INTRODUCTION

A short survey of methods for solving mixed boundary value problems of potential theory is given, and some of their limitations are pointed out. The necessity for developing a new method is justified. A concise description of each chapter is given for the reader's convenience.

Various applications of potential theory in electrostatics, heat transfer, elasticity, diffusion, and other branches of engineering science are well known, and have attracted significant attention of scientists like Laplace, Poisson, Green, Beltrami, Kirchhoff, Lord Kelvin, Hobson, and others who made a significant contribution to the field during past centuries. The boundary value problems with mixed conditions are the most difficult to solve, and, at the same time, they are among the most important in various engineering applications. Recall the celebrated problem of an electrified circular disc. It is next to impossible even to name every scientist who solved the problem by one method or another. We can specify two categories of methods. The first one requires construction of the Green's function, after which each particular solution can be presented in quadratures. The second category encompasses various integral transform methods. Let us briefly discuss both categories.

Hobson (1899) has constructed the Green's function for a circular disc and a spherical bowl using a method due to Sommerfeld. He has used toroidal coordinates τ, σ, ϕ (in our notation), which are related to the cartesian x, y, and z as follows:

$$x = \frac{a\sinh\tau \cos\phi}{\cosh\tau - \cos\sigma}, \qquad y = \frac{a\sinh\tau \sin\phi}{\cosh\tau - \cos\sigma}, \qquad z = \frac{a\sin\sigma}{\cosh\tau - \cos\sigma}.$$

Here a is the disc radius. The potential function V at the point (τ_0,σ_0,ϕ_0)

external to the disc, which on the disc takes the values $v(\tau,\phi)$, can be expressed

$$V(\tau_0,\sigma_0,\phi_0) = \frac{1}{\pi^2}\int\!\!\int_S \left\{ \frac{(1+\cosh\tau)\cos(\sigma_0/2)}{2aR[\cosh^2(\alpha/2) - \sin^2(\sigma_0/2)]^{1/2}} + \frac{z}{R^3}\tan^{-1}\frac{(1-\cos\sigma_0)^{1/2}}{(\cosh\alpha + \cos\sigma_0)^{1/2}} \right\} v(\tau,\phi)\, dS, \tag{0.1}$$

where R is the distance between the points (τ_0,σ_0,ϕ_0) and (τ,σ,ϕ), and α is defined by $\cosh\alpha = \cosh\tau_0\cosh\tau - \sinh\tau_0\sinh\tau\cos(\phi-\phi_0)$. The practical value of expression (0.1) is quite limited, since there seems to be no way to evaluate the integral involved, even in the simplest case when $v(\tau,\phi)$ is constant. Hobson had to use a very ingenious method in order to find the potential function for v=const and $v=\mu x$, μ=const. On the other hand, it turns out that the integral in (0.1) is computable in elementary functions for any polynomial v (in Cartesian coordinates). This was one of the reasons that prompted the author to look for an alternative approach, which would be as general as (0.1), and, on the other hand, would allow elementary and straightforward computation of the integrals involved.

Consideration of the mathematically equivalent problem of a circular punch pressed against an elastic half-space leads to the integral equation (Galin, 1953)

$$\omega(\rho,\phi) = H\int_0^{2\pi}\int_0^{a} \frac{\sigma(\rho_0,\phi_0)}{R}\,\rho_0 d\rho_0 d\phi_0. \tag{0.2}$$

Here a is the punch radius, H is an elastic constant, ω is the normal displacement under the punch (a known function), σ stands for the pressure exerted by the punch (an unknown function), and R is the distance between the points (ρ,ϕ) and (ρ_0,ϕ_0). Leonov (1953) has obtained a closed form exact solution of integral equation (0.2) by a very ingenious method. His result reads in our notation

$$\sigma(\rho,\phi) = -\frac{1}{4\pi^2 H}\left\{\Delta\int_0^{2\pi}\int_0^{a} \frac{\omega(\rho_0,\phi_0)}{R}\,\rho_0 d\rho_0 d\phi_0\right.$$

$$+ \frac{2}{\pi} \int_0^{2\pi} \int_0^{a} \left[\frac{R}{\eta} - \tan^{-1}\left(\frac{R}{\eta}\right) \right] \frac{\omega(\rho_0,\phi_0)}{R^3} \rho_0 d\rho_0 d\phi_0 \Bigg\}. \tag{0.3}$$

Here $\eta=[(a^2-\rho^2)(a^2-\rho_0^2)]^{1/2}/a$ and Δ is the two-dimensional Laplace operator. One can observe the same handicap in (0.3): difficulty to evaluate the integrals directly, even in the simplest case when ω is constant. Again, it is clear that the integrals are computable in elementary functions for any prescribed polynomial displacement. This indicates a gap in our knowledge which needs to be filled.

The integral transform method, involving dual integral equations, was originated, probably by Weber (1873) and Beltrami (1881), and continued by Busbridge (1938) and others. Significant achievements in the systematic application of the method to various problems belong to Sneddon. The reader is referred to the books by Sneddon (1951, 1966) and Ufliand (1967) for additional references. Some quite remarkable results were obtained by Ufliand (1977). Despite this success, it has always been the author's conviction that our use of integral transforms generally indicates our inability to solve problems directly. To this end, two illustrative examples are presented.

Here is how the problem of a circular disc, charged to a potential v_0=const, is solved by the dual integral equation method. It is necessary to find a harmonic function V, vanishing at infinity, and subjected to the mixed boundary conditions on the plane z=0:

$$V = v_0, \text{ for } \rho\leq a; \qquad \frac{\partial V}{\partial z} = 0, \text{ for } \rho>a. \tag{0.4}$$

The solution is presented in the form

$$V(\rho,z) = \int_0^{\infty} A(t)e^{-tz}J_0(t\rho) \frac{dt}{t} . \tag{0.5}$$

Here J_0 is the Bessel function of zero order, and $A(t)$ is the as yet unknown function which should be chosen to satisfy (0.4). Substitution of the boundary conditions (0.4) in (0.5) leads to the dual integral equations

$$\int_0^\infty A(t)J_0(t\rho)\,\frac{dt}{t} = v_0, \quad \text{for} \quad 0\le\rho\le a;$$

$$\int_0^\infty A(t)J_0(t\rho)dt = 0, \quad \text{for} \quad \rho>a. \tag{0.6}$$

By using the discontinuous Weber-Schafheitlin integrals, one can deduce that

$$A(t) = \frac{2}{\pi}\, v_0\sin(at). \tag{0.7}$$

The solution (0.5) can now be rewritten as

$$V(\rho,z) = \frac{2}{\pi}\, v_0\int_0^\infty e^{-tz}\sin(at)\; J_0(t\rho)\,\frac{dt}{t}\;, \quad \text{for} \quad z\ge0, \tag{0.8}$$

and the charge density σ over the disc is given by

$$\sigma(\rho) = \frac{v_0}{\pi^2}\int_0^\infty J_0(t\rho)\;\sin(at)\;dt. \tag{0.9}$$

Both integrals (0.8) and (0.9) are computable in elementary functions. For example, the last integral yields

$$\sigma(\rho) = \frac{v_0}{\pi^2(a^2 - \rho^2)^{1/2}}. \tag{0.10}$$

There seems to be a discrepancy between the simplicity of the final result and the apparatus used to obtain it. The general idea, that an elementary result should be obtained by elementary means, calls for a search for a new and elementary approach.

The second example comes from consideration of the simplest case of a penny-shaped crack of radius a, subjected to axisymmetric pressure $p(\rho)$. The corresponding potential function f can be found in (Kassir and Sih, 1975) as follows:

$$f(\rho,z) = \int_0^\infty A(s)\ J_0(\rho s)\ \exp(-sz)\ \frac{ds}{s}\ , \tag{0.11}$$

where

$$A(s) = -\frac{1}{\pi\mu}\int_0^a \sin st\ dt \int_0^t \frac{rp(r)\ dr}{(t^2 - r^2)^{1/2}}\ . \tag{0.12}$$

The user has to substitute the explicit expression for p in (0.12), and to evaluate two consecutive integrals. The result is to be substituted in (0.11), and an infinite integral with Bessel function is to be evaluated. It seems natural to try to spare one integration by substituting (0.12) in (0.11), changing the order of integration and evaluating the integral (Gradshtein and Ryzhik, 1963, formula 6.752.1):

$$\int_0^\infty \sin st\ J_0(\rho s)\ \exp(-sz)\ \frac{ds}{s} = \sin^{-1}\frac{t}{l_2(t)}\ , \tag{0.13}$$

where the notation

$$l_2(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} + [(\rho - t)^2 + z^2]^{1/2}\}. \tag{0.14}$$

was introduced. Formulae (0.11) and (0.12) can be combined to give

$$f(\rho,z) = -\frac{1}{\pi\mu}\int_0^a \sin^{-1}\frac{t}{l_2(t)}\ dt \int_0^t \frac{p(\rho_0)\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}}\ . \tag{0.15}$$

Note that expression (0.15) contains no trace of the integral transform, and thus gives us a hint that a direct and elementary solution is indeed possible. For example, in the case of a uniform loading, p=const., and the potential function (0.15) will take the form:

$$f(\rho,z) = -\frac{p}{\pi\mu}\int_0^a t\ \sin^{-1}\frac{t}{l_2(t)}\ dt. \tag{0.16}$$

The integral in (0.16), though looking formidable due to (0.14), can be evaluated in elementary functions, namely,

$$f(\rho,z) = -\frac{p}{4\pi\mu}\left[(2a^2 + 2z^2 - \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right) + l_1(\rho^2 - l_1^2)^{1/2} - 2z(a^2 - l_1^2)^{1/2}\right]. \tag{0.17}$$

Here the abbreviations l_1 and l_2 stand for $l_1(a)$ and $l_2(a)$ respectively, with l_2 defined by (0.14), and

$$l_1(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} - [(\rho - t)^2 + z^2]^{1/2}\}. \tag{0.18}$$

One can show that arbitrary polynomial loading in (0.15) will lead to an elementary expression for the potential function, and thus to an elementary complete solution.

The situation can now be summarized. The Green’s function approach is the most general, the main impediment being the inability of direct derivation of results which were usually *constructed* due to some ingenious considerations. By contrast, the integral transform method allows a straightforward derivation of the results, but it is the least general, since each particular problem has to be solved from beginning to the end. The method is best suited to axisymmetric problems. In the general case, separate solutions have to be obtained for each harmonic. Non-axisymmetric problems involving various interactions (several *arbitrarily located* charged discs, interaction of punches and cracks, etc.) are extremely difficult to solve by the integral transform method. A new method has to be found which would be as general as the Green’s function method, and, at the same time, it has to be elementary and straightforward, with no integral transforms or special function expansions involved.

When one problem has been solved by a complicated method, it is often possible to find another method to solve the same problem more simply. The new method would have been of little value if all it could do were to solve more easily the already solved problems. The main advantage of the new method presented here is its ability to solve non-axisymmetric problems as easily as axisymmetric ones, which in turn opens up new horizons, and allows us to solve some problems which were not even considered before, namely, the analytical treatment of nonclassical domains, and the solution to various interaction problems.

The general description of the method is given in Chapter 1. It starts with a derivation of the basic integral representation for the reciprocal of the distance between two points, followed by several generalizations. A closed form exact solution is given to the non-axisymmetric mixed problem of potential theory for a half-space, with Dirichlet conditions prescribed inside a circle, and Neumann conditions given on the outside, and vice-versa. Some integrals, which are of fundamental value to the method, are evaluated in elementary functions.

Chapter 2 is devoted to the mixed boundary value problems of an elastic half-space. The general solution is expressed in terms of three harmonic functions. A classification of internal and external mixed problems of type I and type II is introduced. The problems of the first type are characterized by mixed conditions with respect to normal parameters (the pressure and normal displacement), with the shear stress prescribed all over the boundary. These problems are solved for a non-homogeneous half-space, with the elasticity modulus assumed to be a power function of the depth. The case when the boundary conditions are mixed with respect to tangential displacements and stresses, with the normal stress being prescribed all over the boundary, is classified as the type II problem. Each type is considered separately. Exact closed form solution and various examples of punch and crack problems are presented.

The problem is called mixed-mixed when the boundary conditions are mixed with respect to both normal and tangential parameters. This kind of problem is the most difficult to solve due to the coupling of the governing integral equations, which can no longer be solved separately. The axisymmetric and non-axisymmetric internal and external problems are considered in Chapter 3. The exact solution is given in terms of Fourier series expansions. A flat punch, bonded to a transversely isotropic elastic half-space and subjected to general loading is considered in detail. The interaction of exterior loading with the punch is also investigated.

While the problems solved in Chapter 2 deal primarily with the stresses and displacements in the plane $z=0$, the *complete* solution to various crack problems is the subject of Chapter 4. The solution is called *complete*, when explicit expressions for the field of stresses and displacements is defined in the whole space. The cases of a penny-shaped crack under arbitrary normal and tangential loadings are considered separately. Explicit expressions are derived for all the Green’s functions involved. All the results are given in terms of elementary functions. These solutions enable us to solve more complicated problems of interaction of a penny-shaped crack and an exterior load. A set of non-singular governing integral equations is derived for the interaction between coplanar circular cracks, subjected to normal pressure and shear loading. An approximate analytical solution is presented for the case of a flat crack of general shape, subjected to a uniform pressure or a uniform shear. The solution is exact for an ellipse, and is expected to be satisfactory for a wide variety of shapes. A comparison is made with various numerical results, available in the literature, and a very good accuracy is established in many cases. The method is less accurate for domains with sharp angles and the aspect ratio far away from unity.

A general solution in terms of one harmonic function is given in Chapter 5 to the problem of a smooth punch, penetrating a transversely isotropic elastic half-space. The main potential function and all the Green's functions are expressed in terms of elementary functions for a circular punch of arbitrary profile. It is shown that the complete solution is also presentable in elementary functions for a general polynomial profile. The knowledge of a complete solution, combined with the reciprocal theorem, enables us to solve more complicated problems of interaction of exterior loadings with punches, and the interaction between punches. The general method is applied to the analytical treatment of nonclassical contact problems. Again, the solution is exact for an elliptical punch, and is expected to be satisfactory for a punch of arbitrary planform. The cases of a flat centrally and noncentrally loaded punch, and the case of a curved punch are considered in detail. An extensive comparison is made with the numerical results, available in the literature. As it was in the case of crack problems, the agreement is quite satisfactory, except for the domains with sharp angles and the aspect ratio far away from unity. An analytical solution is given to the problem of a non-smooth punch, subjected to normal and shear loading, with the Coulomb friction law assumed between the punch base and the elastic half-space.

The best way to master a new method is through the exercises. Each chapter contains a certain number of them, the majority of exercises are supplied with an answer, a hint or a complete solution. The reader is encouraged to do them all. The transformations involved are elementary, though sometimes very non-trivial, and require some ingenuity.

CHAPTER 1

DESCRIPTION OF THE NEW METHOD

Some integral representations, which are of fundamental value to the method, are derived. An exact closed form solution is given to the mixed boundary value problem of potential theory for a half-space, with a circular line of division of boundary conditions.

1.1 Integral representation for the reciprocal of the distance between two points

The author has decided to start with a derivation of a new integral representation for the reciprocal of the distance between two points located in the plane z=0 since this quantity is very important in potential theory. Here we repeat the derivation leading to such a representation, as it was given in (Fabrikant 1971e). Consider the expression

$$\frac{1}{R^{1+u}} = \frac{1}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0))^{(1+u)/2}}, \tag{1.1.1}$$

where u is a constant and $-1<u<1$. The standard expansion of (1.1.1) in Fourier series will take the form

$$\frac{1}{R^{1+u}} = \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi} \int_0^{2\pi} \frac{e^{-in\psi}d\psi}{(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\psi)^{(1+u)/2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi\rho_0^{1+u}} \frac{2\pi\Gamma[n + (1 + u)/2]}{\Gamma[(1 + u)/2]\ \Gamma(n + 1)} \left(\frac{\rho}{\rho_0}\right)^n F\left(\frac{1 + u}{2},\ n + \frac{1 + u}{2},\ n+1;\ \frac{\rho^2}{\rho_0^2}\right). \tag{1.1.2}$$

Here F stands for the Gauss hypergeometric function. By using another integral representation

$$F\left(\frac{1+u}{2},\ n+\frac{1+u}{2},\ n+1;\ z\right)$$

$$= \frac{2\Gamma(n+1)}{\Gamma[n+(1+u)/2]\ \Gamma[1-(1+u)/2]} \int_0^1 \frac{t^{2n+u}(1-t^2)^{-(1+u)/2}}{(1-zt^2)^{(1+u)/2}}\, dt,$$

expression (1.1.2) can be transformed into

$$\frac{1}{R^{1+u}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\sum_{n=-\infty}^{\infty}\frac{e^{in(\phi-\phi_0)}}{(\rho\rho_0)^n}\int_0^{min(\rho_0,\rho)}\frac{x^{2n+u}dx}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}\ . \qquad (1.1.3)$$

Summation in (1.1.3) finally gives

$$\frac{1}{R^{1+u}} = \frac{1}{(\ \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0))^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{min(\rho_0,\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0\right)\ x^u dx}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}\ . \qquad (1.1.4)$$

Here the notation was introduced

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\ \cos\psi}\ . \qquad (1.1.5)$$

After one cumbersome derivation is finished, we can always find a way to do it much simpler. Indeed, if we introduce a new variable

$$\eta(x) = [(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}/x, \qquad (1.1.6)$$

expression (1.1.4) may be rewritten as

$$\frac{1}{R^{1+u}} = \frac{2}{\pi} \cos \frac{\pi u}{2} \int_0^\infty \frac{\eta^{-u} d\eta}{R^2 + \eta^2}. \qquad (1.1.7)$$

The integral in (1.1.7) can be evaluated by using formula (3.241.4) from (Gradshtein and Ryzhik, 1963), thus proving the identity. Note that parameter η will be used throughout the book also for the case when $x>\max(\rho,\rho_0)$, and expression (1.1.6) in this case is interpreted as

$$\eta(x) = (x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}/x.$$

One can deduce from (1.1.7) that in the particular case when $u=0$, the integral in (1.1.4) can be evaluated as indefinite, and we have a very important representation

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \ \phi-\phi_0) \ dx}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}} = -\frac{1}{R} \tan^{-1}\left[\frac{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}{xR}\right]. \qquad (1.1.8)$$

All the results above are related to the distance between two points in the plane $z=0$. We need to generalize them to represent

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}} . \qquad (1.1.9)$$

One can observe that representation (1.1.4) remains valid if we formally substitute ρ and ρ_0 by arbitrary quantities l_1 and l_2. We need to choose them so that

$$\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2 = l_1^2 + l_2^2 - 2l_1 l_2\cos(\phi-\phi_0). \qquad (1.1.10)$$

This leads to two equations

$$l_1^2 + l_2^2 = \rho^2 + \rho_0^2 + z^2, \qquad l_1 l_2 = \rho\rho_0. \qquad (1.1.11)$$

The solution will take the form

$$l_1(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho + \rho_0)^2 + z^2]^{1/2} - [(\rho - \rho_0)^2 + z^2]^{1/2}\}, \qquad (1.1.12)$$

$$l_2(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho + \rho_0)^2 + z^2]^{1/2} + [(\rho - \rho_0)^2 + z^2]^{1/2}\}. \tag{1.1.13}$$

Hereafter the following abbreviations will be used:

$$l_1(x) \equiv l_1(x,\rho,z), \qquad l_2(x) \equiv l_2(x,\rho,z), \tag{1.1.14}$$

$$l_1 \equiv l_1(a,\rho,z), \qquad l_2 \equiv l_2(a,\rho,z). \tag{1.1.15}$$

Note the limiting properties

$$\lim_{z\to 0} l_1(x) = \min(x,\rho), \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho). \tag{1.1.16}$$

In view of the properties above, the representation (1.1.4) can be generalized

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ x^u dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{(1+u)/2}}\ . \tag{1.1.17}$$

Formula (1.1.17) simplifies when u=0

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}\ . \tag{1.1.18}$$

Again, one can notice that the integral in (1.1.18) may be evaluated as indefinite

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ dx}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}{xR_0}.$$

(1.1.19)

The last representation is very important and will be widely used throughout the book.

Another series of useful formulae can be obtained from those above by a simple change of variables, namely,

$$\int \frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}} = \frac{1}{R_0}\tan^{-1}\frac{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}}{xR_0},$$

(1.1.20)

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ x^u dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{(1+u)/2}}\ , \qquad (1.1.21)$$

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_{l_2(\rho_0)}^{\infty}\frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{\{[x^2 - l_1^2(\rho_0)][x^2 - l_2^2(\rho_0)]\}^{1/2}}\ , \qquad (1.1.22)$$

$$\int \frac{\lambda\left(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0\right)\ dx}{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}} = \frac{1}{R}\tan^{-1}\left[\frac{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}{xR}\right]. \qquad (1.1.23)$$

The representations above are useful for solving external mixed boundary value problems.

Several modifications of (1.1.19) are available. For example, we can write

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ \mathrm{d}x}{(\rho^2 - x^2)^{1/2}[\rho_0^2 - g^2(x)]^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{(\rho^2 - x^2)^{1/2}[\rho_0^2 - g^2(x)]^{1/2}}{xR_0}. \qquad (1.1.24)$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}. \qquad (1.1.25)$$

It is important to notice that the function $g(x)$ is inverse to l_1 for $x^2<\rho^2$, and is inverse to l_2 for $x^2>\rho^2+z^2$. Introduction of a new variable $x=l_1(y)$, $y=g(x)$ transforms (1.1.24) into

$$\int \frac{[l_2^2(y) - y^2]^{1/2}}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - l_1^2(y)]}\ \lambda\left(\frac{l_1^2(y)}{\rho\rho_0},\ \phi-\phi_0\right)\mathrm{d}y$$

$$= -\frac{1}{R_0}\tan^{-1}\frac{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - y^2]^{1/2}}{yR_0} \qquad (1.1.26)$$

A particular case of (1.1.18), when z=0, reads

$$\frac{1}{R} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\ \phi-\phi_0)\ \mathrm{d}x}{(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}}. \qquad (1.1.27)$$

The same result takes another form due to (1.1.22)

$$\frac{1}{R} = \frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2},\ \phi-\phi_0)\ \mathrm{d}x}{(x^2 - \rho^2)^{1/2}(x^2 - \rho_0^2)^{1/2}}. \qquad (1.1.28)$$

The integral representations of this section can be generalized for a sphere (see Fabrikant, 1987d), and yet another modification in toroidal coordinates is also possible. The book volume limitation does not allow us to go into detail.

Exercise 1.1

1. Prove the identity (η is defined by 1.1.6)

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = -\frac{\rho^2\rho_0^2 - x^4}{x^3\eta}.$$

2. Prove the identity (R is defined by the first line of 1.1.27)

$$\lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right) = -\frac{x\eta}{R^2 + \eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

Hint: use the identity: $x^2 + \rho^2\rho_0^2/x^2 - 2\rho\rho_0\cos(\phi-\phi_0) = R^2 + \eta^2$, and the result above.

3. Prove the identity

$$\lambda\left(\frac{\rho\rho_0}{x^2}, \phi-\phi_0\right) = \frac{x\eta}{R^2 + \eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

4. Prove the identities

$$(l_2^2 - \rho^2)^{1/2}(l_2^2 - a^2)^{1/2} = zl_2, \qquad (a^2 - l_1^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = zl_1,$$

$$(a^2 - l_1^2)^{1/2}(l_2^2 - a^2)^{1/2} = za, \qquad (l_2^2 - \rho^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z\rho.$$

Reminder: l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively. *Hint*: use (1.1.11)

5. Prove that $g(x)$ is inverse to both l_1 and l_2, namely, prove that $g(l_1)=a$, and $g(l_2)=a$.

6. Prove the identities

$$\frac{\partial l_1}{\partial z} = -\frac{zl_1}{l_2^2 - l_1^2}, \qquad \frac{\partial l_2}{\partial z} = \frac{zl_2}{l_2^2 - l_1^2},$$

$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2 - l_1^2} = \frac{\rho(a^2 - l_1^2)}{l_1(l_2^2 - l_1^2)}, \qquad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2 - l_1^2} = \frac{\rho(l_2^2 - a^2)}{l_2(l_2^2 - l_1^2)}.$$

Hint: use the properties above.

7. Evaluate the integral

$$\int \frac{dx}{(\rho_0^2 - x^2)^{1/2}} \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0}, \phi - \psi\right).$$

Answer: $-\frac{1}{R_0} \tan^{-1} \frac{(\rho_0^2 - x^2)^{1/2}[l_2^2(x) - x^2]^{1/2}}{xR_0}$.

Hint: use (1.1.26)

8. Evaluate the integral

$$\int \frac{dx}{(x^2 - \rho_0^2)^{1/2}} \frac{(x^2 - l_1^2(x))^{1/2}}{[l_2^2(x) - l_1^2(x)]} \lambda\left(\frac{\rho\rho_0}{l_2^2(x)}, \phi-\phi_0\right).$$

Answer: $\frac{1}{R_0} \tan^{-1} \frac{(x^2 - l_1^2(x))^{1/2}(x^2 - \rho_0^2)^{1/2}}{xR_0}$.

Hint: use (1.1.20)

9. Establish the integral representation

$$\frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\cos[\kappa(\rho^2 - x^2)^{1/2}(\rho_0^2 - x^2)^{1/2}/x + (\pi\nu/2)]}{[(\rho^2 - x^2)(\rho_0^2 - x^2)]^{(1+\nu)/2}} \lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right) x^\nu dx = \frac{e^{-\kappa R}}{R^{1+\nu}}.$$

Note: try to use this integral representation for solving the Klein-Gordon equation.

1.2 Properties of the $\mathcal{L}$-operators

Introduce the integral $\mathcal{L}$-operator as follows:

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|}e^{in\phi}\int_0^{2\pi} e^{-in\phi_0} f(\phi_0)d\phi_0 = \sum_{n=-\infty}^{\infty} k^{|n|}f_n e^{in\phi}. \quad (1.2.1)$$

Here f_n is the n-th Fourier coefficient of the function f. In the case when $k<1$, formula (1.2.1) can be rewritten as

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi-\phi_0)\ f(\phi_0)\ d\phi_0, \quad (1.2.2)$$

where $\lambda(\cdot,\cdot)$ is defined by (1.1.5). The $\mathcal{L}$-operator may also be called Poisson operator, since it was introduced by Poisson for solving the two-dimensional Dirichlet problem for a circle.

The following properties of the $\mathcal{L}$-operators are valid

$$\mathcal{L}(k_1)\mathcal{L}(k_2) = \mathcal{L}(k_1k_2), \qquad \lim_{k\to 1}\mathcal{L}(k)f = f. \quad (1.2.3)$$

The proof is elementary, and left to the reader. These properties are widely used in various transformations throughout this book, and are essential to the new method.

Exercise 1.2

1. Prove the identity $\mathcal{L}(k)\lambda(m,\phi)=\lambda(km,\phi)$, for $k<1$ and $m<1$.
Hint: use (1.2.3)

2. Evaluate the integral

$$\int_0^{2\pi} \frac{d\phi}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)][r^2 + r_0^2 - 2rr_0\cos(\phi-\psi)]}, \text{ for } \rho>\rho_0 \text{ and } r>r_0.$$

Answer: $\dfrac{2\pi(\rho^2r^2 - \rho_0^2r_0^2)}{(\rho^2 - \rho_0^2)(r^2 - r_0^2)[\rho^2r^2 + \rho_0^2r_0^2 - 2\rho\rho_0rr_0\cos(\phi_0-\psi)]}$.

3. Evaluate the integral

$$\int_0^{2\pi} \frac{d\phi}{[1 + k^2 - 2k\cos(\phi-\phi_0)]^2[1 + k_1^2 - 2k_1\cos(\phi-\psi)]}, \text{ for } k<1 \text{ and } k_1<1.$$

Answer: $$\frac{2\pi}{1 + k^2k_1^2 - 2kk_1\cos(\phi_0-\psi)} \left\{\frac{2k^2}{(1 - k^2)^3} + \frac{1}{1 + k^2k_1^2 - 2kk_1\cos(\phi_0-\psi)} \left[\frac{k_1^2}{1 - k_1^2} + \frac{1 - k^4k_1^2}{(1 - k^2)^2}\right]\right\}.$$

4. Evaluate the integral

$$\int_0^{2\pi} \frac{e^{i\phi}\, d\phi}{[1 + k^2 - 2k\cos(\phi-\phi_0)]^2[1 + k_1^2 - 2k_1\cos(\phi-\psi)]}, \text{ for } k<1 \text{ and } k_1<1.$$

Answer: $$\frac{2\pi}{[1 + k^2k_1^2 - 2kk_1\cos(\phi_0-\psi)]^2} \left\{\frac{k_1^3e^{i\Psi}}{1 - k_1^2} + \frac{ke^{i\phi_0}\,[2(1 + k^4k_1^2) - kk_1(1 + k^2)e^{-i(\psi-\phi_0)}] + k_1e^{i\Psi}(1 - 3k^2)}{(1 - k^2)^3}\right\}.$$

5. Evaluate the integral

$$\int_0^{2\pi} \frac{e^{2i\phi}\, d\phi}{[1 + k^2 - 2k\cos(\phi-\phi_0)]^2[1 + k_1^2 - 2k_1\cos(\phi-\psi)]}, \text{ for } k<1 \text{ and } k_1<1.$$

Answer: $$\frac{2\pi}{[1 + k^2k_1^2 - 2kk_1\cos(\phi_0-\psi)]^2} \left\{\frac{k_1^2e^{2i\Psi}}{1 - k_1^2} + \frac{2kk_1e^{-i(\psi-\phi_0)}[e^{2i\Psi}(k^4-3k^2+1) - k^2e^{2i\phi_0}] - k^2e^{2i\phi_0}(k^2 + k_1^2-3 - 3k^2k_1^2)}{(1 - k^2)^3}\right\}.$$

1.3 Some further integral representations

An integral representation for z/R_0^3 is presented here, for $R_0=[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}$. Consider the integral:

$$I_1 = \frac{1}{\rho_0}\,\mathcal{L}\!\left(\frac{1}{\rho_0}\right)\frac{d}{d\rho_0}\int_0^{\rho_0}\frac{x\,dx}{(\rho_0^2 - x^2)^{1/2}}\,\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\!\left(\frac{l_1(x)x}{l_2(x)}\,,\ \phi-\phi_0\right) \qquad (1.3.1)$$

The integral (1.3.1) looks cumbersome, and the first impression is that one should be lucky just to express it in elliptic integrals. It will be shown below that the integral is quasi-elliptic, and is computable in elementary functions. By using the rule of differentiation

$$\frac{d}{d\rho_0}\int_0^{\rho_0}\frac{f(x)\,dx}{(\rho_0^2 - x^2)^{1/2}} = \lim_{x\to 0}\left[\frac{f(x)}{x}\right] + \rho_0\int_0^{\rho_0}\frac{dx}{(\rho_0^2 - x^2)^{1/2}}\,\frac{d}{dx}\left[\frac{f(x)}{x}\right], \qquad (1.3.2)$$

the integral (1.3.1) can be rewritten as

$$I_1 = \int_0^{\rho_0}\frac{dx}{(\rho_0^2 - x^2)^{1/2}}\,\frac{d}{dx}\left[\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\!\left(\frac{l_1(x)x}{l_2(x)\rho_0}\,,\ \phi-\phi_0\right)\right]. \qquad (1.3.3)$$

Introduce a new variable:

$$j = \frac{(\rho_0^2 - x^2)^{1/2}\,[l_2^2(x) - x^2]^{1/2}}{x}\,,$$

$$j' = \frac{dj}{dx} = -\,\frac{[l_2^2(x) - x^2]^{1/2}\,[\rho_0^2 l_2^2(x) - x^2 l_1^2(x)]}{x^2[l_2^2(x) - l_1^2(x)]\,(\rho_0^2 - x^2)^{1/2}}\,. \qquad (1.3.4)$$

Let us transform the expression for λ:

$$\lambda\!\left(\frac{l_1(x)x}{l_2(x)\rho_0}\,,\ \phi-\phi_0\right) = \lambda\!\left(\frac{l_1^2(x)}{\rho\rho_0}\,,\ \phi-\phi_0\right) = -\,\frac{l_1^2(x) - \rho^2\rho_0^2/l_1^2(x)}{l_1^2(x) + \rho^2\rho_0^2/l_1^2(x) - 2\rho\rho_0\cos(\phi-\phi_0)}$$

$$= \frac{-\ l_1^2(x)\ +\ \rho^2\rho_0^2/l_1^2(x)}{l_1^2(x)\ +\ \rho^2\rho_0^2/l_1^2(x)\ -\ 2\rho\rho_0\cos(\phi-\phi_0)\ +\ \rho^2+\rho_0^2+z^2\ -\ l_1^2(x)-l_2^2(x)\ +\ x^2-\rho_0^2}$$

$$= -\ \frac{x^2 l_1^2(x)\ -\ \rho_0^2 l_2^2(x)}{x^2(R_0^2\ +\ j^2)}\ =\ -\ \frac{[l_2^2(x)\ -\ l_1^2(x)]\ (\rho_0^2\ -\ x^2)^{1/2}}{[l_2^2(x)\ -\ x^2]^{1/2}}\ \frac{j'}{R_0^2\ +\ j^2} \tag{1.3.5}$$

The substitution of (1.3.4) and (1.3.5) in (1.3.3) allows us to continue the transformations:

$$I_1\ =\ -\int_0^{\rho_0} \frac{dx}{(\rho_0^2\ -\ x^2)^{1/2}}\ \frac{d}{dx}\left[\ \frac{[x^2\ -\ l_1^2(x)]^{1/2}(\rho_0^2\ -\ x^2)^{1/2}}{[l_2^2(x)\ -\ x^2]^{1/2}}\ \frac{j'}{R_0^2\ +\ j^2}\ \right]$$

$$=\ -\int_0^{\rho_0} \frac{dx}{(\rho_0^2\ -\ x^2)^{1/2}}\ \frac{d}{dx}\left[\ \frac{xz(\rho_0^2\ -\ x^2)^{1/2}}{l_2^2(x)\ -\ x^2}\ \frac{j'}{R_0^2\ +\ j^2}\ \right]$$

$$=\ -z\int_0^{\rho_0} \frac{dx}{(\rho_0^2\ -\ x^2)^{1/2}}\ \frac{d}{dx}\left[\ \frac{(\rho_0^2\ -\ x^2)^{3/2}j'}{xj^2(R_0^2\ +\ j^2)}\ \right]$$

$$=\ -z\left\{\ \left.\frac{(\rho_0^2\ -\ x^2)j'}{xj^2(R_0^2\ +\ j^2)}\right|_0^{\rho_0}\ -\ \int_0^{\rho_0} \frac{dj(x)}{j^2(R_0^2\ +\ j^2)}\ \right\}$$

$$=\ -z\left.\left\{\ \frac{(\rho_0^2\ -\ x^2)j'}{xj^2(R_0^2\ +\ j^2)}\ +\ \frac{1}{R_0^2 j}\ +\ \frac{1}{R_0^3}\ \tan^{-1}\ \frac{j}{R_0}\ \right\}\right|_0^{\rho_0}\ =\ \frac{\pi}{2}\ \frac{z}{R_0^3} \tag{1.3.6}$$

Finally, (1.3.6) allows us to write the required representation:

$$\frac{z}{R_0^3}\ =\ \frac{2}{\pi\rho_0}\ \mathcal{L}(\frac{1}{\rho_0})\ \frac{d}{d\rho_0}\int_0^{\rho_0} \frac{x\,dx}{(\rho_0^2\ -\ x^2)^{1/2}}\ \frac{[x^2\ -\ l_1^2(x)]^{1/2}}{l_2^2(x)\ -\ l_1^2(x)}\ \lambda\left(\frac{l_1(x)x}{l_2(x)}\ ,\ \phi-\phi_0\right). \tag{1.3.7}$$

The second integral to consider is

$$I_2 = -\frac{1}{\rho_0}\,\mathcal{L}(\rho_0)\,\frac{d}{d\rho_0}\int_{\rho_0}^{a}\frac{x\,dx}{(x^2-\rho_0^2)^{1/2}}\,\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho}{l_2^2(x)},\ \phi-\phi_0\right). \qquad (1.3.8)$$

Make use of the rule of differentiation

$$\frac{d}{dx}\int_x^a\frac{F(\rho)\,d\rho}{(\rho^2-x^2)^{1/2}} = -\frac{F(a)x}{a(a^2-x^2)^{1/2}} + x\int_x^a\frac{d\rho}{(\rho^2-x^2)^{1/2}}\,\frac{d}{d\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= -\frac{F(a)a}{x(a^2-x^2)^{1/2}} + \frac{1}{x}\int_x^a\frac{\rho\,d\rho}{(\rho^2-x^2)^{1/2}}\,\frac{d}{d\rho}F(\rho). \qquad (1.3.9)$$

Expression (1.3.8) will take the form

$$I_2 = \frac{(l_2^2-a^2)^{1/2}}{(a^2-\rho_0^2)^{1/2}(l_2^2-l_1^2)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2},\ \phi-\phi_0\right)$$

$$-\int_{\rho_0}^{a}\frac{dx}{(x^2-\rho_0^2)^{1/2}}\,\frac{d}{dx}\left[\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2(x)},\ \phi-\phi_0\right)\right]. \qquad (1.3.10)$$

Introduce a new variable

$$h = \frac{(x^2-\rho_0^2)^{1/2}[x^2-l_1^2(x)]^{1/2}}{x},$$

$$h' = \frac{dh}{dx} = \frac{[x^2-l_1^2(x)]^{1/2}[x^2 l_2^2(x)-\rho_0^2 l_1^2(x)]}{x^2(x^2-\rho_0^2)^{1/2}[l_2^2(x)-l_1^2(x)]}. \qquad (1.3.11)$$

The expression for λ can be presented in the manner, similar to (1.3.5), namely,

$$\lambda\left(\frac{\rho\rho_0}{l_2^2(x)}\ ,\ \phi-\phi_0\right) = \frac{[l_2^2(x) - l_1^2(x)](x^2 - \rho_0^2)^{1/2}}{[x^2 - l_1^2(x)]^{1/2}}\ \frac{h'}{R_0^2 + h^2}. \tag{1.3.12}$$

Substitution of (1.3.11) and (1.3.12) in (1.3.10) yields

$$I_2 = \frac{(l_2^2 - a^2)^{1/2}}{(a^2 - \rho_0^2)^{1/2}(l_2^2 - l_1^2)}\ \lambda\left(\frac{\rho\rho_0}{l_2^2}\ ,\ \phi-\phi_0\right)$$

$$-\int_{\rho_0}^{a}\frac{\mathrm{d}x}{(x^2 - \rho_0^2)^{1/2}}\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xz(x^2 - \rho_0^2)^{1/2}h'}{[x^2 - l_1^2(x)](R_0^2 + h^2)}\right]$$

$$= \frac{(l_2^2 - a^2)^{1/2}}{(a^2 - \rho_0^2)^{1/2}(l_2^2 - l_1^2)}\ \lambda\left(\frac{\rho\rho_0}{l_2^2}\ ,\ \phi-\phi_0\right)$$

$$-\ z\int_{\rho_0}^{a}\frac{\mathrm{d}x}{(x^2 - \rho_0^2)^{1/2}}\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(x^2 - \rho_0^2)^{3/2}h'}{xh^2(R_0^2 + h^2)}\right]. \tag{1.3.13}$$

Integration by parts in (1.3.13) yields

$$-\ \frac{1}{\rho_0}\ \mathcal{L}(\rho_0)\ \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{x\mathrm{d}x}{(x^2 - \rho_0^2)^{1/2}}\ \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)}\ \lambda\left(\frac{\rho}{l_2^2(x)}\ ,\ \phi-\phi_0\right)$$

$$= \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right]. \tag{1.3.14}$$

Here h stands for $h(a)$, as defined by the first expression of (1.3.11). In the limiting case, when $a\to\infty$, expression (1.3.14) gives yet another representation for z/R_0^3, namely,

$$\frac{z}{R_0^3} = -\frac{2}{\pi\rho_0}\mathcal{L}(\rho_0)\frac{d}{d\rho_0}\int_{\rho_0}^{\infty}\frac{x dx}{(x^2-\rho_0^2)^{1/2}}\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{\rho}{l_2^2(x)},\ \phi-\phi_0\right). \quad (1.3.15)$$

The last expression is useful in external problems, while its equivalent (1.3.6) is needed in solving internal ones. Since the integral in (1.3.1) was evaluated earlier as indefinite, we may consider its generalization

$$I_3 = \frac{1}{\rho_0}\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{d}{d\rho_0}\int_a^{\rho_0}\frac{x dx}{(\rho_0^2-x^2)^{1/2}}\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{l_1(x)x}{l_2(x)},\ \phi-\phi_0\right). \quad (1.3.16)$$

Make use of the rule of differentiation

$$\frac{d}{dx}\int_a^x\frac{F(\rho)\,d\rho}{(x^2-\rho^2)^{1/2}} = \frac{F(a)x}{a(x^2-a^2)^{1/2}} + x\int_a^x\frac{d\rho}{(x^2-\rho^2)^{1/2}}\frac{d}{d\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= \frac{F(a)a}{x(x^2-a^2)^{1/2}} + \frac{1}{x}\int_a^x\frac{\rho d\rho}{(x^2-\rho^2)^{1/2}}\frac{d}{d\rho}F(\rho). \quad (1.3.17)$$

Expression (1.3.16) will take the form

$$I_3 = \frac{(a^2-l_1^2)^{1/2}}{(\rho_0^2-a^2)^{1/2}[l_2^2-l_1^2]}\lambda\left(\frac{l_1^2}{\rho\rho_0},\ \phi-\phi_0\right)$$

$$+\int_a^{\rho_0}\frac{dx}{(\rho_0^2-x^2)^{1/2}}\frac{d}{dx}\left[\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0},\ \phi-\phi_0\right)\right]. \quad (1.3.18)$$

By introducing the notation

$$F(y) = \frac{z}{R_0^3}\left[\frac{R_0}{j(y)} + \tan^{-1}\left(\frac{j(y)}{R_0}\right)\right], \quad (1.3.19)$$

where $j(y)$ is defined according to (1.3.4), expression (1.3.18) can be rewritten as

$$I_3 = \frac{\rho_0^2 - a^2}{a}\frac{dF(a)}{da} + \int_a^{\rho_0} \frac{dy}{(\rho_0^2 - y^2)^{1/2}} \frac{d}{dy}\left[\frac{(\rho_0^2 - y^2)^{3/2}}{y}\frac{dF(y)}{dy}\right]. \qquad (1.3.20)$$

Integration in (1.3.20) can be performed by parts, with a simple result $F(a)$, which means establishment of another integral representation

$$\frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right]$$

$$= \frac{1}{\rho_0}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{d}{d\rho_0}\int_a^{\rho_0}\frac{x\,dx}{(\rho_0^2 - x^2)^{1/2}}\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\left(\frac{l_1(x)x}{l_2(x)}, \phi-\phi_0\right). \qquad (1.3.21)$$

Exercise 1.3

1. Prove that h in (1.3.14) can be defined by any of the expressions

$$h \equiv h(a) = \frac{(a^2 - \rho_0^2)^{1/2}(a^2 - l_1^2)^{1/2}}{a} = \frac{z(a^2 - \rho_0^2)^{1/2}}{(l_2^2 - a^2)^{1/2}}$$

$$= \frac{[l_2^2 - l_1^2(\rho_0)]^{1/2}[l_2^2 - l_2^2(\rho_0)]^{1/2}}{l_2} = \frac{(a^2 - \rho_0^2)^{1/2}(l_2^2 - \rho^2)^{1/2}}{l_2}.$$

2. Prove that j in (1.3.4) can be defined in several equivalent ways:

$$j \equiv j(a) = \frac{(\rho_0^2 - a^2)^{1/2}(l_2^2 - a^2)^{1/2}}{a} = \frac{(\rho_0^2 - a^2)^{1/2}(\rho^2 - l_1^2)^{1/2}}{l_1}$$

$$= \frac{z(\rho_0^2 - a^2)^{1/2}}{(a^2 - l_1^2)^{1/2}} = \frac{[l_1^2(\rho_0) - l_1^2]^{1/2}[l_2^2(\rho_0) - l_1^2]^{1/2}}{l_1}.$$

3. Establish (1.3.15)

1.4 Internal mixed boundary value problem for a half-space

The material in this section follows the paper (Fabrikant, 1986h). Introduce a set of cylindrical coordinates (ρ,ϕ,z). Consider the problem of finding a potential function V, harmonic in the half-space $z\geq 0$, vanishing at infinity, and subject to the boundary conditions on the plane $z=0$

$$V = v(\rho,\phi), \qquad \text{for } \rho\leq a, \quad 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho>a, \quad 0\leq\phi<2\pi. \tag{1.4.1}$$

The problem (1.4.1) can be interpreted as an electrostatic one of a charged disc, with a certain potential prescribed on its surface, or it can be interpreted as an elastic contact problem of a circular punch pressed against an elastic half-space; other interpretations are also possible. We call the problem internal because the non-zero conditions are prescribed inside the disc. The potential function V can be represented through the simple layer as follows:

$$V(\rho,\phi,z) = \int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)}{R_0}\,\rho_0 d\rho_0 d\phi_0 + \int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)}{R_0}\,\rho_0 d\rho_0 d\phi_0. \tag{1.4.2}$$

Here

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}, \qquad \text{and } \sigma = -\frac{1}{2\pi}\frac{\partial V}{\partial z}\Big|_{z=0}.$$

Substitution of (1.1.18) and (1.1.22) in (1.4.2) yields, after changing the order of integration

$$V(\rho,\phi,z) = 4\int_0^{l_1}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_{g(x)}^a \frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+ 4\int_{l_2}^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}}\int_a^{g(x)}\frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.3}$$

Here the $\mathcal{L}$-operator is defined by (1.2.1), g is given by (1.1.25), the abbreviations l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a d\rho_0 \int_0^{l_1(\rho_0)} dx = \int_0^{l_1} dx \int_{g(x)}^a d\rho_0, \qquad \int_a^\infty d\rho_0 \int_{l_2(\rho_0)}^\infty dx = \int_{l_2}^\infty dx \int_a^{g(x)} d\rho_0. \tag{1.4.4}$$

The rule is illustrated by Fig. 1.4.1, where the domains of integration are shaded.

Fig. 1.4.1 Domains of integration.

Substitution of the boundary condition (1.4.1) in (1.4.3) leads to the governing integral equation

$$4\int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.4.5}$$

Expression (1.4.5) is now presented as a sequence of two Abel-type operators and one $\mathcal{L}$-operator. We recall that the general Abel integral equation

$$\int_x^a \frac{F(y)\ dy}{(y^2 - x^2)^{(1+u)/2}} = f(x) \tag{1.4.6}$$

has the solution

$$F(r) = -\frac{2\cos(\pi u/2)}{\pi}\frac{d}{dr}\int_r^a \frac{f(x)\ x dx}{(x^2 - r^2)^{(1-u)/2}}. \tag{1.4.7}$$

Since the variables in the Abel operators of (1.4.5) are interwoven with those of the $\mathcal{L}$-operator, we need to apply their combination, in order to invert (1.4.5). The first operator to be applied to both sides of (1.4.5) is

$$\mathcal{L}\left(\frac{\zeta}{t}\right)\frac{d}{dt}\int_0^t \frac{\rho d\rho}{(t^2 - \rho^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho}{\zeta}\right). \tag{1.4.8}$$

Here we introduced a dummy parameter ζ in order to make the parameter of the $\mathcal{L}$-operator dimensionless, and also in order to claim it being less that unity almost everywhere in the interval of integration which is a precondition for usage of the properties (1.2.3). We call the parameter ζ 'dummy' because it was introduced for formal reasons only; it will disappear in the final result, and has no real bearing on the transformations to follow. Of course, the introduction of dummy parameter does not rigorously validate our use of the properties (1.2.3). Such a validation is beyond scope of this book: the author is satisfied by the fact that the final result everywhere is proven to be correct. In order to make the intermediate transformations rigorous, one has to prove the theorem stating that one can use the properties (1.2.3) in the mathematical manipulations with an expression of the type $\mathcal{L}(k_1)\mathcal{M}_1\mathcal{L}(k_2)\mathcal{M}_2\mathcal{L}(k_3)...$, where $\mathcal{M}$ are certain linear operators, if the product $k_1k_2k_3...$ is less than unity, thus allowing for any particular k to be greater than unity. We hope that some readers might be willing and able to prove the theorem.

The result of application of (1.4.8) to both sides of (1.4.5) is

$$2\pi\int_t^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - t^2)^{1/2}}\ \mathcal{L}\left(\frac{t}{\rho_0}\right)\sigma(\rho_0,\phi) = \mathcal{L}\left(\frac{\zeta}{t}\right)\frac{d}{dt}\int_0^t \frac{\rho d\rho}{(t^2 - \rho^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho}{\zeta}\right) v(\rho,\phi). \tag{1.4.9}$$

The second operator to be applied to both sides of (1.4.9) is

$$\mathcal{L}\left(\frac{y}{\zeta}\right)\frac{d}{dy}\int_y^a \frac{t dt}{(t^2 - y^2)^{1/2}}\ \mathcal{L}\left(\frac{\zeta}{t}\right).$$

with the result

$$\sigma(y,\phi) = -\frac{1}{\pi^2 y}\,\mathcal{L}\left(\frac{y}{\zeta}\right)\frac{d}{dy}\int_y^a \frac{t\,dt}{(t^2 - y^2)^{1/2}}\,\mathcal{L}\left(\frac{\zeta^2}{t^2}\right)\frac{d}{dt}\int_0^t \frac{\rho\,d\rho}{(t^2 - \rho^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho}{\zeta}\right)v(\rho,\phi). \tag{1.4.10}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (1.4.10)

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2 - y^2)^{1/2}} - \int_y^a \frac{dt}{(t^2 - y^2)^{1/2}}\,\frac{d}{dt}\Phi(t,y,\phi)\right]. \tag{1.4.11}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t}\int_0^t \frac{\rho\,d\rho}{(t^2 - \rho^2)^{1/2}}\,\frac{d}{d\rho}\left[\rho\,\mathcal{L}\left(\frac{\rho y}{t^2}\right)v(\rho,\phi)\right]. \tag{1.4.12}$$

Note that the dummy parameter ζ has disappeared from the final solution, and the combined parameter of the $\mathcal{L}$-operator is less than unity, as it should be. In the future we shall no longer use the dummy parameter explicitly, assuming that the use of the properties (1.2.3) is justified. Using integration by parts and the fact that $\lambda(k,\psi)$ satisfies the two-dimensional Laplace equation in polar coordinates, the following identity can be established

$$\frac{d}{dt}\Phi(t,y,\phi) = \int_0^t \frac{\rho\,d\rho}{(t^2 - \rho^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi), \tag{1.4.13}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Substitution of (1.4.13) in (1.4.12) leads to another form of solution, namely,

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2 - y^2)^{1/2}} - \int_y^a \frac{dt}{(t^2 - y^2)^{1/2}}\int_0^t \frac{\rho\,d\rho}{(t^2 - \rho^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi)\right], \tag{1.4.14}$$

Interchange of the order of integration in (1.4.14) and integration with respect to t (see (1.1.23)) yields

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left\{\frac{\Phi(a,y,\phi)}{(a^2 - y^2)^{1/2}}\right.$$

$$\left. - \frac{1}{2\pi}\int_0^{2\pi}\int_0^a \tan^{-1}\left[\frac{(a^2 - \rho^2)^{1/2}(a^2 - y^2)^{1/2}}{a[\rho^2 + y^2 - 2\rho y\cos(\phi-\psi)]^{1/2}}\right] \frac{\Delta v(\rho,\psi)\ \rho d\rho d\psi}{[\rho^2 + y^2 - 2\rho y\cos(\phi-\psi)]^{1/2}}\right\}. \tag{1.4.15}$$

The solution obtained here consists of two parts: the first part is singular at the boundary while the second one vanishes at the boundary. In various applications it is required that the solution be nonsingular at the boundary. The necessary and sufficient condition then is $\Phi(a,a,\phi)=0$. In elastic contact problems this condition defines the radius of the contact domain. Notice also that in the case when v is a two-dimensional harmonic function, the non-trivial solution is singular.

Now it is of interest to express the potential V in the half-space directly through its value v prescribed inside the disc $\rho=a$. Substitution of (1.4.10) in (1.4.3) yields, after subsequent integration

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{l_1} \frac{dx}{(\rho^2 - x^2)^{1/2}}\ \mathcal{L}\left(\frac{x^2}{\rho g^2(x)}\right) \frac{d}{dg(x)}\int_0^{g(x)} \frac{r dr}{[g^2(x) - r^2]^{1/2}}\ \mathcal{L}(r)v(r,\phi). \tag{1.4.16}$$

Here the following property of the Abel operators was used

$$\int_y^a \frac{dr}{(r^2 - y^2)^{1/2}}\ \frac{d}{dr}\int_r^a \frac{tf(t)\ dt}{(t^2 - r^2)^{1/2}} = -\frac{\pi}{2} f(y). \tag{1.4.17}$$

Introduction of a new variable $t=g(x)$, $x=l_1(t)$, transforms (1.4.16) into

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^a \frac{dl_1(t)}{[\rho^2 - l_1^2(t)]^{1/2}}\ \mathcal{L}\left(\frac{l_1^2(t)}{\rho t^2}\right) \frac{d}{dt}\int_0^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}}\ \mathcal{L}(\rho_0)v(\rho_0,\phi). \tag{1.4.18}$$

By changing the order of integration in (1.4.18), according to the rule

$$\int_0^a F(r)\mathrm{d}r\ \frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho f(\rho)\ \mathrm{d}\rho}{(r^2-\rho^2)^{1/2}} = -\int_0^a f(\rho)\mathrm{d}\rho\ \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{F(r)r\mathrm{d}r}{(r^2-\rho^2)^{1/2}}, \tag{1.4.19}$$

the following expression can be obtained

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_0^a \left\{ \mathcal{L}(\rho_0)\ \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a \frac{t\mathrm{d}l_1(t)}{(t^2-\rho_0^2)^{1/2}[\rho^2-l_1^2(t)]^{1/2}}\ \mathcal{L}\left(\frac{\rho}{l_2^2(t)}\right)\right\} v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.4.20}$$

The integral in curly brackets can be evaluated using (1.3.14), with the result

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a \left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] \frac{z}{R_0^3}\ v(\rho_0,\phi_0)\ \rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.4.21}$$

Here

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}, \qquad h = (a^2 - l_1^2)^{1/2}(a^2-\rho_0^2)^{1/2}/a. \tag{1.4.22}$$

Formulae (1.4.18) and (1.4.21) define the potential function V in the half-space $z\geq 0$, expressed directly through its value v prescribed inside the disc $\rho=a$, $z=0$. Expression (1.4.18) is useful when an explicit evaluation of the integrals is possible, while expression (1.4.21) is more convenient for numerical integration.

Note that in the limiting case, when $z=0$, equation (1.4.21) transforms into a known result, namely,

$$V(\rho,\phi,0) = v(\rho,\phi), \qquad \text{for } \rho\leq a; \text{ and}$$

$$V(\rho,\phi,0) = \frac{(\rho^2-a^2)^{1/2}}{\pi^2}\int_0^{2\pi}\int_0^a \frac{v(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(a^2-\rho_0^2)^{1/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}, \quad \text{for } \rho>a.$$

The solution of the first mixed boundary value problem is completed. The main results are given by formulae (1.4.10) and (1.4.18).

Consider now another internal problem, characterized by the following mixed conditions on the boundary $z=0$:

$$\frac{\partial V}{\partial z} = -2\pi\sigma(\rho,\phi), \qquad \text{for } \rho\le a, \text{ and } 0\le\phi<2\pi;$$

$$V = 0, \qquad \text{for } \rho>a, \text{ and } 0\le\phi<2\pi. \tag{1.4.23}$$

The problem (1.4.23) can be interpreted as an electrostatic one of a charged disc $\rho\le a$ inside an infinite grounded diaphragm $\rho>a$. Mathematically similar problem arises in the consideration of a penny-shaped crack subjected to an arbitrary pressure σ.

Substitution of (1.4.23) in (1.4.3) leads to the integral equation, for $\rho>a$,

$$\int_0^a \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$+ \int_\rho^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_a^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi) = 0. \tag{1.4.24}$$

Notice that σ in the first term of (1.4.24) is known from (1.4.23), while σ in the second term is yet to be determined. By using the integral representations (1.1.27) and (1.1.28), equation (1.4.24) can be rewritten as

$$\int_\rho^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_a^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi)$$

$$= -\int_\rho^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^a \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi). \tag{1.4.25}$$

Operation on both sides of (1.4.25) by

$$\mathcal{L}(t) \frac{d}{dt} \int_t^\infty \frac{\rho d\rho}{(\rho^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho}\right)$$

leads to

$$\int_a^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right) \sigma(\rho_0,\phi) = -\int_0^a \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right) \sigma(\rho_0,\phi). \tag{1.4.26}$$

The next operator to apply is

$$\mathcal{L}\left(\frac{1}{\rho}\right) \frac{d}{d\rho} \int_a^\rho \frac{t dt}{(\rho^2 - t^2)^{1/2}} \mathcal{L}(t),$$

and the final result takes the form

$$\sigma(\rho,\phi) = -\frac{2}{\pi(\rho^2 - a^2)^{1/2}} \int_0^a \frac{(a^2 - \rho_0^2)^{1/2} \rho_0 d\rho_0}{\rho^2 - \rho_0^2} \mathcal{L}\left(\frac{\rho_0}{\rho}\right) \sigma(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2(\rho^2 - a^2)^{1/2}} \int_0^{2\pi}\int_0^a \frac{(a^2 - \rho_0^2)^{1/2} \sigma(\rho_0,\phi_0) \rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)} . \tag{1.4.27}$$

Formula (1.4.27) defines the value of σ outside the circle $\rho=a$ directly through its value inside. Now σ is known all over the plane $z=0$, and substitution of (1.4.27) in the second term of (1.4.3) allows us to express the potential function V directly through the prescribed value of σ. The first integration yields

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_{g(x)}^a \frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$- 4\int_{l_2}^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^a \frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi). \tag{1.4.28}$$

Here the following integral was employed

$$\int_a^\rho \frac{y dy}{(\rho^2 - y^2)^{1/2}(y^2 - a^2)^{1/2}(y^2 - r^2)} = \frac{\pi}{2(\rho^2 - r^2)^{1/2}(a^2 - r^2)^{1/2}} , \quad \text{for } r<a.$$

(1.4.29)

The first term in (1.4.28) can be transformed by using (1.1.22), in the following manner:

$$\int_0^{l_1} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_{g(x)}^{a} \frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$= \int_0^{a} \rho_0 d\rho_0 \int_{l_2(\rho_0)}^{\infty} \frac{dx}{(x^2 - \rho^2)^{1/2}[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi)$$

$$= \int_{l_2(0)}^{l_2} \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^{g(x)} \frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0\phi)$$

$$+ \int_{l_2}^{\infty} \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^{a} \frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0\phi). \tag{1.4.30}$$

Substitution of (1.4.30) in (1.4.28) yields

$$V(\rho,\phi,z) = 4 \int_{l_2(0)}^{l_2} \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_0^{g(x)} \frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0\phi). \tag{1.4.31}$$

Introduction of a new variable $t=g(x)$, $x=l_2(t)$, transforms (1.4.31) into

$$V(\rho,\phi,z) = 4\int_0^{a} \frac{dl_2(t)}{[l_2^2(t) - \rho^2]^{1/2}} \int_0^{t} \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right) \sigma(\rho_0,\phi). \tag{1.4.32}$$

An interchange of the order of integration in (1.4.32), and integration with respect to t (see 1.1.20 and exercise 1.1.8), yields

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{a} \frac{1}{R_0} \tan^{-1}\left(\frac{h}{R_0}\right) \sigma(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0. \tag{1.4.33}$$

Formulae (1.4.31–33) give three equivalent representations of the potential function V, the first two being more convenient for explicit evaluation of the integrals involved, while the third one has some advantages for numerical integration. Two examples are considered below.

Example 1. Let the potential prescribed inside the disc be $v(\rho,\phi)=v_n\rho^n\cos n\phi$, v_n=const. The solution due to (1.4.16) is

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\ \rho^n} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \cos n\phi \int_0^{l_1} \frac{x^{2n}dx}{(\rho^2 - x^2)^{1/2}}$$

$$= v_n\rho^n\cos n\phi\left[1 - \frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \frac{(l_2^2 - a^2)^{1/2}}{l_2} F\left(\frac{1}{2}-n,\ \frac{1}{2};\ \frac{3}{2};\ \frac{l_2^2 - a^2}{l_2^2}\right)\right]. \tag{1.4.34}$$

The hypergeometric function in (1.4.34) can be expressed in elementary functions (Bateman and Erdelyi, 1955)

$$F(\frac{1}{2} - n,\ \frac{1}{2};\ \frac{3}{2};\ \zeta) = \frac{(1-\zeta)^{n+1/2}}{\Gamma(n+1)} \frac{d^n}{d\zeta^n}\left[\frac{\zeta^{n-1/2}}{\sqrt{1-\zeta}} \sin^{-1}\sqrt{\zeta}\right]. \tag{1.4.35}$$

Example 2. Let the charge distribution be prescribed in the form $\sigma(\rho,\phi)=\sigma_n\rho^n\cos n\phi$, σ_n=const. The solution is given by (1.4.32)

$$V(\rho,\phi,z) = 2\sqrt{\pi}\ \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \sigma_n\rho^n\cos n\phi \int_0^{b} \frac{x^{2n+2}dx}{(x^2 + z^2)^{n+1}}$$

$$= \sqrt{\pi}\ \frac{\Gamma(n+1)}{\Gamma(n+\frac{5}{2})} \sigma_n\rho^n\cos n\phi\ \frac{b^{2n+3}}{z^{2n+2}}\ F(n+1,\ n+\frac{3}{2};\ n+\frac{5}{2};\ -\frac{b^2}{z^2}), \tag{1.4.36}$$

where $b=(a^2 - l_1^2)^{1/2}$, and the hypergeometric function can be expressed in elementary (Bateman and Erdelyi, 1955)

$$F\left(n+1,\ n+\frac{3}{2};\ n+\frac{5}{2};\ \zeta\right) = \frac{2n+3}{\Gamma(n+1)}\,\frac{d^n}{d\zeta^n}\left\{\frac{1}{\zeta}\left[\frac{1}{2\sqrt{\zeta}}\ln\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} - 1\right]\right\}. \tag{1.4.37}$$

Bibliographical note. The degree of effectiveness of the new approach can be established by comparison with the results reported in the literature. Formulae, similar to (1.4.21), can be found in the works of Lord Kelvin. Heine, Hobson and others. As was mentioned before, their practical use was quite limited. The equivalent expressions, like (1.4.18), allow us to evaluate the integrals involved in a straightforward and elementary manner. This was demonstrated by two examples above. The general solution has now become a working tool, available to anyone with even an undergraduate background in mathematics.

Copson (1947) was very close to the discovery of the new method. He established the identity

$$\int_0^{2\pi}\frac{\cos n(\phi-\psi)\,d\phi}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}$$

$$= \frac{4\cos n(\phi_0-\psi)}{(\rho\rho_0)^n}\int_0^{min(\rho_0,\rho)}\frac{x^{2n}dx}{(\rho^2-x^2)^{1/2}(\rho_0^2-x^2)^{1/2}}. \tag{1.4.38}$$

Copson has obtained a solution, similar to (1.4.10), in the form of a Fourier expansion. It remains unclear why he did not perform the summation in (1.4.38) leading to (1.1.27) which, in a way, is the foundation of the new method.

Galin (1953) obtained the nonsingular part (corresponding to the second term in (1.4.15) of the general solution of equation (1.4.5). Leonov (1953) obtained the closed form complete solution (0.3). Both solutions had the same limitation as the previous classical solutions: difficulty in practical use.

Another type of solution of equation (1.4.5) can be expressed through the z-derivative of (1.4.21) for $z=0$, due to the relationship

$$\sigma = -\frac{1}{2\pi}\left.\frac{\partial V}{\partial z}\right|_{z=0}. \tag{1.4.39}$$

The result is

$$\sigma(\rho,\phi) = \frac{1}{2\pi^3}\left\{-\Delta\int_0^{2\pi}\int_0^a \frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)\ \omega(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0\right.$$

$$\left. - \frac{a}{(a^2 - \rho^2)^{3/2}}\int_0^{2\pi}\int_0^a \frac{1 - t\bar{t}}{(1 - t)(1 - \bar{t})}\ \frac{\omega(\rho_0,\phi_0)}{(a^2 - \rho_0^2)^{1/2}}\ \rho_0 d\rho_0 d\phi_0\right\}$$

$$= \frac{1}{\pi^2}\left\{-\Delta\int_\rho^a \frac{dx}{(x^2 - \rho^2)^{1/2}}\int_0^x \frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\ \omega(\rho_0,\phi)\right.$$

$$\left. - \frac{a}{(a^2 - \rho^2)^{3/2}}\int_0^a \frac{\rho_0 d\rho_0}{(a^2 - \rho_0^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{a^2}\right)\ \omega(\rho_0,\phi_0)\right\}, \qquad (1.4.40)$$

where

$$t = \frac{\rho\rho_0}{a^2}\ e^{i(\phi-\phi_0)}, \qquad (1.4.41)$$

and the overbar everywhere indicates the complex conjugate value. Formula (1.4.40) corresponds to the solution by Mossakovskii et al. (1985). The developed apparatus can be used for solving the problem of several charged coaxial (Fabrikant, 1987e) and arbitrarily located (Fabrikant, 1988a) circular discs.

Exercise 1.4

1. A circular conducting disc is kept at the potential v_0. Find the potential function V.

Answer: $V(\rho,z) = \frac{2}{\pi}\ v_0\sin^{-1}(l_1/\rho) = \frac{2}{\pi}\ v_0\sin^{-1}\left(\frac{a}{l_2}\right)$.

Hint: use formula (1.4.18)

2. Subject to the conditions of the previous problem, find the charge distribution σ by using formulae (1.4.10) and (1.4.39). Prove that in both cases the result is the same.

Answer: $\sigma = \frac{v_0}{\pi^2(a^2 - \rho^2)^{1/2}}$.

3. Solve problems 1 and 2 for $v = v_1 \rho\cos\phi$, $v_1 =$ const.

Answer: $V(\rho,\phi,z) = \frac{2}{\pi} v_1 \rho\cos\phi\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{a}{l_2}\sqrt{1 - (a/l_2)^2}\right]$,

$$\sigma(\rho,\phi) = \frac{2v_1\rho\cos\phi}{\pi^2(a^2 - \rho^2)^{1/2}}.$$

4. A uniform charge density $\sigma = \sigma_0 =$const is prescribed over a circular disc of radius a and potential $V=0$ for $\rho \geq a$ and $z=0$. Find the potential function.

Answer: $V(\rho,z) = 4\sigma_0\left[(a^2 - l_1^2)^{1/2} - z\sin^{-1}\left(\frac{a}{l_2}\right)\right]$.

5. Solve the previous problem for the case where $\sigma = \sigma_1\rho\cos\phi$, σ_1=const.

Answer: $V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[(a^2 - l_1^2)^{1/2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}\left(\frac{a}{l_2}\right)\right]$.

6. The potential function is given by the expression

$$V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[(a^2 - l_1^2)^{1/2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}\left(\frac{a}{l_2}\right)\right].$$

Find the charge distribution on the plane $z=0$.

Answer: $\sigma = \sigma_1\rho\cos\phi$, for $\rho \leq a$;

$$\sigma = -\frac{4}{3\pi}\sigma_1\rho\cos\phi\left[\frac{a}{(\rho^2 - a^2)^{1/2}}\left(\frac{3}{2} - \frac{a^2}{2\rho^2}\right) - \frac{3}{2}\sin^{-1}\left(\frac{a}{\rho}\right)\right], \text{ for } \rho > a.$$

Hint: use (1.4.39).

7. Solve the mixed boundary value problem of potential theory for a sphere.
Hint: see (Fabrikant, 1987i).

1.5 External mixed boundary value problem for a half-space

The material in this section follows the paper (Fabrikant, 1986f). The problem is called external when non-zero boundary conditions are prescribed outside the disc. As in the previous section, we consider two types of problem.

Problem 1. It is necessary to find a function, harmonic in the half-space $z\geq 0$, vanishing at infinity, and subject to the mixed boundary conditions on the plane $z=0$, namely,

$$\frac{\partial V}{\partial z}\Big|_{z=0} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi;$$

$$V=v(\rho,\phi), \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi. \tag{1.5.1}$$

The problem (1.5.1) can be interpreted as an electrostatic one of a charged diaphragm, or as an external elastic contact problem. The potential V is presented through a simple layer distribution (1.4.3). Substitution of the boundary conditions (1.5.1) in (1.4.3) leads to the governing integral equation

$$4\int_\rho^\infty \frac{dx}{(x^2-\rho^2)^{1/2}} \int_a^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{1/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.5.2}$$

Its solution is obtained in exactly the same manner as that of (1.4.5), and is

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\rho}\ \mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_a^\rho \frac{x dx}{(\rho^2-x^2)^{1/2}}\ \mathcal{L}(x^2)\ \frac{d}{dx}\int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{1/2}}\ \mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \tag{1.5.3}$$

The rules of differentiation (1.3.2) and (1.3.9) allow us to rewrite (1.5.3) as follows

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2-a^2)^{1/2}} + \int_a^\rho \frac{dx}{(\rho^2-x^2)^{1/2}}\ \frac{\partial}{\partial x}\chi(x,\rho,\phi)\right\}, \tag{1.5.4}$$

where

$$\chi(x,\rho,\phi) = x\int_x^\infty \frac{d\rho_0}{(\rho_0^2-x^2)^{1/2}}\ \frac{\partial}{\partial\rho_0}\left[\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\ v(\rho_0\phi)\right]. \tag{1.5.5}$$

The following transformation can now be performed:

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \frac{\partial}{\partial x}\left[x\int_x^\infty \frac{d\rho_0}{(\rho_0^2 - x^2)^{1/2}} (\mathcal{L}v)'\right]$$

$$= \int_x^\infty \frac{d\rho_0}{(\rho_0^2 - x^2)^{1/2}} [(\mathcal{L}v)' + \rho_0(\mathcal{L}v)'' - 2(\mathcal{L}'\rho_0 v)']$$

$$= \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \left[\mathcal{L}\left(v'' + \frac{1}{\rho_0}v'\right) - \left(\mathcal{L}'' + \frac{1}{\rho_0}\mathcal{L}'\right)v\right]. \tag{1.5.6}$$

Here, for the sake of brevity, the primes (´) indicate the partial derivatives with respect to ρ_0, $\mathcal{L}$ stands for $\mathcal{L}(x^2/\rho\rho_0)$, $v\equiv v(\rho_0,\phi)$, and the following identity was used

$$\frac{\partial}{\partial x} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) = -2\left(\frac{\rho_0}{x}\right)\frac{\partial}{\partial \rho_0} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right).$$

Since

$$\mathcal{L}\frac{1}{\rho_0^2}\frac{\partial^2 v}{\partial\phi^2} = \frac{1}{\rho_0^2}\frac{\partial^2 \mathcal{L}}{\partial\phi^2} v,$$

its addition to and subtraction from (1.5.6) yields

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \left[\mathcal{L}\Delta v - (\Delta\mathcal{L})v\right], \tag{1.5.7}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Since λ is a harmonic function, $\Delta\mathcal{L}=0$, and (1.5.7) simplifies to

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\Delta v. \tag{1.5.8}$$

Substitution of (1.5.8) in (1.5.4) yields

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2 - a^2)^{1/2}} + \int_a^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\Delta v(\rho_0,\phi)\right\}. \tag{1.5.9}$$

It should be noticed that the first term in (1.5.9) becomes singular when $\rho \to a$, while the second term vanishes at the edge of the disc. In the case of v being a harmonic function, the second term in (1.5.9) vanishes, and the solution is represented by the first term only. Further integration with respect to x becomes possible in (1.5.9), after changing the order of integration and using (1.1.8). The result is

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{ \frac{\chi(a,\rho,\phi)}{(\rho^2 - a^2)^{1/2}} \right.$$

$$\left. + \frac{1}{2\pi}\int_0^{2\pi}\int_a^\infty \frac{\Delta v(\rho_0,\phi_0)\ \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}} \tan^{-1}\frac{(\rho^2 - a^2)^{1/2}(\rho_0^2 - a^2)^{1/2}}{a[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}\right\}. \tag{1.5.10}$$

Solutions, like (1.5.3) and (1.5.9), are appropriate for use when an exact evaluation of the integrals is possible, while the solution in the form (1.5.10) has some advantages when numerical integration is to be employed.

Now we can express the potential function V directly through its boundary value v. Since σ=0 inside the circle $\rho=a$, the potential function (1.4.3) takes the form

$$V(\rho,\phi,z) = 4\int_{l_2}^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \int_a^{g(x)} \frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right) \sigma(\rho_0,\phi). \tag{1.5.11}$$

Substitution of (1.5.3) in (1.5.11) yields, after the first integration,

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_{l_2}^\infty \frac{dx}{(x^2 - \rho^2)^{1/2}} \mathcal{L}\left(\frac{\rho g^2(x)}{x^2}\right)\frac{\partial}{\partial g(x)} \int_{g(x)}^\infty \frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}} \mathcal{L}\left(\frac{1}{\rho_0}\right) v(\rho_0,\phi). \tag{1.5.12}$$

Here the properties of the $\mathcal{L}$-operators (1.2.3) were used, along with the following identity, valid for the Abel-type operators

$$\int_a^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(t)\ t\mathrm{d}t}{(x^2 - t^2)^{1/2}} = \frac{\pi}{2} f(\rho). \tag{1.5.13}$$

Introduction of a new variable $y=g(x)$, $x=l_2(y)$, in (1.5.12), allows us to rewrite (1.5.12)

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^\infty \frac{\mathrm{d}l_2(y)}{[l_2^2(y) - \rho^2]^{1/2}} \mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right) \frac{\mathrm{d}}{\mathrm{d}y} \int_y^\infty \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2 - y^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho_0}\right) v(\rho_0,\phi). \tag{1.5.14}$$

Interchange of the order of integration in (1.5.14) yields

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^\infty \left\{ \mathcal{L}\left(\frac{1}{\rho_0}\right) \frac{\mathrm{d}}{\mathrm{d}\rho_0} \int_a^{\rho_0} \frac{y\mathrm{d}l_2(y)}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - \rho^2]^{1/2}} \mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\right\} v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.5.15}$$

Here the general formula was used

$$\int_a^\infty F(\rho)\mathrm{d}\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{xf(x)\mathrm{d}x}{(x^2 - \rho^2)^{1/2}} = -\int_a^\infty f(x)\ \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{\rho F(\rho)\ \mathrm{d}\rho}{(x^2 - \rho^2)^{1/2}} . \tag{1.5.16}$$

The integral in curly brackets of (1.5.15) can be evaluated, according to (1.3.21), with the result

$$V(\rho,\phi,z) = \frac{1}{\pi^2} \int_0^{2\pi}\int_a^\infty \frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right] v(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.5.17}$$

Here R_0 is defined by (1.4.22), and

$$j(x) = \frac{(\rho_0^2 - x^2)^{1/2}[l_2^2(x) - x^2]^{1/2}}{x}. \tag{1.5.18}$$

The abbreviation j in (1.5.17) stands for $j(a)$. In the particular case, when z=0, expression (1.5.17) simplifies to

$$V(\rho,\phi,0) = \frac{1}{\pi^2}\,(a^2 - \rho^2)^{1/2}\int_0^{2\pi}\int_a^{\infty}\frac{v(\rho_0,\phi_0)\;\rho_0 d\rho_0 d\phi_0}{(\rho_0^2 - a^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]},$$
$$\text{for } \rho<a;$$
$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for } \rho\geq a. \tag{1.5.19}$$

The general solution is completed. The charge density σ is given by the two equivalent expressions (1.5.3) and (1.5.10), while the potential is in the two forms (1.5.14) and (1.5.17), the first one being more convenient for exact evaluation of the integrals involved, while the second is better suited for numerical integration.

Problem 2. Consider the problem of finding a harmonic function, vanishing at infinity, and subject to the mixed conditions on the plane z=0

$$V = 0, \quad \text{for } \rho\leq a, \quad 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -\,2\pi\sigma(\rho,\phi), \quad \text{for } \rho>a, \quad 0\leq\phi<2\pi. \tag{1.5.20}$$

The problem may be interpreted as an electrostatic one of a charged infinite diaphragm, with a grounded disc inside, or as an external crack problem in elasticity. Substitution of the boundary conditions (1.5.20) in (1.4.3) leads to the governing integral equation

$$\int_0^{\rho}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_x^{a}\frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= -\int_a^{\infty}\frac{dx}{(x^2 - \rho^2)^{1/2}}\int_a^{x}\frac{\rho_0 d\rho_0}{(x^2 - \rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.5.21}$$

One should notice that σ in the second term of (1.5.21) is known from the boundary condition (1.5.20), while the value of σ in the first term is yet to be determined. The right hand side of (1.5.21) can be transformed, by using (1.1.27) and (1.1.28):

$$\int_0^{\rho}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_x^{a}\frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= -\int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_a^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi),$$

with an immediate result

$$\int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x}{\rho_0}\right) \sigma(\rho_0,\phi) = -\int_a^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x}{\rho_0}\right) \sigma(\rho_0,\phi). \quad (1.5.22)$$

Application of the operator

$$\mathcal{L}(\rho) \frac{d}{d\rho} \int_\rho^a \frac{x dx}{(x^2 - \rho^2)^{1/2}} \mathcal{L}\left(\frac{1}{x}\right)$$

to both sides of (1.5.22) gives, after necessary transformations

$$\sigma(\rho,\phi) = -\frac{2}{\pi(a^2 - \rho^2)^{1/2}} \int_a^\infty \frac{(\rho_0^2 - a^2)^{1/2}}{\rho_0^2 - \rho^2} \mathcal{L}\left(\frac{\rho}{\rho_0}\right) \sigma(\rho_0,\phi)\, \rho_0 d\rho_0, \quad \text{for } \rho<a, \quad (1.5.23)$$

or, interpreting the $\mathcal{L}$-operator, we obtain

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2(a^2 - \rho^2)^{1/2}} \int_0^{2\pi}\int_a^\infty \frac{(\rho_0^2 - a^2)^{1/2}\sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)}. \quad (1.5.24)$$

Now the value of σ is known all over the plane z=0, and (1.4.3) can be used in order to express the potential V directly through the prescribed σ. Substitution of (1.5.23) in (1.4.3) yields, after the first integration

$$V(\rho,\phi,z) = -4\int_0^{l_1} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_a^\infty \frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0,\phi)$$

$$+\ 4\int_{l_2}^{\infty}\frac{dx}{(x^2 - \rho^2)^{1/2}}\int_{a}^{g(x)}\frac{\rho_0 d\rho_0}{[g^2(x) - \rho_0^2]^{1/2}}\ \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\ \sigma(\rho_0,\phi). \qquad (1.5.25)$$

The second term in (1.5.25) is equivalent to the second term in (1.4.2), which, in turn, can be represented by using (1.1.18), as

$$4\int_{a}^{\infty}\left\{\int_{0}^{l_1(\rho_0)}\frac{dx}{(\rho^2 - x^2)^{1/2}[\rho_0^2 - g^2(x)]^{1/2}}\ \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\right\}\ \sigma(\rho_0,\phi)\ \rho_0 d\rho_0.$$

The following scheme of changing the order of integration is enacted

$$\int_{a}^{\infty}d\rho_0\int_{0}^{l_1(\rho_0)}dx\ =\ \int_{0}^{l_1}dx\int_{a}^{\infty}d\rho_0\ +\ \int_{l_1}^{l_1(\infty)}dx\int_{g(x)}^{\infty}d\rho_0,$$

and the second term in (1.5.25) can be rewritten as

$$4\int_{0}^{l_1}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_{a}^{\infty}\frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}}\ \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\ \sigma(\rho_0,\phi)$$

$$+\ 4\int_{l_1}^{l_1(\infty)}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}}\ \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\ \sigma(\rho_0,\phi). \qquad (1.5.26)$$

Substitution of (1.5.26) in (1.5.25) gives, by virtue of $l_1(\infty)=\rho$,

$$V(\rho,\phi,z)\ =\ 4\int_{l_1}^{\rho}\frac{dx}{(\rho^2 - x^2)^{1/2}}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{[\rho_0^2 - g^2(x)]^{1/2}}\ \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\ \sigma(\rho_0,\phi). \qquad (1.5.27)$$

Interchange of the order of integration in (1.5.27), and integration with respect to x, according to (1.1.24), results in

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty} \frac{1}{R_0} \tan^{-1}\left(\frac{j}{R_0}\right) \sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0, \tag{1.5.28}$$

where R_0 is defined by (1.4.22), and j stands for $j(a)$, as defined by (1.3.4).

The second problem is now solved. Expression (1.5.24) defines the charge density σ inside a circle directly in terms of its values outside. The potential V is given by two equivalent expressions (1.5.27) and (1.5.28), the first one to be used for exact evaluation of the integrals, while the second has some advantages in the case of numerical integration. Some specific examples are considered below.

Example 1. Consider an external mixed problem with the following boundary conditions at z=0

$$V = v_0/\rho^n, \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi. \tag{1.5.29}$$

The conditions (1.5.29) correspond to those of Problem 1. The solution is given by (1.5.14) and (1.5.3). Substitution of (1.5.29) in (1.5.14) yields, after the integration

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \int_{l_2}^{\infty} \frac{dx}{(x^2-\rho^2)^{1/2} g^n(x)}, \tag{1.5.30}$$

where $g(x)$ is defined by (1.1.25), and the following integral was employed (Gradshtein and Ryzhik, 1963)

$$\int_x^{\infty} \frac{d\rho}{\rho^n(\rho^2-x^2)^{1/2}} = \frac{\sqrt{\pi}\ \Gamma(n/2)}{2\Gamma[(n+1)/2]\ x^n}. \tag{1.5.31}$$

The integral in (1.5.30) can be evaluated in terms of elementary functions for any integer n, but the procedure is slightly different for even and odd values of n. For example, for even n=2k, the problem reduces to the evaluation of the integral

$$\int_{l_2}^{\infty} \frac{(x^2 - \rho^2)^{k-1/2}\mathrm{d}x}{x^{2k}(x^2 - \rho^2 - z^2)^k} ,$$

which can be evaluated by introduction of a new variable $t=x/(x^2 - \rho^2)^{1/2}$. The final result is

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}} \frac{\Gamma[(n + 1)/2]}{\Gamma(n/2)z^n} \left\{ \sum_{m=1}^{k} \frac{A_m}{2m - 1} [1 - Q_0^{2m-1}] \right.$$

$$\left. + 2B_1 \ln Q + \sum_{m=2}^{k} \frac{B_m}{1 - m} \left[(Q_1^{m-1} - Q_2^{m-1}) - (Q_3^{m-1} - Q_4^{m-1}) \right] \right\}, \tag{1.5.32}$$

where

$$A_{k-m+1} = \frac{1}{(m - 1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\eta^{m-1}} \left[\frac{(\eta - 1)^{k-1}}{(r^2 - \eta)^k} \right], \quad \text{for } \eta=0, \text{ and } r^2=1+\rho^2/z^2;$$

$$B_{k-m+1} = \frac{1}{(m - 1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{(t^2 - 1)^{k-1}}{t^{2k}(r^2 + t)^k} \right], \quad \text{for } t=\sqrt{1+\rho^2/z^2};$$

$$Q_0 = \frac{(l_2^2 - \rho^2)^{1/2}}{l_2} , \qquad Q = \frac{l_2[(\rho^2 + z^2)^{1/2} + (l_2^2 - a^2)^{1/2}]}{a[(\rho^2 + z^2)^{1/2} + z]} ,$$

$$Q_1 = \frac{z[(\rho^2 + z^2)^{1/2} + (l_2^2 - a^2)^{1/2}]}{l_1^2} , \quad Q_2 = \frac{z[(\rho^2 + z^2)^{1/2} - (l_2^2 - a^2)^{1/2}]}{l_1^2} ,$$

$$Q_3 = \frac{z[(\rho^2 + z^2)^{1/2} + z]}{\rho^2} , \quad Q_4 = \frac{z[(\rho^2 + z^2)^{1/2} - z]}{\rho^2}. \tag{1.5.33}$$

For the case of an odd $n=2k+1$, the integration can be performed by using the substitution $t=(x^2-\rho^2-z^2)^{1/2}$, and the final result is

$$V(\rho,\phi,z) = \frac{v_0}{\sqrt{\pi}} \frac{\Gamma[(n + 1)/2]}{\Gamma(n/2)} \left\{ \sum_{m=1}^{k} \frac{C_m}{(2m - 1)(a^2 - l_1^2)^{m-1/2}} + \sum_{m=1}^{k+1} D_m E_m \right\}, \tag{1.5.34}$$

where

$$C_m = \frac{1}{(k-m)!}\frac{d^{k-m}}{dt^{k-m}}\left[\frac{(t+z^2)^k}{(t+\rho^2+z^2)^{k+1}}\right], \quad \text{for } t=0;$$

$$D_m = \frac{1}{(k+1-m)!}\frac{d^{k+1-m}}{dt^{k+1-m}}\left[\frac{(t+z^2)^k}{t^k}\right], \quad \text{for } t=-(\rho^2+z^2);$$

$$E_m = \frac{(-1)^m}{(m-1)!}\frac{d^{m-1}}{dt^{m-1}}\left[\frac{1}{\sqrt{t}}\tan^{-1}\frac{\sqrt{t}}{(a^2-l_1^2)^{1/2}}\right], \quad \text{for } t=\rho^2+z^2. \tag{1.5.35}$$

Substitution of (1.5.29) in (1.5.3) yields, after integration

$$\sigma(\rho,\phi) = \frac{v_0\Gamma[(n+1)/2]}{\pi^{3/2}\Gamma(n/2)}\left\{\frac{1}{a^n(\rho^2-a^2)^{1/2}} - \frac{n(\rho^2-a^2)^{1/2}}{\rho^{n+2}}F\left(\tfrac{1}{2}n+1,\ \tfrac{1}{2};\ \tfrac{3}{2};\ 1-\frac{a^2}{\rho^2}\right)\right\}, \tag{1.5.36}$$

and the Gauss hypergeometric function can be expressed in elementary functions (Bateman and Erdelyi, 1955), namely, for even $n=2k$, $k=1,2,3$ ...,

$$F\left(k+1,\ \tfrac{1}{2};\ \tfrac{3}{2};\ t\right) = \frac{1}{2k!}\frac{d^k}{dt^k}\left[t^{k-1/2}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}\right],$$

and for odd $n=2k+1$, $k=0,1,2,\ ...,$

$$F\left(k+\tfrac{3}{2},\ \tfrac{1}{2};\ \tfrac{3}{2};\ t\right) = \frac{\sqrt{\pi}}{2\sqrt{t}\,\Gamma(k+\frac{3}{2})}\frac{d^k}{dt^k}\left[\frac{t^{k+1/2}}{\sqrt{1-t}}\right]. \tag{1.5.37}$$

The charge density distribution, evaluated due to (1.5.36), is non-negative for $n=1$, and changes sign when $n\geq 2$, its negative maximum increases with n, while the total charge stays at zero.

Example 2. Consider the boundary conditions at $z=0$

$$V = (v_n/\rho^n)\ e^{in\phi}, \quad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho<a,\ 0\leq\phi<2\pi, \tag{1.5.38}$$

where v_n is constant. The solution is given by (1.5.3) and (1.5.14). Substitution of (1.5.38) in (1.5.14) yields, after integration

$$V(\rho,\phi,z) = \frac{2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n)} \rho^n e^{in\phi} \int_{l_2}^{\infty} \frac{dx}{x^{2n}(x^2-\rho^2)^{1/2}} . \tag{1.5.39}$$

The final integration gives

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\rho^n} e^{in\phi}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(n+1/2)}{(2k-1)\Gamma(k)\Gamma(n+1-k)} (1 - Q_0^{2k-1}), \tag{1.5.40}$$

where Q_0 is defined by (1.5.33). Substituting (1.5.38) in (1.5.3), we get, after integration,

$$\sigma(\rho,\phi) = \frac{\Gamma(n+1/2)}{\pi^{3/2}\Gamma(n)} \frac{v_n e^{in\phi}}{\rho^n(\rho^2-a^2)^{1/2}}. \tag{1.5.41}$$

Evidently, expression (1.5.41) can also be obtained by differentiation of (1.5.40) with respect to z for z=0.

Example 3. Consider a case related to Problem 2, with the boundary conditions

$$V = 0, \qquad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi \frac{\sigma_0}{\rho^n} \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi; \tag{1.5.42}$$

The solution is given by (1.5.23) and (1.5.27). Substitution of (1.5.42) in (1.5.27) yields, after integration using (1.5.31),

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \int_{l_1}^{\rho} \frac{dx}{(\rho^2-x^2)^{1/2}g^{n-1}(x)}, \tag{1.5.43}$$

where $g(x)$ is defined by (1.1.25). The technique used in the previous example can be employed here for further integration. The final result depends on the value of n being even or odd. For even n=2k, k=1,2,3, ..., the potential is

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0 \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)} \left\{ 2B_1 \ln Q \right.$$

$$+ \sum_{m=1}^{k-1} \frac{A_m}{(2m-1)\, z^{2m-1}} \left[1 - \left(\frac{(a^2 - l_1^2)^{1/2}}{a} \right)^{2m-1} \right]$$

$$\left. + \sum_{m=1}^{k-1} \frac{B_{m+1}}{mz^m} [Q_1^m - Q_2^m - (Q_3^m - Q_4^m)] \right\}. \quad (1.5.44)$$

Here Q, Q_1, Q_2, Q_3, and Q_4 are defined by (1.5.33), and

$$A_{k-m} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dt^{m-1}} \left[\frac{(t-z^2)^{k-1}}{(\rho^2 + z^2 - t)^k} \right] \quad \text{for } t=0;$$

$$B_{k-m+1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dt^{m-1}} \left[\frac{(t^2 - z^2)^{k-1}}{t^{2k-2}[(\rho^2 + z^2)^{1/2} - t]^k} \right], \quad \text{for } t=-(\rho^2 + z^2)^{1/2}.$$

(1.5.45)

For odd $n=2k+1$, the result is

$$V(\rho,\phi,z) = 2\sqrt{\pi} \frac{\sigma_0 \Gamma[(n-1)/2]}{\Gamma(n/2)\, z^{n-1}} \sum_{m=1}^{k} \left\{ \frac{G_m}{2m-1} \left(\frac{(l_2^2 - a^2)^{1/2}}{a} \right)^{2m-1} - H_m L_m \right\}.$$

(1.5.46)

Here

$$G_{k-m+1} = \frac{1}{\Gamma(m)} \frac{d^{m-1}}{dt^{m-1}} \left[\frac{(1+t)^{k-1}}{(\xi + t)^k} \right], \quad \text{for } t=0,\ \xi=(\rho^2 + z^2)/z^2;$$

$$H_{k-m+1} = \frac{1}{\Gamma(m)} \frac{d^{m-1}}{dt^{m-1}} \left[\frac{(1+t)^{k-1}}{t^k} \right], \quad \text{for } t=-(\rho^2+z^2)/z^2;$$

$$L_m = \frac{(-1)^m}{\Gamma(m)} \frac{d^{m-1}}{dt^{m-1}} \left[\frac{1}{\sqrt{t}} \tan^{-1}\left(\frac{\sqrt{t}}{a} (l_2^2 - a^2)^{1/2} \right) \right], \quad \text{for } t=(\rho^2+z^2)/z^2.$$

(1.5.47)

Example 4. Consider the boundary conditions on the plane $z=0$:

$$V = 0, \qquad \text{for } \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_n/\rho^n)e^{in\phi}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi; \qquad (1.5.48)$$

Substitution of (1.5.48) in (1.5.27) yields, after integration,

$$V(\rho,\phi,z) = 2\sqrt{\pi}\,\frac{\sigma_n\Gamma(n-1/2)}{\Gamma(n)\rho^n}\,e^{in\phi}\Bigg\{(l_2^2-a^2)^{1/2}-z$$

$$-z\sum_{k=2}^{n}\frac{(-1)^k\Gamma(n)}{\Gamma(k)\,\Gamma(n-k+1)(2k-3)}\,[1-(1-l_1^2/a^2)^{k-3/2}]\Bigg\}, \qquad (1.5.49)$$

and on the plane z=0

$$V(\rho,\phi,0) = 2\sqrt{\pi}\,\frac{\sigma_n\Gamma(n-1/2)}{\Gamma(n)\rho^n}\,e^{in\phi}\Re(\rho^2-a^2)^{1/2}.$$

The symbol $\Re$ indicates the real part sign. The charge density is defined, according to (1.5.23),

$$\sigma(\rho,\phi) = \frac{\sigma_n\Gamma(n-1/2)}{\sqrt{\pi}\,\Gamma(n)\rho^n}\,e^{in\phi}\Re\Bigg\{1-\frac{a}{(a^2-\rho^2)^{1/2}}$$

$$+\sum_{k=2}^{n}\frac{(-1)^k\Gamma(n)}{\Gamma(k)\,\Gamma(n-k+1)(2k-3)}\,[1-(1-\rho^2/a^2)^{k-3/2}]\Bigg\}. \qquad (1.5.50)$$

A more general case of boundary conditions, namely,

$$V = 0, \qquad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_{jn}/\rho^j)e^{in\phi}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi; \qquad (1.5.51)$$

can also be considered, by using the same technique as in the previous examples, and the final result can always be expressed in elementary functions. The form of the result will be different for (j+n) even, and for (j+n) odd. As an example, the following expression can be obtained by substituting (1.5.51) in (1.5.23), for the case when $j+n=2k$

$$\sigma(\rho,\phi) = \frac{\sigma_{jn}}{\rho^j} e^{in\phi}\Re\left\{1 - \frac{a}{(a^2-\rho^2)^{1/2}}\left[1 - \sum_{m=2}^{k}\frac{\Gamma(m-3/2)}{2\sqrt{\pi}\Gamma(m)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.5.52}$$

and for odd $j+n=2k+1$

$$\sigma(\rho,\phi) = \frac{2\sigma_{jn}}{\pi\rho^j} e^{in\phi}\Re\left\{\sin^{-1}\left(\frac{\rho}{a}\right) - \frac{a}{(a^2-\rho^2)^{1/2}}\left[1 - \sum_{m=2}^{k}\frac{\sqrt{\pi}\Gamma(m-1)}{4\Gamma(m+1/2)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.5.53}$$

Expressions (1.5.52) and (1.5.53) represent general formulae which cover all the particular cases considered in Examples 3 and 4.

The examples above have demonstrated the simplicity of the method. The generation of the solution is reduced to a straightforward and elementary procedure.

Exercise 1.5

1. The following boundary conditions are prescribed at $z=0$

$$V = v_0/\rho, \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \dfrac{2v_0}{\pi(\rho^2+z^2)^{1/2}} \sin^{-1}\left(\dfrac{(\rho^2+z^2)^{1/2}}{l_2}\right),$

$$\sigma(\rho,\phi) = \frac{v_0 a}{\pi^2\rho^2(\rho^2-a^2)^{1/2}}.$$

The total charge is equal v_0.

2. The following boundary conditions are prescribed at $z=0$

$$V = v_0/\rho^2, \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{v_0}{\rho^2 + z^2}\left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{l_2}\right.$$

$$\left. + \frac{z}{(\rho^2 + z^2)^{1/2}} \ln\frac{l_2[(\rho^2 + z^2)^{1/2} + (l_2^2 - a^2)^{1/2}]}{a[(\rho^2 + z^2)^{1/2} + z]}\right],$$

$$\sigma(\rho,\phi) = \frac{v_0}{2\pi\rho^2}\left[\frac{1}{(\rho^2 - a^2)^{1/2}} - \frac{1}{\rho}\ln\frac{\rho + (\rho^2 - a^2)^{1/2}}{a}\right].$$

3. The following boundary conditions are prescribed at z=0

$$V = v_0/\rho^3, \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{4v_0}{(\rho^2 + z^2)^2}\left[\frac{z^2}{(a^2 - l_1^2)^{1/2}} - \frac{l_1^2(a^2 - l_1^2)^{1/2}}{2a^2}\right.$$

$$\left. + \frac{\rho^2 - 2z^2}{2(\rho^2 + z^2)^{1/2}} \sin^{-1}\left(\frac{(\rho^2 + z^2)^{1/2}}{l_2}\right)\right].$$

Note that the potential at the coordinate origin is finite, namely, $V(0,0,0)=4v_0/(3\pi a^3)$.

$$\sigma(\rho,\phi) = \frac{2v_0(2a^2 - \rho^2)}{\pi^2 a\rho^4(\rho^2 - a^2)^{1/2}}.$$

4. The following boundary conditions are prescribed at z=0

$$V = v_0/\rho^4, \qquad \text{for } \rho\geq a,\ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\leq\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{3v_0}{2(\rho^2+z^2)^2}\left\{\frac{\rho^2-z^2}{\rho^2+z^2}\left[1-\frac{(l_2^2-\rho^2)^{1/2}}{l_2}\right]\right.$$

$$-\frac{1}{3}\left[1-\left(\frac{(l_2^2-\rho^2)^{1/2}}{l_2}\right)^3\right]+\frac{z}{2(\rho^2+z^2)}\left[\frac{l_2^2}{a^2}(l_2^2-a^2)^{1/2}-z\right]$$

$$\left.-\frac{z(2z^2-3\rho^2)}{2(\rho^2+z^2)^{3/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+(l_2^2-a^2)^{1/2}]}{a[(\rho^2+z^2)^{1/2}+z]}\right\},$$

The last expression simplifies at z=0:

$$V(\rho,\phi,0) = \frac{v_0}{\rho^4}\left[1-\frac{(a^2-\rho^2)^{1/2}}{a}-\frac{\rho^2(a^2-\rho^2)^{1/2}}{2a^3}\right], \qquad \text{for } \rho\le a;$$

and $V(\rho,\phi,0) = v_0/\rho^4$, for $\rho>a$. Note that the potential at the coordinate origin is finite, namely, $V(0,0,0)=3v_0/(8a^4)$.

$$\sigma(\rho,\phi) = \frac{3v_0}{8\pi\rho^4}\left\{\frac{3a^2-\rho^2}{a^2(\rho^2-a^2)^{1/2}}-\frac{3}{\rho}\ln\left[\frac{\rho+(\rho^2-a^2)^{1/2}}{a}\right]\right\}.$$

5. The following boundary conditions are prescribed at z=0

$$V = (v_1/\rho)\, e^{i\phi}, \qquad \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function.

Answer: $$V(\rho,\phi,z) = \frac{v_1}{\rho}\, e^{i\phi}\left[1-\frac{(a^2-l_1^2)^{1/2}}{a}\right]$$

6. The following boundary conditions are prescribed at z=0

$$V = (v_2/\rho^2)\, e^{2i\phi}, \qquad \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function.

Answer: $$V(\rho,\phi,z) = \frac{3v_2}{2\rho^2}\, e^{2i\phi} \left\{\left[1 - \frac{(a^2 - l_1^2)^{1/2}}{a}\right] - \frac{1}{3}\left[1 - \left(\frac{(a^2 - l_1^2)^{1/2}}{a}\right)^3\right]\right\}.$$

7. The following boundary conditions are prescribed at z=0

$$V = (v_3/\rho^3)\, e^{3i\phi}, \qquad \text{for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function.

Answer: $$V(\rho,\phi,z) = \frac{15v_3}{4\rho^3}\, e^{3i\phi} \left\{\frac{1}{2}\left[1 - \frac{(a^2 - l_1^2)^{1/2}}{a}\right]\right.$$

$$\left. - \frac{1}{3}\left[1 - \left(\frac{(a^2 - l_1^2)^{1/2}}{a}\right)^3\right] + \frac{1}{10}\left[1 - \left(\frac{(a^2 - l_1^2)^{1/2}}{a}\right)^5\right]\right\}.$$

8. Let the following boundary conditions be prescribed at z=0:

$$V = 0, \qquad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -\, 2\pi\sigma_0/\rho^2,\ \ \sigma_0=\text{const}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{2\pi\sigma_0}{(\rho^2 + z^2)^{1/2}}\ \ln\frac{l_2[(\rho^2 + z^2)^{1/2} + (l_2^2 - a^2)^{1/2}]}{a[(\rho^2 + z^2)^{1/2} + z]},$$

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^2}\ \Re\left[1 - \frac{a}{(a^2 - \rho^2)^{1/2}}\right], \qquad \sigma(0,0) = -\,\frac{\sigma_0}{2a^2}.$$

9. Let the following boundary conditions be prescribed at z=0:

$$V = 0, \qquad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -\, 2\pi\sigma_0/\rho^3,\ \ \sigma_0=\text{const}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{4\sigma_0}{\rho^2 + z^2}\left[\frac{(l_2^2 - a^2)^{1/2}}{a} - \frac{z}{(\rho^2 + z^2)^{1/2}}\sin^{-1}\left(\frac{(\rho^2 + z^2)^{1/2}}{l_2}\right)\right],$$

$$\sigma(\rho,\phi) = \frac{2\sigma_0}{\pi\rho^2}\left[\frac{1}{\rho}\sin^{-1}\left(\frac{\rho}{a}\right) - \frac{1}{(a^2 - \rho^2)^{1/2}}\right],\ \text{for } \rho<a;\ \sigma(0,0) = -2\sigma_0/(3\pi a^3).$$

10. Let the following boundary conditions be prescribed at $z=0$:

$$V = 0, \quad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^4,\ \sigma_0=\text{const}, \quad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $$V(\rho,\phi,z) = \frac{\pi\sigma_0}{2(\rho^2 + z^2)^2}\left\{\frac{2z}{a}(a^2 - l_1^2)^{1/2} - 3z + \frac{l_2^2}{a^2}(l_2^2 - a^2)^{1/2}\right.$$

$$\left.\frac{\rho^2 - 2z^2}{(\rho^2 + z^2)^{1/2}}\ln\frac{l_2[(\rho^2 + z^2)^{1/2} + (l_2^2 - a^2)^{1/2}]}{a[(\rho^2 + z^2)^{1/2} + z]}\right\},$$

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^4}\Re\left[1 - \frac{2a^2 - \rho^2}{2a(a^2 - \rho^2)^{1/2}}\right], \qquad \sigma(0,0) = -\frac{\sigma_0}{8a^4}.$$

11. Consider the boundary conditions on the plane $z=0$:

$$V = 0, \quad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_1/\rho)e^{i\phi}, \quad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = 2\pi(\sigma_1/\rho)\ e^{i\phi}[(l_2^2 - a^2)^{1/2} - z]$,

$$\sigma(\rho,\phi) = (\sigma_1/\rho)e^{i\phi}\Re[1 - a/(a^2 - \rho^2)^{1/2}].$$

12. Consider the boundary conditions on the plane $z=0$:

$$V = 0, \quad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_2/\rho^2)e^{2i\phi}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \pi(\sigma_2/\rho^2)\ e^{2i\phi}[(l_2^2 - a^2)^{1/2} - 2z + z(a^2 - l_1^2)^{1/2}/a]$,

$$\sigma(\rho,\phi) = (\sigma_2/\rho^2)e^{i\phi}\Re\left[1 - \frac{2a^2 - \rho^2}{2a(a^2 - \rho^2)^{1/2}}\right].$$

13. Consider the boundary conditions on the plane $z=0$:

$$V = 0, \qquad \text{for } \rho\le a, \quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_3/\rho^3)e^{3i\phi}, \qquad \text{for } \rho>a, \quad 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \dfrac{3\pi\sigma_3}{4\rho^3}\ e^{3i\phi}\Bigg[(l_2^2 - a^2)^{1/2} - \dfrac{8}{3}z$

$$+ 2\frac{z}{a}\ (a^2 - l_1^2)^{1/2} - \frac{1}{3}z\left(\frac{(a^2 - l_1^2)^{1/2}}{a}\right)^3\Bigg],$$

$$\sigma(\rho,\phi) = \frac{\sigma_3}{\rho^3}\ e^{3i\phi}\Re\left[1 - \frac{3a}{8(a^2 - \rho^2)^{1/2}} - \frac{3(a^2 - \rho^2)^{1/2}}{4a} + \frac{(a^2 - \rho^2)^{3/2}}{8a^3}\right].$$

14. Prove that the total charge Q_T in Problem 2 (1.5.20) can be expressed directly in terms of the given charge density σ as

$$Q_T = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty}\sigma(\rho,\phi)\ \cos^{-1}\left(\frac{a}{\rho}\right)\ \rho d\rho d\phi.$$

Hint: integrate (1.5.23).

15. Solve the problem above in the case when $\sigma=\sigma_0/\rho^n$.

Answer: $Q_T = \dfrac{2\sigma_0\sqrt{\pi}\ \Gamma[(n - 1)/2]}{(n - 2)\ \Gamma(n/2)a^{n-2}}$.

1.6 Some fundamental integrals

The integrals are called fundamental because of their primary importance to the new method, and also because almost all the integral representations, derived earlier, are just particular cases of the fundamental ones to be evaluated here. Consider three points in the system of cylindrical coordinates, namely, $M(\rho,\phi,z)$, $M_0(\rho_0,\phi_0,z_0)$, and $N(r,\psi,0)$. The following notation is introduced:

$$l_1(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} - [(\rho - t)^2 + z^2]^{1/2}\} , \tag{1.6.1}$$

$$l_2(t) = \frac{1}{2}\{[(\rho + t)^2 + z^2]^{1/2} + [(\rho - t)^2 + z^2]^{1/2}\} , \tag{1.6.2}$$

$$l_{10}(t) = \frac{1}{2}\{[(\rho_0 + t)^2 + z_0^2]^{1/2} - [(\rho_0 - t)^2 + z_0^2]^{1/2}\} , \tag{1.6.3}$$

$$l_{20}(t) = \frac{1}{2}\{[(\rho_0 + t)^2 + z_0^2]^{1/2} + [(\rho_0 - t)^2 + z_0^2]^{1/2}\}. \tag{1.6.4}$$

According to the earlier convention, l_{10} stands as an abbreviation for $l_{10}(a)$, etc.; $R(\cdot,\cdot)$ denotes the distance between two points.

Consider the following integral:

$$I_1 = \int_0^{2\pi}\int_0^a \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\left[\frac{h_0}{R(N,M_0)}\right] r\mathrm{d}r\mathrm{d}\psi , \tag{1.6.5}$$

where

$$h_0 = [a^2 - l_{10}^2]^{1/2} [a^2 - r^2]^{1/2}/a \tag{1.6.6}$$

Make use of the integral representation (1.1.23)

$$\frac{1}{R(N,M_0)} \tan^{-1}\left[\frac{h_0}{R(N,M_0)}\right] = \int_r^a \frac{\mathrm{d}l_{20}(x)}{[l_{20}^2(x) - \rho_0^2]^{1/2} (x^2 - r^2)^{1/2}} \lambda\left(\frac{\rho_0 r}{l_{20}^2(x)}, \psi-\phi_0\right) \tag{1.6.7}$$

where $\lambda(\cdot,\cdot)$ is defined by (1.1.5).

The substitution of (1.6.7) in (1.6.5) yields, after changing the order of integration:

$$I_1 = \int_0^{2\pi} d\psi \int_0^a \frac{dl_{20}(x)}{[l_{20}^2(x) - \rho_0^2]^{1/2}} \int_0^x \frac{r dr}{(x^2 - r^2)^{1/2}} \lambda\left(\frac{\rho_0 r}{l_{20}^2(x)}, \psi - \phi_0\right) \frac{z}{R^3(M,N)}. \quad (1.6.8)$$

By substituting the integral representation (1.3.7),

$$\frac{z}{R^3(M,N)} \equiv \frac{z}{[\rho^2 + r^2 - 2r\rho \cos(\phi - \psi) + z^2]^{3/2}}$$

$$= \frac{2}{\pi r} \mathcal{L}\left(\frac{1}{r}\right) \frac{d}{dr} \int_0^r \frac{t dt}{(r^2 - t^2)^{1/2}} \frac{[t^2 - l_1^2(t)]^{1/2}}{l_2^2(t) - l_1^2(t)} \lambda\left(\frac{l_1(t)t}{l_2(t)}, \phi - \psi\right) \quad (1.6.9)$$

in (1.6.8), the following result can be obtained, after integration with respect to ψ:

$$I_1 = 4\int_0^a \frac{dl_{20}(x)}{[l_{20}(x) - \rho_0^2]^{1/2}} \int_0^x \frac{dr}{(x^2 - r^2)^{1/2}} \frac{d}{dr} \int_0^r \frac{t dt}{(r^2 - t^2)^{1/2}}$$

$$\times \frac{[t^2 - l_1^2(t)]^{1/2}}{l_2^2 - l_1^2} \lambda\left(\frac{l_1(t)t\rho_0}{l_{20}^2(x)l_2(t)}, \phi - \phi_0\right) \quad (1.6.10)$$

Here the following property of the $\mathcal{L}$-operators was used:

$$\mathcal{L}(k)\ \lambda(k_1, \cdot) = \lambda(kk_1, \cdot)\ , \text{ for } k,\ k_1 < 1 \quad (1.6.11)$$

The well known property of the Abel-type operators, namely,

$$\int_0^x \frac{dr}{(x^2 - r^2)^{1/2}} \frac{d}{dr} \int_0^r \frac{f(t)\ t dt}{(r^2 - t^2)^{1/2}} = \frac{\pi}{2} f(x)\ , \quad (1.6.12)$$

allows us to simplify (1.6.10) significantly:

$$I_1 = 2\pi\int_0^a \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right)dx. \tag{1.6.13}$$

It is noteworthy that the integrand in (1.6.13) is symmetric with respect to the points M and M_0 while it did not look so in the original expression (1.6.5). The integrand in (1.6.13) is a perfect differential, so that the integral can be evaluated as indefinite:

$$\int \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right) dx$$

$$= \frac{1}{2R_1} \tan^{-1} \frac{\Theta_1(x)}{R_1} + \frac{1}{2R_2} \tan^{-1} \frac{\Theta_2(x)}{R_2}, \tag{1.6.14}$$

where

$$R_1 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + (z - z_0)^2]^{1/2},$$

$$R_2 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + (z + z_0)^2]^{1/2},$$

$$\Theta_1(x) = \theta(x) + zz_0/\theta(x), \qquad \Theta_2(x) = \theta(x) - zz_0/\theta(x),$$

$$\theta(x) = [x^2 - l_1^2(x)]^{1/2} [x^2 - l_{10}^2(x)]^{1/2}/x. \tag{1.6.15}$$

Notice that when $z_0=0$, $\theta(x)$ transforms into $h(x)$ as it is defined by (1.3.11), and the integral (1.6.14) reduces to the one considered in Exercise 1.1.8. Correctness of the integral in (1.6.14) can be verified by differentiation. The algebra involved is not trivial. Here we present some intermediate transformations:

$$\frac{\partial}{\partial x}\theta(x) = \frac{\theta(x) [l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)]}{x [l_2^2(x) - l_1^2(x)] [l_{20}^2(x) - l_{10}^2(x)]}, \tag{1.6.16}$$

$$\lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right) = \frac{l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)}{2x^2}\left[\frac{1}{R_1^2 + \Theta_1^2(x)} + \frac{1}{R_2^2 + \Theta_2^2(x)}\right]$$

(1.6.17)

Formula (1.6.14) allows us to evaluate the integral (1.6.5):

$$\int_0^{2\pi}\int_0^a \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\left[\frac{h_0}{R(N,M_0)}\right] r\,\mathrm{d}r\,\mathrm{d}\psi$$

$$= \pi \frac{|z|}{z} \left\{\frac{1}{R_1}\left[\tan^{-1}\left(\frac{\Theta_1}{R_1}\right) - \frac{\pi}{2}\frac{|zz_0|}{zz_0}\right] + \frac{1}{R_2}\left[\tan^{-1}\left(\frac{\Theta_2}{R_2}\right) + \frac{\pi}{2}\frac{|zz_0|}{zz_0}\right]\right\},$$

(1.6.18)

where the contractions Θ_1 and Θ_2 stand for $\Theta_1(a)$ and $\Theta_2(a)$ respectively. Note an important particular case when $z_0=0$. Formula (1.6.18) in this case transforms into:

$$\int_0^{2\pi}\int_0^a \frac{z}{R^3(M,N)} \frac{1}{R(N,N_0)} \tan^{-1}\left[\frac{(a^2 - r^2)^{1/2} (a^2 - \rho_0^2)^{1/2}}{aR(N,N_0)}\right] r\,\mathrm{d}r\,\mathrm{d}\psi$$

$$= \frac{2\pi}{R(M,N_0)} \tan^{-1}\left[\frac{h}{R(M,N_0)}\right], \text{ for } \rho_0<a, \tag{1.6.19}$$

and the integral vanishes when $\rho_0 \geq a$. Here the point N_0 has the cylindrical coordinates $(\rho_0,\phi_0,0)$, and h is defined by (1.4.22).

The second fundamental integral to be considered is:

$$I_2 = \int_0^{2\pi}\int_0^a \frac{z_0}{R^3(N,M_0)} \left[\frac{R(N,M_0)}{h_0} + \tan^{-1}\frac{h_0}{R(N,M_0)}\right] \frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(M,N)},$$

(1.6.20)

where h_0 is defined by (1.6.6). Make use of the integral representation for the reciprocal distance

$$\frac{1}{R(M,N)} = \frac{2}{\pi}\int_0^r \frac{\mathrm{d}x}{(r^2 - x^2)^{1/2}} \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \lambda\left(\frac{l_1(x)x}{l_2(x)r}, \phi-\psi\right),$$

(1.6.21)

which is a variation of (1.1.26). By substituting (1.6.21) in (1.6.20), the following expression results, after changing the order of integration:

$$I_2 = \frac{2}{\pi} \int_0^{2\pi} d\psi \int_0^a \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} dx \int_x^a \frac{r dr}{(r^2 - x^2)^{1/2}}$$

$$\times \lambda\left(\frac{l_1(x)x}{l_2(x)r} , \phi - \psi \right) \frac{z_0}{R^3(N,M_0)} \left[\frac{R(N,M_0)}{h_0} + \tan^{-1} \frac{h_0}{R(N,M_0)} \right] , \qquad (1.6.22)$$

We use the integral representation (1.3.14):

$$\frac{z_0}{R^3(N,M_0)} \left[\frac{R(N,M_0)}{h_0} + \tan^{-1} \frac{h_0}{R(N,M_0)} \right]$$

$$= - \frac{\mathcal{L}(r)}{r} \frac{d}{dr} \int_r^a \frac{t dt}{(t^2 - r^2)^{1/2}} \frac{[l_{20}^2(t) - t^2]^{1/2}}{l_{20}^2(t) - l_{10}^2(t)} \lambda\left(\frac{l_{10}(t)}{t l_{20}(t)} , \phi_0 - \psi \right). \qquad (1.6.23)$$

The substitution of (1.6.23) in (1.6.22) yields, after integration with respect to ψ:

$$I_2 = -4 \int_0^a \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} dx \int_x^a \frac{dr}{(r^2 - x^2)^{1/2}} \frac{d}{dr} \int_r^a \frac{t dt}{(t^2 - r^2)^{1/2}}$$

$$\times \frac{[l_{20}^2(t) - t^2]^{1/2}}{l_{20}^2(t) - l_{10}^2(t)} \lambda\left(\frac{l_1(x) l_{10}(t) x}{l_2(x) l_{20}(t) t} , \phi - \phi_0 \right). \qquad (1.6.24)$$

We recall another well known property of the Abel operators:

$$\int_x^a \frac{dr}{(r^2 - x^2)^{1/2}} \frac{d}{dr} \int_r^a \frac{f(t) t dt}{(t^2 - r^2)^{1/2}} = - \frac{\pi}{2} f(x). \qquad (1.6.25)$$

Application of (1.6.25) to (1.6.24) yields:

$$I_2 = 2\pi \int_0^a \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[l_{20}^2(x) - x^2]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x) l_{10}(x)}{l_2(x) l_{20}(x)} , \phi - \phi_0 \right) dx.$$

(1.6.26)

Note certain similarity between (1.6.26) and (1.6.13). The integrand in (1.6.26) is a perfect differential, and can be evaluated in elementary functions:

$$\int \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[l_{20}^2(x) - x^2]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)} , \phi - \phi_0 \right) dx$$

$$= - \frac{1}{2R_1} \tan^{-1} \frac{\Xi_1(x)}{R_1} - \frac{1}{2R_2} \tan^{-1} \frac{\Xi_2(x)}{R_2} , \qquad (1.6.27)$$

where R_1 and R_2 are defined by (1.6.15), and

$$\Xi_1(x) = \xi(x) + zz_0/\xi(x) , \qquad \Xi_2(x) = \xi(x) - zz_0/\xi(x) ,$$

$$\xi(x) = [l_2^2(x) - x^2]^{1/2} [l_{20}^2(x) - x^2]^{1/2}/x \qquad (1.6.28)$$

Again, one can notice that in the case when z_0=0, $\xi(x)$ transforms into $j(x)$ as it is defined in (1.3.4), and the integral (1.6.27) coincides with (1.1.26). As before, correctness of the integration can be verified by differentiation, using (1.6.17) and the property:

$$\frac{\partial}{\partial x}\xi(x) = - \frac{\xi(x) [l_2^2(x)l_{20}^2(x) - l_1^2(x)l_{10}^2(x)]}{x [l_2^2(x) - l_1^2(x)] [l_{20}^2(x) - l_{10}^2(x)]} , \qquad (1.6.29)$$

Finally, the integral (1.6.20) can be evaluated as follows:

$$\int_0^{2\pi}\int_0^{a} \frac{z_0}{R^3(N,M_0)} \left[\frac{R(N,M_0)}{h_0} + \tan^{-1} \frac{h_0}{R(N,M_0)} \right] \frac{r dr d\psi}{R(M,N)}$$

$$= \pi \frac{|z_0|}{z_0} \left\{ \frac{1}{R_1} \left[\frac{\pi}{2} - \tan^{-1} \frac{\Xi_1}{R_1} \right] + \frac{1}{R_2} \left[\frac{\pi}{2} - \tan^{-1} \frac{\Xi_2}{R_2} \right] \right\}. \qquad (1.6.30)$$

According to our convention, Ξ_1 and Ξ_2 denote $\Xi_1(a)$ and $\Xi_2(a)$ respectively.

Consider a particular case, when z_0=0, and $\rho_0>a$. Due to the relationship

$$\frac{az_0}{(a^2 - l_{10}^2)^{1/2}} \to (\rho_0^2 - a^2)^{1/2}, \text{ for } z_0 \to 0,$$

the integral (1.6.30) will take the form:

$$\int_0^{2\pi}\int_0^a \frac{(\rho_0^2 - a^2)^{1/2}}{(a^2 - r^2)^{1/2}} \frac{r\mathrm{d}r\mathrm{d}\psi}{R(M,N)\ R^2(N_0,N)}$$

$$= \frac{2\pi}{R_0}\left[\frac{\pi}{2} - \tan^{-1}\frac{(l_2^2 - a^2)^{1/2}(\rho_0^2 - a^2)^{1/2}}{aR_0}\right]. \qquad (1.6.31)$$

Here $R_0=R(M,N_0)=[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}$. The integration in (1.6.30) for $z_0=0$ and $\rho_0<a$ yields π^2/R_0. The case of $z=0$ can be considered in a similar manner.

The integrals evaluated above may be called internal because the domain of integration was the interior of a disc. We can also evaluate relevant external integrals. For example, consider the integral

$$I_3 = \int_0^{2\pi}\int_a^{\infty} \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\frac{j_0}{R(N,M_0)}\ r\mathrm{d}r\mathrm{d}\psi, \qquad (1.6.32)$$

where $j_0 = (r^2 - a^2)^{1/2}(l_{20}^2 - a^2)^{1/2}/a.$ (1.6.33)

The integral representations due to (1.1.26) and (1.3.15), namely,

$$\frac{1}{R(N,M_0)} \tan^{-1}\frac{j_0}{R(N,M_0)} = \int_a^r \frac{[l_{20}^2(x) - x^2]^{1/2}\ \mathrm{d}x}{(r^2 - x^2)^{1/2}[l_{20}^2(x) - l_{10}^2(x)]}\ \lambda\left(\frac{l_{10}^2(x)}{\rho_0 r}, \phi_0-\psi\right), \qquad (1.6.34)$$

$$\frac{z}{R^3(M,N)} = -\frac{2}{\pi r}\mathcal{L}(r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^{\infty}\frac{t\mathrm{d}t}{(t^2 - r^2)^{1/2}}\frac{[l_2^2(t) - t^2]^{1/2}}{l_2^2(t) - l_1^2(t)}\ \lambda\left(\frac{\rho}{l_2^2(t)}, \phi-\psi\right), \qquad (1.6.35)$$

can be substituted in (1.6.32) yielding

$$I_3 = -4\int_a^\infty \frac{[l_{20}^2(x) - x^2]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)}\,dx \int_x^\infty \frac{dr}{(t^2 - r^2)^{1/2}} \frac{[l_2^2(t) - t^2]^{1/2}}{l_2^2(t) - l_1^2(t)} \lambda\left(\frac{l_{10}^2(x)\rho}{l_2^2(t)\rho_0}, \phi-\phi_0\right)$$

$$= 2\pi\int_a^\infty \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)} \frac{[l_{20}^2(x) - x^2]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)} \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}, \phi-\phi_0\right) dx. \tag{1.6.36}$$

This is the integral which was already evaluated in (1.6.27), so we may write the final result

$$\int_0^{2\pi}\int_a^\infty \frac{z}{R^3(M,N)} \frac{1}{R(N,M_0)} \tan^{-1}\frac{j_0}{R(N,M_0)}\, r\,dr\,d\psi$$

$$= \pi\,\frac{|z|}{z}\left\{\frac{1}{R_1}\left[\tan^{-1}\left(\frac{\Xi_1}{R_1}\right) - \frac{\pi}{2}\,\frac{|zz_0|}{zz_0}\right] + \frac{1}{R_2}\left[\tan^{-1}\left(\frac{\Xi_2}{R_2}\right) + \frac{\pi}{2}\,\frac{|zz_0|}{zz_0}\right]\right\}, \tag{1.6.37}$$

Comparison with the relevant internal integral (1.6.18) indicates similarity, except for substitution of Θ by Ξ.

The second external integral is

$$I_4 = \int_0^{2\pi}\int_a^\infty \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{j_0} + \tan^{-1}\frac{j_0}{R(N,M_0)}\right]\frac{r\,dr\,d\psi}{R(M,N)}, \tag{1.6.38}$$

where j_0 is defined by (1.6.33). Make use of the integral representations (see Exercise 1.1.8 and (1.3.21))

$$\frac{1}{R(M,N)} = \frac{2}{\pi}\int_r^\infty \frac{dx}{(x^2 - r^2)^{1/2}} \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)} \lambda\left(\frac{l_1(x)r}{l_2(x)x}, \phi - \psi\right), \tag{1.6.39}$$

$$\frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{j_0} + \tan^{-1}\frac{j_0}{R(N,M_0)}\right]$$

$$= \frac{1}{r}\ \mathcal{L}\left(\frac{1}{r}\right)\ \frac{d}{dr}\int_a^r \frac{x\,dx}{(r^2 - x^2)^{1/2}}\ \frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)}\ \lambda\left(\frac{l_{10}(x)x}{l_{20}(x)}\ ,\ \psi-\phi_0\right). \tag{1.6.40}$$

Substitution of (1.6.39) and (1.6.40) in (1.6.38) leads to

$$I_4 = 2\pi\int_a^\infty \frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\ \frac{[x^2 - l_{10}^2(x)]^{1/2}}{l_{20}^2(x) - l_{10}^2(x)}\ \lambda\left(\frac{l_1(x)l_{10}(x)}{l_2(x)l_{20}(x)}\ ,\ \phi-\phi_0\right)dx. \tag{1.6.41}$$

This integral was evaluated in (1.6.14), and the final result is

$$\int_0^{2\pi}\int_a^\infty \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{j_0} + \tan^{-1}\frac{j_0}{R(N,M_0)}\right]\frac{r\,dr\,d\psi}{R(M,N)}$$

$$= \pi\,\frac{|z_0|}{z_0}\left\{\frac{1}{R_1}\left[\frac{\pi}{2} - \tan^{-1}\frac{\Theta_1}{R_1}\right] + \frac{1}{R_2}\left[\frac{\pi}{2} - \tan^{-1}\frac{\Theta_2}{R_2}\right]\right\}. \tag{1.6.42}$$

One can notice the same similarity between the internal (1.6.30) and the external (1.6.42) integrals. The similarity goes further. By using the property

$$(l_2^2(x) - x^2)(x^2 - l_1^2(x)) = x^2z^2,$$

we deduce, that for $zz_0>0$,

$$\Xi_1(x) = \xi(x) + \theta(x), \qquad \Xi_2(x) = \xi(x) - \theta(x),$$

$$\Theta_1(x) = \theta(x) + \xi(x), \qquad \Theta_2(x) = \theta(x) - \xi(x). \tag{1.6.43}$$

This means that $\Xi_1=\Theta_1$ and $\Xi_2=-\Theta_2$ for $zz_0>0$. In the case when $zz_0<0$, the relationships change, namely, $\Xi_1=-\Theta_1$ and $\Xi_2=\Theta_2$.

Exercise 1.6

Introduce the following points: $M(\rho,\phi,z)$, $M_0(\rho_0,\phi_0,z_0)$, $N(r,\psi,0)$, $N_0(\rho_0,\phi_0,0)$,

$P(\rho,\phi,0)$; as before, $R(\cdot,\cdot)$ stands for the distance between two points.

1. Evaluate the integral

$$\int_0^{2\pi}\int_0^{a} \frac{z_0}{R^3(N,M_0)}\left[\frac{R(N,M_0)}{h_0} + \tan^{-1}\frac{h_0}{R(N,M_0)}\right]\frac{r\mathrm{d}r\mathrm{d}\psi}{R(P,N)}, \quad \text{for } \rho>a.$$

Answer: $\dfrac{2\pi}{R(M_0,P)}\left[\dfrac{\pi}{2} - \tan^{-1}\dfrac{(l_{20}^2 - a^2)^{1/2}(\rho^2 - a^2)^{1/2}}{aR(M_0,P)}\right].$

Hint: use (1.6.30) for $z=0$.

2. Evaluate the integral above for $\rho<a$.

Answer: $\pi^2/R(M_0,P)$.

3. Evaluate the integral

$$\int_0^{2\pi}\int_a^{\infty} \frac{z}{R^3(M,N)}\,\frac{1}{R(N,N_0)}\,\tan^{-1}\left[\frac{(r^2 - a^2)^{1/2}(\rho_0^2 - a^2)^{1/2}}{aR(N,N_0)}\right] r\mathrm{d}r\mathrm{d}\psi.$$

Answer: $\dfrac{2\pi}{R(M,N_0)}\tan^{-1}\left[\dfrac{(\rho_0^2 - a^2)^{1/2}(l_2^2 - a^2)^{1/2}}{aR(M,N_0)}\right].$

4. Evaluate the integral

$$\int_0^{2\pi}\int_a^{\infty} \frac{(a^2 - \rho_0^2)^{1/2}}{(r^2 - a^2)^{1/2}}\,\frac{r\mathrm{d}r\mathrm{d}\psi}{R(M,N)\ R^2(N_0,N)}$$

Answer: $\dfrac{2\pi}{R(M,N_0)}\left[\dfrac{\pi}{2} - \tan^{-1}\dfrac{(a^2 - l_1^2)^{1/2}(a^2 - \rho_0^2)^{1/2}}{aR(M,N_0)}\right].$