CHAPTER 2

MIXED BOUNDARY VALUE PROBLEMS IN ELASTICITY

Elastic half-space has been proven to be a useful mathematical model for consideration of various contact and crack problems in finite bodies, provided that the domain of contact or the crack size is much smaller than the characteristic dimension of the body. A general solution in terms of three harmonic functions is presented for the case of transverse isotropy. Exact closed form solutions are given to the mixed problems of the first and second type, with various applications considered. The material in this Chapter is based on the papers (Fabrikant, 1970, 1971b, 1971c, 1985b, 1986g).

2.1 General solution

Consider a transversely isotropic elastic body which is characterized by five elastic constants A_{ik} defining the following stress-strain relationships:

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x} + (A_{11} - 2A_{66})\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11} - 2A_{66})\frac{\partial u_x}{\partial x} + A_{11}\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x} + A_{13}\frac{\partial u_y}{\partial y} + A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \qquad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau_{zx} = A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z} \right). \tag{2.1.1}$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 , \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 ,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 . \tag{2.1.2}$$

Substitution of (2.1.1) in (2.1.2) yields:

$$A_{11}\frac{\partial^2 u_x}{\partial x^2} + A_{66}\frac{\partial^2 u_x}{\partial y^2} + A_{44}\frac{\partial^2 u_x}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_y}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0 ,$$

$$A_{66}\frac{\partial^2 u_y}{\partial x^2} + A_{11}\frac{\partial^2 u_y}{\partial y^2} + A_{44}\frac{\partial^2 u_y}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_x}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial y \partial z} = 0 ,$$

$$A_{44}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{44} + A_{13})\left[\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} \right] = 0 . \tag{2.1.3}$$

Introduce complex tangential displacements $u=u_x+iu_y$, and $\overline{u}=u_x-iu_y$. This will allow us to reduce the number of equations in (2.1.3) by one, and to rewrite these equations in a more compact manner, namely,

$$\frac{1}{2}(A_{11} + A_{66})\Delta u + A_{44}\frac{\partial^2 u}{\partial z^2} + \frac{1}{2}(A_{11} - A_{66})\Lambda^2\overline{u} + (A_{13} + A_{44})\Lambda\frac{\partial w}{\partial z} = 0,$$

$$A_{44}\Delta w + A_{33}\frac{\partial^2 w}{\partial z^2} + \frac{1}{2}(A_{13}+ A_{44})\frac{\partial}{\partial z}(\overline{\Lambda}u + \Lambda\overline{u}) = 0. \tag{2.1.4}$$

Here the following differential operators were used:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \qquad \Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} , \tag{2.1.5}$$

and the overbar everywhere indicates the complex conjugate value. Note also that $\Delta=\Lambda\bar{\Lambda}$. One can verify that equations (2.1.4) can be satisfied by

$$u = \Lambda(F_1 + F_2 + iF_3), \qquad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z} \tag{2.1.6}$$

where all three functions F_k satisfy the equation (Elliott, 1948):

$$\Delta F_k + \gamma_k^2 \frac{\partial^2 F_k}{\partial z^2} = 0, \qquad \text{for } k = 1,2,3, \tag{2.1.7}$$

and the values of m_k and γ_k are related by the following expressions (Elliott, 1948):

$$\frac{A_{44} + m_k(A_{13}+A_{44})}{A_{11}} = \frac{m_k A_{33}}{m_k A_{44} + A_{13} + A_{44}} = \gamma_k^2, \qquad \text{for } k=1,2;$$

$$\gamma_3 = \left(A_{44}/A_{66}\right)^{1/2}. \tag{2.1.8}$$

Introducing the notation $z_k=z/\gamma_k$, for $k=1,2,3$, we may call function $F_k=F(x,y,z_k)$ harmonic. Note the property $m_1m_2=1$, which seems to have escaped the attention of previous researchers, and which will help us to simplify various expressions to follow. The other elastic constants which will be used throughout the book are:

$$G_1 = \beta + \gamma_1\gamma_2 H , \qquad G_2 = \beta - \gamma_1\gamma_2 H ,$$

$$H = \frac{(\gamma_1 + \gamma_2)A_{11}}{2\pi(A_{11}A_{33} - A_{13}^2)} , \qquad \alpha = \frac{(A_{11}A_{33})^{1/2} - A_{13}}{A_{11}(\gamma_1 + \gamma_2)} , \qquad \beta = \frac{\gamma_3}{2\pi A_{44}}. \tag{2.1.9}$$

Introduce the following inplane stress components:

$$\sigma_1 = \sigma_x + \sigma_y , \qquad \sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy} , \qquad \tau_z = \tau_{zx} + i\tau_{yz}. \tag{2.1.10}$$

This will simplify expressions (2.1.1), namely

$$\sigma_1 = (A_{11} - A_{66})(\bar{\Lambda}u + \Lambda\bar{u}) + 2A_{13}\frac{\partial w}{\partial z} , \qquad \sigma_2 = 2A_{66}\Lambda u ,$$

$$\sigma_z = \frac{1}{2}A_{13}(\overline{\Lambda}u + \Lambda\overline{u}) + A_{33}\frac{\partial w}{\partial z} , \qquad \tau_z = A_{44}\left[\frac{\partial u}{\partial z} + \Lambda w \right]. \tag{2.1.11}$$

We have now only four components of stress, instead of six, as it was in (2.1.1). The substitution of (2.1.6) in (2.1.11) yields:

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2} \{[\gamma_1^2 - (1 + m_1)\gamma_3^2]F_1 + [\gamma_2^2 - (1 + m_2)\gamma_3^2]F_2\} ,$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + \mathrm{i}F_3) ,$$

$$\sigma_z = A_{44} \frac{\partial^2}{\partial z^2} [(1 + m_1)\gamma_1^2F_1 + (1 + m_2)\gamma_2^2F_2]$$

$$= -A_{44} \Delta[(1 + m_1)F_1 + (1 + m_2)F_2] ,$$

$$\tau_z = A_{44}\Lambda \frac{\partial}{\partial z} [(1 + m_1)F_1 + (1 + m_2)F_2 + \mathrm{i}F_3] . \tag{2.1.12}$$

Here we used the fact that each F_k satisfies equation (2.1.7), and the relation: $A_{11}\gamma_k^2 - A_{13}m_k = A_{44}(1+m_k)$, (for k=1,2) which is an immediate consequence of (2.1.8). Expressions (2.1.6) and (2.1.12) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x,y,z) = c_kF(x,y,z_k) , \tag{2.1.13}$$

where $z_k = z/\gamma_k$, and c_k is an as yet unknown complex constant. As we shall see further, this is possible indeed. All the results obtained in the book are valid for isotropic solids, provided that we take

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \qquad H = \frac{1 - \nu^2}{\pi E}, \qquad \alpha = \frac{1 - 2\nu}{2(1 - \nu)},$$

$$\beta = \frac{1 + \nu}{\pi E}, \qquad G_1 = \frac{(2 - \nu)(1 + \nu)}{\pi E}, \qquad G_2 = \frac{\nu(1 + \nu)}{\pi E}, \tag{2.1.14}$$

where E is the elastic modulus, and ν is Poisson coefficient.

Exercise 2.1

1. Establish the equivalence of (2.1.3) and (2.1.4)

2. Prove that the solution (2.1.6) satisfies (2.1.3), provided that the condition (2.1.7) is met.

3. Prove the identity $m_1 m_2=1$.

4. Prove the identity $A_{11}\gamma_k^2 - A_{13}m_k = A_{44}(1+m_k)$.

2.2 Point force solutions

The field of stresses and displacements due to a concentrated load is important for the integral equation formulation of various mixed boundary value problems. Two cases are considered here: an arbitrary point load in a transversely isotropic elastic space, and the action of an arbitrary concentrated force on the boundary of a similar half-space. Though these problems have been solved by many authors, we follow here the results given in (Fabrikant, 1970). The main reason for this is the simplification of the elastic coefficients, which seems to have escaped the attention of other authors. Here is an example: one of the coefficients in (Chen, 1966) reads

$$\frac{A_{13} + A_{44}}{A_{11}A_{44}(\gamma_1^2 - \gamma_2^2)}\left(\frac{A_{33}m_1}{\gamma_1^2} - A_{13}\right).$$

This expression, after simplification, reduces to $1/(1-m_2)$.

Let a point force, with components T_x, T_y, and P in Cartesian coordinates be applied at the point N_0 inside a transversely isotropic elastic space. We may assume, without loss of generality, that the polar cylindrical coordinates of N_0 are $(\rho_0,\phi_0,0)$. We need to find the field of stresses and displacements at the point $M(\rho,\phi,z)$. Introduce the complex tangential force $T=T_x+iT_y$. The general solution can be expressed through the three potential functions:

$$F_1 = \frac{1}{4\pi A_{44}(m_1 - m_2)}\left[\frac{1}{2}\gamma_1 m_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)\right],$$

$$F_2 = -\frac{1}{4\pi A_{44}(m_1 - m_2)}\left[\frac{1}{2}\gamma_2 m_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{8\pi A_{44}}\,(\overline{\Lambda}\chi_3 - \Lambda\overline{\chi}_3). \tag{2.2.1}$$

Here the notation was introduced

$$\chi_k(z) = \chi(z_k),\ R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2},\ \text{for } k=1,2,3;$$

$$\chi(z) = T[z\ln(R_0 + z) - R_0]. \tag{2.2.2}$$

The displacements are defined by (2.1.6) as follows:

$$u = \frac{1}{4\pi A_{44}(m_1 - m_2)}\left\{\frac{1}{2}\gamma_1 m_2\left[-\frac{T}{R_1} + \frac{q^2\overline{T}}{R_1(R_1 + z_1)^2}\right]\right.$$

$$\left. - \frac{1}{2}\gamma_2 m_1\left[-\frac{T}{R_2} + \frac{q^2\overline{T}}{R_2(R_2 + z_2)^2}\right] - \frac{P}{\overline{q}}\left[\frac{z_1}{R_1} - \frac{z_2}{R_2}\right]\right\}$$

$$+ \frac{\gamma_3}{8\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3 + z_3)^2}\right], \tag{2.2.3}$$

$$w = \frac{1}{4\pi A_{44}(m_1 - m_2)}\left\{\frac{1}{2}\left[\frac{T}{q} + \frac{\overline{T}}{\overline{q}}\right]\left(-\frac{z_1}{R_1} + \frac{z_2}{R_2}\right) + P\left[\frac{m_1}{\gamma_1 R_1} - \frac{m_2}{\gamma_2 R_2}\right]\right\}. \tag{2.2.4}$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{2.2.5}$$

The stress field is defined by (2.1.12). We shall need only the expressions for σ_z and τ_z. Here they are:

$$\sigma_z = -\frac{1}{4\pi}\left\{\frac{1}{2}(T\overline{q} + \overline{T}q)\left[\frac{\gamma_1}{(m_1 - 1)R_1^3} + \frac{\gamma_2}{(m_2 - 1)R_2^3}\right]\right.$$

$$\left. + P\left[\frac{m_1 z_1}{(m_1 - 1)R_1^3} + \frac{m_2 z_2}{(m_2 - 1)R_2^3}\right]\right\},$$

(2.2.6)

$$\tau_z = \frac{T}{8\pi}\left[\frac{z_1}{(m_1-1)R_1^3} + \frac{z_2}{(m_2-1)R_2^3} - \frac{z_3}{R_3^3}\right]$$

$$-\frac{\bar{T}q^2}{8\pi}\left[\frac{2R_1+z_1}{(m_1-1)R_1^3(R_1+z_1)^2} + \frac{2R_2+z_2}{(m_2-1)R_2^3(R_2+z_2)^2} + \frac{2R_3+z_3}{R_3^3(R_3+z_3)^2}\right]$$

$$-\frac{Pq}{4\pi}\left[\frac{m_1}{\gamma_1(m_1-1)R_1^3} + \frac{m_2}{\gamma_2(m_2-1)R_2^3}\right]. \tag{2.2.7}$$

Consider a transversely isotropic elastic half-space $z \geq 0$. Let a concentrated force, with components T_x, T_y, and P, be applied at the point $N_0(\rho_0,\phi_0,0)$. We need to find the field of stresses and displacements in the half-space. The potential functions are defined by

$$F_1 = \frac{H\gamma_1}{m_1-1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1+z_1)\right],$$

$$F_2 = \frac{H\gamma_2}{m_2-1}\left[\frac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2+z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \tag{2.2.8}$$

Substitution of (2.2.8) in (2.1.6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\bar{T}}{R_3(R_3+z_3)^2}\right]$$

$$+\frac{H\gamma_2}{m_2-1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\bar{T}}{R_2(R_2+z_2)^2}\right] + \frac{Pq}{R_2(R_2+z_2)}\right\}$$

$$+\frac{H\gamma_1}{m_1-1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\bar{T}}{R_1(R_1+z_1)^2}\right] + \frac{Pq}{R_1(R_1+z_1)}\right\},$$

(2.2.9)

$$w = H\left\{\frac{1}{2}(T\bar{q} + \bar{T}q)\left[\frac{\gamma_2 m_1}{(m_1 - 1)R_1(R_1 + z_1)} + \frac{\gamma_1 m_2}{(m_2 - 1)R_2(R_2 + z_2)}\right]\right.$$

$$\left. + P\left[\frac{m_1}{(m_1 - 1)R_1} + \frac{m_2}{(m_2 - 1)R_2}\right]\right\}. \tag{2.2.10}$$

We shall need expressions for the following stress components:

$$\sigma_z = \frac{1}{2\pi(\gamma_1 - \gamma_2)}\left\{\left[\frac{1}{2}\gamma_1\gamma_2(T\bar{q} + \bar{T}q) + Pz\right]\left(-\frac{1}{R_1^3} + \frac{1}{R_2^3}\right)\right\},$$

$$\tau_z = \frac{\gamma_2}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{Tz_1}{R_1^3} - \frac{\bar{T}q^2(2R_1 + z_1)}{R_1^3(R_1 + z_1)^2}\right] - \frac{\gamma_1}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{Tz_2}{R_2^3} - \frac{\bar{T}q^2(2R_2 + z_2)}{R_2^3(R_2 + z_2)^2}\right]$$

$$- \frac{1}{4\pi}\left[\frac{Tz_3}{R_3^3} + \frac{\bar{T}q^2(2R_3 + z_3)}{R_3^3(R_3 + z_3)^2}\right] + \frac{Pq}{2\pi(\gamma_1 - \gamma_2)}\left[-\frac{1}{R_1^3} + \frac{1}{R_2^3}\right]. \tag{2.2.11}$$

Expressions (2.2.9) and (2.2.10) simplify for the case when z=0

$$u = \frac{1}{2}G_1\,\frac{T}{R} + \frac{1}{2}G_2\,\frac{\bar{T}q^2}{R^3} - H\alpha\,\frac{P}{\bar{q}}\ , \tag{2.2.12}$$

$$w = H\alpha\Re\left(\frac{T}{q}\right) + H\frac{P}{R}. \tag{2.2.13}$$

Here H, α, G_1, and G_2 are defined by (2.1.9), and

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]^{1/2}. \tag{2.2.14}$$

Expressions (2.2.12) and (2.2.13) will be used for the integral equation formulation of various mixed boundary value problems in an elastic half-space.

The following classification of mixed boundary value problems may be suggested. The problem is called *internal mixed* when the normal/tangential displacements are prescribed *inside* a finite domain, while the relevant tractions are prescribed on the rest of the half-space boundary. In the case when the displacements are given *outside* a finite domain, the problem is called *external.*

We can specify two types of internal problems. The internal problem of *type I*: the *normal* displacements are prescribed *inside* a finite domain S, the normal traction is given *outside* the domain S, while the tangential tractions are known all over the plane z=0. The internal problem of *type II*: the *tangential* displacements are prescribed *inside* S, and the shear tractions are given outside, while the normal traction is known all over the plane z=0. The external problems of types I and II are defined in the same way as the internal ones above, with an interchange of the terms *traction* and *displacement*. These are the four types of problems which will be considered in this chapter. We shall call a problem mixed-mixed when the boundary conditions are mixed with respect to both normal and tangential components. These problems will be considered in the next chapter.

Exercise 2.2

1. Establish (2.2.3) and (2.2.4).

2. Verify (2.2.6) and (2.2.7).

3. Derive expressions for σ_1 and σ_2 in both cases of concentrated load, considered in section 2.2.

4. Derive the equivalent solutions for an isotropic body.

5. Consider the case of arbitrary point loading applied *inside* an elastic half-space.

2.3 Internal mixed problem of type I.

Consider a transversely isotropic elastic half-space $z\geq 0$. Introduce a set of polar cylindrical coordinates (ρ,ϕ,z). Let the following boundary conditions be prescribed on the plane z=0:

$$w = w(\rho,\phi), \quad \rho\leq a, \; 0\leq\phi<2\pi,$$

$$\sigma = \sigma(\rho,\phi), \quad \rho>a, \; 0\leq\phi<2\pi,$$

$$\tau = \tau(\rho,\phi), \quad 0\leq\rho<\infty, \; 0\leq\phi<2\pi, \tag{2.3.1}$$

Here σ stands for the normal loading, and τ is the complex shear loading, namely, $\tau=\tau_{zx}+i\tau_{yz}$. The governing integral equation can be written by using (2.2.13), namely,

$$H\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}} = f(\rho,\phi). \tag{2.3.2}$$

We use the same notation σ for the unknown normal loading inside the circle $\rho\leq a$, as well as for the prescribed function σ outside the circle. This should not create any confusion since the argument (ρ_0,ϕ_0) provides a clear distinction. Function f is known from the conditions (2.3.1), and is

$$f(\rho,\phi) = w(\rho,\phi) - \int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$- H\alpha\Re\int_0^{2\pi}\int_0^{\infty}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}. \tag{2.3.3}$$

As soon as equation (2.3.2) is solved, and the value of σ inside the circle becomes known, the tangential displacements in the plane z=0 can be defined by (2.2.12):

$$u = \frac{1}{2}G_1\int_0^{2\pi}\int_0^{\infty}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$+ \frac{1}{2}G_2\int_0^{2\pi}\int_0^{\infty}\frac{[\rho e^{i\phi}-\rho_0 e^{i\phi_0}]^2\overline{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{3/2}} - H\alpha\int_0^{2\pi}\int_0^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi}-\rho_0 e^{-i\phi_0}}. \tag{2.3.4}$$

Integral equation (2.3.2) was solved in section 1.4. It seems useful to consider here a more general case:

$$H\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}} = f(\rho,\phi), \tag{2.3.5}$$

where $-1<\kappa<1$. This type of equation arises in the problems of nonhomogeneous elastic half-space, with the modulus of elasticity E being a power function of z, namely, $E=E_0z^{\kappa}$. Of course, in the nonhomogeneous case H is no longer defined by (2.1.9). The reader is referred to the paper by Rostovtsev (1964) for details. Rostovtsev (1964) obtained an exact solution of (2.3.5) in Fourier series. Here we present a closed form solution.

By using the integral representation (1.1.4), integral equation (2.3.5) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2}\int_0^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) = f(\rho,\phi). \tag{2.3.6}$$

Integral equation (2.3.6) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of (1.4.5). The first operator to be applied to both sides of (2.3.6) is

$$\mathcal{L}\left(\frac{1}{t}\right) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\rho \mathrm{d}\rho}{(t^2 - \rho^2)^{(1-\kappa)/2}} \mathcal{L}(\rho). \tag{2.3.7}$$

The result of application of (2.3.7) to both sides of (2.3.6) is

$$2\pi H t^\kappa \int_t^a \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - t^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{t}{\rho_0}\right) \sigma(\rho_0,\phi) = \mathcal{L}\left(\frac{1}{t}\right) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\rho \mathrm{d}\rho}{(t^2 - \rho^2)^{(1-\kappa)/2}} \mathcal{L}(\rho) f(\rho,\phi). \tag{2.3.8}$$

The second operator to be applied to both sides of (2.3.8) is

$$\mathcal{L}(y) \frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2 - y^2)^{(1-\kappa)/2}} \mathcal{L}\left(\frac{1}{t}\right),$$

with the result

$$\sigma(y,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2 H y} \mathcal{L}(y) \frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2 - y^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}\left(\frac{1}{t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2 - \rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho)f(\rho,\phi). \tag{2.3.9}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (2.3.9) in the form

$$\sigma(y,\phi) = \frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(a^2 - y^2)^{(1-\kappa)/2}} - \int_y^a \frac{\mathrm{d}t}{(t^2 - y^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{2.3.10}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t^{1+\kappa}}\int_0^t \frac{\rho^{1-\kappa}\mathrm{d}\rho}{(t^2 - \rho^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^{1+\kappa}\mathcal{L}\left(\frac{\rho y}{t^2}\right)f(\rho,\phi)\right]. \tag{2.3.11}$$

Yet another form of solution can be found in (Fabrikant, 1971e). The problem solved above has two major applications: contact problems of a smooth punch pressed against an elastic half-space, and that of an external circular crack in an infinite elastic body. Let us consider both cases in more detail.

Example 1. The smooth punch problem. In elastic contact problems, we have $\sigma=0$, for $\rho>a$, and $\tau=0$ all over the plane $z=0$, so that function $f=w$ (see 2.3.9). It becomes possible to compute the resultant force P and the tilting moments M_x and M_y directly in terms of the prescribed displacement w. Since

$$P = \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\ \rho\mathrm{d}\rho\mathrm{d}\phi, \tag{2.3.12}$$

substitution of (2.3.9) in (2.3.12) yields directly the resultant force

$$P = \frac{\cos(\pi\kappa/2)}{\pi^2 H}\int_0^{2\pi}\int_0^a \frac{w(\rho,\phi)\ \rho\mathrm{d}\rho\mathrm{d}\phi}{(a^2 - \rho^2)^{(1-\kappa)/2}}. \tag{2.3.13}$$

For computation of the tilting moments M_x and M_y, it is convenient to introduce the complex parameter

$$M = M_x + iM_y = -i \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\ e^{i\phi}\ \rho^2 d\rho d\phi. \tag{2.3.14}$$

By using (2.3.9), the final expression for the tilting moment is found to be

$$M = -i\ \frac{2\cos(\pi\kappa/2)}{\pi^2 H(1+\kappa)} \int_0^{2\pi}\int_0^a \frac{w(\rho,\phi)\ e^{i\phi}\rho^2 d\rho d\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}. \tag{2.3.15}$$

Expressions (2.3.14) and (2.3.15) are in agreement with similar results of Rostovtsev (1964).

By reviewing the derivation of expression (2.3.6), one may find that it is valid for evaluating the normal displacements *outside* the contact region, if the upper limit of integration ρ is replaced by a. Substitution of (2.3.9) into the modified form of (2.3.6) results in

$$w(\rho,\phi) = \frac{2\cos(\pi\kappa/2)}{\pi} \int_0^a \frac{dx}{(\rho^2-x^2)^{(1+\kappa)/2}} \frac{d}{dx}\int_0^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1-\kappa)/2}}\ \mathcal{L}\left(\frac{\rho_0}{\rho}\right) w(\rho_0,\phi),$$

$$\text{for } \rho>a. \tag{2.3.16}$$

Performing differentiation of the integrand, and then integrating by parts, we obtain

$$w(\rho,\phi) = \frac{1}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)(\rho^2-a^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_0^a \frac{w(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{(1-\kappa)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]},$$

$$\text{for } \rho>a. \tag{2.3.17}$$

Here the following identities were employed (Bateman and Erdélyi, 1955)

$$\frac{d}{d\zeta}\left[\zeta^{(1+\kappa)/2}\ F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; \zeta\right)\right] = \frac{1+\kappa}{2}\ \zeta^{-(1-\kappa)/2}(1-\zeta)^{-(1+\kappa)/2},$$

$$\frac{d}{dx}\int_0^x \frac{f(t)t\ dt}{(x^2-t^2)^{(1-\kappa)/2}} = f(0)x^{\kappa} + x\int_0^x \frac{df(t)}{(x^2-t^2)^{(1-\kappa)/2}}. \tag{2.3.18}$$

All the quantities of interest, namely, the pressure exerted by the punch σ, the resultant force P, the tilting moment M, and the normal displacement outside the punch, can be expressed directly through the prescribed normal displacement w by formulae (2.3.9), (2.3.13), (2.3.15), and (2.3.17) respectively.

Example 2. External crack in non-homogeneous elasticity. Consider a non-homogeneous elastic space with modulus of elasticity $E=E_0|z|^\kappa$, E_0=const, $|\kappa|<1$. This space is weakened by a circular external crack $\rho\geq a$. An arbitrary pressure $\sigma(\rho,\phi)$ is applied to both faces of the crack in opposite directions. The problem is to find the normal stress in the crack neck, the normal displacements of the crack faces, the stress intensity factor, and the work required to open up the crack.

Due to the symmetry of the problem, it may be reduced to the mixed boundary value problem of a half-space, subject to the boundary conditions at z=0:

$$w = 0, \qquad \tau = 0, \qquad \text{for } 0\leq\rho\leq a, \quad 0\leq\phi<2\pi;$$

$$\sigma = \sigma(\rho,\phi), \qquad \tau = 0, \qquad \text{for } a<\rho<\infty, \quad 0\leq\phi<2\pi. \tag{2.3.19}$$

The governing integral equation takes the form (2.3.5), with the known function

$$f(\rho,\phi) = -H\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \tag{2.3.20}$$

Its solution can be found in exactly the same manner as that of (1.5.21), and is

$$\sigma(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2(a^2 - \rho^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_a^{\infty}\frac{(\rho_0^2 - a^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)}. \tag{2.3.21}$$

Expression (2.3.21) gives the normal stress in the crack neck through the pressure applied to the crack faces. Note that (1.5.24) may be considered as a particular case of (2.3.21), when κ=0.

The normal displacement of the crack faces can be evaluated as a superposition of the displacement caused by the applied pressure, and the displacement due to the normal stress in the crack neck. By using a procedure analogous to the one described in section 1.5, we can obtain the expression

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\left\{\int_\rho^{\infty}\frac{x^\kappa dx}{(x^2 - \rho^2)^{(1+\kappa)/2}}\int_a^{x}\frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{(1+\kappa)/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$+\int_0^a \frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\Bigg\}, \quad \text{for } \rho>a. \tag{2.3.22}$$

Substitution of (2.3.21) in (2.3.22) leads, after simplification, to

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2} \int_a^\rho \frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}} \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi),$$

$$\text{for } \rho>a. \tag{2.3.23}$$

The normal displacements of the crack faces are now defined in terms of the applied pressure.

Introduce the stress intensity factor as

$$k_1(\phi) = \lim_{\rho\to a}[(a-\rho)^{(1-\kappa)/2}\sigma(\rho,\phi)]. \tag{2.3.24}$$

Substitution of (2.3.21) in (2.3.24) gives

$$k_1(\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2a)^{(1-\kappa)/2}} \int_a^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-a^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{a}{\rho_0}\right)\sigma(\rho_0,\phi).$$

Introduce the *stress intensity function*:

$$K_1(\rho,\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_\rho^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{2.3.25}$$

It is obvious that the limiting case of the stress intensity function, when $\rho\to a$, is the stress intensity factor. By using the property of the $\mathcal{L}$-operators (1.2.3) we may rewrite (2.3.23) as

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2} \int_a^\rho \frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho}\right) \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_a^\rho \frac{x^{(1+\kappa)/2}\mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho}\right) K_1(x,\phi). \tag{2.3.26}$$

The energy W may be defined by the integral

$$W = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)w(\rho,\phi)\rho \ \mathrm{d}\rho\mathrm{d}\phi. \tag{2.3.27}$$

Substitution of (2.3.26) in (2.3.27) gives

$$W = 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} \mathrm{d}\phi \int_a^\infty \sigma(\rho,\phi)\rho \ \mathrm{d}\rho \int_a^\rho \frac{x^{(1+\kappa)/2} \ \mathrm{d}x}{(\rho^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} \mathrm{d}\phi \int_a^\infty x^{(1+\kappa)/2}\mathrm{d}x \int_x^\infty \frac{\sigma(\rho,\phi)\rho \ \mathrm{d}\rho}{(\rho^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} \mathrm{d}\phi \int_a^\infty K_1(x,\phi)x^{(1+\kappa)/2}\mathrm{d}x \int_x^\infty \frac{\rho \ \mathrm{d}\rho}{(\rho^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x}{\rho}\right) \sigma(\rho,\phi).$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (2.3.25) yields the final result

$$W = \frac{2^{1-\kappa}\pi^2 H}{\cos(\pi\kappa/2)} \int_0^{2\pi}\int_a^\infty [K_1(\rho,\phi)]^2\rho \ \mathrm{d}\rho \ \mathrm{d}\phi. \tag{2.3.28}$$

Expression (2.3.28) interprets the stress intensity function squared as being proportional to the energy per unit area required to open up the crack. In the case of axial symmetry, formula (2.3.28) simplifies to

$$W = \frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)} \int_a^\infty [K_1(\rho)]^2\rho \ \mathrm{d}\rho,$$

with

$$K_1(\rho) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_\rho^\infty \frac{\sigma(\rho_0,\phi_0)\rho_0 \; d\rho_0}{(\rho_0^2 - \rho^2)^{(1+\kappa)/2}} .$$

All the results obtained become valid for a transversely isotropic space provided that $\kappa=0$, and H is defined by (2.1.9). When H is defined by (2.1.14), we have the results for isotropic body, namely,

$$W = 2\pi \frac{1 - \nu^2}{E} \int_0^{2\pi}\int_a^\infty [K_1(\rho,\phi)]^2\rho \; d\rho \; d\phi, \tag{2.3.29}$$

with

$$K_1(\rho,\phi) = \frac{\sqrt{2}}{\pi\sqrt{\rho}} \int_\rho^\infty \frac{x \; dx}{(x^2 - \rho^2)^{1/2}} \mathcal{L}\left(\frac{\rho}{x}\right) \sigma(x,\phi). \tag{2.3.30}$$

The case of axial symmetry simplifies (2.3.29) and (2.3.30) as follows:

$$W = 4\pi^2 \frac{1 - \nu^2}{E} \int_a^\infty [K_1(\rho)]^2\rho \; d\rho, \tag{2.3.31}$$

with

$$K_1(\rho) = \frac{\sqrt{2}}{\pi\sqrt{\rho}} \int_\rho^\infty \frac{\sigma(x)x \; dx}{(x^2 - \rho^2)^{1/2}} . \tag{2.3.32}$$

Exercise 2.3.

1. The normal displacements under a flat circular punch are given by $w(\rho,\phi)=w_0+\theta\rho\cos\phi$, with w_0=const, and θ=const. Find the traction distribution σ exerted by the punch.

Answer: $\sigma(\rho,\phi) = \dfrac{\cos(\pi\kappa/2)}{\pi^2 H} \dfrac{w_0 + (2\theta\rho\cos\phi)/(1 + \kappa)}{(a^2 - \rho^2)^{(1-\kappa)/2}}.$

2. In the problem above find the relationships between the applied force P, the tilting moment M, and the punch settlement w_0 and the inclination angle θ.

Answer: $P = \dfrac{2w_0 a^{1+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)}$, $\quad M = \dfrac{4\theta a^{3+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}$.

3. The normal displacements under a paraboloidal punch are $w(\rho,\phi)=w_0-c\rho^2$, with w_0=const, and c=const. Find the traction distribution σ and the radius of contact a.

Solution: utilization of (2.3.9) yields

$$\sigma(\rho) = \frac{\cos(\pi\kappa/2)}{\pi^2 H(a^2-\rho^2)^{(1-\kappa)/2}}\left\{w_0 + \frac{2c[a^2(1-\kappa)-2\rho^2]}{(1+\kappa)^2}\right\}.$$

The radius of contact a is found from the condition $\sigma(a)=0$. The result is $a=[(1+\kappa)w_0/(2c)]^{1/2}$, and the final expression for the traction is

$$\sigma(\rho) = \frac{2w_0\cos(\pi\kappa/2)}{\pi^2 a^2 H(1+\kappa)}\,(a^2-\rho^2)^{(1+\kappa)/2}.$$

4. In the problem above find the relationship between the punch settlement w_0 and the applied force P.

Answer: $P = \dfrac{4a^{1+\kappa}w_0\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}$.

5. Consider an external circular crack $\rho>a$. Find the traction distribution σ in the crack neck due to the action of a pair of equal concentrated forces P, applied normally to the crack faces in opposite directions at the point (b,ψ).

Answer: $\sigma(\rho,\phi) = -\dfrac{P}{\pi^2}\cos\left(\dfrac{\pi\kappa}{2}\right)\left(\dfrac{b^2-a^2}{a^2-\rho^2}\right)^{(1-\kappa)/2}\dfrac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}$.

6. Consider an external circular crack $\rho>a$. Let a uniform pressure σ_0 be applied in opposite directions to the annulus $b\le\rho\le c$, $(b>a)$, the rest of the crack faces being traction free. Find the stress distribution σ in the crack neck and the stress intensity factor k_1.

Answer: $\sigma(\rho) = -\dfrac{2\sigma_0\cos(\pi\kappa/2)}{\pi(a^2-\rho^2)^{(1-\kappa)/2}}\displaystyle\int_b^c \frac{(\rho_0^2-a^2)^{(1-\kappa)/2}\rho_0\, d\rho_0}{\rho_0^2-\rho^2}$, for $\rho<a$.

In general, the last integral can be computed in terms of hypergeometric functions. In the particular case of an isotropic body ($\kappa=0$), the integral is computable in elementary functions:

$$\sigma(\rho) = -\frac{2}{\pi}\sigma_0\left[\frac{(c^2 - a^2)^{1/2} - (b^2 - a^2)^{1/2}}{(a^2 - \rho^2)^{1/2}}\right.$$

$$\left. - \tan^{-1}\left(\frac{c^2 - a^2}{a^2 - \rho^2}\right)^{1/2} + \tan^{-1}\left(\frac{b^2 - a^2}{a^2 - \rho^2}\right)^{1/2}\right].$$

The stress intensity factor is

$$k_1 = -\frac{2\sigma_0\cos(\pi\kappa/2)[(c^2 - a^2)^{(1-\kappa)/2} - (b^2 - a^2)^{(1-\kappa)/2}]}{\pi(2a)^{(1-\kappa)/2}(1 - \kappa)}.$$

7. In the preceding problem find the crack opening displacement w.

Answer: $w(\rho) = \dfrac{4\cos(\pi\kappa/2)}{1 - \kappa}\sigma_0\left\{\displaystyle\int_a^{\min(\rho,c)} \frac{(c^2 - x^2)^{(1-\kappa)/2}}{(\rho^2 - x^2)^{(1+\kappa)/2}}\, x^\kappa\, dx\right.$

$$\left. - \int_a^{\min(\rho,b)} \frac{(b^2 - x^2)^{(1-\kappa)/2}}{(\rho^2 - x^2)^{(1+\kappa)/2}}\, x^\kappa\, dx\right\}, \text{ for } \rho>a.$$

8. By using formula (2.3.9), prove the identity

$$\int_0^{2\pi}\int_0^{a} \sigma(\rho,\phi)\rho^{1+|n|}\, e^{in\phi}\, d\rho\, d\phi$$

$$= \frac{(1 + \kappa)\Gamma(1 + |n|)}{2\pi^2\Gamma[|n| + (1 + \kappa)/2]}\cos\frac{\pi\kappa}{2}\int_0^{2\pi}\int_0^{a}\frac{f(\rho,\phi)\rho^{1+|n|}\, e^{in\phi}\, d\rho\, d\phi}{(a^2 - \rho^2)^{(1-\kappa)/2}}.$$

Note: in the particular cases n=0 and n=–1, the last identity transforms into (2.3.13) and (2.3.15) respectively.

9. Define the stress intensity function $K_1(\rho,\phi)$ in terms of the displacement w.

Answer: $K_1(\rho,\phi) = \dfrac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2 H\rho^{(1+\kappa)/2}}\,\mathcal{L}\left(\dfrac{1}{\rho}\right)\dfrac{d}{d\rho}\displaystyle\int_a^{\rho}\frac{x\, dx}{(\rho^2 - x^2)^{(1-\kappa)/2}}\,\mathcal{L}(x)w(x,\phi).$

Hint: perform the inversion of (2.3.26).

10. Prove the identity

$$\lim_{\rho\to a}\left\{\frac{d}{d\rho}\int_a^\rho \frac{(x^2-a^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1-\kappa)/2}}\, f(x)\, dx\right\} = \frac{\pi(1-\kappa)f(a)}{2\cos(\pi\kappa/2)}.$$

Hint: use the substitution $t=(\rho^2-x^2)/(x^2-a^2)$.

11. Use the identity above to express the stress intensity factor in terms of the displacement w.

Answer: $k_1(\phi) = \dfrac{1-\kappa}{2^{2-\kappa}\pi H}\lim_{\rho\to a}\left[\dfrac{w(\rho,\phi)}{(\rho-a)^{(1-\kappa)/2}}\right].$

Hint: compute the limit $\rho\to a$ of the result in Exercise 9.

2.4 External mixed problem of type I.

The problem is characterized by the following mixed boundary conditions on the plane $z=0$:

$$w = w(\rho,\phi), \qquad \rho>a,\ 0\le\phi<2\pi,$$

$$\sigma = \sigma(\rho,\phi), \qquad \rho\le a,\ 0\le\phi<2\pi,$$

$$\tau = \tau(\rho,\phi), \qquad 0\le\rho<\infty,\ 0\le\phi<2\pi, \tag{2.4.1}$$

Here the same notation is used as in the previous section. The governing integral equation can be written by using (2.2.13), namely,

$$H\int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}} = f(\rho,\phi). \tag{2.4.2}$$

We use again the same notation σ for the unknown normal loading inside the circle $\rho\le a$, as well as for the prescribed function σ outside the circle. Function f is known from the second condition (2.4.1), and is

$$f(\rho,\phi) = w(\rho,\phi) - \int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$- H\alpha\Re\int_0^{2\pi}\int_0^\infty \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}. \tag{2.4.3}$$

As soon as equation (2.4.2) is solved, and the value of σ inside the circle becomes known, the tangential displacements in the plane z=0 can be defined by (2.3.4). Integral equation (2.4.2) was solved in section 1.5. We consider again a more general case:

$$H\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}} = f(\rho,\phi), \tag{2.4.4}$$

where $-1<\kappa<1$. The new method allows us to present a closed form solution.

By using the integral representation (1.1.21), for z=0, the integral equation (2.4.4) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2}\int_\rho^\infty\frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi) = f(\rho,\phi). \tag{2.4.5}$$

Integral equation (2.4.5) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of (1.5.2). The first operator to be applied to both sides of (2.4.5) is

$$\mathcal{L}(t)\ \frac{d}{dt}\int_t^\infty\frac{\rho d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\ \mathcal{L}\left(\frac{1}{\rho}\right). \tag{2.4.6}$$

The result of application of (2.4.6) to both sides of (2.4.5) is

$$-2\pi H t^\kappa\int_a^t\frac{\rho_0 d\rho_0}{(t^2-\rho_0^2)^{(1+\kappa)/2}}\ \mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi)$$

$$= \mathcal{L}(t)\ \frac{d}{dt}\int_t^\infty\frac{\rho d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\ \mathcal{L}\left(\frac{1}{\rho}\right) f(\rho,\phi). \tag{2.4.7}$$

The second operator to be applied to both sides of (2.4.7) is

$$\mathcal{L}\left(\frac{1}{y}\right) \frac{d}{dy}\int_a^y \frac{t^{1-\kappa}dt}{(y^2 - t^2)^{(1-\kappa)/2}} \mathcal{L}(t),$$

with the result

$$\sigma(y,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2 Hy} \mathcal{L}\left(\frac{1}{y}\right) \frac{d}{dy}\int_a^y \frac{t^{1-\kappa}dt}{(y^2 - t^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}(t^2) \frac{d}{dt}\int_t^\infty \frac{\rho d\rho}{(\rho^2 - t^2)^{(1-\kappa)/2}} \mathcal{L}\left(\frac{1}{\rho}\right) f(\rho,\phi). \tag{2.4.8}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (2.4.8) in the form

$$\sigma(y,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(y^2 - a^2)^{(1-\kappa)/2}} - \int_a^y \frac{dt}{(y^2 - t^2)^{(1-\kappa)/2}} \frac{d}{dt}\Phi(t,y,\phi)\right]. \tag{2.4.9}$$

Here

$$\Phi(t,y,\phi) = t^{1-\kappa}\int_t^\infty \frac{d\rho}{(\rho^2 - t^2)^{(1-\kappa)/2}} \frac{d}{d\rho}\left[\mathcal{L}\left(\frac{t^2}{\rho y}\right) f(\rho,\phi)\right]. \tag{2.4.10}$$

We can consider again two major applications: contact problems of a smooth punch pressed against an elastic half-space, and that of a penny-shaped crack in an infinite elastic body. Let us consider both cases in more detail.

Example 1. The smooth punch problem. In elastic contact problems we have σ=0, for $\rho<a$, and τ=0 all over the plane z=0, so that function $f=w$. It becomes possible to compute the resulting force P and the tilting moments M_x and M_y directly in terms of the prescribed displacement w. Since

$$P = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi) \rho d\rho d\phi, \tag{2.4.11}$$

substitution of (2.4.8) in (2.4.11) yields directly the resultant force

$$P = \lim_{y\to\infty}\left\{-\frac{\cos(\pi\kappa/2)}{\pi^2 H}\int_a^y \frac{t^{2-\kappa}\mathrm{d}t}{(y^2 - t^2)^{(1-\kappa)/2}}\int_t^\infty \frac{\mathrm{d}\rho}{(\rho^2 - t^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^{2\pi} w(\rho,\phi)\ \mathrm{d}\phi\right\}. \tag{2.4.12}$$

The tilting moment can be found in a similar manner. We can also express the normal displacement inside the circle $\rho\le a$ directly in terms of the prescribed displacement w outside the circle. We substitute (2.4.8) in (2.4.5) keeping in mind that, for $\rho\le a$, the lower limit of integration of the first integral will be a instead of ρ. By using the properties of Abel operators and the $\mathcal{L}$-operators, the following expression can be obtained

$$w(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\int_a^\infty \frac{\mathrm{d}x}{(x^2 - \rho^2)^{(1+\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{(1-\kappa)/2}}\ \mathcal{L}\left(\frac{\rho}{\rho_0}\right)w(\rho_0,\phi). \tag{2.4.13}$$

Carrying out the differentiation of the integrand, interchanging the order of integration, and then integrating with respect to x yields

$$w(\rho,\phi) = -\frac{2\cos(\pi\kappa/2)}{\pi(1 + \kappa)}\int_a^\infty \left(\frac{\rho_0^2 - a^2}{\rho_0^2 - \rho^2}\right)^{(1+\kappa)/2}$$

$$\times F\left(\frac{1 + \kappa}{2},\ \frac{1 + \kappa}{2};\ \frac{3 + \kappa}{2};\ \frac{\rho_0^2 - a^2}{\rho_0^2 - \rho^2}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\left[\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\ w(\rho_0,\phi)\right]\ \mathrm{d}\rho_0.$$

Integration by parts and the differential properties of the Gauss hypergeometric functions (2.3.18) allow us to simplify the last expression, namely,

$$w(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\ (a^2 - \rho^2)^{(1-\kappa)/2}\int_a^\infty \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - a^2)^{(1-\kappa)/2}(\rho_0^2 - \rho^2)}\ \mathcal{L}\left(\frac{\rho}{\rho_0}\right)w(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2}\ \cos\left(\frac{\pi\kappa}{2}\right)(a^2-\rho^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_a^\infty \frac{w(\rho_0,\phi_0)\rho_0\ \mathrm{d}\rho_0\ \mathrm{d}\phi_0}{(\rho_0^2-a^2)^{(1-\kappa)/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]}. \tag{2.4.14}$$

Expression (2.4.14) gives the normal displacements inside a circle $\rho\le a$ directly in terms of the prescribed displacement outside the circle. We note a certain similarity between (2.3.17) and (2.4.14).

Example 2. Penny-shaped crack in non-homogeneous elasticity. Consider a non-homogeneous elastic space with modulus of elasticity $E=E_0|z|^\kappa$, E_0=const, $|\kappa|<1$. This space is weakened by a penny-shaped crack $\rho\le a$. The crack is opened by arbitrary pressure $\sigma(\rho,\phi)$. The problem is to find the normal stress on the plane $z=0$ outside the crack, the crack opening displacement, the stress intensity factor, and the work required to open up the crack.

Due to the symmetry of the problem, it may be reduced to the mixed boundary value problem of a half-space, subject to the boundary conditions at $z=0$:

$$w = 0, \qquad \tau = 0, \qquad \text{for } a<\rho<\infty,\ 0\le\phi<2\pi;$$

$$\sigma = \sigma(\rho,\phi), \qquad \tau = 0, \qquad \text{for } 0\le\rho\le a,\ 0\le\phi<2\pi. \tag{2.4.15}$$

The governing integral equation takes the form (2.4.2), with the known function

$$f(\rho,\phi) = -H\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \tag{2.4.16}$$

Its solution can be found in exactly the same manner as that of (1.4.25), and is

$$\sigma(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2(\rho^2 - a^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)}. \tag{2.4.17}$$

Again, one should notice the similarity between (2.3.21) and (2.4.17). Expression (2.4.17) gives the normal stress in the plane $z=0$ outside the crack in terms of the pressure applied to the crack faces. Note that (1.4.27) may be considered as a particular case of (2.4.17), when $\kappa=0$.

The crack opening displacement can be evaluated as a superposition of the displacement caused by the applied pressure, and the displacement due to the normal stress (2.4.17) outside the crack. By using a procedure analogous to the one described in section 1.4, we obtain the expression

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\left\{\int_a^\infty \frac{x^\kappa dx}{(x^2 - \rho^2)^{(1+\kappa)/2}}\int_a^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{(1+\kappa)/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$+ \int_0^\rho \frac{x^\kappa dx}{(\rho^2 - x^2)^{(1+\kappa)/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\Bigg\}, \text{ for } \rho<a. \tag{2.4.18}$$

Substitution of (2.4.17) in (2.4.18) leads, after simplification, to

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2} \int_\rho^a \frac{x^\kappa dx}{(x^2 - \rho^2)^{(1+\kappa)/2}} \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi),$$

$$\text{for } \rho<a. \tag{2.4.19}$$

The crack opening displacements are now defined in terms of the applied pressure.

Introduce the stress intensity factor as

$$k_1(\phi) = \lim_{\rho\to a}[(\rho - a)^{(1-\kappa)/2}\sigma(\rho,\phi)]. \tag{2.4.20}$$

Substitution of (2.4.17) in (2.4.20) gives

$$k_1(\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2a)^{(1-\kappa)/2}} \int_0^a \frac{\rho_0 d\rho_0}{(a^2 - \rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho_0}{a}\right)\sigma(\rho_0,\phi). \tag{2.4.21}$$

Introduce the *stress intensity function*:

$$K_1(\rho,\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_0^\rho \frac{\rho_0 d\rho_0}{(\rho^2 - \rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi). \tag{2.4.22}$$

One can see that the limiting case of the stress intensity function, when $\rho\to a$, is the stress intensity factor. By using the property of the $\mathcal{L}$-operators (1.2.3) we may rewrite (2.4.19) as

$$w(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2} \int_\rho^a \frac{x^\kappa dx}{(x^2 - \rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma(\rho_0,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_{\rho}^{a} \frac{x^{(1+\kappa)/2}\mathrm{d}x}{(x^2 - \rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) K_1(x,\phi). \tag{2.4.23}$$

The energy W may be defined by the integral

$$W = \int_0^{2\pi}\int_0^{a} \sigma(\rho,\phi)w(\rho,\phi)\rho \; \mathrm{d}\rho\mathrm{d}\phi. \tag{2.4.24}$$

Substitution of (2.4.23) in (2.4.24) gives

$$W = 2^{(3-\kappa)/2}\pi H \int_0^{2\pi}\mathrm{d}\phi \int_0^{a} \sigma(\rho,\phi)\rho \; \mathrm{d}\rho \int_{\rho}^{a} \frac{x^{(1+\kappa)/2} \; \mathrm{d}x}{(x^2 - \rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi}\mathrm{d}\phi \int_0^{a} x^{(1+\kappa)/2} \; \mathrm{d}x \int_0^{x} \frac{\sigma(\rho,\phi)\rho \; \mathrm{d}\rho}{(x^2 - \rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi}\mathrm{d}\phi \int_0^{a} K_1(x,\phi) \; x^{(1+\kappa)/2} \; \mathrm{d}x \int_0^{x} \frac{\rho \; \mathrm{d}\rho}{(x^2 - \rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) \sigma(\rho,\phi).$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (2.4.22) yields the final result

$$W = \frac{2^{1-\kappa}\pi^2 H}{\cos(\pi\kappa/2)} \int_0^{2\pi}\int_0^{a} [K_1(\rho,\phi)]^2\rho \; \mathrm{d}\rho \; \mathrm{d}\phi. \tag{2.4.25}$$

Expression (2.4.25) interprets the stress intensity function squared as being proportional to the energy per unit area required to open up the crack. In the case of axial symmetry, formula (2.4.25) simplifies to

$$W = \frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)} \int_0^a [K_1(\rho)]^2 \rho \, d\rho,$$

with

$$K_1(\rho) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_0^\rho \frac{\sigma(\rho_0,\phi_0)\rho_0 \, d\rho_0}{(\rho^2 - \rho_0^2)^{(1+\kappa)/2}} .$$

All the results obtained are valid for a transversely isotropic space provided that $\kappa=0$, and H is defined by (2.1.9).

Exercise 2.4

1. Find the equivalent of (2.4.25) for an isotropic body.

Answer: $W = 2\pi \dfrac{1 - \nu^2}{E} \displaystyle\int_0^{2\pi}\int_0^a [K_1(\rho,\phi)]^2 \rho \, d\rho \, d\phi,$

with

$$K_1(\rho,\phi) = \frac{\sqrt{2}}{\pi\sqrt{\rho}} \int_0^\rho \frac{x \, dx}{(\rho^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x}{\rho}\right) \sigma(x,\phi).$$

2. Solve the problem above for the case of axial symmetry.

Answer: $W = 4\pi^2 \dfrac{1 - \nu^2}{E} \displaystyle\int_0^a [K_1(\rho)]^2 \rho \, d\rho,$

with

$$K_1(\rho) = \frac{\sqrt{2}}{\pi\sqrt{\rho}} \int_0^\rho \frac{\sigma(x)x \, dx}{(\rho^2 - x^2)^{1/2}} .$$

Note: these results were obtained by Sneddon (1965).

3. A penny-shaped crack in a non-homogeneous elastic space is opened by the pressure $\sigma(\rho,\phi)=\sigma_0+\sigma_1\rho\cos\phi$, with σ_0=const and σ_1=const. Find the normal stress in the plane z=0 outside the crack.

Answer: $\sigma(\rho,\phi) = -\dfrac{2\cos(\pi\kappa/2)}{\pi(3 - \kappa)} \left(\dfrac{a}{\rho}\right)^{3-\kappa} \left\{\sigma_0 F\left(\dfrac{3-\kappa}{2}, \dfrac{3-\kappa}{2}; \dfrac{5-\kappa}{2}; \left(\dfrac{a}{\rho}\right)^2\right)\right.$

$$+ \frac{2}{5-\kappa}\left(\frac{a}{\rho}\right)^2 F\left(\frac{3-\kappa}{2}, \frac{5-\kappa}{2}; \frac{7-\kappa}{2}; \left(\frac{a}{\rho}\right)^2\right)\sigma_1\rho\cos\phi\Bigg\}, \quad \text{for } \rho>a.$$

The result can be expressed in elementary functions for a homogeneous body:

$$\sigma(\rho,\phi) = -\frac{2}{\pi}\Bigg\{\left[\frac{a}{(\rho^2-a^2)^{1/2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\sigma_0$$

$$+\left[\frac{3\rho^2-a^2}{3\rho(\rho^2-a^2)^{1/2}} - 3\frac{\rho}{a}\sin^{-1}\left(\frac{a}{\rho}\right)\right]\sigma_1\rho\cos\phi\Bigg\}.$$

4. In the problem above find the crack opening displacements w.

Answer: $w = \dfrac{4H\cos(\pi\kappa/2)}{(1-\kappa)^2}(a^2-\rho^2)^{(1-\kappa)/2}\left[\sigma_0 + \dfrac{2}{3-\kappa}\sigma_1\rho\cos\phi\right].$

5. Consider a penny-shaped crack $\rho\le a$. Find the stress distribution σ in the plane z=0 outside the crack due to the action of a pair of equal concentrated forces P, applied normally to the crack faces in opposite directions at the point (b,ψ), $b<a$.

Answer: $\sigma(\rho,\phi) = -\dfrac{P}{\pi^2}\cos\left(\dfrac{\pi\kappa}{2}\right)\left(\dfrac{a^2-b^2}{\rho^2-a^2}\right)^{(1-\kappa)/2}\dfrac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}.$

6. Prove the identity for a penny-shaped crack:

$$\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\rho^{|n|+1}e^{in\phi}\,d\rho\,d\phi = -\int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)\rho^{|n|+1}e^{in\phi}\,d\rho\,d\phi.$$

Note: the identity states that the normal stress in the plane z=0 is in equilibrium, which is *not* the case for an external crack.

7. Express the stress intensity function $K_1(\rho,\phi)$ in terms of the displacement w.

Answer: $K_1(\rho,\phi) = -\dfrac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2 H\rho^{(1+\kappa)/2}}\mathcal{L}(\rho)\dfrac{d}{d\rho}\displaystyle\int_\rho^a \dfrac{x\,dx}{(x^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}\left(\dfrac{1}{x}\right)w(x,\phi).$

Hint: perform the inversion of (2.4.23).

8. Prove the identity

$$\lim_{\rho\to a}\left\{\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{(a^2-x^2)^{(1-\kappa)/2}}{(x^2-\rho^2)^{(1-\kappa)/2}}\,f(x)\,\mathrm{d}x\right\} = -\frac{\pi(1-\kappa)f(a)}{2\cos(\pi\kappa/2)}.$$

Hint: use the substitution $t=(x^2-\rho^2)/(a^2-x^2)$.

9. Use the identity above to express the stress intensity factor in terms of the displacement w.

Answer: $k_1(\phi) = \dfrac{1-\kappa}{2^{2-\kappa}\pi H}\lim_{\rho\to a}\left[\dfrac{w(\rho,\phi)}{(a-\rho)^{(1-\kappa)/2}}\right].$

Hint: compute the limit $\rho\to a$ of the result in Exercise 7.

10. Consider a transversely isotropic elastic space weakened by a penny-shaped crack $\rho\le a$ in the plane $z=0$, subjected to arbitrary pressure $\sigma(\rho,\phi)$. Find the complex tangential displacements u in the plane $z=0$.

Answer: $$u = -H\alpha\int_0^{2\pi}\int_0^a \frac{(a^2-\rho_0^2)^{1/2}}{(a^2-\rho\rho_0 e^{-i(\phi-\phi_0)})^{1/2}}\,\frac{\sigma(\rho_0,\phi_0)\;\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho e^{-i\phi}-\rho_0 e^{-i\phi_0}}, \quad \text{for } \rho\le a;$$

$$u = -\frac{2}{\pi}H\alpha\int_0^{2\pi}\int_0^a \frac{(a^2-\rho_0^2)^{1/2}}{(a^2-\rho\rho_0 e^{-i(\phi-\phi_0)})^{1/2}}$$

$$\times\tan^{-1}\left[\frac{(a^2-\rho\rho_0 e^{-i(\phi-\phi_0)})^{1/2}}{(\rho^2-a^2)^{1/2}}\right]\frac{\sigma(\rho_0,\phi_0)\;\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho e^{-i\phi}-\rho_0 e^{-i\phi_0}}, \quad \text{for } \rho>a.$$

Hint: see (Fabrikant, 1987a) for details.

2.5 Integral representation for q^2/R^3

While it was sufficient to know the integral representation for $1/R$ in order to solve the mixed boundary value problems of type I, this is no longer the case for the problems of type II. The need to know the integral representation for q^2/R^3 is quite obvious from (2.2.12). We recall that q is defined by (2.2.5), and R is given by (2.2.14). The original derivation was made by the author many years ago. It was very long and cumbersome. Here we present only the idea used, and the final result.

Since

$$q^2/R^3 = q/(\bar{q}R), \tag{2.5.1}$$

we may use the following expansion:

$$\frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} = e^{2i\phi_0}\left\{\left[1 - \left(\frac{\rho}{\rho_0}\right)^2\right]\sum_{k=0}^{\infty}\left(\frac{\rho}{\rho_0}\right)^k e^{-ik(\phi-\phi_0)} - \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}\right\}, \text{ for } \rho<\rho_0;$$

$$\frac{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} = e^{2i\phi}\left\{\left[1 - \left(\frac{\rho_0}{\rho}\right)^2\right]\sum_{k=0}^{\infty}\left(\frac{\rho_0}{\rho}\right)^k e^{ik(\phi-\phi_0)} - \frac{\rho_0}{\rho}e^{-i(\phi-\phi_0)}\right\}, \text{ for } \rho>\rho_0;$$

(2.5.2)

Now we have to substitute (2.5.2) and (1.1.27) in (2.5.1). This procedure yields, for $\rho<\rho_0$,

$$\frac{q^2}{R^3} = e^{2i\phi_0}\left\{\left[1 - \left(\frac{\rho}{\rho_0}\right)^2\right]\sum_{k=0}^{\infty}\left(\frac{\rho}{\rho_0}\right)^k e^{-ik(\phi-\phi_0)} - \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}\right\}$$

$$\times\sum_{n=-\infty}^{\infty}\frac{2}{\pi}\, e^{in(\phi-\phi_0)}\int_0^{\rho}\frac{(x^2/\rho\rho_0)^{|n|}\,dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\ .$$

(2.5.3)

A very tedious procedure follows: grouping together the terms belonging to each harmonic. The derivation does not end there: we shall also need to use the identities:

$$\int_0^{min(\rho_0,\rho)}\frac{(\rho_0^2 - x^2)\sum_{k=0}^{n}\rho^{2(n-k)}x^{2k} - \rho^{2n}x^2}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\,dx$$

$$= \int_0^{min(\rho_0,\rho)}\frac{x^{2n}[(2n + 1)\rho_0^2 - (2n + 2)x^2]}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\,dx,$$

(2.5.4)

and

$$\rho_0^2\int_0^{min(\rho_0,\rho)} \frac{(2n-1)\rho^2 - 2nx^2}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\, x^{2n-2}dx$$

$$= \int_0^{min(\rho_0,\rho)} \frac{2n\rho^2 - (2n+1)x^2}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\, x^{2n}dx, \quad \text{for } n=1,2,3, \ldots \qquad (2.5.5)$$

The first identity may be proven by the mathematical induction method, the second one may be established by using the property

$$\int_0^{min(\rho_0,\rho)} d[x^{2n-1}(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}] = 0.$$

A similar procedure is required for the case $\rho>\rho_0$. The final result is

$$\frac{q^2}{R^3} = \frac{2}{\pi}\Bigg\{\sum_{n=0}^{\infty}\frac{e^{2i\phi}}{\rho^2}\Bigg[\frac{e^{in(\phi-\phi_0)}}{(\rho\rho_0)^n}\int_0^{min(\rho_0,\rho)} \frac{(2n+1)\rho^2 - (2n+2)x^2}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\, x^{2n}dx\Bigg]$$

$$+ \sum_{n=0}^{\infty}\frac{e^{2i\phi_0}}{\rho_0^2}\Bigg[\frac{e^{-in(\phi-\phi_0)}}{(\rho\rho_0)^n}\int_0^{min(\rho_0,\rho)}\frac{(2n+1)\rho_0^2 - (2n+2)x^2}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\, x^{2n}\, dx\Bigg]$$

$$- \frac{e^{i(\phi+\phi_0)}}{\rho\rho_0}\int_0^{min(\rho_0,\rho)} \frac{x^2\, dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\Bigg\}. \qquad (2.5.6)$$

Expression (2.5.6), though looking cumbersome, will prove very useful for solving internal mixed boundary value problem of the type II. We need yet another integral representation which is useful in external problems. The procedure is as tedious as the one described above, with the final result

$$\frac{q^2}{R^3} = \frac{2}{\pi}\Bigg\{\sum_{n=0}^{\infty}e^{2i\phi}\Bigg[(e^{i(\phi-\phi_0)}\rho\rho_0)^n\int_{max(\rho_0,\rho)}^{\infty} \frac{(2n+1)x^2 - (2n+2)\rho_0^2}{x^{2n+2}(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}\, dx\Bigg]$$

$$+\sum_{n=0}^{\infty} e^{2i\phi_0}\left[(e^{-i(\phi-\phi_0)}\rho\rho_0)^n \int_{max(\rho_0,\rho)}^{\infty} \frac{(2n+1)x^2-(2n+2)\rho^2}{x^{2n+2}(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\,dx\right]$$

$$\left.- e^{i(\phi+\phi_0)}\rho\rho_0 \int_{max(\rho_0,\rho)}^{\infty} \frac{dx}{x^2(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\right\}. \quad (2.5.7)$$

Here the following identities were used:

$$\int_{max(\rho_0,\rho)}^{\infty} \frac{(x^2-\rho^2)\sum_{k=0}^{n} x^{2k}\rho_0^{2(n-k)} - \rho^2 x^{2n}}{x^{2n+2}(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\,dx$$

$$= \rho_0^{2n}\int_{max(\rho_0,\rho)}^{\infty} \frac{(2n+1)x^2-(2n+2)\rho^2}{x^{2n+2}(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\,dx,$$

and

$$\int_{max(\rho_0,\rho)}^{\infty} \frac{2n\rho_0^2-(2n-1)x^2}{x^{2n}(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\,dx$$

$$= \rho^2\int_{max(\rho_0,\rho)}^{\infty} \frac{(2n+1)\rho_0^2-2nx^2}{x^{2n+2}(x^2-\rho^2)^{1/2}(x^2-\rho_0^2)^{1/2}}\,dx.$$

These identities can be derived from (2.5.4) and (2.5.5) by a formal substitution of x by $\rho\rho_0/x$. As usual, when the final result is achieved, one can find an easier way to do it. We shall discuss further some generalizations of the integral representations derived here (see section 2.7).

Exercise 2.5

1. Prove the identities (2.5.4) and (2.5.5).

2. Establish the representation (2.5.6).

3. Establish the representation (2.5.7).

2.6 Internal mixed problem of type II

The material in this section follows essentially the paper (Fabrikant, 1971c). Consider a transversely isotropic elastic half-space $z\geq 0$. Let the normal traction σ be prescribed all over the plane $z=0$. An arbitrary tangential displacement $u=u_x+iu_y$ is specified inside a circle $\rho=a$, while the complex shear loading τ is known outside the circle. The problem is to find the shear traction inside the circle. The mathematical formulation of the boundary conditions is

$$\sigma = \sigma(\rho,\phi), \quad \text{for } 0\leq\rho<\infty, \quad 0\leq\phi<2\pi;$$

$$\tau = \tau(\rho,\phi), \quad \text{for } a<\rho<\infty, \quad 0\leq\phi<2\pi;$$

$$u = u(\rho,\phi), \quad \text{for } 0\leq\rho\leq a, \quad 0\leq\phi<2\pi; \tag{2.6.1}$$

The governing integral equation can be written due to (2.2.12):

$$\frac{1}{2}G_1\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} + \frac{1}{2}G_2\int_0^{2\pi}\int_0^a \frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R} = \chi(\rho,\phi). \tag{2.6.2}$$

Function χ is known from the boundary conditions (2.6.1):

$$\chi(\rho,\phi) = u + H\alpha\int_0^{2\pi}\int_0^\infty \frac{\sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}}$$

$$- \frac{1}{2}G_1\int_0^{2\pi}\int_a^\infty \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{R} - \frac{1}{2}G_2\int_0^{2\pi}\int_a^\infty \frac{q\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\bar{q}R}. \tag{2.6.3}$$

Though a closed form exact solution of (2.6.2) is possible, we present first its exact solution in Fourier series. Assume validity of the expansions:

$$\tau(\rho,\phi) = \sum_{n=-\infty}^{\infty}\tau_n(\rho)\ e^{in\phi}, \qquad \chi(\rho,\phi) = \sum_{n=-\infty}^{\infty}\chi_n(\rho)\ e^{in\phi}. \tag{2.6.4}$$

Substitution of (2.5.6) and (2.6.4) in (2.6.2) leads to an infinite set of integral equations

$$\frac{2G_1}{\rho^{n+1}}\int_0^{\rho}\frac{x^{2n+2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{\tau_{n+1}(\rho_0)\ \mathrm{d}\rho_0}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}$$

$$+\frac{2G_2}{\rho^{n+1}}\int_0^{\rho}\frac{x^{2n-2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{(2n - 1)\rho^2 - 2nx^2}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}}\ \bar{\tau}_{-n+1}(\rho_0)\ \mathrm{d}\rho_0 = \chi_{n+1}(\rho), \tag{2.6.5}$$

$$\frac{2G_1}{\rho^{n-1}}\int_0^{\rho}\frac{x^{2n-2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{\tau_{-n+1}(\rho_0)\ \mathrm{d}\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}}$$

$$+\frac{2G_2}{\rho^{n-1}}\int_0^{\rho}\frac{x^{2n-2}\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{(2n - 1)\rho_0^2 - 2nx^2}{\rho_0^{n}(\rho_0^2 - x^2)^{1/2}}\ \bar{\tau}_{n+1}(\rho_0)\ \mathrm{d}\rho_0 = \chi_{-n+1}(\rho), \tag{2.6.6}$$

Equations (2.6.5) and (2.6.6) are valid for n=1,2,3, ... In the case of axial symmetry, n=0, and the integral equation takes the form

$$\frac{2}{\rho}\int_0^{\rho}\frac{x^2\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{G_1\tau_1(\rho_0) - G_2\bar{\tau}_1(\rho_0)}{(\rho_0^2 - x^2)^{1/2}}\ \mathrm{d}\rho_0 = \chi_1(\rho). \tag{2.6.7}$$

Its solution is elementary, namely,

$$\tau_1(\rho) = -\frac{2}{\pi^2(G_1^2 - G_2^2)}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{a}\frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[x\int_0^x\frac{G_1\chi_1(\rho_0) + G_2\bar{\chi}_1(\rho_0)}{(x^2 - \rho_0^2)^{1/2}}\ \mathrm{d}\rho_0\right]. \tag{2.6.8}$$

The general solution of the system (2.6.5) and (2.6.6) can be presented in the form

$$\tau_{-n+1}(\rho) = \rho^{n-1}\int_{\rho}^{a}\frac{f_{-n+1}(t)\ \mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + \left(\frac{G_2}{G_1}\ \bar{C}_n + D_n\right)\frac{\rho^{n-1}}{(a^2 - \rho^2)^{1/2}},$$

$$\tau_{n+1}(\rho) = \rho^{n-1}\int_{\rho}^{a} \frac{f_{n+1}(t)\ dt}{(t^2 - \rho^2)^{1/2}} + \frac{2n}{\rho^{n+1}}\int_{\rho}^{a} y^{2n-1}dy \int_{y}^{a} \frac{f_{n+1}(t)\ dt}{(t^2 - y^2)^{1/2}}$$

$$+ C_n\left[\frac{\rho^{n-1}}{(a^2 - \rho^2)^{1/2}} + \frac{2n}{\rho^{n+1}} \int_{\rho}^{a} \frac{y^{2n-1}\ dy}{(a^2 - y^2)^{1/2}}\right]. \tag{2.6.9}$$

Here f_k are the as yet unknown complex functions, and C_n and D_n are the as yet unknown constants. By substitution of (2.6.9) in (2.6.5) and (2.6.6), we obtain, after interchanging the order of integration and integration with respect to ρ_0,

$$\frac{\pi G_1}{\rho^{n+1}}\left[\rho \int_{0}^{a} t^{2n-1} f_{n+1}(t)\ dt + C_n \rho a^{2n-1} - \int_{0}^{\rho} (\rho^2 - t^2)^{1/2} t^{2n-1}\ f_{n+1}(t)\ dt\right]$$

$$+ \frac{\pi G_2}{\rho^{n+1}} \int_{0}^{\rho} (\rho^2 - t^2)^{1/2}\ t^{2n-1}\bar{f}_{-n+1}(t)\ dt = \chi_{n+1}(\rho), \tag{2.6.10}$$

$$\frac{\pi G_1}{\rho^{n-1}}\left[\int_{0}^{\rho} \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_{x}^{a} f_{-n+1}(t)\ dt + D_n \frac{\Gamma(\frac{1}{2})\,\Gamma(n - \frac{1}{2})}{2\Gamma(n)} \rho^{2n-2}\right]$$

$$- \frac{\pi G_2}{\rho^{n-1}} \int_{0}^{\rho} \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_{x}^{a} \bar{f}_{n+1}(t)\ dt = \chi_{-n+1}(\rho). \tag{2.6.11}$$

Expression (2.6.10) can be simplified if we define C_n as

$$C_n = -a^{-2n+1}\int_{0}^{a} t^{2n-1} f_{n+1}(t)\ dt. \tag{2.6.12}$$

Application of the operator

$$r^{-2n+2}\frac{d}{dr}\int_0^r \frac{\rho^n d\rho}{(r^2-\rho^2)^{1/2}}$$

to both sides of (2.6.11) yields

$$\frac{\pi^2}{2}\left[G_1\int_r^a f_{-n+1}(t)\,dt - G_2\int_r^a \bar{f}_{n+1}(t)\,dt + G_1 D_n\right] = r^{-2n+2}\frac{d}{dr}\int_0^r \frac{\chi_{-n+1}(\rho)\rho^n d\rho}{(r^2-\rho^2)^{1/2}}. \tag{2.6.13}$$

Since expression (2.6.13) should stay valid in the limiting case of $r \to a$, this defines D_n as follows:

$$D_n = \frac{2}{\pi^2 G_1 a^{2n-2}}\frac{d}{da}\int_0^a \frac{\chi_{-n+1}(\rho)\rho^n d\rho}{(a^2-\rho^2)^{1/2}}. \tag{2.6.14}$$

Both constants are now defined. Inversion of (2.6.10) and differentiation of (2.6.13) lead to the system

$$-G_1 f_{n+1}(r) + G_2\bar{f}_{-n+1}(r) = \psi_{n+1}(r),$$

$$-G_1 f_{-n+1}(r) + G_2\bar{f}_{n+1}(r) = \psi_{-n+1}(r). \tag{2.6.15}$$

Here the notations were introduced

$$\psi_{n+1}(r) = \frac{2}{\pi^2 r^{2n-1}}\frac{d}{dr}\int_0^r \frac{d[\rho^{n+1}\chi_{n+1}(\rho)]}{(r^2-\rho^2)^{1/2}},$$

$$\psi_{-n+1}(r) = \frac{2}{\pi^2}\frac{d}{dr}\left[r^{-2n+2}\frac{d}{dr}\int_0^r \frac{\rho^n\chi_{-n+1}(\rho)\,d\rho}{(r^2-\rho^2)^{1/2}}\right]. \tag{2.6.16}$$

Solution of the system (2.6.15) yields

$$f_{-n+1}(r) = -\frac{G_1\psi_{-n+1}(r) + G_2\overline{\psi}_{n+1}(r)}{G_1^2 - G_2^2},$$

$$f_{n+1}(r) = -\frac{G_1\psi_{n+1}(r) + G_2\overline{\psi}_{-n+1}(r)}{G_1^2 - G_2^2}. \tag{2.6.17}$$

The general solution is now completed. It is given by formulae (2.6.9), with the constants C and D defined by (2.6.12) and (2.6.14), and functions f defined by (2.6.17) and (2.6.16).

Example. Consider a transversely isotropic elastic space weakened by an external circular crack $\rho \geq a$ in the plane $z=0$. Two *identically oriented* equal concentrated forces P are applied normally to the crack faces at the points $(\rho_0,\phi_0,0^{\pm})$, $\rho_0>a$. Let us find the tangential stress in the crack neck. Due to the antisymmetry of the applied load, the problem can be reduced to the one of a half-space, with the tangential displacements and the normal stress vanishing in the crack neck. The mathematical formulation of the boundary conditions is

$$\sigma = P\delta(\rho-\rho_0)\delta(\phi-\phi_0)/\rho \qquad \text{for } 0\leq\rho<\infty, \qquad 0\leq\phi<2\pi;$$

$$\tau = 0, \qquad \text{for } a<\rho<\infty, \qquad 0\leq\phi<2\pi;$$

$$u = 0, \qquad \text{for } 0\leq\rho\leq a, \qquad 0\leq\phi<2\pi; \tag{2.6.18}$$

Here $\delta(\cdot)$ is the Dirac delta-function. The governing integral equation corresponds to (2.6.2), with the right hand side χ, defined by (2.6.3), i.e.

$$\chi(\rho,\phi) = \frac{PH\alpha}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}} = -\frac{PH\alpha}{\rho_0 e^{-i\phi_0}}\sum_{n=0}^{\infty}\left(e^{-i(\phi-\phi_0)}\frac{\rho}{\rho_0}\right)^n.$$

The general solution, presented above, yields the following results

$$\chi_{-n+1}(\rho) = -\frac{PH\alpha e^{i\phi_0}}{\rho_0}\left(e^{i\phi_0}\frac{\rho}{\rho_0}\right)^{n-1}, \qquad \chi_{n+1}(\rho) = 0,$$

$$f_{-n+1} = f_{n+1} = C_n = 0, \qquad D_n = -\frac{2PH\alpha\Gamma(n)}{\pi^{3/2}G_1\Gamma(n-\frac{1}{2})(\rho_0 e^{-i\phi_0})^n} \tag{2.6.19}$$

Substitution of (2.6.19) in (2.6.17) and (2.6.9) leads to the solution

$$\tau(\rho,\phi) = -\frac{2PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{(\rho_0 e^{-i\phi_0})^{n+1}\Gamma(n+\frac{1}{2})}\frac{\rho^n e^{-in\phi}}{(a^2-\rho^2)^{1/2}}.$$

The summation can be performed, according to the scheme

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\zeta^n = \pi^{-1/2}F(1, 1; \tfrac{1}{2}; \zeta) = \frac{\pi^{-1/2}}{1-\zeta}\left[1+\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\sin^{-1}\sqrt{\zeta}\right].$$

Here we have used the well known property of the hypergeometric functions (Bateman and Erdélyi, 1955). Now the final result will take the form

$$\tau(\rho,\phi) = -\frac{2PH\alpha}{\pi^2 G_1\rho_0 e^{-i\phi_0}(a^2-\rho^2)^{1/2}}\frac{1}{1-b}\left[1+\left(\frac{b}{1-b}\right)^{1/2}\sin^{-1}\sqrt{b}\right],$$

where $b=(\rho/\rho_0)e^{-i(\phi-\phi_0)}$. In the case of isotropy, the last formula coincides with the result of Ufliand (1967).

Exercise 2.6

1. Consider a transversely isotropic elastic half-space $z\geq 0$. The tangential displacement $u=u_0=\text{const}$ is prescribed inside a circle $\rho=a$. The shear traction outside the circle is equal to zero, and the normal pressure vanishes all over the plane $z=0$. Find the shear traction inside the circle.

Answer: $\tau(\rho) = \dfrac{2u_0}{\pi^2 G_1(a^2-\rho^2)^{1/2}}$.

2. In the problem above find the normal displacement w in the plane $z=0$.

Answer: $w = \dfrac{4\Re(u_0 e^{-i\phi})H\alpha a}{\pi G_1\rho}$, for $\rho>a$;

$$w = \frac{4\Re(u_0 e^{-i\phi})H\alpha[a-(a^2-\rho^2)^{1/2}]}{\pi G_1\rho}, \quad \text{for } \rho\leq a;$$

3. Subject to the conditions of the first problem, find the tangential displacement outside the circle $\rho=a$.

Answer: $u = \dfrac{2}{\pi}\left[u_0\sin^{-1}\left(\dfrac{a}{\rho}\right)+\bar{u}_0\dfrac{G_2}{G_1}\dfrac{a(\rho^2-a^2)^{1/2}}{\rho^2}e^{2i\phi}\right]$.

2.7 External mixed problem of type II

Consider a transversely isotropic elastic half-space $z\geq 0$. Let the normal traction σ be prescribed all over the plane $z=0$. An arbitrary tangential displacement $u=u_x+iu_y$ is specified outside a circle $\rho=a$, while the complex shear loading τ is known inside the circle. The problem is to find the shear traction outside the circle, the tangential displacement inside, and the normal displacement at the plane $z=0$. The mathematical statement of the boundary conditions is

$$\begin{aligned}
\sigma &= \sigma(\rho,\phi), &&\text{for}\quad 0\leq\rho<\infty, && 0\leq\phi<2\pi;\\
\tau &= \tau(\rho,\phi), &&\text{for}\quad 0\leq\rho<a, && 0\leq\phi<2\pi;\\
u &= u(\rho,\phi), &&\text{for}\quad a\leq\rho<\infty, && 0\leq\phi<2\pi.
\end{aligned}\qquad (2.7.1)$$

The governing integral equation can be written due to (2.2.12):

$$\frac{1}{2}\,G_1\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{R} + \frac{1}{2}\,G_2\int_0^{2\pi}\int_a^{\infty}\frac{q\bar{\tau}(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{\bar{q}R} = \chi(\rho,\phi). \qquad (2.7.2)$$

Function χ is known due to the boundary conditions (2.7.1):

$$\chi(\rho,\phi) = u + H\alpha\int_0^{2\pi}\int_0^{\infty}\frac{\sigma(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{\rho e^{-i\phi} - \rho_0 e^{-i\phi_0}}$$

$$- \frac{1}{2}\,G_1\int_0^{2\pi}\int_0^{a}\frac{\tau(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{R} - \frac{1}{2}\,G_2\int_0^{2\pi}\int_0^{a}\frac{q\bar{\tau}(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{\bar{q}R}. \qquad (2.7.3)$$

Once equation (2.7.2) has been solved, the normal displacement is found from

$$w(\rho,\phi) = H\alpha\;\Re\left\{\int_0^{2\pi}\int_0^{\infty}\frac{\tau(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}\right\} + H\int_0^{2\pi}\int_0^{\infty}\frac{\sigma(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{R}\;. \qquad (2.7.4)$$

We present the exact solution of (2.7.2) in Fourier series. Assume validity of the expansions:

$$\tau(\rho,\phi) = \sum_{n=-\infty}^{\infty} \tau_n(\rho)\ e^{in\phi}, \qquad \chi(\rho,\phi) = \sum_{n=-\infty}^{\infty} \chi_n(\rho)\ e^{in\phi}. \tag{2.7.5}$$

Substitution of the integral representation (2.5.7) and (2.7.5) in (2.7.2) leads to an infinite set of coupled integral equations

$$2G_1\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}} \int_a^x \frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}\ d\rho_0}{(x^2-\rho_0^2)^{1/2}} + 2G_2\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}$$

$$\times\int_a^x \frac{2nx^2-(2n+1)\rho_0^2}{(x^2-\rho_0^2)^{1/2}}\ \bar{\tau}_{-n+1}(\rho_0)\rho_0^n\ d\rho_0 = \chi_{n+1}(\rho), \quad \text{for } n=0,1,2,\ \dots \tag{2.7.6}$$

$$2G_1\rho^{n-1}\int_\rho^\infty \frac{dx}{x^{2n-2}(x^2-\rho^2)^{1/2}} \int_a^x \frac{\tau_{-n+1}(\rho_0)\rho_0^{n}\ d\rho_0}{(x^2-\rho_0^2)^{1/2}} + 2G_2\rho^{n-1}\int_\rho^\infty \frac{(2n-1)x^2-2n\rho^2}{x^{2n}(x^2-\rho^2)^{1/2}}dx$$

$$\times\int_a^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n\ d\rho_0}{(x^2-\rho_0^2)^{1/2}} = \chi_{-n+1}(\rho), \quad \text{for } n=1,2,3,\ \dots \tag{2.7.7}$$

Here

$$\chi_{n+1}(\rho) = u_{n+1}(\rho) + 2\pi H\alpha\rho^{-n-1}\int_0^\rho \sigma_n(\rho_0)\ \rho_0^{n+1}d\rho_0 - \frac{2}{\rho^{n+1}}\int_0^a \frac{x^{2n}dx}{(\rho^2-x^2)^{1/2}}$$

$$\times\int_x^a \frac{G_1x^2\tau_{n+1}(\rho_0) + G_2[2n\rho^2-(2n+1)x^2]\bar{\tau}_{-n+1}(\rho_0)}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\ d\rho_0, \tag{2.7.8}$$

and

$$\chi_{-n+1}(\rho) = u_{-n+1}(\rho) + 2\pi H\alpha\rho^{n-1}\int_\rho^\infty \sigma_{-n}(\rho_0)\ \rho_0^{-n+1}d\rho_0 - \frac{2}{\rho^{n-1}}\int_0^a \frac{x^{2n-2}dx}{(\rho^2-x^2)^{1/2}}$$

$$\times\int_x^a \frac{G_1\rho_0^2\tau_{-n+1}(\rho_0) + G_2[(2n-1)\rho_0^2 - 2nx^2]\bar{\tau}_{n+1}(\rho_0)}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}\,d\rho_0, \tag{2.7.9}$$

The case of axial symmetry corresponds to n=0, and we have only one equation (2.7.6) to solve. The general solution of the system (2.7.6) and (2.7.7) may be presented in the form

$$\tau_{n+1}(\rho) = \frac{1}{\rho^{n+1}}\left[\int_a^\rho \frac{f_{n+1}(t)\,dt}{(\rho^2 - t^2)^{1/2}} + \frac{D_n}{(\rho^2 - a^2)^{1/2}}\right], \text{ for } n=1,2, \ldots$$

$$\tau_{-n+1}(\rho) = \frac{1}{\rho^{n+1}}\int_a^\rho \frac{y\,dy}{(\rho^2 - y^2)^{1/2}}\frac{d}{dy}\left[y^{2n}\int_a^y \frac{f_{-n+1}(t)}{t^{2n+1}}\,dt\right]$$

$$+ C_n\left[\frac{1}{\rho^{n+1}(\rho^2 - a^2)^{1/2}} + \frac{2n}{\rho^{n+1}a^{2n+1}}\int_a^\rho \frac{y^{2n}dy}{(\rho^2 - y^2)^{1/2}}\right], \text{ for } n=1,2,3, \ldots \tag{2.7.10}$$

Here functions f are to be determined, and C and D are the as yet unknown complex constants. We used the same notation as in the previous section hoping that this would not produce any confusion, and that the reader would understand clearly that, for example, C_n of this section is not equal to C_n from the previous one. Substitution of (2.7.10) in equations (2.7.6) yields, after simplification,

$$\pi\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}}\left[G_1\left(\int_a^x f_{n+1}(t)\,dt + D_n\right)\right.$$

$$\left. - G_2\left(\int_a^x \bar{f}_{-n+1}(t)\,dt + \bar{C}_n\right)\right] = \chi_{n+1}(\rho), \tag{2.7.11}$$

$$\pi\rho^{n-1}G_1\int_\rho^\infty \frac{x\mathrm{d}x}{(x^2-\rho^2)^{1/2}}\left[\int_a^x \frac{f_{-n+1}(t)\ \mathrm{d}t}{t^{2n+1}} + \frac{C_n}{a^{2n+1}}\right]$$

$$+\ \pi\rho^{n-1}G_2\int_\rho^\infty \frac{(x^2-\rho^2)^{1/2}}{x^{2n+1}}\ \bar{f}_{n+1}(x)\ \mathrm{d}x = \chi_{-n+1}(\rho). \tag{2.7.12}$$

The following identity was used during the procedure of integration by parts

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(x^2-\rho^2)^{1/2}}{x^{2n}}\right] = \frac{(2n-1)x^2-2n\rho^2}{x^{2n+1}(x^2-\rho^2)^{1/2}}.$$

Looking at the first integral in (2.7.12), one may conclude that it converges only if the term in square brackets tends to zero when $x\to\infty$. This condition defines C_n as follows:

$$C_n = -a^{2n+1}\int_a^\infty f_{-n+1}(t)\ t^{-2n-1}\mathrm{d}t. \tag{2.7.13}$$

By dividing both sides of (2.7.11) by $\rho^n(\rho^2-r^2)^{1/2}$, integrating with respect to ρ from r to ∞, differentiating with respect to r, and multiplying the result by r^{2n+2}/π, we obtain

$$G_1\left[\int_a^r f_{n+1}(t)\ \mathrm{d}t + D_n\right] - G_2\left[\int_a^r \bar{f}_{-n+1}(t)\ \mathrm{d}t + \overline{C}_n\right]$$

$$= -\frac{2}{\pi^2}\ r^{2n+2}\ \frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{\chi_{n+1}(\rho)\ \mathrm{d}\rho}{\rho^n(\rho^2-r^2)^{1/2}}. \tag{2.7.14}$$

Proceeding to the limit as $r\to a$ in (2.7.14) gives us the formula for evaluating D_n, namely,

$$D_n = \frac{1}{G_1}\left[G_2\overline{C}_n - \frac{2}{\pi^2}\, a^{2n+2}\,\frac{d}{da}\int_a^\infty \frac{\chi_{n+1}(\rho)\ d\rho}{\rho^n(\rho^2 - a^2)^{1/2}}\right]. \tag{2.7.15}$$

Inversion of (2.7.12) and differentiation of (2.7.14) lead to the system of equations

$$G_1 f_{-n+1}(r) - G_2\bar{f}_{n+1}(r) = \psi_{-n+1}(r),$$

$$G_1 f_{n+1}(r) - G_2\bar{f}_{-n+1}(r) = \psi_{n+1}(r). \tag{2.7.16}$$

Here

$$\psi_{-n+1}(r) = -\frac{2}{\pi^2}\, r^{2n+1}\,\frac{d}{dr}\int_r^\infty \frac{d\rho}{(\rho^2 - r^2)^{1/2}}\,\frac{d}{d\rho}\left[\frac{\chi_{-n+1}(\rho)}{\rho^{n-1}}\right],$$

$$\psi_{n+1}(r) = -\frac{2}{\pi^2}\,\frac{d}{dr}\left[r^{2n+2}\,\frac{d}{dr}\int_r^\infty \frac{\chi_{n+1}(\rho)\ d\rho}{\rho^n(\rho^2 - r^2)^{1/2}}\right]. \tag{2.7.17}$$

The following rule of differentiation was used during the above transformations:

$$\frac{d}{d\rho}\int_\rho^\infty \frac{f(x)\ dx}{(x^2 - \rho^2)^{1/2}} = \rho\int_\rho^\infty \frac{d[f(x)/x]}{(x^2 - \rho^2)^{1/2}}. \tag{2.7.18}$$

Solution of the system (2.7.16) is

$$f_{n+1}(r) = \frac{G_1\psi_{n+1}(r) + G_2\overline{\psi}_{-n+1}(r)}{G_1^2 - G_2^2},$$

$$f_{-n+1}(r) = \frac{G_1\psi_{-n+1}(r) + G_2\overline{\psi}_{n+1}(r)}{G_1^2 - G_2^2}. \tag{2.7.19}$$

The set of expressions (2.7.10), (2.7.13), (2.7.15), (2.7.17), and (2.7.19) gives a complete solution to the problem. Since the shear traction is now known throughout the plane z=0, the tangential displacements inside the circle can be defined by (2.7.2), and the normal displacement is defined by (2.7.4). We consider further the case of a penny-shaped crack in more detail, including the derivation of closed form solutions.

Example: Penny-shaped crack. Let the crack of radius a be located in the plane z=0. Both crack faces are loaded by arbitrary shear tractions acting in opposite directions. The boundary conditions are antisymmetric, so the problem can be reduced to a mixed one for a half-space, with the boundary conditions:

$$\sigma = 0, \quad \text{for } 0\leq\rho<\infty, \quad 0\leq\phi<2\pi;$$

$$\tau = \tau(\rho,\phi), \quad \text{for } 0\leq\rho<a, \quad 0\leq\phi<2\pi;$$

$$u = 0, \quad \text{for } a\leq\rho<\infty, \quad 0\leq\phi<2\pi. \tag{2.7.20}$$

The shear tractions outside the crack, the tangential displacement inside, and the normal displacement in the plane z=0 are to be determined. The crack problem can be considered as a particular case of a more general one solved above. However, we can show that the crack problem has a simpler solution. Indeed, the integral equations (2.7.6) and (2.7.7) remain unchanged, but instead of (2.7.8) and (2.7.9) we have

$$\chi_{n+1}(\rho) = -\frac{2}{\rho^{n+1}}\int_0^a \frac{x^{2n}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{G_1 x^2\tau_{n+1}(\rho_0)+G_2[2n\rho^2-(2n+1)x^2]\bar{\tau}_{-n+1}(\rho_0)}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}d\rho_0, \tag{2.7.21}$$

and

$$\chi_{-n+1}(\rho) = -\frac{2}{\rho^{n-1}}\int_0^a \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{G_1 \rho_0^2\tau_{-n+1}(\rho_0)+G_2[(2n-1)\rho_0^2-2nx^2]\bar{\tau}_{n+1}(\rho_0)}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}d\rho_0. \tag{2.7.22}$$

The structure of integral equations indicates the possibility of expression the yet unknown τ outside the crack through the prescribed τ inside. Such a representation takes the form

$$\tau_{n+1}(\rho) = -\frac{2}{\pi\rho^{n+1}(\rho^2 - a^2)^{1/2}}\int_0^a \frac{(a^2 - t^2)^{1/2}\tau_{n+1}(t)t^{n+2}dt}{\rho^2 - t^2} + \frac{A_n}{\rho^{n+1}(\rho^2 - a^2)^{1/2}},$$

$$\text{for } n=0,1,2, \ldots$$

$$\tau_{-n+1}(\rho) = -\frac{2}{\pi\rho^{n-1}(\rho^2 - a^2)^{1/2}}\int_0^a \frac{(a^2 - t^2)^{1/2}\tau_{-n+1}(t)t^{n}dt}{\rho^2 - t^2}, \quad \text{for } n=1,2,3, \ldots \tag{2.7.23}$$

Here A_n is the as yet unknown complex constant. Substitution of (2.7.23) in (2.7.6) yields, after intermediate integration,

$$2G_1\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\left[-\int_0^a \frac{\tau_{n+1}(t)t^{n+2}dt}{(x^2-t^2)^{1/2}} + \frac{\pi}{2}A_n\right]$$

$$+\ 2G_2\rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\left[-\int_0^a \frac{2nx^2-(2n+1)t^2}{(x^2-t^2)^{1/2}}\,\bar{\tau}_{-n+1}(t)t^n\,dt\right.$$

$$\left.+\ (2n+1)\int_0^a (a^2-t^2)^{1/2}\,\bar{\tau}_{-n+1}(t)t^n\,dt\right] = \chi_{n+1}(\rho). \tag{2.7.24}$$

The following identity can be verified by a formal substitution of x by $\rho t/x$:

$$\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}(x^2-t^2)^{1/2}} = \frac{1}{(\rho t)^{2n+2}}\int_0^t \frac{x^{2n+2}dx}{(\rho^2-x^2)^{1/2}(t^2-x^2)^{1/2}}. \tag{2.7.25}$$

Interchanging the order of integration in (2.7.24), substituting (2.7.25), and yet another interchange lead to the expression

$$-\ \frac{2G_1}{\rho^{n+1}}\int_0^a \frac{x^{2n+2}dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_{n+1}(t)\,dt}{t^n(t^2-x^2)^{1/2}}$$

$$-\ \frac{2G_2}{\rho^{n+1}}\int_0^a \frac{2n\rho^2-(2n+1)x^2}{(\rho^2-x^2)^{1/2}}\,x^{2n}dx\int_x^a \frac{\bar{\tau}_{-n+1}(t)\,dt}{t^n(t^2-x^2)^{1/2}}$$

$$+\ \rho^{n+1}\int_\rho^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\left[\pi G_1 A_n\right.$$

$$+ \; 2G_2(2n + 1)\int_0^a (a^2 - t^2)^{1/2} \; \bar{\tau}_{-n+1}(t)t^n dt\Bigg] = \chi_{n+1}(\rho). \tag{2.7.26}$$

Comparison of (2.7.26) and (2.7.21) shows that the equation is satisfied if A_n is defined by

$$A_n = -\frac{2}{\pi}(2n + 1)(G_2/G_1) \int_0^a \bar{\tau}_{-n+1}(\rho)\rho^n(a^2 - \rho^2)^{1/2}d\rho. \tag{2.7.27}$$

Since τ_{-n+1} is defined for $n\geq 1$ only, we may conclude that $A_0=0$. A similar procedure of substitution of (2.7.23) in (2.7.7) leads to

$$2G_1\rho^{n-1} \int_\rho^\infty \frac{dx}{x^{2n-2}(x^2 - \rho^2)^{1/2}} \left[-\int_0^a \frac{\tau_{-n+1}(t)t^n \; dt}{(x^2 - t^2)^{1/2}}\right]$$

$$+ \; 2G_2\rho^{n-1} \int_\rho^\infty \frac{(2n - 1)x^2 - 2n\rho^2}{x^{2n}(x^2 - \rho^2)^{1/2}} \; dx\Bigg[-\int_0^a \frac{\bar{\tau}_{n+1}(t)t^n}{(x^2 - t^2)^{1/2}} \; dt$$

$$+ \; \frac{1}{ax}\left(\int_0^a (a^2 - t^2)^{1/2} \; \bar{\tau}_{n+1}(t)t^{n+2}dt + \frac{\pi}{2} A_n\right)\Bigg] = \chi_{-n+1}(\rho). \tag{2.7.28}$$

It may be noted that the integral

$$\int_\rho^\infty \frac{(2n - 1)x^2 - 2n\rho^2}{x^{2n+1}(x^2 - \rho^2)^{1/2}} \; dx = \int_\rho^\infty d\left[-\frac{(x^2 - \rho^2)^{1/2}}{x^{2n}}\right] = 0, \text{ for } n\geq 1. \tag{2.7.29}$$

By using (2.7.25) and (2.7.29), equation (2.7.28) can be transformed into

$$-\frac{2G_1}{\rho^{n-1}} \int_0^a \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\tau_{-n+1}(t) \; dt}{t^{n-2}(t^2 - x^2)^{1/2}}$$

$$-\frac{2G_2}{\rho^{n-1}}\int_0^a \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{(2n - 1)t^2 - 2nx^2}{t^n(t^2 - x^2)^{1/2}}\,\bar{\tau}_{n+1}(t)\,dt = \chi_{-n+1}(\rho).$$

Comparison of the last expression with (2.7.22) proves that equation (2.7.7) is satisfied, and that (2.7.23) is indeed the solution to the crack problem. A closed form solution can be obtained by summation of (2.7.23) and (2.7.27), with the result

$$\tau(\rho,\phi) = -\frac{1}{\pi^2(\rho^2 - a^2)^{1/2}}\int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{1/2}\tau(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)}$$

$$-\frac{G_2 e^{2i\phi}}{\pi^2 G_1\rho^2(\rho^2 - a^2)^{1/2}}\int_0^{2\pi}\int_0^a \frac{3 - (\rho_0/\rho)e^{i(\phi-\phi_0)}}{[1 - (\rho_0/\rho)e^{i(\phi-\phi_0)}]^2}(a^2 - \rho_0^2)^{1/2}\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0. \quad (2.7.30)$$

One can notice that the first integral in (2.7.30) corresponds to the solution for the case of normal loading of a penny-shaped crack. Define the complex stress intensity factor as

$$k(\phi) = \lim_{\rho\to a}[(\rho - a)^{1/2}\tau(\rho,\phi)]. \quad (2.7.31)$$

Substitution of (2.7.30) in (2.7.31) yields

$$k(\phi) = -\frac{1}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{1/2}\tau(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{a^2 + \rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)}$$

$$-\frac{G_2 e^{2i\phi}}{\pi^2 G_1 a^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{3 - (\rho_0/a)e^{i(\phi-\phi_0)}}{[1 - (\rho_0/a)e^{i(\phi-\phi_0)}]^2}\,(a^2 - \rho_0^2)^{1/2}\bar{\tau}(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0. \quad (2.7.32)$$

Since our definition of τ contains both x- and y-components, so will the expression for the stress intensity factor $k=k_x+ik_y$. If we require the expression for the radial and tangential components, we have to use the relationship

$$\tau_{zx} + i\tau_{yz} = (\tau_{z\rho} + i\tau_{\phi z})e^{i\phi}.$$

This allows us to rewrite (2.7.32) in terms of the second and third mode stress intensity factors as follows:

$$k_2 + ik_3 = -\frac{e^{-i\phi}}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{1/2}\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{a^2 + \rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)}$$

$$-\frac{G_2 e^{i\phi}}{\pi^2 G_1 a^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{3 - (\rho_0/a)e^{i(\phi-\phi_0)}}{[1 - (\rho_0/a)e^{i(\phi-\phi_0)}]^2}(a^2 - \rho_0^2)^{1/2}\ \bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0. \tag{2.7.33}$$

While using (2.7.33), one should remember that τ is still defined in terms of the cartesian coordinate components.

In order to define the tangential displacements inside the crack directly in terms of the prescribed shear loading, equations (2.7.6) and (2.7.7) have to be rewritten for $\rho \le a$. They will have a similar form, with a difference only in the limits of integration, namely,

$$u_{n+1}(\rho) = 2G_1\rho^{n+1}\int_a^\infty \frac{dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{(x^2 - \rho_0^2)^{1/2}}$$

$$+ 2G_2\rho^{n+1}\int_a^\infty \frac{dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}}\int_a^x \frac{2nx^2 - (2n + 1)\rho_0^2}{(x^2 - \rho_0^2)^{1/2}}\ \bar{\tau}_{-n+1}(\rho_0)\rho_0^n\ d\rho_0$$

$$+ \frac{2G_1}{\rho^{n+1}}\int_0^\rho \frac{x^{2n+2}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}$$

$$+ \frac{2G_2}{\rho^{n+1}}\int_0^\rho \frac{x^{2n}\ dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{2n\rho^2 - (2n + 1)x^2}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}\ \bar{\tau}_{-n+1}(\rho_0)\ d\rho_0,$$

$$\text{for } n=0,1,2,\ \dots \tag{2.7.34}$$

$$u_{-n+1}(\rho) = 2G_1\rho^{n-1}\int_a^\infty \frac{dx}{x^{2n-2}(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\tau_{-n+1}(\rho_0)\rho_0^n d\rho_0}{(x^2 - \rho_0^2)^{1/2}}$$

$$+\ 2G_2\rho^{n-1}\int_a^\infty \frac{dx}{x^{2n}(x^2-\rho^2)^{1/2}}\int_a^x \frac{(2n-1)x^2-2n\rho^2}{(x^2-\rho_0^2)^{1/2}}\,\bar{\tau}_{n+1}(\rho_0)\rho_0^n\ d\rho_0$$

$$+\ \frac{2G_1}{\rho^{n-1}}\int_0^\rho \frac{x^{2n-2}dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_{-n+1}(\rho_0)\ d\rho_0}{\rho_0^{n-2}(\rho_0^2-x^2)^{1/2}}$$

$$+\ \frac{2G_2}{\rho^{n-1}}\int_0^\rho \frac{x^{2n-2}\ dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{(2n-1)\rho_0^2-2nx^2}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\bar{\tau}_{n+1}(\rho_0)\ d\rho_0,$$

$$\text{for } n=1,2,3,\ \ldots \tag{2.7.35}$$

The first two terms in (2.7.34) and (2.7.35) represent the displacement inside the crack due to the shear traction outside, while the remaining terms give the displacement caused by the shear tractions inside. Substitution of (2.7.23) in (2.7.34) yields, after integration with respect to ρ_0,

$$u_{n+1}(\rho) = -2G_1\rho^{n+1}\int_a^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^a \frac{\tau_{n+1}(t)t^{n+2}dt}{(x^2-t^2)^{1/2}}$$

$$-\ 2G_2\rho^{n+1}\int_a^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^a \frac{2nx^2-(2n+1)t^2}{(x^2-t^2)^{1/2}}\,\bar{\tau}_{-n+1}(t)t^n dt$$

$$+\ \frac{2G_1}{\rho^{n+1}}\int_0^\rho \frac{x^{2n+2}dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n(\rho_0^2-x^2)^{1/2}}$$

$$+\ \frac{2G_2}{\rho^{n+1}}\int_0^\rho \frac{x^{2n}dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{2n\rho^2-(2n+1)x^2}{\rho_0^n(\rho_0^2-x^2)^{1/2}}\,\bar{\tau}_{-n+1}(\rho_0)\ d\rho_0. \tag{2.7.36}$$

Transform the third term in (2.7.36) by using (2.7.25). The procedure is as follows:

$$\frac{2G_1}{\rho^{n+1}}\int_0^{\rho}\frac{x^{2n+2}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}$$

$$= \frac{2G_1}{\rho^{n+1}}\Bigg[\int_0^{\rho}\frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n}\int_0^{\rho_0}\frac{x^{2n+2}dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}$$

$$+ \int_{\rho}^{a}\frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n}\int_0^{\rho}\frac{x^{2n+2}dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}\Bigg]$$

$$= \frac{2G_1}{\rho^{n+1}}\Bigg[\int_0^{\rho}\frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n}\int_{\rho}^{\infty}\frac{(\rho\rho_0)^{2n+2}dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}$$

$$+ \int_{\rho}^{a}\frac{\tau_{n+1}(\rho_0)\ d\rho_0}{\rho_0^n}\int_{\rho_0}^{\infty}\frac{(\rho\rho_0)^{2n+2}dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}\Bigg]$$

$$= 2G_1\rho^{n+1}\Bigg[\int_a^{\infty}\frac{dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}}\int_0^a\frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{(x^2 - \rho_0^2)^{1/2}}$$

$$+ \int_{\rho}^{a}\frac{dx}{x^{2n+2}(x^2 - \rho^2)^{1/2}}\int_0^x\frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{(x^2 - \rho_0^2)^{1/2}}\Bigg]. \qquad (2.7.37)$$

An analogous transformation of the last term in (2.7.36) leads to the identity

$$\frac{2G_2}{\rho^{n+1}}\int_0^{\rho}\frac{x^{2n}dx}{(\rho^2 - x^2)^{1/2}}\int_x^a\frac{2n\rho^2 - (2n + 1)x^2}{\rho_0^n(\rho_0^2 - x^2)^{1/2}}\ \bar{\tau}_{-n+1}(\rho_0)\ d\rho_0$$

$$= 2G_2\rho^{n+1}\left[\int_a^\infty \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^a \frac{2nx^2-(2n+1)\rho_0^2}{(x^2-\rho_0^2)^{1/2}}\,\bar{\tau}_{-n+1}(\rho_0)\rho_0^n\,d\rho_0\right.$$

$$\left.+\int_\rho^a \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^x \frac{2nx^2-(2n+1)\rho_0^2}{(x^2-\rho_0^2)^{1/2}}\,\bar{\tau}_{-n+1}(\rho_0)\rho_0^n\,d\rho_0\right]. \tag{2.7.38}$$

Back substitution of (2.7.37) and (2.7.38) in (2.7.36) results in

$$u_{n+1}(\rho) = 2G_1\rho^{n+1}\int_\rho^a \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^x \frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{(x^2-\rho_0^2)^{1/2}}$$

$$+ 2G_2\rho^{n+1}\int_\rho^a \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^x \frac{2nx^2-(2n+1)\rho_0^2}{(x^2-\rho_0^2)^{1/2}}\,\bar{\tau}_{-n+1}(\rho_0)\rho_0^n\,d\rho_0,$$

$$\text{for } n=0,1,2,\ \ldots,\ \text{and } \rho\le a. \tag{2.7.39}$$

A similar procedure can be applied to (2.7.35). Substitution of (2.7.23) in (2.7.35) yields, after an integration with respect to ρ_0,

$$u_{-n+1}(\rho) = -2G_1\rho^{n-1}\int_a^\infty \frac{dx}{x^{2n-2}(x^2-\rho^2)^{1/2}}\int_0^a \frac{\tau_{-n+1}(t)t^n\,dt}{(x^2-t^2)^{1/2}}$$

$$+ 2G_2\rho^{n-1}\int_a^\infty \frac{(2n-1)x^2-2n\rho^2}{x^{2n}(x^2-\rho^2)^{1/2}}\,dx\left[-\int_0^a \frac{\bar{\tau}_{n+1}(t)\,dt}{(x^2-t^2)^{1/2}}\right.$$

$$\left.+\frac{1}{ax}\left(\int_0^a (a^2-t^2)^{1/2}\bar{\tau}_{n+1}(t)t^{n+2}\,dt+\frac{\pi}{2}A_n\right)\right]$$

$$+\frac{2G_1}{\rho^{n-1}}\int_0^\rho \frac{x^{2n-2}dx}{(\rho^2-x^2)^{1/2}}\int_x^a \frac{\tau_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}(\rho_0^2-x^2)^{1/2}}$$

$$+ \frac{2G_2}{\rho^{n-1}} \int_0^\rho \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{(2n-1)\rho_0^2 - 2nx^2}{\rho_0^n(\rho_0^2 - x^2)^{1/2}} \bar{\tau}_{n+1}(\rho_0) \, d\rho_0. \tag{2.7.40}$$

The following identities may be established by using the procedures identical to those used for deriving (2.7.37) and (2.7.38)

$$\frac{2G_1}{\rho^{n-1}} \int_0^\rho \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\tau_{-n+1}(\rho_0) \, d\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}}$$

$$= 2G_1\rho^{n-1}\left[\int_a^\infty \frac{dx}{x^{2n-2}(x^2 - \rho^2)^{1/2}} \int_0^a \frac{\tau_{-n+1}(\rho_0)\rho_0^n}{(x^2 - \rho_0^2)^{1/2}} \, d\rho_0 \right.$$

$$\left. + \int_\rho^a \frac{dx}{x^{2n-2}(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\tau_{-n+1}(\rho_0)\rho_0^n}{(x^2 - \rho_0^2)^{1/2}} \, d\rho_0 \right], \tag{2.7.41}$$

$$\frac{2G_2}{\rho^{n-1}} \int_0^\rho \frac{x^{2n-2}dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{(2n-1)\rho_0^2 - 2nx^2}{\rho_0^n(\rho_0^2 - x^2)^{1/2}} \bar{\tau}_{n+1}(\rho_0) \, d\rho_0$$

$$= 2G_2\rho^{n-1}\left[\int_a^\infty \frac{(2n-1)x^2 - 2n\rho^2}{x^{2n}(x^2 - \rho^2)^{1/2}} \, dx \int_0^a \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n \, d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \right.$$

$$\left. + \int_\rho^a \frac{(2n-1)x^2 - 2n\rho^2}{x^{2n}(x^2 - \rho^2)^{1/2}} \, dx \int_0^x \frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n \, d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \right]. \tag{2.7.42}$$

The back substitution of (2.7.41) and (2.7.42) in (2.7.40) yields

$$u_{-n+1}(\rho) = 2G_1\rho^{n-1} \int_\rho^a \frac{dx}{x^{2n-2}(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\tau_{-n+1}(\rho_0)\rho_0^n}{(x^2 - \rho_0^2)^{1/2}} \, d\rho_0$$

$$+\ 2G_2\rho^{n-1}\left[\int_\rho^a \frac{(2n-1)x^2 - 2n\rho^2}{x^{2n}(x^2-\rho^2)^{1/2}}\,dx \int_0^x \frac{\bar\tau_{n+1}(\rho_0)\rho_0^n\,d\rho_0}{(x^2-\rho_0^2)^{1/2}} + B_n(a^2-\rho^2)^{1/2}\right]. \qquad (2.7.43)$$

Here

$$B_n = a^{-2n-1}\int_0^a t^n(a^2-t^2)^{1/2}[\bar\tau_{n+1}(t) - (2n+1)(G_2/G_1)\tau_{-n+1}(t)]\,dt. \qquad (2.7.44)$$

Formulae (2.7.39), (2.7.43), and (2.7.44) give the tangential displacement of the crack faces in terms of the prescribed shear tractions. Note that the displacement vanishes outside the crack. The complex tangential displacement u can be represented in terms of its harmonics as

$$u(\rho,\phi) = \sum_{n=0}^{\infty} u_{n+1}(\rho)e^{i(n+1)\phi} + \sum_{n=1}^{\infty} u_{-n+1}(\rho)e^{-i(n-1)\phi}, \qquad (2.7.45)$$

where u_{n+1} and u_{-n+1} are defined by (2.7.39) and (2.7.44). The summation will be performed for the terms containing G_1 and G_2 separately. Substitution of (2.7.39) in (2.7.45) and summation of the terms with G_1 gives

$$\frac{G_1}{\pi}\int_0^{2\pi} d\phi_0 \int_\rho^a \frac{\lambda(\rho\rho_0/x^2,\ \phi-\phi_0)\,dx}{(x^2-\rho^2)^{1/2}} \int_0^x \frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\ . \qquad (2.7.46)$$

Here we should change the order of integration according to the scheme

$$\int_\rho^a dx \int_0^x d\rho_0 = \int_\rho^a d\rho_0 \int_{\rho_0}^a dx + \int_0^\rho d\rho_0 \int_\rho^a dx. \qquad (2.7.47)$$

Integration with respect to x in (2.7.46) can be performed as in (1.1.23), and yields

$$\frac{G_1}{\pi}\int_0^{2\pi}\int_0^a \left[\ \frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)\ \right]\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0. \qquad (2.7.48)$$

where R and $\eta(x)$ are defined by (2.2.14) and (1.1.6) respectively. We recall that, according to our convention, the abbreviation η stands for $\eta(a)$. Summation of the terms with G_2 in (2.7.39) and (2.7.43) is slightly more complicated, but generally it reduces to the series

$$\sum_{n=0}^{\infty} nz^n = \frac{z}{(1 - z)^2} .$$

The result of summation is

$$\frac{G_2}{\pi} \int_0^{2\pi} e^{2i\phi_0}\, d\phi_0 \int_\rho^a \frac{\lambda(\rho\rho_0/x^2,\ \phi-\phi_0)\ dx}{(x^2 - \rho^2)^{1/2}}$$

$$\times \int_0^x \left\{ 1 + \frac{4iq\rho e^{-i\phi_0} \sin(\phi - \phi_0)}{x^2(1 - \zeta)(1 - \bar{\zeta})} \right\} \frac{\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} .$$

(2.7.49)

Here $\zeta=(\rho\rho_0/x^2)e^{i(\phi-\phi_0)}$, and $\bar{\zeta}$ is a complex conjugate of ζ. The integral in (2.7.49), though looking formidable, is a perfect differential and can be computed as indefinite

$$\int \frac{\lambda(\rho\rho_0/x^2,\ \phi-\phi_0)\ dx}{(x^2 - \rho^2)^{1/2}\ (x^2 - \rho_0^2)^{1/2}} \left\{ 1 + \frac{4iq\rho e^{-i\phi_0} \sin(\phi - \phi_0)}{x^2(1 - \zeta)(1 - \bar{\zeta})} \right\}$$

$$= e^{-2i\phi_0} \frac{q}{\bar{q}R} \tan^{-1}\left(\frac{\eta(x)}{R}\right) + \frac{2i\rho e^{-i\phi_0} \sin(\phi - \phi_0)\ \eta(x)}{x^2\ \bar{q}\ (1 - \zeta)(1 - \bar{\zeta})} .$$

(2.7.50)

It is noteworthy, that (2.7.50) can be considered as a generalization of the integral representation for q^2/R^3, given by (2.5.7). Indeed, such a representation can be obtained from (2.7.50) by taking a definite integral, namely,

$$\frac{2}{\pi} \int_{max(\rho_0,\rho)}^{\infty} \frac{e^{2i\phi_0}\lambda(\rho\rho_0/x^2,\ \phi-\phi_0)\ dx}{(x^2 - \rho^2)^{1/2}\ (x^2 - \rho_0^2)^{1/2}} \left\{ 1 + \frac{4iq\rho e^{-i\phi_0} \sin(\phi - \phi_0)}{x^2(1 - \zeta)(1 - \bar{\zeta})} \right\} = \frac{q}{\bar{q}R}.$$

By changing the order of integration in (2.7.49) and integrating with respect to

x, according to (2.7.50), we obtain

$$\frac{G_2}{\pi}\int_0^{2\pi}\int_0^a \left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{2i\rho e^{i\phi_0}\sin(\phi-\phi_0)\eta}{a^2\bar{q}(1-t)(1-\bar{t})}\right]\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0, \tag{2.7.51}$$

where t is defined by (1.4.41). The remaining step is the summation of B_n (2.7.44), with the result

$$\frac{G_2}{\pi a^2}\int_0^{2\pi}\int_0^a \left[\frac{\bar{\tau}(\rho_0,\phi_0)\, e^{2i\phi_0}}{(1-\bar{t})} - \frac{G_2}{G_1}\frac{(3-\bar{t})\,\tau(\rho_0,\phi_0)}{(1-\bar{t})^2}\right]\eta\ \rho_0 d\rho_0 d\phi_0. \tag{2.7.52}$$

Finally, the summation of (2.7.48), (2.7.51) and (2.7.52) leads to

$$u(\rho,\phi) = \frac{G_1}{\pi}\int_0^{2\pi}\int_0^a \left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\frac{(3-\bar{t})\,\eta}{a^2(1-\bar{t})^2}\right]\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0$$

$$+\frac{G_2}{\pi}\int_0^{2\pi}\int_0^a \left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta[(q/\bar{q}) - te^{2i\phi_0}]}{a^2(1-t)(1-\bar{t})}\right]\bar{\tau}(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0. \tag{2.7.53}$$

Since the normal tractions vanish in the plane $z=0$, the normal displacement w will be defined by (2.7.4) as

$$w(\rho,\phi) = H\alpha\ \Re\int_0^{2\pi}\int_0^{\infty}\frac{\tau(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{\rho e^{i\phi} - \rho_0 e^{i\phi_0}}\ ,$$

which is equivalent to the following series representation

$$w(\rho,\phi) = 2\pi H\alpha\ \Re\sum_{n=0}^{\infty}\left\{\frac{e^{-i(n+1)\phi}}{\rho^{n+1}}\int_0^{\rho}\tau_{-n}(\rho_0)\ \rho_0^{n+1} d\rho_0\right.$$

$$-\ e^{in\phi}\rho^{n}\left[\int_{\rho}^{a}\frac{\tau_{n+1}(\rho_0)}{\rho_0^{n}}\,d\rho_0+\int_{a}^{\infty}\frac{\tau_{n+1}(\rho_0)}{\rho_0^{n}}\,d\rho_0\right]\Bigg\}\ ,\qquad \text{for}\ \rho\le a;$$

(2.7.54)

$$w(\rho,\phi)=2\pi H\alpha\ \Re\sum_{n=0}^{\infty}\Bigg\{\frac{e^{-i(n+1)\phi}}{\rho^{n+1}}\left[\int_0^a\tau_{-n}(\rho_0)\,\rho_0^{n+1}d\rho_0\right.$$

$$\left.+\int_a^{\rho}\tau_{-n}(\rho_0)\,\rho_0^{n+1}\,d\rho_0\right]-e^{in\phi}\rho^{n}\int_{\rho}^{\infty}\frac{\tau_{n+1}(\rho_0)}{\rho_0^{n}}\,d\rho_0\Bigg\},\qquad\text{for}\ \rho\ge a.$$

(2.7.55)

By substituting formulae (2.7.23) and (2.7.27) in (2.7.54) and (2.7.55), and performing the summation and integration, we obtain

$$w=H\alpha\ \Re\Bigg\{\int_0^{2\pi}\!\!\int_0^a\frac{(a^2-\rho_0^2)^{1/2}}{a(1-t)^{1/2}}\,\frac{\tau(\rho_0,\phi_0)}{q}\,\rho_0d\rho_0d\phi_0$$

$$+\frac{G_2}{G_1a}\int_0^{2\pi}\!\!\int_0^a\left(\frac{1}{(1-t)^{3/2}}-1\right)(a^2-\rho_0^2)^{1/2}\,\bar{\tau}(\rho_0,\phi_0)e^{i\phi_0}\,d\rho_0\,d\phi_0\Bigg\},\ \text{for}\ \rho\le a;$$

and

$$w=\frac{2}{\pi}H\alpha\ \Re\Bigg\{\int_0^{2\pi}\!\!\int_0^a\frac{(a^2-\rho_0^2)^{1/2}}{a(1-t)^{1/2}}\tan^{-1}\!\left(\frac{a(1-t)^{1/2}}{(\rho^2-a^2)^{1/2}}\right)\frac{\tau(\rho_0,\phi_0)}{q}\,\rho_0d\rho_0d\phi_0$$

$$+\frac{G_2}{G_1}\int_0^{2\pi}\!\!\int_0^a\left[\frac{1}{a(1-t)^{3/2}}\tan^{-1}\!\left(\frac{a(1-t)^{1/2}}{(\rho^2-a^2)^{1/2}}\right)-\frac{1}{a}\sin^{-1}\frac{a}{\rho}\right.$$

$$\left.-\frac{\rho_0e^{-i\phi_0}(\rho^2-a^2)^{1/2}}{\bar{q}a^2(1-t)}\right](a^2-\rho_0^2)^{1/2}\,\bar{\tau}(\rho_0,\phi_0)\,e^{i\phi_0}\,d\rho_0d\phi_0\Bigg\},\ \text{for}\ \rho>a.$$

Exercise 2.7

1. Uniform shear tractions $\tau=\tau_0$, where τ_0 is a complex constant, are applied antisymmetrically inside a penny-shaped crack of radius a. Find the tangential displacements of the crack faces.

Answer: $u = 2\tau_0[(G_1^2 - G_2^2)/G_1](a^2 - \rho^2)^{1/2}$.

2. In the example above find the shear tractions in the plane z=0 outside the crack.

Answer: $\tau(\rho,\phi) = \dfrac{2}{\pi}\left\{\left[\sin^{-1}\left(\dfrac{a}{\rho}\right) - \dfrac{a}{(\rho^2 - a^2)^{1/2}}\right]\tau_0 - \dfrac{G_2 a^3 e^{2i\phi}}{G_1\rho^2(\rho^2 - a^2)^{1/2}}\bar{\tau}_0\right\}$.

3. Subject to the conditions of the first example, find the normal displacement w in the plane z=0.

Answer: $w(\rho,\phi) = \pi H\alpha\rho\Re\left[\tau_0 e^{-i\phi} + (G_2/G_1)\bar{\tau}_0 e^{i\phi}\right]$, for $\rho\le a$;

$$w(\rho,\phi) = 2H\alpha\Re\left[\tau_0 e^{-i\phi} + (G_2/G_1)\bar{\tau}_0 e^{i\phi}\right]\left[\rho\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{a}{\rho}(\rho^2 - a^2)^{1/2}\right], \text{ for } \rho\ge a.$$

2.8 Inverse crack problem in elasticity

The usually considered elastic crack problems assume that the stress distribution on the crack faces is known, and the crack opening displacements are to be determined. Investigation of materials with rigid inclusions leads to another formulation of the crack problem, namely, the displacements are prescribed on the crack faces, and the stress distribution is to be determined. The problem so formulated is called *the inverse crack problem*. We shall consider two types of problem: the case of a smooth rigid inclusion (normal displacements are prescribed, tangential stresses vanish) is called the inverse crack problem of type I. The second type corresponds to the case when the normal stress is equal to zero over the plane z=0, and the antisymmetric tangential displacements are prescribed inside the crack. The stress distribution is to be determined in each case. Strictly speaking, the inverse crack problem does not belong to the class of the mixed problems, and its exact solution is known for a general crack. We show below how some specific results can be obtained by the new method.

Smooth rigid inclusion problem. Consider a penny-shaped crack of radius a in a transversely isotropic space. Let the crack be opened by a rigid smooth inclusion. The crack opening displacements are prescribed as

$$w = w(\rho,\phi), \qquad \text{for } z = 0^+, \qquad 0\leq\rho\leq a;$$

$$w = -w(\rho,\phi), \qquad \text{for } z = 0^-, \qquad 0\leq\rho\leq a;$$

Due to symmetry, the problem can be reduced to the one of a half-space, with the boundary conditions at the plane z=0

$$w = w(\rho,\phi), \qquad \text{for } \rho\leq a, \qquad 0\leq\phi<2\pi;$$

$$w = 0, \qquad \text{for } \rho\geq a, \qquad 0\leq\phi<2\pi;$$

$$\tau = 0, \qquad \text{for } 0\leq\rho<\infty, \qquad 0\leq\phi<2\pi. \tag{2.8.1}$$

The general relationship between the crack opening displacement w and the applied pressure σ was established in (2.4.19), and in this case may be written as

$$w(\rho,\phi) = 4H\int_\rho^a \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{2.8.2}$$

Since in our case the displacement w is known, and σ is unknown, we can interpret (2.8.2) as an integral equation. It can be solved exactly. The first operator to be applied is

$$\mathcal{L}(t) \frac{d}{dt} \int_t^a \frac{\rho \; d\rho}{(\rho^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho}\right),$$

with the result

$$2\pi H\int_0^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi) = -\mathcal{L}(t)\frac{d}{dt} \int_t^a \frac{\rho \; d\rho}{(\rho^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho}\right)w(\rho,\phi). \tag{2.8.3}$$

The next operator to be applied is

$$\mathcal{L}\left(\frac{1}{y}\right) \frac{d}{dy} \int_0^y \frac{t \; dt}{(y^2 - t^2)^{1/2}} \mathcal{L}(t).$$

The final result reads

$$\sigma(y,\phi) = -\frac{1}{\pi^2 H y}\,\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{t\ \mathrm{d}t}{(y^2-t^2)^{1/2}}\,\mathcal{L}(t^2)\frac{\mathrm{d}}{\mathrm{d}t}\int_t^a \frac{\rho\ \mathrm{d}\rho}{(\rho^2-t^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)w(\rho,\phi). \tag{2.8.4}$$

Formula (2.8.4) is valid inside the crack only. The normal traction outside the crack can be expressed directly in terms of the prescribed crack opening displacement due to the relationship between the normal stresses inside and outside the crack (2.4.17), which in this case takes the form

$$\sigma(\rho,\phi) = -\frac{2}{\pi(\rho^2-a^2)^{1/2}}\int_0^a \frac{(a^2-y^2)^{1/2}}{\rho^2-y^2}\,\mathcal{L}\left(\frac{y}{\rho}\right)\sigma(y,\phi)y\ \mathrm{d}y, \qquad \text{for } \rho>a. \tag{2.8.5}$$

Substitution of (2.8.4) in (2.8.5) yields, after integration with respect to y,

$$\sigma(\rho,\phi) = \frac{\mathcal{L}(1/\rho)}{\pi^2 H\rho^2(\rho^2-a^2)^{1/2}}\int_0^a\left[a - t\left(\frac{\rho^2-a^2}{\rho^2-t^2}\right)^{1/2}\right]\mathrm{d}t$$

$$\times\frac{\mathrm{d}}{\mathrm{d}t}\left[t\mathcal{L}(t^2)\frac{\mathrm{d}}{\mathrm{d}t}\int_t^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-t^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)w(\rho_0,\phi)\right].$$

Integration by parts in the last expression leads to

$$\sigma(\rho,\phi) = \frac{\mathcal{L}(1/\rho)}{\pi^2 H}\int_0^a \frac{x\mathrm{d}x}{(\rho^2-x^2)^{3/2}}\,\mathcal{L}(x^2)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)w(\rho_0,\phi). \tag{2.8.6}$$

Expression (2.8.6) can also be represented in the form

$$\sigma(\rho,\phi) = -\frac{\mathcal{L}(1/\rho)}{\pi^2 H\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^a \frac{x\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\,\mathcal{L}(x^2)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)w(\rho_0,\phi). \tag{2.8.7}$$

Comparison of (2.8.7) and (2.8.4) indicates that the stress outside the crack can be represented in almost the same form as the stress inside, with the only difference in the upper limit of the first integral. By using the rule (1.3.9) and the condition $w(a,\phi)=0$, expression (2.8.6) can be rewritten as

$$\sigma(\rho,\phi) = \frac{1}{\pi^2 H}\int_0^a \frac{\mathrm{d}x}{(\rho^2 - x^2)^{3/2}}\int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho_0}\left[\rho_0 \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) w(\rho_0,\phi)\right]. \tag{2.8.8}$$

Interchange of the order of integration and integration by parts yields

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2 H}\int_0^{2\pi}\int_0^a \frac{w(\rho_0,\phi_0)\ \rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{3/2}}. \tag{2.8.9}$$

The last expression can also be rewritten as

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2 H}\ \Delta\int_0^{2\pi}\int_0^a \frac{w(\rho_0,\phi_0)\ \rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}. \tag{2.8.10}$$

Here Δ is the two-dimensional Laplace operator. It will be shown later that formula (2.8.10) is of general nature: it is valid for an arbitrarily shaped flat crack all over the plane z=0.

We can also express some integral characteristics in terms of the crack opening displacement. The resultant force P is defined by

$$P = \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\ \rho \mathrm{d}\rho \mathrm{d}\phi. \tag{2.8.11}$$

Substitution of (2.8.4) in (2.8.11) leads to

$$P = -\frac{1}{\pi^2 H}\int_0^{2\pi} \mathrm{d}\phi \int_0^a \frac{x \mathrm{d}x}{(a^2 - x^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a \frac{w(\rho,\phi)\rho\ \mathrm{d}\rho}{(\rho^2 - x^2)^{1/2}}. \tag{2.8.12}$$

Integration by parts in (2.8.12) yields yet another representation

$$P = \frac{a^2}{\pi^2 H}\int_0^{2\pi} \mathrm{d}\phi \int_0^a \frac{\mathrm{d}x}{(a^2 - x^2)^{3/2}}\int_x^a \frac{w(\rho,\phi)\rho\ \mathrm{d}\rho}{(\rho^2 - x^2)^{1/2}}. \tag{2.8.13}$$

Several additional forms can be obtained by integration in (2.8.13) with respect to x, and a consequent use of various integral representations for the complete elliptic integrals (see Exercise 2.8.2).

One can also evaluate the resultant moment of the traction exerted by the rigid inclusion. Introducing the complex moment $M=M_x+iM_y$, we can deduce

$$M = -i\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)e^{i\phi}\rho^2 \, d\rho d\phi. \tag{2.8.14}$$

Substitution of (2.8.4) in (2.8.14) yields

$$M = \frac{i}{\pi^2 H}\int_0^{2\pi} e^{i\phi}\, d\phi \int_0^a \frac{x^3 dx}{(a^2 - x^2)^{1/2}} \frac{d}{dx}\int_x^a \frac{w(\rho,\phi)\, d\rho}{(\rho^2 - x^2)^{1/2}}. \tag{2.8.15}$$

Yet another representation can be obtained from (2.8.15) (see Exercise 2.8.3).

Example. It is of interest to consider a general case where the displacements can be presented as an expansion

$$w(\rho,\phi) = (a^2 - \rho^2)^{1/2}\sum_{n=-\infty}^{\infty} w_n \rho^{|n|} e^{in\phi}. \tag{2.8.16}$$

Substitution of (2.8.16) in (2.8.4) and (2.8.7) yields

$$\sigma(\rho,\phi) = \frac{1}{2\sqrt{\pi}H}\sum_{n=-\infty}^{\infty} \frac{\Gamma(|n| + 3/2)}{\Gamma(|n| + 1)}\, w_n \rho^{|n|} e^{in\phi}, \qquad \text{for } \rho\le a.$$

$$\sigma(\rho,\phi) = -\frac{1}{2\pi H}\sum_{n=-\infty}^{\infty} \frac{a^{2|n|+3} w_n e^{in\phi}}{(2|n| + 3)\rho^{|n|+3}}\, F\left(\frac{3}{2},\ \frac{3}{2}+|n|;\ \frac{5}{2}+|n|;\ \frac{a^2}{\rho^2}\right), \qquad \text{for } \rho>a. \tag{2.8.17}$$

The Gauss hypergeometric function can be expressed in terms of elementary functions (Bateman and Erdélyi, 1955)

$$F\left(\frac{3}{2},\ \frac{3}{2}+n;\ \frac{5}{2}+n;\ \zeta\right) = \frac{(-1)^n(3 + 2n)}{n!\ \ (1 - \zeta)^{1/2}}\frac{d^n}{d\zeta^n}\left\{\frac{(1-\zeta)^n}{\zeta}\left[1 - \left(\frac{1-\zeta}{\zeta}\right)^{1/2}\sin^{-1}\sqrt{\zeta}\right]\right\}. \tag{2.8.18}$$

The traction in (2.8.17) becomes singular when $\rho\to a^+$. Using integration by parts in (2.8.6), the singular and nonsingular parts can be separated as follows:

$$\sigma(\rho,\phi) = -\frac{1}{2\pi H}\sum_{n=-\infty}^{\infty}\frac{a^{2|n|+1}}{\rho^{|n|+1}}\left[\frac{\rho}{(\rho^2 - a^2)^{1/2}} - F\left(\frac{1}{2},\ \frac{1}{2}+|n|;\ \frac{3}{2}+|n|;\ \frac{a^2}{\rho^2}\right)\right]w_n e^{in\phi}, \tag{2.8.19}$$

which yields the stress intensity factor as

$$k_1 = \frac{\sqrt{a}}{2\sqrt{2}\pi H}\sum_{n=-\infty}^{\infty} w_n a^{|n|} e^{in\phi}. \tag{2.8.20}$$

The hypergeometric function in (2.8.19) can be expressed in elementary functions:

$$F\left(\frac{1}{2},\ \frac{1}{2}+|n|;\ \frac{3}{2}+|n|;\ \zeta\right) = (-1)^n\,\frac{2n+1}{n!}\,\sqrt{1-\zeta}\,\frac{d^n}{d\zeta^n}\left[\frac{(1-\zeta)^{n-1/2}}{\sqrt{\zeta}}\,\sin^{-1}\sqrt{\zeta}\right]. \tag{2.8.21}$$

The total force P and the tilting moment M can be determined from (2.8.12) and (2.8.15) respectively

$$P = \frac{\pi a^2}{4H}\, w_0,$$

$$M = -i\,\frac{3\pi a^4}{16H}\, w_{-1}$$

Inverse crack problem of type II. Let the tangential displacements be prescribed on the crack faces

$$u = u(\rho,\phi), \qquad \text{for } \rho\le a \qquad z = 0^+;$$

$$u = -u(\rho,\phi), \qquad \text{for } \rho\le a \qquad z = 0^-. \tag{2.8.22}$$

It is required to find the shear stress distribution in the plane z=0. Due to antisymmetry, the problem can be reduced to that of a half-space, subjected to the boundary conditions:

$$u = u(\rho,\phi), \qquad \text{for } \rho\le a, \qquad 0\le\phi<2\pi;$$

$$u = 0, \qquad \text{for } \rho\ge a, \qquad 0\le\phi<2\pi;$$

$$\sigma = 0, \qquad \text{for } 0\le\rho<\infty, \qquad 0\le\phi<2\pi. \tag{2.8.23}$$

The relationship between the tangential displacement u and the shear traction τ

was established in (2.7.39) and (2.7.43). Since in the inverse problems u is known, and τ is to be determined, we should treat these expressions as a set of integral equations to be solved for the two unknowns τ_{n+1} and τ_{-n+1}. Assume the solution in the form

$$\tau_{n+1}(\rho) = G_1 f_{n+1}(\rho) + G_2\left[\bar{f}_{-n+1}(\rho) - \frac{2n}{\rho^{n+1}}\int_0^{\rho} \bar{f}_{-n+1}(x)x^{n}\mathrm{d}x\right], \quad \text{for } n = 0, 1, 2 \ldots$$

$$\tau_{-n+1}(\rho) = G_1 f_{-n+1}(\rho) + G_2\left[\bar{f}_{n+1}(\rho) + 2n\rho^{n-1}\int_0^{\rho} \frac{\bar{f}_{n+1}(x)}{x^{n}}\mathrm{d}x\right] + C_n\rho^{n-1}, \quad \text{for } n = 1, 2, 3 \ldots$$

(2.8.24)

Here f is an as yet unknown complex function and C_n is an as yet unknown complex constant. Substitution of (2.8.24) in equation (2.7.39) and subsequent integration results in

$$u_{n+1}(\rho) = 2(G_1^2 - G_2^2)\rho^{n+1}\int_{\rho}^{a} \frac{\mathrm{d}x}{x^{2n+2}(x^2-\rho^2)^{1/2}}\int_0^{x} \frac{f_{n+1}(\rho_0)\rho_0^{n+2}\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}} \tag{2.8.25}$$

from which the value of f_{n+1} can be found as

$$f_{n+1}(\rho) = -\frac{2}{\pi^2(G_1^2-G_2^2)\rho^{n+2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^{\rho} \frac{x^{2n+3}\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{a} \frac{u_{n+1}(\rho_0)\,\mathrm{d}\rho_0}{\rho_0^{n}\,(\rho_0^2-x^2)^{1/2}} \tag{2.8.26}$$

Substitution of (2.8.24) in equation (2.7.43) gives, after simplification

$$u_{-n+1}(\rho) = 2(G_1^2 - G_2^2)\rho^{n-1}\int_{\rho}^{a} \frac{\mathrm{d}x}{x^{2n-2}(x^2-\rho^2)^{1/2}}\int_0^{x} \frac{f_{-n+1}(\rho_0)\rho_0^{n}\,\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}} + 2D_n\rho^{n-1}(a^2-\rho^2)^{1/2}$$

(2.8.27)

Here D_n is a complex constant defined by

$$D_n = G_2\left[2nG_1\int_0^{a} \frac{\mathrm{d}x}{x^{2n}}\int_0^{x} \frac{\bar{f}_{n+1}(\rho)\,\rho^{n}\,\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}}\right.$$

$$+ 2nG_2 \int_0^a f_{-n+1}(\rho) \frac{\rho^n}{a^{2n}} \cosh^{-1}(\frac{a}{\rho})\, d\rho + B_n \Big] + G_1 C_n \frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+1/2)} \tag{2.8.28}$$

Now the purpose of introducing the constant C_n in (2.8.24) becomes clear: we can choose C_n so that D_n vanish. In this case expression (2.8.27) can be inverted, with the result

$$f_{-n+1}(\rho) = -\frac{2}{\pi^2(G_1^2 - G_2^2)\rho^n} \frac{d}{d\rho} \int_0^\rho \frac{x^{2n-1} dx}{(\rho^2 - x^2)^{1/2}} \frac{d}{dx} \int_x^a \frac{u_{-n+1}(\rho_0)\, d\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}} \tag{2.8.29}$$

We can substitute (2.8.26) and (2.8.29) in (2.8.28) in order to find the value of the as yet unknown constant C_n from the condition $D_n = 0$. The mathematical transformations involved are cumbersome though elementary. Some of the integrals are presented below.

$$\int_0^a \frac{dx}{x^{2n}} \int_0^x \frac{\bar{f}_{n+1}(\rho)\, \rho^n\, d\rho}{(x^2 - \rho^2)^{1/2}} = \frac{1}{\pi(G_1^2 - G_2^2)} \left[\frac{2n+1}{2n} \int_0^a \frac{u_{n+1}(\rho)\, d\rho}{\rho^{n+1}} - \frac{\sqrt{\pi}\,\Gamma(n)}{2a^{2n}\,\Gamma(n+1/2)} \int_0^a u_{n+1}(\rho)\, \rho^{n-1} d\rho \right], \tag{2.8.30}$$

$$\int_0^a f_{-n+1}(\rho)\, \rho^n \cosh^{-1}(\frac{a}{\rho})\, d\rho = \frac{\sqrt{\pi}\,\Gamma(n)}{\pi(G_1^2 - G_2^2)\,\Gamma(n-1/2)} \int_0^a u_{-n+1}(\rho)\, \rho^{n-1} d\rho. \tag{2.8.31}$$

Substitution of (2.8.24) in (2.7.44) gives the following expression for B_n

$$B_n = \frac{1}{a^{2n+1}} \int_0^a \left\{ -2nG_2 f_{-n+1}(\rho) + \left[G_1 - (2n+1)\frac{G_2^2}{G_1} \right] \bar{f}_{n+1}(\rho) \right.$$

$$\left. - \frac{2nG_2}{\rho^{n+1}} \int_0^\rho x^n f_{-n+1}(x)\, dx - 2n(2n+1) \frac{G_2^2}{G_1} \rho^{n-1} \int_0^\rho \frac{\bar{f}_{n+1}(x)}{x^n}\, dx \right\} (a^2 - \rho^2)^{1/2} \rho^n d\rho \tag{2.8.32}$$

The following integrals need to be computed

$$\int_0^a f_{-n+1}(\rho)\rho^n(a^2-\rho^2)^{1/2}d\rho = \frac{1}{\pi(G_1^2-G_2^2)}\frac{\sqrt{\pi}\,\Gamma(n+1/2)}{\Gamma(n)}\int_0^a u_{-n+1}(\rho)\rho^n d\rho \tag{2.8.33}$$

$$\int_0^a f_{n+1}(\rho)\rho^n(a^2-\rho^2)^{1/2}d\rho = \frac{a}{\pi(G_1^2-G_2^2)}\frac{\sqrt{\pi}\,\Gamma(n+1)}{\Gamma(n+1/2)}\int_0^a u_{n+1}(\rho)\rho^{n-1} d\rho \tag{2.8.34}$$

$$\int_0^a (a^2-\rho^2)^{1/2}\frac{d\rho}{\rho}\int_0^\rho f_{-n+1}(x)x^n dx = \frac{1}{\pi(G_1^2-G_2^2)}\left[\frac{a\sqrt{\pi}\,\Gamma(n)}{\Gamma(n-1/2)}\int_0^a u_{-n+1}(\rho)\rho^{n-1}d\rho\right.$$

$$\left.-\frac{\sqrt{\pi}\,\Gamma(n+1/2)}{\Gamma(n)}\int_0^a u_{-n+1}(\rho)\rho^n d\rho\right] \tag{2.8.35}$$

$$\int_0^a \rho^{2n-1}(a^2-\rho^2)^{1/2}\,d\rho\int_0^\rho \frac{f_{n+1}(x)}{x^n}\,dx = \frac{1}{\pi(G_1^2-G_2^2)}\left[\frac{a^{2n+1}}{2n}\int_0^a \frac{u_{n+1}(\rho)}{\rho^{n+1}}\,d\rho - \right.$$

$$\left.-\frac{a\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+1/2)}\int_0^a u_{n+1}(\rho)\rho^{n-1}d\rho\right] \tag{2.8.36}$$

Substitution of (2.8.33-2.8.36) in (2.8.32) yields

$$B_n = \frac{1}{\pi a^{2n}(G_1^2-G_2^2)}\left[-2G_2\frac{\sqrt{\pi}\,\Gamma(n+1)}{\Gamma(n-1/2)}\int_0^a u_{-n+1}(\rho)^{n-1}d\rho + G_1\frac{\sqrt{\pi}\,\Gamma(n+1)}{\Gamma(n+1/2)}\int_0^a \bar{u}_{n+1}(\rho)\rho^{n-1}d\rho\right.$$

$$\left.-(2n+1)a^{2n}\frac{G_2^2}{G_1}\int_0^a \frac{\bar{u}_{n+1}(\rho)}{\rho^{n+1}}\,d\rho\right] - C_n\frac{G_2}{G_1}\frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+1/2)}. \tag{2.8.37}$$

Substitution of (2.8.30), (2.8.31) and (2.8.37) in (2.8.28) gives a fairly simple formula for C_n, namely,

$$C_n = -\frac{4G_2\Gamma(n+3/2)}{\pi^{3/2}(G_1^2-G_2^2)\,\Gamma(n)}\int_0^a \frac{\bar{u}_{n+1}(\rho)\,d\rho}{\rho^{n+1}} \tag{2.8.38}$$

The problem, in principle, is now solved but a certain simplification is possible due to the following integral:

$$\int_0^\rho \frac{\bar{f}_{n+1}(x)}{x^n}\,dx = -\frac{2}{\pi^2(G_1^2-G_2^2)}\left\{\frac{1}{\rho^{2n}}\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\right.$$

$$\left.\times\frac{d}{dx}\left[x^{2n+1}\int_x^a \frac{\bar{u}_{n+1}(\rho_0)\,d\rho_0}{\rho_0^n\,(\rho_0^2-x^2)^{1/2}}\right] - \frac{\sqrt{\pi}\,\Gamma(n+3/2)}{\Gamma(n+1)}\int_0^a \frac{\bar{u}_{n+1}(\rho)\,d\rho}{\rho^{n+1}}\right\} \tag{2.8.39}$$

Note that the last term in (2.8.39) will cancel C_n when substituted in (2.8.24). Finally, formulae (2.8.26), (2.8.29) and (2.8.38) lead to

$$\tau_{n+1}(\rho) = -\frac{2}{\pi^2(G_1^2-G_2^2)}\left\{\frac{G_1}{\rho^{n+2}}\frac{d}{d\rho}\int_0^\rho \frac{x^{2n+3}\,dx}{(\rho^2-x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{u_{n+1}(\rho_0)\,d\rho_0}{\rho_0^n\,(\rho_0^2-x^2)^{1/2}}\right.$$

$$\left.+\,G_2\rho^n\frac{d}{d\rho}\left[\frac{1}{\rho^{2n}}\int_0^\rho \frac{x^{2n-1}dx}{(\rho^2-x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{\bar{u}_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}(\rho_0^2-x^2)^{1/2}}\right]\right\}$$

$$\text{for}\ \ \rho \le a,\quad n = 0, 1, 2, ... \tag{2.8.40}$$

$$\tau_{-n+1}(\rho) = -\frac{2}{\pi^2(G_1^2-G_2^2)}\left[\frac{G_1}{\rho^n}\frac{d}{d\rho}\int_0^\rho \frac{x^{2n-1}\,dx}{(\rho^2-x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{u_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}(\rho_0^2-x^2)^{1/2}}\right.$$

$$\left.+\,\frac{G_2}{\rho^n}\frac{d}{d\rho}\int_0^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\frac{d}{dx}\left(x^{2n+1}\int_x^a \frac{\bar{u}_{n+1}(\rho_0)d\rho_0}{\rho_0^n\,(\rho_0^2-x^2)^{1/2}}\right)\right]$$

$$\text{for}\ \ \rho \le a,\quad n = 1, 2, 3, ... \tag{2.8.41}$$

Expressions (2.8.40) and (2.8.41) are valid inside the crack only. In order to express the shear traction outside the crack, we recall formulae (2.7.23) and

(2.7.27) which relate one to the other. Substitution of (2.8.41) in (2.7.27) allows us to compute

$$A_n = -\frac{4G_2\Gamma(n+3/2)}{\pi^{3/2}(G_1^2 - G_2^2)\,\Gamma(n)}\int_0^a \bar{u}_{-n+1}(\rho)\,\rho^n d\rho \tag{2.8.42}$$

Finally, substitution of (2.8.40-2.8.42) in (2.7.23) yields

$$\tau_{n+1}(\rho) = -\frac{2}{\pi^2(G_1^2 - G_2^2)}\left[\frac{G_1}{\rho^{n+2}}\frac{d}{d\rho}\int_0^a \frac{x^{2n+3}\,dx}{(\rho^2 - x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{u_{n+1}(\rho_0)\,d\rho_0}{\rho_0^n\,(\rho_0^2 - x^2)^{1/2}}\right.$$

$$\left. + \frac{G_2}{\rho^n}\frac{d}{d\rho}\int_0^a \frac{x^{2n-1}dx}{(\rho^2 - x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{\bar{u}_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}}\right]$$

$$\text{for } \rho > a, \quad n = 0, 1, 2, \ldots \tag{2.8.43}$$

$$\tau_{-n+1}(\rho) = -\frac{2}{\pi^2(G_1^2 - G_2^2)}\left[\frac{G_1}{\rho^n}\frac{d}{d\rho}\int_0^a \frac{x^{2n-1}\,dx}{(\rho^2 - x^2)^{1/2}}\frac{d}{dx}\int_x^a \frac{u_{-n+1}(\rho_0)\,d\rho_0}{\rho_0^{n-2}(\rho_0^2 - x^2)^{1/2}}\right.$$

$$\left. + \frac{G_2}{\rho^n}\frac{d}{d\rho}\int_0^a \frac{dx}{(\rho^2 - x^2)^{1/2}}\frac{d}{dx}\left(x^{2n+1}\int_x^a \frac{\bar{u}_{n+1}(\rho_0)d\rho_0}{\rho_0^n\,(\rho_0^2 - x^2)^{1/2}}\right)\right]$$

$$\text{for } \rho > a, \quad n = 1, 2, 3, \ldots \tag{2.8.44}$$

Formulae (2.8.43) and (2.8.44) can be used to define the stress intensity factor directly in terms of the tangential displacements prescribed *inside* the crack. We recall that the stress intensity factor was given by (2.7.31). By using the properties

$$\lim_{\rho\to a}\left[(\rho - a)^{1/2}\frac{d}{d\rho}\int_0^\rho \frac{f(x)\,dx}{(\rho^2 - x^2)^{1/2}}\right] = -\frac{f(a)}{\sqrt{2a}},$$

$$\lim_{\rho\to a}\frac{d}{d\rho}\int_\rho^a \frac{f(x)\,dx}{(x^2 - \rho^2)^{1/2}} = -\frac{\pi}{2}\lim_{\rho\to a}\frac{f(\rho)}{(a^2 - \rho^2)^{1/2}}, \tag{2.8.45}$$

the appropriate harmonics of the stress intensity factor can be written as

$$k_{n+1} = -\frac{a}{\pi(G_1^2 - G_2^2)\sqrt{2a}} \lim_{\rho\to a}\left[\frac{G_1 u_{n+1}(\rho) + G_2 \overline{u}_{-n+1}(\rho)}{(a^2-\rho^2)^{1/2}}\right],$$

$$k_{-n+1} = -\frac{a}{\pi(G_1^2 - G_2^2)\sqrt{2a}} \lim_{\rho\to a}\left[\frac{G_1 u_{-n+1}(\rho) + G_2 \overline{u}_{n+1}(\rho)}{(a^2-\rho^2)^{1/2}}\right].$$

Their summation can be performed in an elementary manner, to give finally

$$k(\phi) = -\frac{a}{\pi(G_1^2 - G_2^2)\sqrt{2a}} \lim_{\rho\to a}\left[\frac{G_1 u(\rho,\phi) + G_2 e^{2i\phi}\overline{u}(\rho,\phi)}{(a^2-\rho^2)^{1/2}}\right]. \tag{2.8.46}$$

The simple expression (2.8.46) for the stress intensity factor will be very useful in the investigation of crack interaction since it is much easier to solve the integral equation involved in terms of the displacements than in terms of the stresses which are singular at the crack boundary. Expression (2.8.46) can be made more symmetric by introduction of the polar displacements $u^{(\rho)} = u_\rho + iu_\phi = e^{-i\phi}u$, and similar shear traction $\tau^{(\rho)} = \tau_{\rho z} + i\tau_{\phi z} = e^{-i\phi}\tau$. The corresponding stress intensity factor $k^{(\rho)}$ will take the form

$$k^{(\rho)} = -\frac{a}{\pi(G_1^2 - G_2^2)\sqrt{2a}} \lim_{\rho\to a}\left[\frac{G_1 u^{(\rho)}(\rho,\phi) + G_2 \overline{u}^{(\rho)}(\rho,\phi)}{(a^2-\rho^2)^{1/2}}\right]. \tag{2.8.47}$$

Note that $k^{(\rho)}$ is proportional to the combination $k_2 + ik_3$ of the second and the third mode stress intensity factors.

Summation of (2.8.40-2.8.41) and (2.8.43-2.8.44) leads to another simple result

$$\tau = -\frac{1}{2\pi^2(G_1^2 - G_2^2)}\left[G_1\Delta\int\!\!\int_S \frac{u}{R}\,dS + G_2\Lambda^2\int\!\!\int_S \frac{\overline{u}}{R}\,dS\right]. \tag{2.8.48}$$

It will be shown further, that the last expression is valid not just for a penny-shaped crack, but for a general flat crack all over the plane $z = 0$.

Exercise 2.8

1. The crack opening displacement w is prescribed by expression

$w = w_0(a^2 - \rho^2)^{1/2}$, with $w_0 = \text{const.}$ Find the normal stress distribution σ in the plane $z = 0$.

Answer: $\sigma = w_0/(4H)$, for $\rho \le a$;

$$\sigma = -\frac{w_0}{2\pi H}\left[\frac{a}{(a^2-\rho^2)^{1/2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right], \text{ for } \rho > a$$

2. Prove that the total force P exerted by an inclusion can be defined by

$$P = \frac{1}{\pi^2 H}\int_0^{2\pi} \mathrm{d}\phi \int_0^a (a^2 - x^2)^{1/2}\mathrm{d}x \int_x^a \frac{w(\rho,\phi)\rho\,\mathrm{d}\rho}{(a^2-\rho^2)(\rho^2-x^2)^{1/2}}.$$

Hint: use (2.8.13)

3. Prove that the tilting moment M can be defined by

$$M = -\frac{i}{\pi^2 H}\int_0^{2\pi} \mathrm{e}^{i\phi}\mathrm{d}\phi \int_0^a \frac{x^2(3a^2-2x^2)}{(a^2-x^2)^{3/2}}\,\mathrm{d}x \int_x^a \frac{w(\rho,\phi)\mathrm{d}\rho}{(\rho^2-x^2)^{1/2}}.$$

Hint: integrate (2.8.15) by parts.