)^n}\int_0^{\rho}\frac{x^{2n}\mathrm{d}x}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\,. \tag{3.6.18}$$

We present now the explicit solutions for some specific values of n. In the axisymmetric case $n=0$, the following results may be obtained:

$$\sigma_0(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^{\rho}\frac{f_0(t)t\mathrm{d}t}{(\rho^2-t^2)^{1/2}}, \qquad \tau_1(\rho) = -\frac{\alpha}{\gamma_1\gamma_2}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{a}\frac{f_0(t)\mathrm{d}t}{(t^2-\rho^2)^{1/2}},$$

$$f_0(t) = \frac{2}{\pi^3}P\cosh^2(\pi\theta)\left[tY_c(t)\int_0^a\frac{Y_c(r)\mathrm{d}r}{(b^2-r^2)^{1/2}(r^2-t^2)}\right.$$

$$\left.+\,Y_s(t)\int_0^a\frac{Y_s(r)r\mathrm{d}r}{(b^2-r^2)^{1/2}(r^2-t^2)}\right]. \tag{3.6.19}$$

The results for $n=1$ are:

$$\sigma_1(\rho) = \frac{d}{d\rho}\int_0^\rho \frac{f_1(t)dt}{(\rho^2 - t^2)^{1/2}},$$

$$\tau_0(\rho) = -\frac{\alpha}{\gamma_1\gamma_2}\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{f_1(t)tdt}{(t^2 - \rho^2)^{1/2}} + \frac{D_1}{(a^2 - \rho^2)^{1/2}},$$

$$\tau_2(\rho) = -\frac{\alpha}{\gamma_1\gamma_2}\rho\frac{d}{d\rho}\left[\frac{1}{\rho^2}\int_\rho^a \frac{f_1(t)tdt}{(t^2 - \rho^2)^{1/2}}\right] - D_1\frac{2a^2 - \rho^2}{\rho^2(a^2 - \rho^2)^{1/2}},$$

$$f_1(t) = \frac{t}{b}f_0(t) + A_1Y_c(t),$$

$$D_1 = -\frac{P\theta}{\pi b\sqrt{\gamma_1\gamma_2}}\left(1 - \frac{\cosh(\pi\theta)}{\pi a\theta}I_s(b)\right)\left[1 + \frac{\pi\theta(G_1 + G_2)}{\tanh(\pi\theta)(G_1 - G_2)}\right]^{-1},$$

$$A_1 = -\frac{P\cosh^2(\pi\theta)}{\pi^3\theta ab}[I_s(b) - 2a\theta I_c(b)] + \frac{\cosh(\pi\theta)}{\pi\theta}D_1\sqrt{\gamma_1\gamma_2}. \tag{3.6.20}$$

We recall that the notations $I_{c,s}$ are defined by (3.2.73) and (3.2.74) respectively.

The case of n=2 is more cumbersome:

$$\sigma_2(\rho) = \frac{1}{\rho^3}\frac{d}{d\rho}\int_0^\rho \frac{4t^2 - 3\rho^2}{(\rho^2 - t^2)^{1/2}}t^2f_2(t)dt),$$

$$\tau_{-1}(\rho) = \frac{d}{d\rho}\left[-\frac{\alpha}{\gamma_1\gamma_2}\int_\rho^a \frac{f_2(t)tdt}{(t^2 - \rho^2)^{1/2}} - D_2(a^2 - \rho^2)^{1/2}\right],$$

$$\tau_3(\rho) = \rho^2\frac{d}{d\rho}\left\{\frac{1}{\rho^4}\left[\frac{\alpha}{\gamma_1\gamma_2}\int_\rho^a \frac{\rho^2 - 2t^2}{(t^2 - \rho^2)^{1/2}}f_2(t)tdt + D_2\frac{2a^2 + \rho^2}{3}(a^2 - \rho^2)^{1/2}\right]\right\},$$

$$f_2(t) = -\frac{4}{\pi^2}\cosh^2(\pi\theta)\; t\left[tY_c(t)\int_0^a \frac{\chi_2(r)Y_c(r)\,\mathrm{d}r}{r^2 - t^2} + Y_s(t)\int_0^a \frac{\chi_2(r)Y_s(r)r\,\mathrm{d}r}{r^2 - t^2}\right]$$

$$+ A_2Y_c(t) + \frac{2\coth(\pi\theta)}{\pi a}\left[\frac{2\theta}{a}\,Y_c(t) - \frac{1}{t}\,Y_s(t)\right]B_2.$$

The constant B_2 corresponds to the degenerate part of the kernel:

$$B_2 = \int_0^a \frac{2t^2 - a^2}{(a^2 - t^2)^{1/2}}\,f_2(t)\,\mathrm{d}t.$$

All the constants are determined as follows:

$$D_2 = \frac{3\alpha}{\gamma_1\gamma_2 a^3}\left[L_1 + A_2\frac{\pi a^3\theta}{\sinh(\pi\theta)}\,\frac{1 - 2\theta^2}{3} - B_2\frac{2a\theta^2(1 + 4\theta^2)\cosh(\pi\theta)}{3\sinh^2(\pi\theta)}\right],$$

$$A_2 = \frac{\cosh(\pi\theta)}{2\pi a^2\theta^2}\left[L_3 - B_2\frac{4\theta(1 + 2\theta^2)}{\sinh(\pi\theta)}\right],$$

$$B_2 = \left\{2\theta^2\left(\frac{Pa^2}{2\pi b^2} + L_2 + L_1\frac{3\pi(G_1 + G_2)}{4a(G_1 - G_2)}\right) + L_3\left[\frac{1}{4} - \theta^2\right.\right.$$

$$\left.\left. + \frac{(G_1 + G_2)}{4(G_1 - G_2)}(1 - 2\theta^2)\right]\right\}\left\{\frac{\theta}{\sinh(\pi\theta)}\left[1 + \frac{\pi\theta(1 + \theta^2)(G_1 + G_2)}{\tanh(\pi\theta)(G_1 - G_2)}\right]\right\}^{-1},$$

$$L_1 = \frac{2\cosh^2(\pi\theta)}{\pi\sinh(\pi\theta)}\int_0^a\left[2a\theta\left(r^2 + a^2\frac{1 - 2\theta^2}{3}\right)Y_c(r)\right.$$

$$\left. + r(2a^2\theta^2 - r^2)Y_s(r)\right]\chi_2(r)\,\mathrm{d}r,$$

$$L_2 = \frac{4}{\pi}\cosh(\pi\theta)\int_0^a\left\{\left[a^2\left(\frac{1}{4} - \theta^2\right) + \frac{r^2}{2}\right]Y_c(r) + \theta a r Y_s(r)\right\}\chi_2(r)\mathrm{d}r,$$

$$L_3 = \frac{4}{\pi}\cosh(\pi\theta)\int_0^a[(r^2 - 2a^2\theta^2)Y_c(r) + 2\theta a r Y_s(r)]\chi_2(r)\mathrm{d}r,$$

$$\chi_2(r) = -\frac{P}{2\pi b^2}\left[\frac{1}{(b^2 - r^2)^{1/2}} + \frac{1}{b + (b^2 - r^2)^{1/2}}\right].$$

In order to provide zero displacements, the punch should be loaded by a normal force N, a shearing force T, directed in the negative Ox direction, and a tilting moment M about the Oy axis. Their relationship with the force P can be established by the statics equations, namely,

$$T = 2\pi\int_0^a\tau_0(\rho)\rho\mathrm{d}\rho = 2\pi\left[\frac{\alpha}{\gamma_1\gamma_2}\int_0^a f_1(t)\mathrm{d}t + D_1 a\right],$$

$$N = 2\pi\int_0^a\sigma_0(\rho)\rho\mathrm{d}\rho = 2\pi\int_0^a\frac{f_0(t)t\mathrm{d}t}{(a^2 - t^2)^{1/2}},$$

$$M = -\int_0^{2\pi}\int_0^a[\sigma_1(\rho)\mathrm{e}^{i\phi} + \sigma_{-1}(\rho)\mathrm{e}^{-i\phi}]\rho^2\cos\phi\mathrm{d}\rho\mathrm{d}\phi = 2\pi\int_0^a\frac{a^2 - 2t^2}{(a^2 - t^2)^{1/2}}f_1(t)\mathrm{d}t.$$

Performing the integrations, we get

$$T = 4\pi a D_1, \qquad N = -\frac{2P}{\pi}\cosh(\pi\theta)\int_0^a\frac{Y_c(r)\mathrm{d}r}{(b^2 - r^2)^{1/2}},$$

$$M = \frac{4P}{\pi b}\cosh(\pi\theta)\int_0^a\frac{r^2Y_c(r) + \theta a r Y_s(r)}{(b^2 - r^2)^{1/2}}\mathrm{d}r + 4\pi a^2\theta\sqrt{\gamma_1\gamma_2}D_1. \tag{3.6.21}$$

If the punch is not loaded then it will undergo the translational, normal, and angular displacements. Their values can be determined from (3.2.71), (3.5.10) and (3.6.21).

Exercise 3.6

1. Subject to the conditions in the example above (page 200), find the normal traction at the punch centre.

Answer: $$\sigma(0) = \frac{P\cosh^2(\pi\theta)}{\pi^2 ab}\left[\int_0^a \frac{[aY_c(x) + 2\theta x Y_s(x)]\mathrm{d}x}{[b + (b^2-x^2)^{1/2}](b^2 - x^2)^{1/2}} - \frac{\pi\theta}{\sinh(\pi\theta)}\right].$$

2. Subject to the conditions in the example above, find the shear traction at the punch centre.

Answer: $$\tau(0) = \frac{P\alpha}{2\pi b^2\gamma_1\gamma_2}\left[1 - \frac{2}{\pi}b\coth(\pi\theta)\int_0^a \frac{Y_s(x)\mathrm{d}x}{x(b^2 - x^2)^{1/2}}\right] + \frac{D_1}{a} + \frac{\pi\theta A_1}{\sqrt{\gamma_1\gamma_2}}.$$

3. Subject to the conditions in the example above, find the traction distribution under the punch in the limiting case α=0.

Answer: $$\sigma_n(\rho) = -\frac{P}{\pi^2}\frac{(b^2 - a^2)^{1/2}}{(a^2 - \rho^2)^{1/2}(b^2 - \rho^2)}\left(\frac{\rho}{b}\right)^n, \qquad \tau_n(\rho) = 0.$$

4. No loading is applied to a flat circular punch of radius *a* bonded to a transversely isotropic half-space. A normal concentrated force *P* is applied at the point ρ=*b*, ϕ=π/2. Find the normal component *w* of the punch displacement.

Answer: $$w = \frac{PH\sinh(\pi\theta)}{\pi\theta a}\int_0^a \frac{Y_c(r)\mathrm{d}r}{(b^2 - r^2)^{1/2}}.$$

Hint: use (3.2.71) and (3.6.21)

5. Subject to the conditions of Exercise 4, find the tangential component *u* of the punch displacement and its direction.

Answer: $$u = \frac{3PH\alpha\cosh(\pi\theta)}{\pi b a^2(1 + \theta^2)}\int_0^a \frac{x^2Y_c(x) + \theta a x Y_s(x)}{(b^2 - x^2)^{1/2}}\,\mathrm{d}x + PH\alpha\,\frac{1}{b}\left[1\right.$$

$$\left. - \frac{\cosh(\pi\theta)}{\pi\theta a}\int_0^a \frac{xY_s(x)\mathrm{d}x}{(b^2 - x^2)^{1/2}}\right], \qquad \text{in the } Oy \text{ direction.}$$

6. Subject to the conditions of Exercise 4, find the tilting angle δ of the punch and its direction.

Answer: $$\delta = \frac{3PH\sinh(\pi\theta)}{\pi a^3 b\theta(1+\theta^2)}\int_0^a \frac{Y_c(x)x^2 + \theta a x Y_s(x)}{(b^2 - x^2)^{1/2}}\,\mathrm{d}x,$$ in the positive direction about the Ox axis.

7. No loading is applied to a flat circular punch of radius a bonded to a transversely isotropic half-space. A tangential concentrated force T is applied at the point $\rho=b$, $\phi=0$ in the positive Ox direction. Find the normal component w of the punch displacement.

Answer: $$w = \frac{TH\sinh(\pi\theta)}{\pi\theta ab}\sqrt{\gamma_1\gamma_2}\left[I_s(b) - \frac{\pi\theta a}{\cosh(\pi\theta)}\right],$$ with I_s defined by (3.2.74).

Hint: use (3.2.72) and the reciprocal theorem.

8. Subject to the general boundary conditions (3.1.1), find the tangential displacements for $\rho>a$, expressed in terms of the stress function f_n.

Answer: $$u_{-n+1}(\rho) = \frac{\pi}{\rho^{n-1}}\int_0^a \frac{(\alpha/\gamma_1\gamma_2)(G_1 - G_2)\bar{f}_n(x) + (G_1 + G_2)\overline{D}_n}{(\rho^2 - x^2)^{1/2}}\,x^{2n-2}\mathrm{d}x,$$

$$u_{n+1}(\rho) = \frac{2\pi H\alpha}{\rho^{n+1}}\int_0^a f_n(x)\mathrm{d}\{x^{2n-1}[(a^2 - x^2)^{1/2} - (\rho^2 - x^2)^{1/2}]\}$$

$$+ \pi(G_1 + G_2)D_n a^{2n-1}\frac{(\rho^2 - a^2)^{1/2}}{\rho^{n+1}}.$$

9. Subject to the general boundary conditions (3.1.1), find the normal displacements for $\rho>a$, expressed in terms of the stress function f_n.

Answer: $$\Re\{w_n(\rho)e^{in\phi} + w_{-n}(\rho)e^{-in\phi}\} = 2\pi^{3/2}H\alpha\,\frac{\Gamma(n)a^{2n-1}}{\Gamma(n+1/2)\rho^n}\,\Re\{D_n e^{in\phi}\}$$

$$+ \frac{8H}{\rho^n}\,\Re\left\{e^{in\phi}\int_0^a \frac{x^{2n}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\mathrm{d}\rho_0}{\rho_0^{2n-1}(\rho_0^2 - x^2)^{1/2}}\int_0^{\rho_0}\frac{t^{2n-1}\mathrm{d}f_n(t)}{(\rho_0^2 - t^2)^{1/2}}\right\}.$$

10. Investigate the interaction of an arbitrary tangential force with a bonded

axisymmetric punch.

3.7 Non-axisymmetric external mixed-mixed problem

The general formulation of the problem is given in section 3.1, with the boundary conditions (3.1.6), and the governing integral equations (3.1.7–3.1.10). We assume that all the parameters involved can be expanded in a Fourier series. An exact solution for the *n*-th harmonic is presented below. The governing integral equations for the *n*-th harmonic are

$$2\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_a^x \frac{G_1\rho_0^2\tau_{n+1}(\rho_0) + G_2[2nx^2-(2n+1)\rho_0^2]\bar{\tau}_{-n+1}(\rho_0)}{(x^2-\rho_0^2)^{1/2}}\,\rho_0^n d\rho_0$$

$$-\frac{2\pi H\alpha}{\rho^{n+1}}\int_a^\rho \sigma_n(\rho_0)\rho_0^{n+1}d\rho_0 = F_{n+1}(\rho),\ \text{for } n\geq 0.$$

$$2\rho^{n-1}\int_\rho^\infty \frac{dx}{x^{2n}(x^2-\rho^2)^{1/2}}\int_a^x \frac{G_1x^2\tau_{-n+1}(\rho_0) + G_2[(2n-1)x^2-2n\rho^2]\bar{\tau}_{n+1}(\rho_0)}{(x^2-\rho_0^2)^{1/2}}\,\rho_0^n d\rho_0$$

$$+\ 2\pi H\alpha\rho^{n-1}\int_\rho^\infty \frac{\sigma_{-n}(\rho_0)d\rho_0}{\rho_0^{n-1}} = F_{-n+1}(\rho),\qquad \text{for } n\geq 1;$$

$$2\pi H\alpha\ \Re\left\{\frac{e^{-in\phi}}{\rho^n}\int_a^\rho \tau_{-n+1}(\rho_0)\rho_0^n d\rho_0 - \rho^n e^{in\phi}\int_\rho^\infty \frac{\tau_{n+1}(\rho_0)d\rho_0}{\rho_0^n}\right\}$$

$$+\ 4H\rho^n\int_\rho^\infty \frac{dx}{x^{2n}(x^2-\rho^2)^{1/2}}\int_a^x \frac{\sigma_{-n}(\rho_0)e^{-in\phi} + \sigma_n(\rho_0)e^{in\phi}}{(x^2-\rho_0^2)^{1/2}}\,\rho_0^{n+1}d\rho_0 = \Re\{e^{in\phi}\Phi_n(\rho)\},$$

$$\text{for } n\geq 0. \tag{3.7.1}$$

The right hand sides in (3.7.1) are known from the boundary conditions, and are

$$F_{n+1}(\rho) = u_{n+1}(\rho) + \frac{2\pi H\alpha}{\rho^{n+1}}\int_0^a \sigma_n(\rho_0)\rho_0^{n+1}d\rho_0$$

$$-\frac{2}{\rho^{n+1}}\int_0^a \frac{x^{2n}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{G_1x^2\tau_{n+1}(\rho_0)+G_2[2n\rho^2-(2n+1)x^2]\bar{\tau}_{-n+1}(\rho_0)}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0,$$

$$F_{-n+1}(\rho) = u_{-n+1}(\rho) - \frac{2}{\rho^{n-1}}\int_0^a \frac{x^{2n-2}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}$$

$$\times\int_x^a \frac{G_1\rho_0^2\tau_{-n+1}(\rho_0)+G_2[(2n-1)\rho_0^2-2nx^2]\bar{\tau}_{n+1}(\rho_0)}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0,$$

$$\Re\{\Phi_n(\rho)e^{in\phi}\} = w_n(\rho)e^{in\phi} + w_{-n}(\rho)e^{-in\phi} + 2\pi H\alpha\,\Re\left\{\frac{e^{-in\phi}}{\rho^n}\int_0^a \tau_{-n+1}(\rho_0)\rho_0^n\mathrm{d}\rho_0\right\}$$

$$-\frac{4H}{\rho^n}\int_0^a \frac{x^{2n}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\sigma_n(\rho_0)e^{in\phi}+\sigma_{-n}(\rho_0)e^{-in\phi}}{\rho_0^{n-1}(\rho_0^2-x^2)^{1/2}}\,\mathrm{d}\rho_0. \tag{3.7.2}$$

The case of axial symmetry (n=0) was considered in detail in section 3.3, and is not discussed here. The solution is sought for $n \geq 1$. We may assume, without loss of generality, that the first two equations (3.7.1) are homogeneous. This can be achieved by the addition of some special solutions to the parameters τ_{-n+1} and τ_{n+1}. These special solutions, satisfying the right hand sides of the first two equations (3.7.1), can be obtained from the results of section 2.7. Of course, this procedure will make the right hand side of the third equation (3.7.1) more complicated.

Assume the solution of (3.7.1), with the first two equations transformed into homogeneous ones, in the form

$$\sigma_n(\rho) = \bar{\sigma}_{-n}(\rho) = \rho^n\int_\rho^\infty \frac{\mathrm{d}f_n(t)}{t^{2n}(t^2-\rho^2)^{1/2}}\ ,$$

$$\tau_{n+1}(\rho) = \frac{C}{\rho^n}\frac{d}{d\rho}\int_a^{\rho}\frac{f_n(t)dt}{(\rho^2 - t^2)^{1/2}} - \frac{D_n}{\rho^{n+1}(\rho^2 - a^2)^{1/2}},$$

$$\tau_{-n+1}(\rho) = \frac{C}{\rho^n}\frac{d}{d\rho}\left\{\rho^{2n}\int_a^{\rho}\frac{dy}{y^{2n}}\frac{d}{dy}\int_a^{y}\frac{\bar{f}_n(t)dt}{(y^2 - t^2)^{1/2}}\right\}$$

$$+ \frac{\bar{D}_n}{\rho^n}\frac{d}{d\rho}\left\{\rho^{2n}\int_a^{\rho}\frac{dt}{t^{2n+1}(t^2 - a^2)^{1/2}}\right\}. \tag{3.7.3}$$

Here f_n is the as yet unknown complex stress function, and C, D_n are the constants to be determined. Substitution of (3.7.3) in the first two equations (3.7.1) satisfies them identically, provided that the following conditions hold

$$C = \frac{\alpha}{\gamma_1\gamma_2}, \qquad D_n = -2nCa^{2n+1}\int_a^{\infty}\frac{f_n(t)}{t^{2n+1}}\,dt. \tag{3.7.4}$$

$$(G_1 + G_2)\frac{\pi^{3/2}\Gamma(n + 1)}{2\Gamma(n + 3/2)}D_n + 2\pi H\alpha a^{2n+2}\int_a^{\infty}\left[xf_n(x)\right.$$

$$\left. - \int_a^{x} f_n(t)dt\right]\frac{dx}{x^{2n+2}(x^2 - a^2)^{1/2}} = 0. \tag{3.7.5}$$

Some general formulae from Appendix A3.3 were used in transformations. The conditions (3.7.4) and (3.7.5) might appear contradictory. It will be shown below (see 3.7.11) that this is not so, because an additional constant will appear in the expression for f_n, due to a homogeneous solution of the integral equation (3.7.8).

So far, we have satisfied the first two equations (3.7.1). Substitution of (3.7.3) in the third of equations (3.7.1) requires the following transformation:

$$\int_a^x \frac{\rho_0^{2n+1}\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}} \int_{\rho_0}^{\infty} \frac{\mathrm{d}f_n(t)}{t^{2n}(t^2-\rho_0^2)^{1/2}} = \int_a^{\infty} \Bigg[(x^2-a^2)^{1/2}(t^2-a^2)^{1/2}Q_n(x,t)$$

$$+\ \psi_n(x,t)\ \ln\frac{|(x^2-a^2)^{1/2}+(t^2-a^2)^{1/2}|}{|t^2-x^2|^{1/2}}\Bigg]\frac{\mathrm{d}f_n(t)}{t^{2n}}. \tag{3.7.6}$$

Here $Q_n(x,t)$ is a polynomial in even powers of x and t. Although we cannot write the explicit expression for $Q_n(x,t)$, it can be computed in elementary manner for any particular n. The explicit expression for ψ_n is

$$\psi_n(x,t) = \frac{x^{2n}}{\pi}\sum_{k=0}^{n}\frac{\Gamma(n-k+1/2)\ \Gamma(k+1/2)}{\Gamma(n-k+1)\ \Gamma(k+1)}\left(\frac{t}{x}\right)^{2k}. \tag{3.7.7}$$

Divide both sides of the third of equations (3.7.1) by ρ^n, differentiate with respect to ρ, multiply the result by $\rho^{2n}/(\rho^2-r^2)^{1/2}$ and integrate with respect to ρ from r to ∞. This procedure allows us to split the kernel of the integral equation into two parts: a singular one and a degenerate one. The result takes the form

$$-\frac{(r^2-a^2)^{1/2}}{r}\int_a^{\infty}\frac{f_n(t)t\,\mathrm{d}t}{(t^2-r^2)(t^2-a^2)^{1/2}} + \frac{\alpha^2}{\gamma_1\gamma_2}\int_a^{\infty}\frac{f_n(t)\mathrm{d}t}{t^2-\rho^2} = \chi_n(r). \tag{3.7.8}$$

Here

$$\chi_n(r) = \frac{1}{4\pi H}\int_r^{\infty}\frac{\rho^{2n}\mathrm{d}\rho}{(\rho^2-r^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\Phi_n(\rho)}{\rho^n}\right]$$

$$-\ \frac{2}{\pi}\int_r^{\infty}\frac{\rho^{2n}\mathrm{d}\rho}{(\rho^2-r^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{\infty}\frac{(x^2-a^2)^{1/2}\mathrm{d}x}{x^{2n}(x^2-\rho^2)^{1/2}}\int_a^{\infty}(t^2-a^2)^{1/2}Q_n(x,t)\ \frac{\mathrm{d}f_n(t)}{t^{2n}}$$

$$+\ \frac{2}{\pi^{3/2}}\int_r^{\infty}\frac{\rho^{2n-1}\mathrm{d}\rho}{(\rho^2-r^2)^{1/2}}\int_{\rho}^{\infty}\frac{\mathrm{d}x}{(x^2-\rho^2)^{1/2}(x^2-a^2)^{1/2}}\int_a^{\infty}q_n(\rho,x,t)(t^2-a^2)^{1/2}\frac{\mathrm{d}f_n(t)}{t^{2n-2}},$$

(3.7.9)

with

$$q_n(\rho,x,t) = \sum_{k=0}^{n-1} \frac{\Gamma(n-k+1/2)}{\Gamma(n-k+1)} \left(\frac{t}{\rho}\right)^{2k} F\left(1-n+k,\ \frac{1}{2};\ \frac{1}{2}-n+k;\ \frac{t^2}{x^2}\right). \tag{3.7.10}$$

Note that the hypergeometric function in (3.7.10) is, in fact, a polynomial, and that all the integrals with respect to x and ρ in the degenerate part of the kernel (3.7.9) are computable in elementary function for any n. The integral equation (3.7.8) was solved in paragraph 3.3, and the solution is

$$f_n(t) = \frac{4}{\pi^2}\cosh^2(\pi\theta)t\left[tY_c(t)\int_a^\infty \frac{\chi_n(r)Y_c(r)\mathrm{d}r}{r^2 - t^2} + Y_s(t)\int_a^\infty \frac{\chi_n(r)Y_s(r)r\mathrm{d}r}{r^2 - t^2}\right] + A_n Y_c(t). \tag{3.7.11}$$

The last term in (3.7.11) represents the homogeneous solution, with A_n being an arbitrary constant. Its value, along with the constant D_n and others which appear due to the degenerate part of the kernel, can be found from the appropriate system of linear algebraic equations and the conditions (3.7.4) and (3.7.5). The general solution may be considered completed.

Example. Consider the action of a normal concentrated load P applied at some point inside the circle $\rho=a$, with the rest of the plane z=0 being clamped. We may assume, without loss of generality, that the force is applied at the point $\rho=b$, $\phi=0$ ($b<a$). Thus, the boundary conditions are

$$u = w = 0, \qquad \text{for } \rho>a;$$

$$\sigma = P\delta(\rho-b)\delta(\phi-0)/\rho, \qquad \tau = 0, \qquad \text{for } \rho<a.$$

We determine in this case

$$F_{-n+1}(\rho) = 0, \quad \text{for } n\geq 1; \qquad F_{n+1} = PH\alpha\,\frac{b^n}{\rho^{n+1}}, \qquad \text{for } n\geq 0;$$

$$\Phi_n(\rho) = -\frac{4}{\pi}PH(\rho b)^n \int_\rho^\infty \frac{\mathrm{d}x}{x^{2n}(x^2 - \rho^2)^{1/2}(x^2 - b^2)^{1/2}}\ . \tag{3.7.12}$$

We present now the explicit solutions for some specific values of n. In the axisymmetric case n=0, the following results may be obtained:

$$\sigma_0(\rho) = \int_\rho^\infty \frac{\mathrm{d}f_0(t)}{(t^2 - \rho^2)^{1/2}} = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{f_0(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}},$$

$$\tau_1(\rho) = \frac{\alpha}{\gamma_1\gamma_2}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{f_0(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} - \frac{D_0}{\rho(\rho^2 - a^2)^{1/2}},$$

$$f_0(t) = \frac{2P}{\pi^3}\cosh^2(\pi\theta)\int_a^\infty \frac{Y_c(t)rY_c(r) + tY_s(t)Y_s(r)}{(r^2 - t^2)(r^2 - b^2)^{1/2}}\,\mathrm{d}r$$

$$- D_0\frac{\gamma_1\gamma_2}{a\alpha}[1 - Y_c(t)],$$

$$D_0 = -\frac{P\alpha\sinh(\pi\theta)}{4\pi^2\theta\gamma_1\gamma_2}\left[1 - \frac{2}{\pi}\coth(\pi\theta)\int_a^\infty \frac{Y_s(r)\mathrm{d}r}{(r^2 - b^2)^{1/2}}\right].$$

The results for the first harmonic (n=1) are

$$\sigma_1(\rho) = \bar{\sigma}_{-1}(\rho) = \rho\int_\rho^\infty \frac{\mathrm{d}f_1(t)}{t^2(t^2 - \rho^2)^{1/2}},$$

$$\tau_2(\rho) = \frac{\alpha}{\gamma_1\gamma_2}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{f_1(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}} - \frac{D_1}{\rho^2(\rho^2 - a^2)^{1/2}},$$

$$\tau_0(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\rho^2\left[\frac{\alpha}{\gamma_1\gamma_2}\int_a^\rho \frac{\mathrm{d}y}{y^2}\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{\bar{f}_1(t)\mathrm{d}t}{(y^2 - t^2)^{1/2}} + \bar{D}_1\int_a^\rho \frac{\mathrm{d}y}{y^3(y^2 - a^2)^{1/2}}\right]\right\},$$

$$f_1(t) = \frac{4}{\pi^2}\cosh^2(\pi\theta)t\left[tY_c(t)\int_a^\infty \frac{\chi_1(r)Y_c(r)\mathrm{d}r}{r^2 - t^2} + Y_s(t)\int_a^\infty \frac{\chi_1(r)Y_s(r)r\mathrm{d}r}{r^2 - t^2}\right] + A_1Y_c(t),$$

$$\chi_1(r) = \frac{Pb}{2\pi r}\left[\frac{1}{(r^2 - b^2)^{1/2}} + \frac{1}{r + (r^2 - b^2)^{1/2}}\right] + \frac{2}{\pi}\left[1 - \frac{(r^2 - a^2)^{1/2}}{r}\right]B_1.$$

The constants D_1, A_1, and B_1 are to be determined from the set of linear algebraic equations

$$D_1 = -\frac{2\alpha a^3}{\gamma_1\gamma_2}\int_a^\infty \frac{f_1(t)\mathrm{d}t}{t^3}, \qquad B_1 = \int_a^\infty \frac{(t^2 - a^2)^{1/2}}{t^2}\,\mathrm{d}f_1(t),$$

$$-\frac{2\pi}{3H\alpha}(G_1 + G_2)D_1 + 2\pi\int_a^\infty \left[xf_1(x) - \int_a^x f_1(t)\mathrm{d}t\right]\frac{\mathrm{d}x}{x^4(x^2 - a^2)^{1/2}} - Pb = 0.$$

It should be noted that the system of tractions in the clamped part is such that its resultant vector is exactly equal to P. This can be shown by a direct integration.

Exercise 3.7

1. Find the solution of the example above (page 211) in the limiting case b=0. *Answer*: the only non-zero stress function is

$$f_0(t) = \frac{P}{2\pi}\left\{\frac{2\cosh^2(\pi\theta)Y_s(t)}{\pi\sinh(\pi\theta)t} - \frac{\sinh(2\pi\theta)[1 - Y_c(t)]}{2\pi\theta[1 + \cosh(\pi\theta)]a}\right\}.$$

2. Find the solution of the example above in the limiting case α=0.
Answer: $\sigma_n(\rho) = -\dfrac{P(a^2 - b^2)^{1/2}}{\pi^2(\rho^2 - a^2)^{1/2}(\rho^2 - b^2)}\left(\dfrac{b}{\rho}\right)^n, \qquad \tau_n = 0.$

3. Prove that the tractions in the plane z=0 are in equilibrium when the exterior is clamped.
Note: this is not the case in *internal* problems.

4. Try to find an exact *closed form* solution to the mixed-mixed boundary value problem.

Appendix A3.1

Some integrals, related to solving internal mixed-mixed problem, are presented here. The notations $Y_{\mathrm{c,s}}$ are defined by (3.2.45). It is assumed that $0<r<a$.

Integrals, containing Y_{c}:

$$\int_0^a \frac{Y_{\mathrm{c}}(x)\mathrm{d}x}{x^2 - r^2} = -\frac{\pi}{2r}\coth(\pi\theta)Y_{\mathrm{s}}(r),$$

$$\int_0^a \left(\frac{a^2 - r^2}{a^2 - x^2}\right)^{1/2} \frac{Y_{\mathrm{c}}(x)\mathrm{d}x}{x^2 - r^2} = -\frac{\pi}{2r}\tanh(\pi\theta)Y_{\mathrm{s}}(r),$$

$$\int_0^a \frac{Y_{\mathrm{c}}(x)x^2\mathrm{d}x}{x^2 - r^2} = \frac{\pi\theta a}{\sinh(\pi\theta)} - \frac{\pi}{2}\,r\coth(\pi\theta)Y_{\mathrm{s}}(r),$$

$$\int_0^a \frac{Y_{\mathrm{c}}(x)x^4\mathrm{d}x}{x^2 - r^2} = -\frac{\pi}{2}r^3\coth(\pi\theta)Y_{\mathrm{s}}(r) + \frac{\pi\theta a}{\sinh(\pi\theta)}\left[r^2 + a^2\frac{1 - 2\theta^2}{3}\right],$$

$$\int_0^a \frac{(a^2 - x^2)^{1/2}Y_{\mathrm{c}}(x)\mathrm{d}x}{x^2 - r^2} = -\frac{\pi}{2r}(a^2 - r^2)^{1/2}\tanh(\pi\theta)Y_{\mathrm{s}}(r) - \frac{\pi}{2\cosh(\pi\theta)},$$

$$\int_0^a \frac{x^2(a^2 - x^2)^{1/2}Y_{\mathrm{c}}(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi a^2}{\cosh(\pi\theta)}\left(\frac{1}{4} + \theta^2\right)$$

$$- \frac{\pi}{2}\left[r(a^2 - r^2)^{1/2}\tanh(\pi\theta)Y_{\mathrm{s}}(r) + \frac{r^2}{\cosh(\pi\theta)}\right],$$

$$\int_0^a \frac{x^2Y_{\mathrm{c}}(x)\mathrm{d}x}{(a^2 - x^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2\cosh(\pi\theta)} - \frac{\pi}{2}\tanh(\pi\theta)\frac{r}{(a^2 - r^2)^{1/2}}\;Y_{\mathrm{s}}(r)\;,$$

$$\int_0^a Y_c(x)\,dx = \frac{\pi\theta a}{\sinh(\pi\theta)}, \qquad \int_0^a x^2 Y_c(x)\,dx = \frac{\pi\theta(1 - 2\theta^2)a^3}{3\sinh(\pi\theta)},$$

$$\int_0^a (a^2 - x^2)^{1/2} Y_c(x)\,dx = \frac{\pi a^2}{\cosh(\pi\theta)}\Big(\frac{1}{4} + \theta^2\Big), \qquad \int_0^a \frac{Y_c(x)\,dx}{(a^2 - x^2)^{1/2}} = \frac{\pi}{2\cosh(\pi\theta)},$$

$$\int_0^a \frac{x^2 Y_c(x)\,dx}{(a^2 - x^2)^{1/2}} = \frac{\pi a^2}{\cosh(\pi\theta)}\Big(\frac{1}{4} - \theta^2\Big),$$

$$\int_0^a \frac{1 - Y_c(x)}{x^2}\,dx = \frac{\pi\theta\coth(\pi\theta) - 1}{a}.$$

Integrals, containing Y_s:

$$\int_0^a \frac{xY_s(x)\,dx}{x^2 - r^2} = \frac{\pi}{2}\coth(\pi\theta)Y_c(r) - \frac{\pi}{2\sinh(\pi\theta)},$$

$$\int_0^a \frac{Y_s(x)\,dx}{x(x^2 - r^2)} = \frac{\pi}{2}\coth(\pi\theta)\frac{Y_c(r) - 1}{r^2},$$

$$\int_0^a \frac{x^3 Y_s(x)\,dx}{x^2 - r^2} = \frac{\pi\theta^2 a^2}{\sinh(\pi\theta)} + \frac{\pi r^2}{2\sinh(\pi\theta)}[\cosh(\pi\theta)Y_c(r) - 1],$$

$$\int_0^a \frac{Y_s(x)\,dx}{x(a^2 - x^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2r^2}\tanh(\pi\theta)\left[\frac{Y_c(r)}{(a^2 - r^2)^{1/2}} - \frac{1}{a}\right],$$

$$\int_0^a \frac{x^3 Y_s(x)\mathrm{d}x}{(a^2 - x^2)^{1/2}(x^2 - r^2)} = \frac{\pi\theta a}{\cosh(\pi\theta)} + \frac{\pi r^2}{2(a^2 - r^2)^{1/2}} \tanh(\pi\theta) Y_c(r),$$

$$\int_0^a \left(\frac{a^2 - r^2}{a^2 - x^2}\right)^{1/2} \frac{x Y_s(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi}{2} \tanh(\pi\theta) Y_c(r),$$

$$\int_0^a \frac{x(a^2 - x^2)^{1/2} Y_s(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi}{2} \tanh(\pi\theta) \ (a^2 - r^2)^{1/2} Y_c(r) - \frac{\pi\theta a}{\cosh(\pi\theta)},$$

$$\int_0^a x Y_s(x)\mathrm{d}x = \frac{\pi\theta^2 a^2}{\sinh(\pi\theta)}, \qquad \int_0^a x^3 Y_s(x)\mathrm{d}x = \frac{\pi\theta^2(2 - \theta^2)a^4}{3\sinh(\pi\theta)},$$

$$\int_0^a \frac{x Y_s(x)\mathrm{d}x}{(a^2 - x^2)^{1/2}} = \frac{\pi\theta a}{\cosh(\pi\theta)}, \qquad \int_0^a \frac{x^3 Y_s(x)\mathrm{d}x}{(a^2 - x^2)^{1/2}} = \frac{\pi\theta(5 - 4\theta^2)a^3}{6\cosh(\pi\theta)},$$

$$\int_0^a \frac{(a^2 - x^2)^{1/2} Y_s(x)\mathrm{d}x}{x} = \frac{\pi}{2} a\tanh(\pi\theta) - \frac{\pi\theta a}{\cosh(\pi\theta)},$$

$$\int_0^a \frac{Y_s(x)\mathrm{d}x}{x(a^2 - x^2)^{1/2}} = \frac{\pi}{2a} \tanh(\pi\theta), \qquad \int_0^a x(a^2 - x^2)^{1/2} Y_s(x)\mathrm{d}x = \frac{\pi\theta(1 + 4\theta^2)a^3}{6\cosh(\pi\theta)},$$

$$\int_0^a \frac{Y_c(x)\mathrm{d}x}{x^2 + r^2} = \frac{\pi\sinh[2\theta\tan^{-1}(a/r)]}{2r\sinh(\pi\theta)},$$

$$\int_0^a \frac{Y_s(x)x\mathrm{d}x}{x^2 + r^2} = \frac{\pi\{\cosh[2\theta\tan^{-1}(a/r)] - 1\}}{2\sinh(\pi\theta)},$$

Integrals, containing combination $Y_c + iY_s$:

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{\mathrm{d}x}{x-r} = \frac{\pi}{i\sinh(\pi\theta)}\left[1 - \cosh(\pi\theta)\left(\frac{a+r}{a-r}\right)^{i\theta}\right],$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{\mathrm{d}x}{(a+x)(x-r)} = \frac{\pi i\coth(\pi\theta)}{a+r}\left(\frac{a+r}{a-r}\right)^{i\theta},$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{\mathrm{d}x}{(x-r)(a^2-x^2)^{1/2}} = \frac{\pi i\tanh(\pi\theta)}{(a^2-r^2)^{1/2}}\left(\frac{a+r}{a-r}\right)^{i\theta},$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{(a^2-x^2)^{1/2}\mathrm{d}x}{x-r} = \pi i\tanh(\pi\theta)(a^2-r^2)^{1/2}\left(\frac{a+r}{a-r}\right)^{i\theta} - \frac{\pi(2ia\theta + r)}{\cosh(\pi\theta)},$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \mathrm{d}x = \frac{2\pi\theta a}{\sinh(\pi\theta)}\text{’} \qquad \int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} x\mathrm{d}x = \frac{2i\pi\theta^2 a^2}{\sinh(\pi\theta)}\text{’}$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} x^2\mathrm{d}x = \frac{2\pi\theta(1-2\theta^2)a^3}{3\sinh(\pi\theta)},$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{\mathrm{d}x}{x+a} = -\frac{i\pi}{\sinh(\pi\theta)}\text{’}$$

$$\int_{-a}^{a} \left(\frac{a+x}{a-x}\right)^{i\theta} \frac{x\mathrm{d}x}{x+a} = \frac{\pi a(i+2\theta)}{\sinh(\pi\theta)},$$

$$\int_{-a}^{a}\left(\frac{a + x}{a - x}\right)^{i\theta}\frac{x^2\mathrm{d}x}{x + a} = -\frac{\pi a^2[2\theta + i(1 - 2\theta^2)]}{\sinh(\pi\theta)}.$$

Appendix A3.2

Some integrals, used in solving the external mixed-mixed problem, are presented here. It is assumed that $a<r<\infty$.

Integrals containing Y_c:

$$\int_{a}^{\infty}\frac{Y_c(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi}{2r}\coth(\pi\theta)Y_s(r),$$

$$\int_{a}^{\infty}\frac{Y_c(x)\mathrm{d}x}{x^2(x^2 - r^2)} = \frac{\pi}{2r^3}\coth(\pi\theta)Y_s(r) - \frac{\pi\theta}{ar^2\sinh(\pi\theta)},$$

$$\int_{a}^{\infty}\frac{xY_c(x)\mathrm{d}x}{(x^2 - a^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2}\tanh(\pi\theta)\frac{Y_s(r)}{(r^2 - a^2)^{1/2}}$$

$$\int_{a}^{\infty}\frac{Y_c(x)\mathrm{d}x}{x(x^2 - a^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2r^2}\left[-\frac{1}{a\cosh(\pi\theta)} + \tanh(\pi\theta)\frac{Y_s(r)}{(r^2 - a^2)^{1/2}}\right],$$

$$\int_{a}^{\infty}\frac{Y_c(x)\mathrm{d}x}{x^3(x^2 - a^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2r^4}\left[-\frac{1}{a\cosh(\pi\theta)}\right.$$

$$\left. + \tanh(\pi\theta)\frac{Y_s(r)}{(r^2 - a^2)^{1/2}}\right] - \frac{\pi[(1/4) - \theta^2]}{a^3r^2\cosh(\pi\theta)},$$

$$\int_a^\infty \frac{(x^2 - a^2)^{1/2} Y_c(x)\,dx}{x(x^2 - r^2)} = \frac{\pi}{2r^2}\left[\frac{a}{\cosh(\pi\theta)} + \tanh(\pi\theta)(r^2 - a^2)^{1/2} Y_s(r)\right],$$

$$\int_a^\infty \frac{Y_c(x)\,dx}{x^2} = \frac{\pi\theta}{a\sinh(\pi\theta)}, \qquad \int_a^\infty \frac{Y_c(x)\,dx}{x(x^2 - a^2)^{1/2}} = \frac{\pi}{2a\cosh(\pi\theta)},$$

$$\int_a^\infty \frac{Y_c(x)\,dx}{x^3(x^2 - a^2)^{1/2}} = \frac{\pi[(1/4) - \theta^2]}{a^3\cosh(\pi\theta)},$$

$$\int_a^\infty [1 - Y_c(x)]\,dx = a[\pi\theta\coth(\pi\theta) - 1],$$

$$\int_a^\infty [1 - Y_c(x)]\,\frac{x\,dx}{(x^2 - a^2)^{1/2}} = \pi\theta a\tanh(\pi\theta),$$

$$\int_a^\infty \left[1 - \frac{xY_c(x)}{(x^2 - a^2)^{1/2}}\right] dx = a[\pi\theta\tanh(\pi\theta) - 1],$$

$$\int_a^\infty (x^2 - a^2)^{1/2}\,d[Y_c(x)] = \pi\theta a\tanh(\pi\theta),$$

$$\int_a^\infty \left(\frac{x}{(x^2 - a^2)^{1/2}} - 1\right) Y_c(x)\,dx = \frac{2\pi\theta a}{\sinh(2\pi\theta)},$$

$$\int_a^\infty \left(1 - \frac{(x^2 - a^2)^{1/2}}{x}\right) Y_c(x)\,dx = \frac{\pi a}{2\cosh(\pi\theta)} - \frac{2\pi\theta a}{\sinh(2\pi\theta)},$$

Integrals containing Y_s:

$$\int_a^\infty \frac{xY_\mathrm{s}(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi}{2}\coth(\pi\theta)[1 - Y_\mathrm{c}(r)],$$

$$\int_a^\infty \frac{Y_\mathrm{s}(x)\mathrm{d}x}{x(x^2 - r^2)} = \frac{\pi}{2r^2}\left[\frac{1}{\sinh(\pi\theta)} - \coth(\pi\theta)Y_\mathrm{c}(r)\right],$$

$$\int_a^\infty \frac{Y_\mathrm{s}(x)\mathrm{d}x}{(x^2 - a^2)^{1/2}(x^2 - r^2)} = -\frac{\pi\tanh(\pi\theta)}{2r(r^2 - a^2)^{1/2}}Y_\mathrm{c}(r),$$

$$\int_a^\infty \frac{x^2Y_\mathrm{s}(x)\mathrm{d}x}{(x^2 - a^2)^{1/2}(x^2 - r^2)} = \frac{\pi}{2}\tanh(\pi\theta)\left[1 - \frac{r}{(r^2 - a^2)^{1/2}}Y_\mathrm{c}(r)\right],$$

$$\int_a^\infty \frac{Y_\mathrm{s}(x)\mathrm{d}x}{x^2(x^2 - a^2)^{1/2}(x^2 - r^2)} = -\frac{\pi}{r^2}\left[\frac{\tanh(\pi\theta)}{2r(r^2 - a^2)^{1/2}}Y_\mathrm{c}(r) + \frac{\theta}{a^2\cosh(\pi\theta)}\right],$$

$$\int_a^\infty \frac{(x^2 - a^2)^{1/2}Y_\mathrm{s}(x)\mathrm{d}x}{x^2 - r^2} = \frac{\pi}{2}\tanh(\pi\theta)\left[1 - \frac{(r^2 - a^2)^{1/2}}{r}Y_\mathrm{c}(r)\right],$$

$$\int_a^\infty Y_\mathrm{s}(x)\frac{\mathrm{d}x}{x} = \frac{\pi}{2}\tanh\left(\frac{\pi\theta}{2}\right), \qquad \int_a^\infty Y_\mathrm{s}(x)\frac{\mathrm{d}x}{x^3} = \frac{\pi\theta^2}{a^2\sinh(\pi\theta)},$$

$$\int_a^\infty Y_\mathrm{s}(x)\frac{\mathrm{d}x}{x^5} = \frac{\pi\theta^2(2 - \theta^2)}{3a^4\sinh(\pi\theta)}, \qquad \int_a^\infty \frac{Y_\mathrm{s}(x)\mathrm{d}x}{(x^2 - a^2)^{1/2}} = \frac{\pi}{2}\tanh(\pi\theta),$$

$$\int_a^\infty \frac{Y_\mathrm{s}(x)\mathrm{d}x}{x^2(x^2 - a^2)^{1/2}} = \frac{\pi\theta}{a^2\cosh(\pi\theta)},$$

$$\int_a^\infty (x^2 - a^2)^{1/2} \mathrm{d}[Y_s(x)] = \pi a^2 \tanh(\pi\theta)(\theta^2 - 1/4),$$

$$\int_a^\infty (x^2 - a^2)^{1/2} Y_s(x) \frac{\mathrm{d}x}{x^2} = \frac{\pi}{2}\left[\tanh(\pi\theta) - \frac{2\theta}{\cosh(\pi\theta)}\right]$$

$$\int_a^\infty (x^2 - a^2)^{1/2} Y_s(x) \frac{\mathrm{d}x}{x^4} = \frac{2\pi\theta(\theta^2 + 1/4)}{3a^2\cosh(\pi\theta)}$$

$$\int_a^\infty \left[\frac{x}{(x^2 - a^2)^{1/2}} - 1\right] Y_s(x) x \mathrm{d}x = \frac{\pi a^2}{4}\left[\tanh(\pi\theta) + \frac{8\theta^2}{\sinh(2\pi\theta)}\right],$$

Integrals containing both Y_c and Y_s:

$$\int_a^\infty [xY_s(x) - 2a\theta Y_c(x)] \mathrm{d}x = \pi a^2 \theta^2 \coth(\pi\theta),$$

$$\int_a^\infty [xY_s(x) - 2a\theta Y_c(x)] \frac{x\mathrm{d}x}{(x^2 - a^2)^{1/2}} = \frac{\pi a^2}{4} \tanh(\pi\theta)(1 + 4\theta^2),$$

$$\int_a^\infty \left[1 - \frac{(x^2 - a^2)^{1/2}}{x}\right] [xY_s(x) - 2a\theta Y_c(x)] \mathrm{d}x$$

$$= \frac{2\pi\theta^2 a^2}{\sinh(2\pi\theta)} + \frac{\pi a^2}{4} \tanh(\pi\theta) - \frac{\pi\theta a^2}{\cosh(\pi\theta)}.$$

Appendix A3.3

Some formulae, related to transformation and computation of integrals, are presented here.

Transformations of the limits of integration:

$$\int_0^{\rho} \frac{f(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{2}{\pi}\,\rho \int_0^{a} \frac{\mathrm{d}y}{\rho^2 - y^2} \int_y^{a} \frac{f(x)\mathrm{d}x}{(x^2 - y^2)^{1/2}}$$

$$\int_{\rho}^{a} \frac{f(x)\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} = \frac{2}{\pi}(a^2 - \rho^2)^{1/2} \int_0^{a} \frac{y\mathrm{d}y}{(y^2 - \rho^2)(a^2 - y^2)^{1/2}} \int_0^{y} \frac{f(x)\mathrm{d}x}{(y^2 - x^2)^{1/2}}$$

$$= \frac{2}{\pi} \int_0^{\rho} \frac{\mathrm{d}y}{(\rho^2 - y^2)^{1/2}} \int_0^{a} \frac{xf(x)\mathrm{d}x}{x^2 - y^2} + \frac{\pi}{2\rho}\lim_{x\to 0}[xf(x)],$$

$$\int_0^{a} \frac{f(x)\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} = \frac{2}{\pi}(\rho^2 - a^2)^{1/2} \int_0^{a} \frac{y\mathrm{d}y}{(\rho^2 - y^2)(a^2 - y^2)^{1/2}} \int_0^{y} \frac{f(x)\mathrm{d}x}{(y^2 - x^2)^{1/2}}$$

Simplification of two consecutive integrals:

$$\int_{\rho}^{a} \frac{x^2\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^{a} \frac{f(y)\mathrm{d}y}{(y^2 - x^2)^{1/2}} = -\frac{\pi}{2}\Bigg[\rho f(\rho) + \int_{\rho}^{a} f(y)\mathrm{d}y$$

$$- \frac{a}{(a^2 - \rho^2)^{1/2}} \lim_{r\to a}[f(r)(a^2 - r^2)^{1/2}]\Bigg];$$

$$\int_{\rho}^{a} \frac{\mathrm{d}x}{x^2(x^2 - \rho^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^{a} \frac{f(y)\mathrm{d}y}{(y^2 - x^2)^{1/2}} = \frac{\pi}{2\rho}\left[- \frac{f(\rho)}{\rho^2} + \int_{\rho}^{a} \frac{f(y)\mathrm{d}y}{y^3}\right].$$

$$\int_b^x \frac{t\,dt}{(x^2 - t^2)^{1/2}} \int_t^a \frac{f(y)\,dy}{(y^2 - t^2)^{1/2}} = \int_b^a f(y) \ln \frac{(x^2 - b^2)^{1/2} + (y^2 - b^2)^{1/2}}{|x^2 - y^2|^{1/2}}\, dy,$$

$$\int_x^a \frac{t\,dt}{(t^2 - x^2)^{1/2}} \int_b^t \frac{f(y)\,dy}{(t^2 - y^2)^{1/2}} = \int_b^a f(y) \ln \frac{(a^2 - x^2)^{1/2} + (a^2 - y^2)^{1/2}}{|x^2 - y^2|^{1/2}}\, dy,$$

$$\int_\rho^a \frac{dx}{(x^2 - \rho^2)^{(1+\kappa)/2}} \frac{d}{dx} \int_x^a \frac{f(r)\,r\,dr}{(r^2 - x^2)^{(1-\kappa)/2}} = -\frac{\pi}{2\cos(\pi\kappa/2)}\, f(\rho),$$

$$\int_0^\rho \frac{dx}{(\rho^2 - x^2)^{(1+\kappa)/2}} \frac{d}{dx} \int_0^x \frac{f(r)\,r\,dr}{(x^2 - r^2)^{(1-\kappa)/2}} = \frac{\pi}{2\cos(\pi\kappa/2)} \left[f(\rho) - \frac{1}{\rho} \lim_{r\to 0} [r f(r)] \right]$$

$$\int_b^x \frac{dt}{(x^2 - t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{f(y)\,dy}{(y^2 - t^2)^{1/2}} = (x^2 - b^2)^{1/2} \int_b^a \frac{f(y)\,dy}{(y^2 - x^2)(y^2 - b^2)^{1/2}},$$

$$\int_x^a \frac{dt}{(t^2 - x^2)^{1/2}} \frac{d}{dt} \int_b^t \frac{f(y)\,dy}{(t^2 - y^2)^{1/2}} = (a^2 - x^2)^{1/2} \int_b^a \frac{f(y)\,dy}{(y^2 - x^2)(a^2 - y^2)^{1/2}},$$

$$\frac{d}{dx} \int_b^x \frac{t\,dt}{(x^2 - t^2)^{1/2}} \int_t^a \frac{f(y)\,dy}{(y^2 - t^2)^{1/2}} = \frac{x}{(x^2 - b^2)^{1/2}} \int_b^a \frac{(y^2 - b^2)^{1/2} f(y)\,dy}{y^2 - x^2},$$

$$\frac{d}{dx} \int_x^a \frac{t\,dt}{(t^2 - x^2)^{1/2}} \int_b^t \frac{f(y)\,dy}{(t^2 - y^2)^{1/2}} = \frac{x}{(a^2 - x^2)^{1/2}} \int_b^a \frac{(a^2 - y^2)^{1/2} f(y)\,dy}{y^2 - x^2},$$

$$\int_0^a \frac{dx}{(\rho^2 - x^2)^{1/2}} \frac{d}{dx} \int_0^x \frac{f(y)\,dy}{(x^2 - y^2)^{1/2}} = (\rho^2 - a^2)^{1/2} \int_0^a \frac{f(y)\,dy}{(\rho^2 - y^2)(a^2 - y^2)^{1/2}},$$

$$\int_a^\infty \frac{x\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} \int_x^\infty \frac{\mathrm{d}f(t)}{(t^2 - x^2)^{1/2}} = \int_a^\infty \frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^\infty \frac{f(t)t\mathrm{d}t}{(t^2 - x^2)^{1/2}}$$

$$= (a^2 - \rho^2)^{1/2} \int_a^\infty \frac{f(t)t\mathrm{d}t}{(t^2 - a^2)^{1/2}(t^2 - \rho^2)}.$$

Computation and/or transformation of some integrals:

$$\int_x^a \frac{\mathrm{d}\rho}{\rho^{2n+1}(\rho^2 - x^2)^{1/2}} \int_\rho^a \frac{t^{2n-1}\mathrm{d}t}{(a^2 - t^2)^{1/2}} = \frac{\pi(a^{2n} - x^{2n})}{4nax^{2n+1}},$$

$$\int_x^a \frac{\mathrm{d}\rho}{\rho^{2n-1}(\rho^2 - x^2)^{1/2}} \int_\rho^a \frac{t^{2n-1}\mathrm{d}t}{(a^2 - t^2)^{1/2}} = \frac{\pi(a^{2n-1} - x^{2n-1})}{2(2n - 1)x^{2n-1}},$$

$$\int_x^a \frac{\mathrm{d}\rho}{\rho^{2n-1}(a^2 - \rho^2)^{1/2}(\rho^2 - x^2)^{1/2}} = \frac{1}{(ax)^{2n-1}} \int_x^a \frac{\rho^{2n-1}\mathrm{d}\rho}{(a^2 - \rho^2)^{1/2}(\rho^2 - x^2)^{1/2}}$$

$$= \frac{\pi}{2ax^{2n-1}} F(1-n,\ \tfrac{1}{2};\ 1;\ 1-\frac{x^2}{a^2}) = \frac{\sqrt{\pi}\Gamma(n - 1/2)}{2ax^{2n-1}\Gamma(n)} F(1-n,\ \tfrac{1}{2};\ \tfrac{3}{2}-n;\ \frac{x^2}{a^2}),$$

$$\int_x^a \frac{(a^2 - \rho^2)^{1/2}\mathrm{d}\rho}{\rho^{2n+1}(\rho^2 - x^2)^{1/2}} = \frac{1}{a^{2n-1}x^{2n+1}} \int_x^a \frac{(\rho^2 - x^2)^{1/2}\rho^{2n-1}\mathrm{d}\rho}{(a^2 - \rho^2)^{1/2}} = \frac{\pi(a^2 - x^2)}{4ax^{2n+1}}$$

$$\times F(1-n,\ \tfrac{1}{2};\ 2;\ 1-\frac{x^2}{a^2}) = \frac{\sqrt{\pi}\Gamma(n + 1/2)(a^2 - x^2)}{2n!ax^{2n+1}} F(1-n,\ \tfrac{1}{2};\ \tfrac{1}{2}-n;\ \frac{x^2}{a^2})$$

$$\int_x^a \frac{(\rho^2 - x^2)^{1/2}\mathrm{d}\rho}{\rho^{2n+1}(a^2 - \rho^2)^{1/2}} = \frac{1}{a^{2n+1}x^{2n-1}} \int_x^a \frac{(a^2 - \rho^2)^{1/2}\rho^{2n-1}\mathrm{d}\rho}{(\rho^2 - x^2)^{1/2}}$$

$$= \frac{\pi(a^2 - x^2)}{4a^3x^{2n-1}} \; F\left(1-n, \frac{3}{2}; 2; 1-\frac{x^2}{a^2}\right).$$

$$\int_a^\rho \frac{x^{2n}\,dx}{(\rho^2 - x^2)^{1/2}} = a^{2n+1}\rho^{2n}\int_a^\rho \frac{dx}{x^{2n+1}(x^2 - a^2)^{1/2}},$$

$$\int_\rho^a \frac{x^{2n}\,dx}{(x^2 - \rho^2)^{1/2}} = a^{2n+1}\rho^{2n}\int_\rho^a \frac{dx}{x^{2n+1}(a^2 - x^2)^{1/2}}.$$

$$\int_x^a \frac{d\rho}{\rho^{2n-1}(\rho^2 - x^2)^{1/2}(\rho^2 - t^2)^{1/2}} = \frac{(a^2 - x^2)^{1/2}(a^2 - t^2)^{1/2}}{a^2x^{2n-2}(x^2 - t^2)} \sum_{m=0}^{n-1}\left(\frac{a^2 - x^2}{a^2}\right)^m$$

$$\times\frac{\Gamma(n)}{\Gamma(n - m)(m!)^2}\,\frac{d^m}{d\zeta^m}\left[(1 - \zeta)^{m-1/2}\,\frac{\sin^{-1}\sqrt{\zeta}}{\sqrt{\zeta}}\right], \text{ with } \zeta = -\frac{t^2(a^2 - x^2)}{a^2(x^2 - t^2)}.$$

$$\int_a^\infty \frac{dx}{x^{2n+1}(x^2 - a^2)^{1/2}(x^2 - \rho^2)} = \frac{1}{a^{2n+1}\rho^{2n+2}}\int_0^\rho \frac{x^{2n+2}\,dx}{(\rho^2 - x^2)^{1/2}(a^2 - x^2)}$$

$$= \frac{1}{a^{2n+1}\rho^{2n+2}}\left[\frac{\pi a^{2n+1}}{2(a^2 - \rho^2)^{1/2}} - \frac{\sqrt{\pi}}{2}\sum_{k=0}^{n}\frac{\Gamma(k+1/2)}{\Gamma(k+1)}\,a^{2(n-k)}\rho^{2k}\right],$$

Rules of interchanging the order of integration:

$$\int_0^a F(r)\,dr\,\frac{d}{dr}\int_0^r \frac{f(\rho)\rho\,d\rho}{(r^2 - \rho^2)^{1/2}} = -\int_0^a f(\rho)\,d\rho\,\frac{d}{d\rho}\int_\rho^a \frac{F(r)r\,dr}{(r^2 - \rho^2)^{1/2}} +$$

$$\lim_{\rho\to 0}\left\{\rho f(\rho)\frac{d}{d\rho}\int_\rho^a \frac{F(r)r\,dr}{(r^2 - \rho^2)^{1/2}}\right\} + \lim_{\rho\to a}\left\{f(\rho)\int_\rho^a \frac{F(r)r\,dr}{(r^2 - \rho^2)^{1/2}}\right\},$$

$$\int_a^\infty F(\rho)\mathrm{d}\rho\ \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{f(x)x\mathrm{d}x}{(x^2-\rho^2)^{1/2}} = -\int_a^\infty f(x)\mathrm{d}x\ \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{F(\rho)\rho\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}},$$

$$\int_0^a \frac{\mathrm{d}t}{t^2-x^2}\int_0^a \frac{f(t,r)\mathrm{d}r}{r^2-t^2} = -\frac{\pi^2 f(x,x)}{4x^2} + \int_0^a \mathrm{d}r\int_0^a \frac{f(t,r)\mathrm{d}t}{(t^2-x^2)(r^2-t^2)},$$

$$\int_0^a \mathrm{d}t\int_0^a \frac{f(t,r)\mathrm{d}r}{r^2-t^2} = -\frac{\pi^2}{4}\ f(0,0) + \int_0^a \mathrm{d}r\int_0^a \frac{f(t,r)\mathrm{d}t}{(r^2-t^2)}.$$