CHAPTER 7

SINGULAR INTEGRAL EQUATIONS

7.1. Approximate solution of singular integral equations

A method for obtaining the approximate solution of singular integral equations of the first and second kind is suggested. The solution is represented in the form of power series with undetermined coefficients multiplied by a function in which the essential features of the singularity of the solution are preserved. The method of collocations is used to determine the unknown coefficients. The examples show that the method suggested is more general and gives good results even in the case when the form of solution does not exactly preserve the essential features of singularity. The method is simpler than others which use the properties of orthogonal polynomials, and is applicable for the solution of single equations as well as systems of simultaneous equations.

Singular integral equations have numerous applications in many branches of mechanics as well as in electrostatics. Such applications include problems of plane flow around an arc, problems of rock mechanics, composite materials and layered media containing cuts, crack propagation in elastic plates, contact problems of the plane theory of elasticity with or without friction, diffusion and wave diffraction problems in media containing geometric or physical singularities and aerofoil problems, etc.

The main results concerning exact solutions of singular integral equations and methods of reducing them to a Fredholm integral equation with continuous and bounded kernel are presented by Muskhelishvili (1953). But even when the exact solution does exist, it is difficult to obtain numerical results because such a solution has the form of a singular integral which is often not suitable for numerical integration. Therefore approximate methods of solution of singular integral equations are of great importance. In 1941 Hildebrand suggested an approximate solution of singular integral equations in the form of a sum of power series with unknown coefficients and a function with prescribed singular properties. Here, the coefficients were determined by the method of least squares. This method is simple but not sufficiently accurate. In more general terms the approximate solution of singular integral equations was also discussed in Ivanov (1956) and Kalandiia (1959). Useful results were obtained by using the orthogonal polynomials properties in Karpenko (1966), Popov (1966) and Erdogan (1969). This method is accurate but is more complicated than Hildebrand's (1941) since it requires a good knowledge of special functions such as Chebyshev and Jacobi polynomials. The finite element approach (D. Q. Dang, D. H. Norrie 1979) to the solution of a singular integral equation proved to be simple but it requires many points of collocations to obtain good accuracy and it fails for example in the case when the stress intensity factor is to be calculated.

The method proposed in this section differs form that of Hildebrand on two points namely: (a) instead of summing, the solution of a singular integral equation is sought in the form of a product of a power series with unknown coefficients and a function with the prescribed singular properties, and (b) the method of collocation is used for the determination of unknown coefficients. These two important differences make the method as accurate as the one using the properties of the orthogonal polynomials and much simpler than any of the other methods available for generating numerical results.

Description of the method. We consider the singular integral equation

$$a\, f(x) + b \int_{-1}^{1} \frac{f(t)\, dt}{t-x} + \int_{-1}^{1} k(x,t)\, f(t)\, dt = G(x). \qquad (7.1.1)$$

Here a and b are constants, $k(x,t)$ is bounded, and $G(x)$ satisfies all the necessary conditions (see Muskhelishvili, 1953) in the interval $-1 \le x \le 1$.

In the case when the solution $f(x)$ at $x = \pm 1$ is unbounded but integrable, one can assume the solution of (1) to take the form

$$f(x) = \sum_{n=0}^{N} \frac{C_n x^n}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}}. \qquad (7.1.2)$$

Here $\alpha = \pi^{-1}\cot^{-1}(\pi b/a)$.

Substitution of solution (2) into (1) yields after integrating

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l\, x^{n-l-1} + \sum_{n=0}^{N} C_n \phi_n(x) = G(x). \qquad (7.1.3)$$

Here the following integrals (Gradshtein and Ryzhik, 1965) have been employed

$$\int_{-1}^{1} \frac{x^k\,dx}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}} = (-1)^k B(\frac{1}{2}-\alpha, \frac{1}{2}+\alpha)\, F(-k, \frac{1}{2}-\alpha; 1; 2),$$

$$\int_{-1}^{1} \frac{dx}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}(x-y)} = -\frac{\pi \tan(\pi\alpha)}{(1+y)^{1/2+\alpha}(1-y)^{1/2-\alpha}},$$

where B denotes the Beta-function; and F denotes the Gauss hypergeometric function. Also

$$\phi_n(x) = \int_{-1}^{1} \frac{k(x,t)t^n\,dt}{(1+t)^{1/2+\alpha}(1-t)^{1/2-\alpha}}, \quad B_l = \frac{\pi(-1)^l}{\cos\pi\alpha} \sum_{m=0}^{l} \frac{(-l)_m\,(\frac{1}{2}-\alpha)_m}{m!\; m!}\, 2^m.$$

Here $(z)_m = z(z+1)\ldots(z+m-1)$; $(z)_0 = 1$.

When the numerical results are generated, the following approach is recommended for calculating $\phi_n(x)$

$$\int_{a}^{b} \frac{f(t)\,dt}{(a-t)^\beta} = \int_{b}^{a-\varepsilon} \frac{f(t)\,dt}{(a-t)^\beta} + \int_{a-\varepsilon}^{a} \frac{f(t)\,dt}{(a-t)^\beta} \approx \int_{b}^{a-\varepsilon} \frac{f(t)\,dt}{(a-t)^\beta} + f(a)\frac{\varepsilon^{1+\beta}}{1+\beta}.$$

This approach permits the avoidance of singularities, and gives accurate results when ε is small.

Now the interval $-1 \le x \le 1$ can be divided into N arbitrary finite sections and the coefficients C_n can be chosen in such a way that (3) will become an identity at all the nodal points x_0, $x_1, \ldots$, x_N. This leads to the following system of $N+1$ linear algebraic equations for the determination of the unknown coefficients C_n

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l\, x_i^{n-l-1} + \sum_{n=0}^{N} C_n \phi_n(x_i) = G(x_i), \quad i = 0,1,\ldots,N. \qquad (7.1.4)$$

When $a=0$, (1) represents the integral equation of the first kind and therefore $\alpha = 0$. Eq. (4) will then simplify to

$$b\sum_{n=1}^{N} C_n \sum_{l=0}^{[(n+1)/2]} \sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)}\, x_i^{\,n-2l-1} + \sum_{n=0}^{N} C_n \phi_n(x_i) = G(x_i),$$
$$i = 0,1,\ldots,N.$$

Here $[r]$ denotes an integer not exceeding r.

When the solution of (1) equals zero at $x = \pm 1$, one can use the following form of approximate solution

$$f(x) = (1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}\sum_{n=0}^{N} C_n\, x^n. \tag{7.1.5}$$

Substitution of solution (5) into (1) yields

$$\frac{\pi b}{\cos\pi\alpha}\sum_{n=0}^{N} C_n\left[\left(\frac{1}{2}-2\alpha^2\right)\sum_{l=0}^{n-1} D_l x^{n-l-1} + x^n(2\alpha - x)\right] + \sum_{n=0}^{N} C_n \psi_n(x) = G(x). \tag{7.1.6}$$

Here the following integrals have been employed

$$\int_{-1}^{1} (1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}x^k\,\mathrm{d}x = 2(-1)^k B\!\left(\frac{3}{2}+\alpha, \frac{3}{2}-\alpha\right) F\!\left(-k, \frac{3}{2}-\alpha; 3; 2\right),$$

$$\int_{-1}^{1} \frac{(1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}}{x-y}\,\mathrm{d}x = \frac{\pi}{\cos\pi\alpha}(2\alpha - y) - \pi\tan\pi\alpha\,(1+y)^{1/2-\alpha}(1-y)^{1/2+\alpha}.$$

Further,

$$\psi_n(x) = \int_{-1}^{1} k(x,t)(1+t)^{1/2-\alpha}(1-t)^{1/2+\alpha}\, t^n\,\mathrm{d}t,$$

$$D_l = (-1)^l \sum_{m=0}^{l} \frac{(-l)_m\left(\frac{3}{2}-\alpha\right)_m}{(3)_m m!}\, 2^m. \tag{7.1.7}$$

As previously, it can be stated that $a = 0$ leads to $\alpha = 0$, and the approximate solution will have the form

$$f(x) = \sqrt{1 - x^2} \sum_{n=0}^{N} C_n x^n. \tag{7.1.8}$$

In this case (6) will be simplified to the form

$$b \sum_{n=0}^{N} C_n \sum_{l=0}^{[(n+1)/2]} D_{2l} x^{n+1-2l} + \sum_{n=0}^{N} C_n \psi_n(x) = G(x). \tag{7.1.9}$$

All the D_k become zero when k is odd, and for even values of k, D_k becomes

$$D_{2l} = + \frac{\sqrt{\pi}}{2} \frac{\Gamma(l - 1/2)}{\Gamma(l + 1)}.$$

By substituting x_0, x_1, ..., x_N instead of x in (6) or (9), a system of $N + 1$ linear algebraic equations can be obtained for the determination of the coefficients C_n in the solution of singular integral equations of the first and second kind respectively.

Special consideration is necessary for the case when $k(x, t) = 0$, and the solution is unbounded at the points $x = \pm 1$, because C_0 becomes arbitrary as function $C_0(1 + x)^{-1/2-\alpha}(1 - x)^{-1/2+\alpha}$ is the solution of the homogeneous singular integral equation. Usually one has an additional equation for the determination of C_0, for example,

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = P, \tag{7.1.10}$$

which in the case of contact problems means that the integral of stresses under the die must be equal to the resulting force P.

Substitution of (2) into (10) yields after integration

$$\sum_{l=0}^{N} B_l C_l = P.$$

Hence,

$$C_0 = \frac{1}{B_0}\left[P - \sum_{l=1}^{N} B_l C_l \right]. \tag{7.1.11}$$

For the determination of the remaining N coefficients C_l, instead of (4), there will be a system of N linear algebraic equations, namely,

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{n-1} B_l x_i^{n-l-1} = G(x_i), \quad i = 1,2,...,N. \tag{7.1.12}$$

And in the case $\alpha = 0$ the system of linear algebraic equations takes the form

$$b \sum_{n=1}^{N} C_n \sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\,\frac{\Gamma(l+1/2)}{\Gamma(l+1)} x_i^{n-1-2l} = G(x_i), \quad i = 1,2,...,N. \tag{7.1.13}$$

Simplified approximate solution of singular integral equations. The method described in the previous section is to be used when a very accurate solution is needed in the whole interval $-1 \le x \le 1$ including edge points. However, as is shown in the following sections, the method can be simplified further so that the solution can be obtained readily even without employing a computer. Some accuracy may be sacrificed at points close to the edge points but the solution is sufficiently accurate at all other points.

Suppose the solution of (1) can be represented in the form,

$$f(x) = \sum_{n=0}^{N} \frac{C_n x^n}{\sqrt{1-x^2}}. \tag{7.1.14}$$

Substitution of solution (14) into (1) leads to the following system of $N+1$ linear algebraic equations for the determination of the coefficients C_n,

$$\frac{aC_0}{\sqrt{1-x_i^2}} + \sum_{n=1}^{N} C_n \left[\frac{a x_i^n}{\sqrt{1-x_i^2}} + b \sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\frac{\Gamma(l+1/2)}{\Gamma(l+1)} x_i^{n-1-2l} \right] + \sum_{n=0}^{N} C_n \chi_n(x_i) = G(x_i), \quad i = 0,1,...,N. \tag{7.1.15}$$

Here,

$$\chi_n(x) = \int_{-1}^{1} \frac{k(x,t)t^n \,\mathrm{d}t}{\sqrt{1-t^2}},$$

and the ratio of the gamma functions can be given in the form

$$\sqrt{\pi}\frac{\Gamma(l+1/2)}{\Gamma(l+1)} = \pi\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots(l-\frac{1}{2})}{1\cdot 2\cdot 3\cdots l}, \tag{7.1.16}$$

and i represents the nodes in the division of the interval $-1 \le x \le 1$ into N sections.

For the case $k(x,t)=0$ one additional arbitrary constant is needed to satisfy the additional condition imposed by (10), in which case, the approximate solution is to be represented in the form

$$f(x) = \sum_{n=0}^{N} \frac{C_n x^n}{\sqrt{1-x^2}} + \frac{C}{(1+x)^{1/2+\alpha}(1-x)^{1/2-\alpha}}. \tag{7.1.17}$$

The following formula may be obtained for the determination of C after substituting solution (17) into (10) and integrating.

$$C = \frac{\cos\pi\alpha}{\pi}\left[P - \sum_{l=0}^{[N/2]} C_{2l}\sqrt{\pi}\frac{\Gamma(l+1/2)}{\Gamma(l+1)}\right]. \tag{7.1.18}$$

Here again coefficients C_n are to be determined from the system of $N+1$ linear algebraic equations

$$\frac{aC_0}{\sqrt{1-x_i^2}} + \sum_{n=1}^{N} C_n\left[\frac{a x_i^n}{\sqrt{1-x_i^2}} + b\sum_{l=0}^{[(n-1)/2]} \sqrt{\pi}\frac{\Gamma(l+1/2)}{\Gamma(l+1)} x_i^{n-1-2l}\right] = G(x_i),$$

$$i = 0,1,\ldots,N. \tag{7.1.19}$$

It will be shown later that this approach gives reliable results not only for the case when the solution $f(x)$ is unbounded, but also when $f(x)$ is equal to zero at points $x=\pm 1$. The results are valid for the whole interval except at edge points close to $x=\pm 1$.

In the examples which follow it is shown that when the number of points of collocation is 11, numerical results are accurate enough in 95% of the interval. It is possible to look for a solution bounded at $x = \pm 1$ in the form

$$f(x) = \sqrt{1 - x^2} \sum_{n=0}^{N} C_n x^n. \tag{7.1.20}$$

The system of $N + 1$ linear algebraic equations in this case is

$$a \sum_{n=0}^{N} C_n x_i^n \sqrt{1 - x_i^2} + b \sum_{n=0}^{N} C_n \frac{\sqrt{\pi}}{2} \sum_{l=0}^{[(n+1)/2]} \frac{\Gamma(l - 1/2)}{\Gamma(l + 1)} x_i^{n - 2l + 1}$$

$$+ \sum_{n=0}^{N} C_n [\chi_{n+2}(x_i) - \chi_n(x_i)] = G(x_i), \quad i = 0, 1, ..., N. \tag{7.1.21}$$

from which the coefficients C_n are evaluated.

Example 1. As the first example consider the case where $k(x,t) = 0$ and $G(x) = \text{constant}$. Then,

$$f(x) + \frac{\cot \pi\alpha}{\pi} \int_{-1}^{1} \frac{f(t)\,\mathrm{d}t}{t - x} = E.$$

This equation corresponds to the plane contact problem of an inclined punch acting on an elastic half-plane with friction where $f(x)$ is the stress at the contact area, E is proportional to the angle of inclination and α is dependent on the coefficient of friction and elastic constants of the material of the half-plane.

The exact solution of this equation is given by Muskhelishvili (1953)

$$f(x) = \frac{E \sin\pi\alpha\,(x + 2\alpha) + P(\cos\pi\alpha)/\pi}{(1 + x)^{1/2 + \alpha}(1 - x)^{1/2 - \alpha}}.$$

The approximate solution was represented in (17). The coefficients C_n were determined by solving the systems of equations (19), and C was calculated using (18). The results for the case where $E = 1$, $P = 1$, are shown in Table 7.1.1. The results agree with the exact results within 3%.

Example 2. In this example, we consider the case where $k(x,t) = 0$ and $G(x)$ is a linear function of x, giving

$$f(x)+\frac{\cot\pi\alpha}{\pi}\int_{-1}^{1}\frac{f(t)\,dt}{t-x}=2\alpha-x. \qquad (7.1.22)$$

The exact solution is bounded in $x=\pm 1$ and has the form

$$f(x)=\sin\pi\alpha\,(1+x)^{1/2-\alpha}(1-x)^{1/2+\alpha}.$$

This equation corresponds to the plane contact problem of paraboloidal punch acting on an elastic half-plane.

The approximate solution was sought using equations (17) and (20). In this case, P is fixed and equals $\pi(0.5-2\alpha^2)\tan\pi\alpha$. The results using both methods are shown in Table 7.1.2. The approximate solution using (17) agrees with the exact solution to 3 significant figures except at the point of collocation x_i = 0.95 where the agreement is within 2%. The other approximate solution using (20) agrees with the exact solution within 1% except at the points of collocation of $x_i=\pm 0.95$ where agreement is within 5%.

Table 7.1.1. Comparison of exact and approximate results in Example 1; x_i = points of collocation, C = coefficient (17), C_n = coefficients, f_a = approximate solution, f_e = exact solution.

α	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
	f_e	0.328	0.244	0.257	0.286	0.322	0.365	0.418	0.489	0.598	0.818	1.46
0.1	f_a	0.307	0.244	0.256	0.286	0.321	0.364	0.417	0.489	0.597	0.818	1.41
	C_n	0.247	0.259	–0.062	–0.004	0.064	–0.059	–0.436	0.132	0.820	–0.115	–0.539
	C	0.117										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.2	f_e	–0.438	0.0579	0.231	0.333	0.415	0.493	0.574	0.670	0.801	1.03	1.62
	f_a	–0.458	0.0573	0.229	0.331	0.414	0.491	0.572	0.668	0.796	1.03	1.52
	C_n	0.577	0.355	–0.197	–0.017	0.059	–0.176	–0.635	0.392	1.20	–0.398	–0.840
	C	–0.086										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.3	f_e	–0.923	0.081	0.354	0.491	0.589	0.673	0.754	0.843	0.955	1.14	1.54
	f_a	–0.937	0.078	0.351	0.488	0.586	0.670	0.751	0.839	0.949	1.13	1.45
	C_n	0.810	0.320	–0.345	–0.026	–0.019	–0.192	–0.461	0.421	0.872	–0.383	–0.686
	C	–0.14										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.4	f_e	–0.614	0.395	0.628	0.733	0.803	0.859	0.911	0.964	1.03	1.12	1.30
	f_a	–0.625	0.390	0.625	0.731	0.801	0.857	0.908	0.961	1.02	1.11	1.26

	C_n	0.952	0.187	–0.458	–0.020	–0.085	–0.115	–0.270	0.250	0.496	–0.233	0.477
	C	–0.095										
	x_i	–0.95	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8	0.95
0.49	f_e	0.771	0.929	0.961	0.975	0.983	0.990	0.995	1.00	0.101	1.02	1.03
	f_a	0.769	0.928	0.961	0.974	0.983	0.989	0.995	1.00	0.101	1.02	1.03
	C_n	1.00	0.0196	–0.500	–0.002	–0.109	–0.012	–0.199	0.026	0.354	–0.025	–0.394
	C	–0.01										

Example 3. As an illustration on the applicability of the simplified method to obtain an approximate solution for singular integro-differential equations, the following is considered.

$$f(x) + \frac{\cot \pi\alpha}{\pi} \int_{-1}^{1} \frac{f'(t)\,\mathrm{d}t}{t - x} = 10 \cos x.$$

Table 7.1.2. Comparison of exact and two approximate solutions in Example 2; x_i = points of collocation, f_e = exact solution, C_n, C = coefficients, f_{a1} = approximate solution (eq.(17)), f_{a2} = approximate solution (eq.(20))

α	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.139	0.231	0.284	0.308	0.315	0.309	0.291	0.260	0.215	1.49	0.0669
	f_{a1}	0.141	0.231	0.284	0.308	0.315	0.309	0.291	0.260	0.215	0.149	0.0668
0.1	C_{n1}	0.320	–0.064	–0.303	0.040	–0.0034	–0.0095	0.0067	–0.0027	–0.014	0.0087	0.0072
	f_{a2}	0.132	0.230	0.283	0.307	0.314	0.308	0.290	0.259	0.214	0.148	0.063
	C_{n2}	0.308	–0.062	0.0018	–0.016	0.061	–0.055	–0.291	0.113	0.556	–0.119	–0.336
	C	–0.011										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.382	0.547	0.620	0.638	0.625	0.588	0.531	0.455	0.356	0.227	0.088
	f_{a1}	0.389	0.547	0.621	0.638	0.625	0.588	0.531	0.455	0.356	0.227	0.089
0.2	C_{n1}	0.600	–0.24	–0.539	0.147	–0.019	0.041	0.016	–0.023	–0.040	0.046	0.012
	f_{a2}	0.360	0.544	0.617	0.636	0.622	0.586	0.529	0.453	0.354	0.226	0.083
	C_{n2}	0.586	–0.235	0.041	–0.067	0.101	–0.199	–0.316	0.405	0.587	–0.465	–0.254
	C	–0.012										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	0.758	0.938	0.981	0.956	0.895	0.809	0.702	0.575	0.427	0.251	0.0842
	f_{a1}	0.771	0.938	0.981	0.956	0.895	0.809	0.702	0.575	0.427	0.251	0.0861
0.3	C_{n1}	0.881	–0.491	–0.661	0.289	–0.049	0.096	0.0002	–0.076	–0.024	0.124	0.028
	f_{a2}	0.716	0.934	0.976	0.953	0.853	0.807	0.700	0.573	0.425	0.250	0.080
	C_{n2}	0.807	–0.485	0.138	–0.144	0.163	–0.532	–0.103	1.16	0.016	–1.32	0.462
	C	–0.008										

	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	1.29	1.37	1.32	1.22	1.10	0.951	0.792	0.621	0.437	0.237	0.0686
	f_{a1}	1.30	1.37	1.32	1.22	1.10	0.951	0.792	0.621	0.437	0.237	0.0705
0.4	C_{n1}	0.955	–0.764	–0.645	0.416	–0.086	0.176	–0.058	–0.189	0.078	0.262	–0.149
	f_{a2}	1.24	1.37	1.32	1.22	1.09	0.950	0.791	0.620	0.436	0.236	0.0663
	C_{n2}	0.950	–0.763	0.297	–0.195	2.20	–1.45	0.661	0.342	–2.02	–3.56	2.58
	C	–0.004										
	x_i	–0.95	–0.80	–0.60	–0.40	–0.20	0.0	0.20	0.40	0.60	0.80	0.95
	f_e	1.88	1.76	1.58	1.39	1.19	1.00	0.803	0.605	0.405	0.204	0.0518
	f_{a1}	1.88	1.76	1.58	1.39	1.19	1.00	0.803	0.605	0.405	0.204	0.0522
0.5	C_{n1}	1.00	–0.979	–0.520	0.479	–0.108	0.273	–0.179	0.374	0.314	0.450	–0.362
	f_{a2}	1.87	1.76	1.58	1.39	1.19	0.999	0.803	0.605	0.405	0.204	0.0515
	C_{n2}	0.999	–0.989	0.487	–0.123	0.047	–3.41	3.06	8.37	–7.75	–8.05	7.51
	C	0.0003										

The approximate solution of the equation is sought in the form

$$f(x) = \sqrt{1-x^2}\sum_{n=0}^{N} C_n x^n.$$

The coefficients C_n were determined from the system of linear algebraic equations,

$$\sum_{n=0}^{N} C_n x_i^n \sqrt{1-x_i^2} + \frac{\cot \pi\alpha}{\pi}\sum_{n=0}^{N} C_n \sum_{l=0}^{[n/2]} \sqrt{\pi}\left(\frac{n+1}{2} - l\right)\frac{\Gamma(l-1/2)}{\Gamma(l+1)}\, x_i^{n-2l} = 10\cos x_i.$$

The exact solution of the singular integral equation under consideration is not known at this time. As a partial check on the validity of the method two different sets of points of collocations $N = 10$ and $N = 20$ were used in the calculations in order to investigate the convergence of the procedure. The results are presented in Table 7.1.3. A comparison of f_1 and f_2 proves good convergence of the procedure proposed.

Table 7.1.3. Convergence of two approximate solutions of singular integro-differential equations in Example 3; x_i = points of collocations, f_1 = solution for $N = 20$, f_2 = solution for $N = 10$.

α	x	–0.95	–0.80	–0.60	–0.40	–0.2	0.0	0.20	0.40	0.60	0.80	0.95
0.1	f_1	–1.06	–2.18	–3.10	–3.71	–4.06	–4.18	–4.06	–3.71	–3.10	–2.18	–1.06
	f_2	–1.06	–2.18	–3.10	–3.71	–4.06	–4.18	–4.06	–3.71	–3.10	–2.18	–1.06
0.2	f_1	–4.34	–9.15	–13.26	–16.03	–17.67	–18.21	–17.67	–16.03	–13.26	–9.15	–4.34
	f_2	–4.35	–9.15	–13.26	–16.03	–17.67	–18.22	–17.67	–16.03	–13.26	–9.15	–4.35

0.3	f_1	14.50	32.17	48.48	60.05	67.10	69.47	67.10	60.05	48.48	32.17	14.50
	f_2	14.53	32.15	48.46	60.03	67.07	69.45	67.07	60.03	48.46	32.15	14.53
0.4	f_1	1.41	4.66	9.92	15.26	19.24	20.71	19.24	15.26	9.92	4.66	1.41
	f_2	1.42	4.66	9.92	15.26	19.24	20.71	19.24	15.26	9.92	4.66	1.42
0.49	f_1	11.75	6.94	8.49	9.48	10.09	10.30	10.09	9.48	8.49	6.94	11.75
	f_2	8.70	6.98	8.55	9.48	10.10	10.30	10.10	9.48	8.55	6.98	8.70

Tables 7.1.1 and 7.1.2 show that the simplified method gives good results with a small number (eleven) of points of collocations. It is noteworthy that results are accurate even in the case when the form of the approximate solution (17) contradicts the condition that the solution is to be bounded in $x = \pm 1$, as it was in (22). The approximate solution (17) contains a singularity of ∞ at the edges $x = \pm 1$ whereas (22) asks for a singularity of 0 at the edges. Values of C calculated for a wide range of α are very small and may be neglected (Table 7.1.2). The simplified method can be use easily by practicing engineers. The method also proved to be stable with respect to the choice of the points of collocations. When good accuracy is necessary at the points very close to $x = \pm 1$, the more exact general approach (4) must be applied.

7.2. One-dimensional integro-differential equations

A method for the numerical solution of singular integro-differential equations is proposed. The approximate solution is sought in the form of the sum of a power series with unknown coefficients multiplied by a special term which controls the appropriate solution behavior near and at the edges of the interval. The coefficients are to be determined from a system of linear algebraic equations. The method is applied to the solution of a contact problem of a disk inserted in an infinite elastic plane. Exact analytical solution is obtained for the particular case when the disk is of the same material as the plane. Comparison is made between the exact and the approximate solution as well as with the solutions previously available in literature. The stability and the accuracy of the present method is investigated under variation of the parameters involved. The application of the method to the case when the boundary conditions for the unknown function are nonzero is discussed along with an illustrative example. A FORTRAN subroutine for the numerical solution of singular integro-differential equations is also provided.

Applications of singular integro-differential equations in various branches of mechanics are well known. Among these are elastic contact problems, stresses in composite materials, airfoil problems, etc. The exact solution of these equations is usually not available and in such cases approximate methods have been commonly used. As early as in 1938 Multhopp suggested a method using Chebyshev polynomials and applied it to the solution of the Prandtl type equation. Later the convergence of Multhopp's method was established by Kalandiia (1957). Methods for the reduction of singular equations to systems of linear algebraic equations by use of different approximate integration rules of Gauss, Radau, Lobatto, etc., were developed by Krenk (1975), Theocaris and Ioakimidis (1979). Methods using the properties of Chebyshev and Jacobi polynomials to obtain the exact evaluation of singular integrals were suggested by Morar and Popov (1976), and Erdogan (1969). In these investigations, the least square method is employed to obtain a system of linear algebraic equations for the determination of the unknown coefficients.

The method proposed in this section uses a simpler power series formulation instead of orthogonal polynomials and subsequently the collocation method (instead of the least square method) for obtaining the system of equations. Such a procedure simplifies the calculations considerably without sacrificing accuracy.

Description of the method. We consider the singular integro-differential equation

$$A(x)f(x) + A_1(x)f'(x) + B(x)\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{t-x} + B_1(x)\int_{-1}^{1}\frac{f'(t)\mathrm{d}t}{t-x}$$

$$+\int_{-1}^{1}K(x,t)\,f(t)\,\mathrm{d}t + \int_{-1}^{1}K_1(x,t)f'(t)\,\mathrm{d}t = G(x) \quad (-1\le x\le 1) \qquad (7.2.1)$$

$$f(-1) = \mathrm{f}(1) = 0.$$

Here A, A_1, B, B_1, G are known continuous and differentiable functions; K, K are known kernels, continuous with respect to both variables x and t, and may have week singularities with respect to t at the points t=±1. For simplicity equation (1) may be written as follows:

$$Lf(t) = G(x) \quad (-1\le x,t\le 1). \qquad (7.2.2)$$

Here L denotes the left-hand operator of equation (1).

It is proven correct to use orthogonal polynomials for the solution of singular integro-differential equations. Since any linear combination of orthogonal polynomials is a polynomial of general form, one may seek an approximate solution of (1) in the form

$$f(x) = S_\delta(x)\sum_{n=0}^{N} C_n x^n, \qquad (7.2.3)$$

where $S_\delta(x) = (1+x)^{1/2-\delta}(1-x)^{1/2+\delta}$, and C_n are coefficients to be determined. In general, $-0.5 < \delta < 0.5$; in the particular case when $A_1(x)/B_1(x)$=const the value of δ is

$$\delta = \frac{1}{\pi}\tan^{-1}\left(\frac{A_1}{\pi B_1}\right). \tag{7.2.4}$$

The influence of the value of δ on the accuracy of the solution is investigated further. The solution in the form (3) provides zero boundary values of the function f. The case of $f(\pm 1) \neq 0$ will be considered later.

Substitution of (3) into (1) yields

$$\sum_{n=0}^{N} C_n Y_n(x) = G(x)$$

where

$$Y_n(x) = A(x)S_\delta(x)x^n + [A_1(x) - \pi B_1(x)\tan(\pi\delta)]\,\frac{n - 2\delta x - (n-1)x^2}{(1+x)^{1/2+\delta}(1-x)^{1/2-\delta}}\,x^{n-1}$$

$$+ B(x)\,\phi_n(x) + B_1(x)\,\xi_n(x) + R_n(x), \tag{7.2.5}$$

where

$$\phi_n(x) = \int_{-1}^{1} \frac{t^n(1+t)^{1/2-\delta}(1-t)^{1/2+\delta}}{t-x}\,\mathrm{d}t = \sum_{k=0}^{n-1} D_k x^{n-1-k}$$

$$- \frac{\pi}{\cos\pi\delta}(x - 2\delta)\,x^n - \pi x^n S_\delta(x)\tan\pi\delta$$

$$\phi_n'(x) = \int_{-1}^{1} \frac{[t^n(1+t)^{1/2-\delta}(1-t)^{1/2+\delta}]'}{t-x}\,\mathrm{d}t, \tag{7.2.6}$$

$$\xi_n(x) = \sum_{k=0}^{n-2} (n-1-k)\,D_k\,x^{n-2-k}, \tag{7.2.7}$$

$$D_k = \frac{\pi(1-4\delta^2)}{2\cos\pi\delta}\,{}_2F_1(-k, \tfrac{3}{2}+\delta, 3; 2), \tag{7.2.8}$$

$$R_n(x) = \int_{-1}^{1} K(x,t)\, S_\delta(t) t^n \,\mathrm{d}t + \int_{-1}^{1} K_1(x,t)[S_\delta(t)t^n]'\,\mathrm{d}t.$$

As usual, ${}_2F_1$ denotes the Gauss hypergeometric function; since $(-k)$ is an integer and negative, ${}_2F_1$ represents a polynomial in δ of the highest power equal to k. When the ratio $A(x)/B(x)=\text{const}$, and the value of δ is defined by (4), equation (5) simplifies since the second term vanishes. When δ equals zero, equation (8) yields

$$D_k = 0 \quad \text{for odd} \quad k = 2m+1$$

and

$$D_k = \frac{\sqrt{\pi}}{2}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+2)}, \quad \text{for even} \quad k = 2m. \tag{7.2.9}$$

The solution involves the choice of an appropriate set of points of collocations x_i, $i=0,1,...,N$, such that equation (5) becomes an identity at every point x_i; hence the following system of $N+1$ linear algebraic equations will be obtained for the determination of the coefficients C_n:

$$\sum_{n=0}^{N} C_n Y_n(x_i) = G(x_i) \quad (i=0,1,...N). \tag{7.2.10}$$

The approximate solution $f(x)$ can be obtained from equation (3) once values of the coefficients C_n are known.

Assessment of accuracy. The accuracy of the approximate solutions is easily evaluated when the exact solution is known. However, the exact solution is usually not available and therefore it is necessary to evaluate the accuracy of the solution indirectly. We consider the error function

$$q(x) = \sum_{n=0}^{N} C_n Y_n(x) - G. \tag{7.2.11}$$

One can determine the average square error as

$$Q_s = \left\{ \int_{-1}^{1} [q(x)]^2 \, dx \right\}^{1/2}, \tag{7.2.12}$$

the maximum absolute error as

$$Q_a = \max \left| q(x) \right| \quad (-1 \le x \le 1), \tag{7.2.13}$$

and the maximum relative error as

$$Q_r = \max \left\{ \frac{|q(x)|}{\sum_{n=0}^{N} \left| C_n Y_n(x) \right| + \left| G(x) \right|} \right\} \quad (-1 \le x \le 1). \tag{7.2.14}$$

It is obvious that $q(x) \equiv 0$ means that the solution is exact but it is not at all obvious that one approximate solution $f_{a1}(x)$ is more accurate than another solution $f_{a2}(x)$ if, for example, $Q_{a1} < Q_{a2}$, i.e. this inequality does not imply $\max \left| f_e(x) - f_{a1}(x) \right| < \max \left| f_e(x) - f_{a2}(x) \right|$ where f_e denotes exact solution. That is why it is necessary to introduce several error parameters. If all three error parameters Q_s, Q_a, Q_r are decreasing simultaneously, that means a more and more accurate solution is obtained in most cases. The utilization of these error parameters will be illustrated through examples in later sections.

Application of the method. Here, the method described is applied to a plane contact problem. Exact solution to the problem is obtained in a particular case. The dependence of the accuracy of the approximate solution on variation of δ and location of the points of collocations is discussed.

Consider a disk of radius ρ_2 made of a material with elastic modulus E_2 and Poisson's ratio v_2. The disk is inserted in a hole of radius ρ_1 in an infinite elastic plane (Fig. 7.2.1) with elastic modulus E_1 and Poisson's ratio v_1. The difference $\varepsilon = \rho_1 - \rho_1$ is assumed to be of the same order as the elastic displacements; therefore one may assume that $\rho_1 = \rho_2 = \rho$ and that the polar angle of the hole ζ_1 is equal to the one of the disk $\zeta_2 = \zeta_1 = \zeta$. A force P_0 and a twisting moment M_0 are applied at the centre of the disk. It is assumed that in the contact region $-\alpha \le \zeta \le \alpha$ the normal radial stresses $\sigma(\zeta)$ and tangential stresses $\tau(\zeta)$ both exist, and the following relationship between them is valid: $\tau(\zeta) = \lambda\sigma(\zeta)$, where λ is the friction coefficient and is taken as a constant. It is appropriate to orient the polar axis Or so that it passes through the centre of contact region. Let P_r and P_t be the radial and tangential components of the force P_0, respectively. The angle between P_0 and P_r is denoted by β. For prescribed M_0, P_0, ρ, ε and elastic properties, the problem is to find the stress

Fig. 7.2.1. Plane contact problem of a disk inserted in a plate

distribution σ in the contact region and the size of the contact region 2α.

This problem was solved in (Morar’ and Popov, 1976) by the orthogonal polynomial method. Here it will be shown that the solution to the problem by the method proposed here leads to much simpler calculations without sacrifices of accuracy. According to their results, the following singular integro-differential equation is to be solved:

$$-\gamma_1\sigma(x) + \gamma_1\lambda\frac{1+a^2x^2}{2a}\sigma'(x) - 2\lambda\int_{-1}^{1}\frac{\sigma(t)\,dt}{t-x} - \frac{1+a^2x^2}{a}\int_{-1}^{1}\frac{\sigma'(t)\,dt}{t-x}$$

$$+2\lambda a^2 b_1 - \gamma_2\rho^{-1}\left(P_r\frac{1-a^2x^2}{1+a^2x^2} + P_t\frac{2ax}{1+a^2x^2}\right) - \gamma_2 P_1 = \varepsilon\gamma_4\rho^{-1}. \quad (7.2.15)$$

Here,

$$b_1 = \int_{-1}^{1}\frac{t\sigma(t)dt}{1+a^2t^2}; \quad P_1 = a\int_{-1}^{1}\frac{\sigma(t)dt}{1+a^2t^2}, \quad (7.2.16)$$

$$P_r = 2a\rho\int_{-1}^{1}\frac{1-a^2t^2-2\lambda at}{(1+a^2t^2)^2}\sigma(t)\,dt, \quad (7.2.17)$$

$$P_t = 2a\rho \int_{-1}^{1} \frac{\lambda(1 - a^2t^2) + 2at}{(1 + a^2t^2)^2} \sigma(t)\, \mathrm{d}t, \tag{7.2.18}$$

$$\gamma_1 = 2\pi\gamma_5[(1 - \kappa_1)(1 + \nu_1)E_2 - (1 - \kappa_2)(1 + \nu_2)E_1]$$

$$\gamma_2 = 4\gamma_5[\kappa_1(1 + \nu_1)E_2 + (1 + \nu_2)E_1], \quad \gamma_3 = 2\gamma_5 E_2(1 + \kappa_1)(1 + \nu_1),$$

$$\gamma_4 = 4\pi\gamma_5 E_1 E_2, \quad \gamma_5 = [(1 + \kappa_1)(1 + \nu_1)E_2 + (1 + \kappa_2)(1 + \nu_2)E_1]^{-1},$$

$$x = \frac{1}{a}\tan\frac{\zeta}{2}, \quad a = \tan\frac{\alpha}{2}, \quad \kappa_j = 3 - 4\nu_j, \quad \text{for } j = 1,2.$$

Substitution of the solution (3) into (15) leads to the following system of linear algebraic equations, analogous to (10):

$$\sum_{n=0}^{N} C_n \Big[-\gamma_1 S_\delta(x)\, x_i^n] + \left[\frac{\gamma_1\lambda}{2} + \pi\tan\pi\delta\right]\frac{1 + a^2x^2}{a}\, \frac{n - 2\delta x_i - (n + 1)x_i^2}{(1 + x_i)^{1/2+\delta}(1 - x_i)^{1/2-\delta}}\, x_i^{n-1}$$

$$- 2\lambda\phi_n(x_i) - \frac{1 + a^2x^2}{a}\,\xi_n(x_i) + 2\lambda a^2\eta_{n+1} - 2\,a\gamma_2\Big[(\chi_n - a^2\chi_{n+2}$$

$$- 2\lambda a\chi_{n+1})\frac{1 - a^2x_i^2}{1 + a^2x_i^2} + (\lambda\chi_n - \lambda a^2\chi_{n+2} + 2a\chi_{n+1})\frac{2ax_i}{1 + a^2x_i^2}\Big] - \gamma_2 a\eta_n = \varepsilon\gamma_4\rho^{-1},$$

$$\text{for } i = 0,1, \ldots, N. \tag{7.2.19}$$

Here functions S_δ, ϕ_n, ξ_n were determined by (3), (6), and (7), and the parameters η_n and χ_n can be expressed in elementary functions as follows:

$$\eta_n = \int_{-1}^{1} \frac{S_\delta(t)t^n\,\mathrm{d}t}{1 + a^2t^2} = \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{a^{2k+2}} D_{n-2-2k} - \frac{\pi(-a^2)^{-[n/2]-1}}{\cos\pi\delta}[\cos H$$

$$+ a(-a^2)^{-m}\sin H + m(1 + 2\delta) - 1], \tag{7.2.20}$$

$$\chi_n = \int_{-1}^{1} \frac{S_\delta(t)t^n\,\mathrm{d}t}{(1 + a^2t^2)^2} = \sum_{k=0}^{[n/2]-2} \frac{(-1)^k(1 + k)}{a^{2k+4}} D_{n-4-2k} + \left[\frac{n}{2}\right]\frac{\pi(-a^2)^{-[n/2]-1}}{\cos\pi\delta}[\cos H$$

$$+\,a(-a^2)^{-m}\sin H + m(1+2\delta) - 1] + \frac{\pi(-a^2)^{-[n/2]}}{2a\cos\pi\delta}\Bigg[\Bigg(a^{-2m}$$

$$-\frac{1+2\delta}{1+a^2}\Bigg)\sin H + \frac{a}{(-a^2)^m}\,\frac{1+2\delta}{1+a^2}\cos H\Bigg], \qquad (7.2.21)$$

where

$$H = (1+2\delta)\tan^{-1}a = [(1+2\delta)\alpha]/2, \quad m = n - 2[n/2],$$

D_k is determined by (8), and $[n/2]$ denotes a maximum integer not exceeding $n/2$.

The integral in (20) was computed by using formula 3.228 from (Gradshtein and Ryzhik, 1965), and the following representation was used:

$$\frac{1}{1+a^2t^2} = \frac{1}{2}\left[\frac{1}{1-iat} + \frac{1}{1+iat}\right].$$

Expression (21) was derived from (20) by means of differentiation of both sides of (20) with respect to a. The system (19) may be simplified by assuming

$$\delta = -\frac{1}{\pi}\tan^{-1}\left(\frac{\gamma_1\lambda}{2\pi}\right). \qquad (7.2.22)$$

In this case the second term in the system (19) will vanish. Certain simplifications are then possible if $\delta = 0$. Besides obvious simplification (19), the expressions (20) and (21) will also be simplified hence for odd $n = 2l+1$, we have $\eta_n = \chi_n = 0$; and for even $n = 2l$ hold

$$\eta_{2l} = \int_{-1}^{1}\frac{\sqrt{1-t^2}\,t^{2l}}{1+a^2t^2}\,\mathrm{d}t = -\frac{\sqrt{\pi}}{2}\sum_{k=0}^{l-1}\frac{\Gamma(k+1/2)}{\Gamma(k+2)}(-a^2)^{k-l}$$

$$+\,\frac{(-1)^l\pi}{a^{2l+2}}\left(\sqrt{1+a^2} - 1\right), \qquad (7.2.23)$$

$$\chi_{2l} = \int_{-1}^{1}\frac{\sqrt{1-t^2}\,t^{2l}}{(1+a^2t^2)^2}\,\mathrm{d}t = -\frac{\sqrt{\pi}}{2}\sum_{k=0}^{l-2}(k-l+1)\frac{\Gamma(k+1/2)}{\Gamma(k+2)}(-a^2)^{k-l}$$

$$+\frac{(-1)^l\pi}{2a^{2l}}\left[\frac{1}{\sqrt{1+a^2}}-2l\,\frac{\sqrt{1+a^2}-1}{a^2}\right]. \tag{7.2.24}$$

After obtaining the values of coefficients C_n from the system (19), all other parameters can be easily calculated as follows:

$$P_0=\sqrt{P_r^2+P_t^2},\quad M_0=2\lambda\rho^2 a\sum_{n=0}^{N} C_n\,\eta_n,$$

$$P_r=2a\rho\sum_{n=0}^{N} C_n(\chi_n-a^2\chi_{n+2}-2\lambda a\chi_{n+1})$$

$$P_t=2a\rho\sum_{n=0}^{N} C_n(\lambda\chi_n-\lambda a^2\chi_{n+2}+2a\chi_{n+1}),\quad \beta=\tan^{-1}(P_t/P_r). \tag{7.2.25}$$

It is obvious that greater force corresponds to greater contact angle α, and it is obvious that there exists a maximum contact angle α_m called the critical angle which can not be exceeded under action of any force. Its value can be obtained by equating to zero the determinant of the system of equations (19).

Before presentation of numerical results, it is worthwhile to note that in the case when the disk and the plate are made of the same material ($E_1=E_2$, $\nu_1=\nu_2$), equation (15) can be solved exactly as $\gamma_1=0$. Having the exact solution for this particular case, one can estimate the accuracy of the approximate solution.

In the case of $\gamma_1=0$, equation (15) can be transformed as follows:

$$\int_{-1}^{1}\frac{w(t)\,\mathrm{d}t}{x-t}=W(x),\qquad (-1\le x\le 1). \tag{7.2.26}$$

Here

$$w(t)=\frac{1+a^2t^2}{a}\,\sigma'(t)+2\lambda\sigma(t),\qquad \sigma(1)=\sigma(-1)=0. \tag{7.2.27}$$

$$W(x)=d_1\frac{1-a^2x^2}{1+a^2x^2}+d_2\frac{2ax}{1+a^2x^2}-d_3,\quad d_1=\gamma_2\rho^{-1}P_r,\quad d_2=\gamma_2\rho^{-1}P_t, \tag{7.2.28}$$

$$d_3 = \gamma_3 P_1 + \varepsilon\gamma_4\rho^{-1} + a\int_{-1}^{1}\left(1 - \frac{2\lambda at}{1+a^2t^2}\right)\sigma(t)\,dt. \tag{7.2.29}$$

The solution of equation (26) is known (Muskhelishvili, 1953), and has the form

$$w(t) = -\frac{1}{\pi^2\sqrt{1-t^2}}\int_{-1}^{1}\frac{W(x)\sqrt{1-x^2}\,dx}{x-t} + \frac{d_4}{\pi\sqrt{1-t^2}}, \tag{7.2.30}$$

and d_4 is an arbitrary constant. Substitution of (28) into (30) yields

$$w(t) = \frac{d_1 t}{\pi\sqrt{1-t^2}}\left[1 - \frac{2\sqrt{1+a^2}}{1+a^2t^2}\right] + \frac{2d_2}{\pi\sqrt{1-t^2}}\left[\frac{\sqrt{1+a^2}}{1+a^2t^2} - 1\right]$$

$$-\frac{d_3 t}{\pi\sqrt{1-t^2}} + \frac{d_4}{\pi\sqrt{1-t^2}}. \tag{7.2.31}$$

Now equation (27) is an ordinary differential equation with respect to σ, and its exact solution can be written in a standard form:

$$\sigma(x) = \exp[-2\lambda\tan^{-1}(ax)]\left\{\int_{-1}^{x}\frac{aw(t)\exp[2\lambda\tan^{-1}(at)]}{1+a^2t^2}\,dt + d_5\right\}. \tag{7.2.32}$$

From the condition $\sigma(-1)=0$, it follows that $d_5=0$. It is not so easy to define other constants d_k, k=1,2,3, and hence they may be expressed through some integrals of σ. The parameter d_4 is to be determined from the condition $\sigma(1)=0$. This is why Panasiuk and Teplyi (1972) determine the constants d_k approximately using the approximation of the exponents involved in (32) as linear functions. An exact solution for the particular case $\lambda=1$ can be found in (Stippes *et. al.*, 1962).

Here, the exact solution is given for a general case. We introduce the following parameters:

$$Z_1 = \int_{-1}^{1}\frac{\sigma(t)\,dt}{1+a^2t^2}, \quad Z_2 = \int_{-1}^{1}\frac{\sigma(t)\,dt}{(1+a^2t^2)^2},$$

$$Z_3 = \int_{-1}^{1} \frac{\sigma(t)\, t\, \mathrm{d}t}{(1+a^2t^2)^2}, \quad Z_4 = \int_{-1}^{1} \left(1 - \frac{2\lambda a t}{(1+a^2t^2)^2}\right) \sigma(t)\, \mathrm{d}t. \tag{7.2.33}$$

The constants d_k may be expressed through Z_k by comparison of (28), (29), (17), and (18) with (33). The result is

$$d_1 = \gamma_2[(2Z_2 - Z_1)2a - 4\lambda a^2 Z_3],$$

$$d_2 = \gamma_2[4a^2 Z_3 + 2a\lambda(2Z_2 - Z_1)],$$

$$d_3 = \gamma_3 a Z_1 + \varepsilon\gamma_4\rho^{-1} + aZ_4. \tag{7.2.34}$$

Direct calculation of Z_k using (33) and (32) is cumbersome, but we can express Z_k directly through some integrals of the function w, multiplying both parts of (27) by an appropriate term and integrating, with the result:

$$\int_{-1}^{1} \frac{w(t)\, \mathrm{d}t}{1+a^2t^2} = 2\lambda Z_1, \quad \int_{-1}^{1} \frac{w(t)\, t\, \mathrm{d}t}{1+a^2t^2} = -\frac{Z_4}{a},$$

$$\int_{-1}^{1} \frac{w(t)\, t\, \mathrm{d}t}{(1+a^2t^2)^2} = \frac{1}{a}(Z_1 - 2Z_2 + 2\lambda a Z_3),$$

$$\int_{-1}^{1} \frac{w(t)\, \mathrm{d}t}{(1+a^2t^2)^2} = 2\lambda Z_2 + 2aZ_3. \tag{7.2.35}$$

Substituting (31) in (35), we obtain, after integration,

$$2\lambda Z_1 = \frac{d_2}{a}\left[\frac{2+a^2}{1+a^2} - \frac{2}{\sqrt{1+a^2}}\right] + \frac{d_4}{\sqrt{1+a^2}},$$

$$Z_4 = \frac{d_1}{a\sqrt{1+a^2}}\left[1 - \frac{1}{\sqrt{1+a^2}}\right] + \frac{d_3}{a}\left[1 - \frac{1}{\sqrt{1+a^2}}\right],$$

$$\frac{1}{a}(Z_1 - 2Z_2 + 2\lambda a Z_3) = \frac{d_1}{2(1+a^2)^{3/2}}\left[1 - \frac{4+a}{2\sqrt{1+a^2}}\right] - \frac{d_3}{2(1+a^2)^{3/2}},$$

$$2aZ_3 + 2\lambda Z_2 = \frac{d_2}{a\,(1+a^2)^{3/2}}\left[\frac{2+2a^2+(3/4)a^4}{\sqrt{1+a^2}} - 2 - a^2\right] + \frac{d_4(2+a^2)}{2(1+a^2)^{3/2}}. \tag{7.2.36}$$

Now considering together the system of equations (34) and (36), we can eliminate the Z-parameters, and obtain the following system of linear algebraic equations for the determination of the constants d_k, $k = 1,2,3,4$:

$$d_1\left[1 + \frac{a^2\gamma_2}{(1+a^2)^{3/2}}\left(1 - \frac{4+a^2}{2\sqrt{1+a^2}}\right)\right] - d_3\frac{a^2\gamma_2}{(1+a^2)^{3/2}} = 0,$$

$$d_2\left[1 - \frac{\gamma_2}{(1+a^2)^{3/2}}\left(\frac{2+a^2+(1/2)a^4}{\sqrt{1+a^2}} - 2\right)\right] - d_4\frac{a\gamma_2}{(1+a^2)^{3/2}} = 0,$$

$$\frac{d_1}{\sqrt{1+a^2}}\left[-1 + \frac{1}{\sqrt{1+a^2}}\right] - d_2\frac{\gamma_3}{2\lambda}\left(\frac{2+a^2}{1+a^2} - \frac{2}{\sqrt{1+a^2}}\right)$$

$$+\frac{d_3}{\sqrt{1+a^2}} - d_4\frac{\gamma_3 a}{2\lambda\sqrt{1+a^2}} = \varepsilon\gamma_4\rho^{-1},$$

$$d_1 I_1 + d_2 I_2 - d_3 I_3 + d_4 I_4 = 0. \tag{7.2.37}$$

Here the fourth equation was obtained from the condition $\sigma(1) = 0$, and

$$I_1 = \frac{1}{\pi}\int_{-1}^{1}\frac{t}{\sqrt{1-t^2}}\left[1 - \frac{2\sqrt{1+a^2}}{1+a^2t^2}\right]\frac{\exp[2\lambda\tan^{-1}(at)]}{1+a^2t^2}\,\mathrm{d}t,$$

$$I_2 = \frac{2}{\pi a}\int_{-1}^{1}\left[\frac{\sqrt{1+a^2}}{1+a^2t^2} - 1\right]\frac{\exp[2\lambda\tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,\mathrm{d}t,$$

$$I_3 = \frac{1}{\pi}\int_{-1}^{1}\frac{t\exp[2\lambda\tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,\mathrm{d}t,$$

$$I_4 = \frac{1}{\pi}\int_{-1}^{1} \frac{\exp[2\lambda \tan^{-1}(at)]}{\sqrt{1-t^2}\,(1+a^2t^2)}\,\mathrm{d}t. \tag{7.2.38}$$

Now the constants d_k are determined by (37), and together with expressions ((31) and (32) give the exact solution to equation (26). For the case of $\lambda=0$, system (37) is not valid since for this case the first of equations (35) does not determine the value of Z_1 any more; instead, we shall have

$$Z_1 = \int_{-1}^{1} \frac{\sigma(t)\,\mathrm{d}t}{1+a^2t^2} = -\int_{-1}^{1} \frac{w(t)\tan^{-1}(at)\,\mathrm{d}t}{1+a^2t^2}.$$

The exact solution for $\lambda=0$ can be expressed in terms of elementary functions and has the form

$$\sigma(x) = \frac{1}{2\pi\sqrt{1+a^2}}\left\{2d_1\frac{a\sqrt{1-x^2}}{1+a^2x^2} + \left[d_1\left(1-\frac{1}{\sqrt{1+a^2}}\right)\right.\right.$$

$$\left.\left.-\,d_3\right]\ln\frac{|a\sqrt{1-x^2}-\sqrt{1+a^2}|}{a\sqrt{1-x^2}+\sqrt{1+a^2}}\right\}. \tag{7.2.39}$$

Parameters $d_2=d_4=0$, and the remaining d_1 and d_3 are determined from the set of equations

$$d_1\left[1+\frac{a^2\gamma_2}{(1+a^2)^{3/2}}\left(1-\frac{4+a^2}{2\sqrt{1+a^2}}\right)\right]-d_3\frac{a^2\gamma_2}{(1+a^2)^{3/2}}=0,$$

$$d_1\left[\left(\frac{1}{1+a^2}-\frac{1}{\sqrt{1+a^2}}\right)\left(1-\frac{\gamma_3}{2}\ln(1+a^2)\right)-\frac{a^2\gamma_3}{2(1+a^2)}\right]$$

$$+\frac{d_3}{\sqrt{1+a^2}}\left(1-\frac{\gamma_3}{2}\ln(1+a^2)\right)=\varepsilon\gamma_4\rho^{-1}. \tag{7.2.40}$$

Here the following integrals were employed:

$$\int_{-1}^{1} \frac{t \tan^{-1}(at)\,dt}{(1+a^2t^2)\sqrt{1-t^2}} = \frac{\pi \ln(1+a^2)}{2a\sqrt{1+a^2}},$$

$$\int_{-1}^{1} \frac{t \tan^{-1}(at)\,dt}{(1+a^2t^2)^2\sqrt{1-t^2}} = \frac{\pi[\ln(1+a^2)+a^2]}{4a(1+a^2)^{3/2}}.$$

In order to compare the numerical results obtained by our method with those reported Morar' and Popov (1976), the calculations were made using the following values of parameters involved: $\nu_1=\nu_2=0.3$, $(E_2/E_1)=0.5, 1.0, 2.0, \infty$. All calculations were carried out assuming, according to (4), $\delta=-\tan^{-1}(\gamma_1\lambda/2\pi)$. All the results are in excellent agreement with those of Morar' and Popov, except for the value of critical angle of contact. To verify which method gives more accurate results, similar values were computed from the exact solution for $E_2=E_1$ by equating to zero the determinant of the system (37) for $\lambda\neq 0$. The same procedure was performed with the system (40) for $\lambda=0$. The results are presented in Table 7.2.1, where α_e denotes the exact value of the critical angle, α_a is the result of our numerical solution, α_M and α_E are the results of Morar' *et. al.* and Erdogan (1969) respectively.

Table 7.2.1. Comparison of values of the critical angle of contact (in degrees) obtained by our exact and approximate method with those reported by other authors

λ	0.0	0.05	0.10	0.15	0.20	0.30	0.40
α_e	169.66	169.66	169.69	169.73	169.79	169.95	170.18
α_a	169.66	169.66	169.69	169.73	169.79	169.95	170.18
α_M	169.6	–	–	–	170.0	171.2	–
α_E	169.66	170.02	171.01	172.50	174.33	–	–

In order to investigate the influence of δ on the accuracy of approximate solution, a comparison was made of the exact solution σ_e for $E_1=E_2$ (which was calculated by solving system (37), and using (31) and (32)), with approximate solution σ_a, corresponding to different values of δ. Since equation (4) in this case implies that $\delta=0$, the approximate solution for this case is identical to the exact one. As should be expected, the more δ differs from the one given by (4), the greater is the error of the approximate solution. The error parameters introduced in (12)–(14) were proven to be very sensitive to accuracy of the solution. When an approximate solution differs from the exact ones by 30%, the error parameters increase by the order of 10^5 times. A similar picture emerges for the common case $E_1 \neq E_2$, i.e. the best accuracy is obtained with the values of δ prescribed by (4).

It is also of interest to find out how strongly the choice of the points of collocations can affect the accuracy of solution. The test computations were performed with three sets of points: $\{x\}_1=-\cos[(i+1/2)\pi/(N+1)]$, $i=0,1,\ ...,N$; $\{x\}_2=-0.99$, -0.9, -0.7, -0.5, -0.25, 0.0, 0.25, 0.5, 0.7, 0.9, 0.99; $\{x\}_3=-0.95$, -0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8, 0.95. All three error parameters were computed for various values of δ. All the calculations were performed for $E_1=2E_2$, $\lambda=0.4$, $\alpha=150$ degrees. The first set gives slightly better accuracy than the second one. The third set proved to be the least accurate, and its inaccuracy sharply increases when the value of δ does not correspond to the one given by (4).

Non-zero boundary conditions. Here we consider the case of singular integro-differential equation (1), with the following boundary conditions:

$$f(-1) = h_1, \quad f(1) = h_2. \tag{7.2.41}$$

We can look for an approximate solution in the form

$$f(x) = S_\delta(x) \sum_{n=0}^{N} C_n x^n + x(h_2 - h_1)/2 + (h_2 + h_1)/2, \tag{7.2.42}$$

with

$$S_\delta(x) = (1+x)^{(1/2)-\delta}(1-x)^{(1/2)+\delta}, \quad -\frac{1}{2}<\delta<\frac{1}{2}.$$

The solution (42) satisfies the boundary conditions (41) exactly, and its substitution in (1) leads to the solution of the same system of equations (10), with the only one difference: instead of $G(x)$ we should use

$$G^*(x) = G(x) - L\{x(h_2 - h_1)/2 + (h_2 + h_1)/2\}. \tag{7.2.43}$$

We consider as an example the following equation

$$f(x) + 1000\, f'(x) + \int_{-1}^{1} \frac{f(t)\,dt}{t-x} + \int_{-1}^{1} \frac{f'(t)\,dt}{t-x} + \int_{-1}^{1} xtf(t)\,dt = G(x), \qquad (7.2.44)$$

with

$$G(x) = \frac{14}{3} + 2008.4x + 3003x^2 + x^3 + (2x + 4x^2 + x^3)\ln\frac{|1-x|}{|1+x|},$$
$$f(-1) = 0, \quad f(1) = 2.$$

It is easy to verify that the exact solution of (44) is

$$f(x) = x^2 + x^3. \qquad (7.2.45)$$

The approximate solution has the form

$$f(x) = S_\delta(x) \sum_{n=0}^{N} C_n x^n + x + 1. \qquad (7.2.46)$$

Substitution of (46) into (1) leads to the system of equations (10) with

$$G^*(x) = x^3 + 3003x^2 + 2006\tfrac{11}{15}x - 998\tfrac{1}{3} + (x^3 + 4x^2 + x - 2)\ln\frac{|1-x|}{|1+x|}.$$

As in previous example, all the calculations were made for the same three sets of the points of collocations for different values of δ. The qualitative picture is the same: the first set is slightly better than the second, and considerably better than the third one. The highest accuracy is obtained for the value of δ implied by (4), namely,

$$\delta = \frac{1}{\pi}\tan^{-1}(1000/\pi) = 0.499$$

This example proves that the method is effective also for the case of non-zero boundary conditions. The method is more simple than the method of orthogonal polynomials, with no less accuracy in results. The method can use arbitrary points of collocations (not necessarily the roots of orthogonal polynomials), hence, there is no need for roots evaluation. The method allows us to compute the unknown function at arbitrary points directly, without cumbersome interpolation.

The software description. The simplicity of the method allowed us to develop an efficient FORTRAN subroutine for numerical solution of singular integro-differential equations. The procedure makes it possible to find an optimal set of points of collocations and an optimal value of δ. This optimum is indicated by the smallest values of the error parameters Q and by the convergence of the coefficients C_n.

The subroutine solves singular integro-differential equation (1), with zero boundary conditions. The case of non-zero boundary conditions is treated according to (42)–(43). The output of the subroutine gives not only the values of unknown function, but also the values of coefficients C_n, the error parameters Q_s, Q_a, and Q_r, and the abscissae x_a and x_r where the maximum absolute error Q_a and the maximum relative error Q_r occur. The knowledge of coefficients C_n allows us to compute the unknown function at arbitrary point, and convergence of those coefficients is an indirect indication of a good accuracy of the solution. The values of x_a and x_r shows the way of changing the set of points of collocations or indicate the necessity of including an additional point. The subroutine listing follows

```
      SUBROUTINE  IDIFEQ(A,A1,B,B1,R,G,D,X,NX,F,C,QS,QA,QR,XR)
C
C     COMPUTER                     –CDC/SINGLE
C
C     PURPOSE                      –APPROXIMATE  SOLUTION  OF  SINGULAR
C                                   INTEGRO-DIFFERENTIAL  EQUATIONS
C                                   THE  SOLUTION  IS  SOUGHT  IN
C                                   THE  FORM  OF  THE  SUM
C                                   OF  TERMS  X**J*(1–X)**(0.5+D)*(1+X)**(0.5–D),
C                                   0≤J≤NX–1,     –0.5<D<0.5
C
C     USAGE           CALL  IDIFEQ(A,A1,B,B1,R,G,D,X,NX,F,C,QS,QA,QR,XR)
C
C     ARGUMENTS       A,A1,B,B1    –COEFFICIENTS  OF  THE  SINGULAR  EQUATION
C                                   (ARE  TO  BE  DETERMINED
C                                   AS  EXTERNAL  FUNCTIONS)

C                     R            –RESULT  OF  INTEGRATION  OF
C                                   REGULAR  PART  OF  THE
C                                   EQUATION  WITH  THE  WEIGHT
C                                   X**J*(1–X)**(0.5+D)*(1+X)**(0.5–D),
C                                   IS  TO  BE  DETERMINED  AS
C                                   EXTERNAL  FUNCTION  R(J,X)

C                     G            –RIGHT-HAND  PART  OF  THE  EQUATION;
C                                   IS  TO  BE  DETERMINED  AS
C                                   EXTERNAL  FUNCTION  G(X)

C                     D            –POWER  PARAMETER  IN  THE
C                                   SOLUTION  SOUGHT

C                     X            –INPUT  VECTOR  OF  DIMENSION  NX
C                                   (POINTS  OF  COLLOCATIONS
C                                   IN  THE  INTERVAL  –1<X<1

C                     NX           –DIMENSION  OF  VECTORS  X,  C,  F  (NX<41)
```

```
C                       F            –OUTPUT VECTOR OF DIMENSION NX
C                                    (SOLUTION OF EQUATION
C                                    AT THE POINTS X(I))

C                       C            –OUTPUT VECTOR OF DIMENSION NX
C                                    (COEFFICIENTS IN THE
C                                    APPROXIMATE SOLUTION)

C                       QS           –AVERAGE SQUARE ERROR (OUTPUT)

C                       QA           –MAXIMUM ABSOLUTE ERROR (OUTPUT)

C                       QR           –MAXIMUM RELATIVE ERROR (OUTPUT)

C                       XA           –ABSCISSA OF MAXIMUM ABSOLUTE ERROR

C                       XR           –ABSCISSA OF MAXIMUM RELATIVE ERROR
C
C     REMARK                         –THE FOLLOWING PROCEDURES FROM
C                                    LIBRARY IMSLIB ARE USED:
C                                    LEQT2F, IQHSCU, DCSQDU.
C                                    LEQT2F – SUBROUTINE FOR SOLVING
C                                          A SYSTEM OF LINEAR
C                                          ALGEBRAIC EQUATIONS
C                                    IQHSCU –SUBROUTINE FOR
C                                          ONE-DIMENSIONAL QUASI-CUBIC
C                                          HER–ITE INTERPOLATION
C                                    DCSQDU – SUBROUTINE FOR CUBIC
C                                          SPLINE QUADRATURE

C     ALL THESE SUBROUTINES CAN BE REPLACED BY EQUIVALENT
C
C
        EXTERNAL A,A1,B,B1,R,G
        DIMENSION X(1),F(1),C(1),WK(2000),CC(161,3),XX(161),
      1 YY(161),ZZ(161),AA(41,41)
        EQUIVALENCE (WK(1),CC(1,1)),(WK(500),XX(1)),(WK(700),YY(1))
        SQ(X)=(1.+X)**(.5–D)*(1.–X)**(.5+D)
        SQ1(X)=(1.+X)**(.5+D)*(1.–X)**(.5–D)
C
C       CALCULATION OF THE COEFFICIENTS OF THE SYSTEM
C       OF LINEAR ALGEBRAIC EQUATIONS
C
        M=1 $ ID=6 $ PI=3.1415926
        DO 1 K=1,NX
        XK=X(K) $ IF(XK.EQ.–1.) XK=–.999 $ IF(XK.EQ.1.) XK=.999
        S=SQ(XK)
        DO 3 J=1,NX
        IF(J.EQ.1) GO TO 2
        S=S*XK
     2  IF(J.EQ.1)S2=–2.*D–XK
        IF(J.GT.1)S2=J–1–2.*D*XK–J*XK**2
        IF(J.GT.2)S2=S2*XK**(J–2)
        R3=A(XK)*S
```

```
      R5=(A1(XK)–B1(XK)*PI*TAN(PI*D)*S2/SQ1(XK)
      R2=B(XK)*FI(J–1,XK,D)
      R1=B1(XK)*SK(J–1,XK,D)
      R4=R(J–1,XK)
    3 AA(K,J)=R1+R2+R3+R4+R5
   20 FORMAT(1X,11G10.3)
    1 C(K)=G(XK)
C
C     SOLUTION OF THE SYSTEM OF LINEAR EQUATIONS
C
      CALL LEQT2F(AA,M,NX,41,C,ID,WK,IE)
      N=3*NX $ IF(N.LT.41)N=41
C
C     CALCULATION OF THE APPROXIMATE SOLUTION F(X)
C
      DO 4 K=1,NX
      XK=X(K) $ F(K)=C(1)
      DO 5 J=2,NX
    5 F(K)=F(K)+C(J)*XK**(J–1)
    4 F(K)=F(K)*SQ(XK)
C
C     CALCULATION OF THE ERRORS OF THE SOLUTION
C
      DO 6 K=1,N
      YY(K)=0.   $ XK=X(K)=(K–1)*2./(N–1)–1.
      IF(K.EQ.1)XK=XX(1)=–.999 $ IF(K.EQ.N)XK=XX(N)=.999
      ZZ(K)=ABS(G(XK)) $ S=SQ(XK)
      DO 7 J=1,NX
      IF(J.EQ.1)GO TO 8
      S=S*XK
    8 IF(J.EQ.1)S2=–2.*D–XK
      IF(J.GT.1)S2=J–1.–2.*D*XK–J*XK**2
      IF(J.GT.2)S2=S2*XK**(J–2)
      AA(1,1)=A(XK)*S+(A1(XK)–B1(XK)*PI*TAN(PI*D))*S2/SQ1(XK)+
       –20+B(XK)*FI(J–1,XK,D)+B1(XK)*SK(J–1,XK,D)+R(J–1,XK)
      YY(K)=YY(K)+C(J)*AA(1,1)
    7 ZZ(K)=ZZ(K)+ABS(C(J)*AA(1,1))
    6 YY(K)=YY(K)–G(XK)
      QA=YY(1) $ QR=YY(1)/ZZ(1) $ MY=MR=1.
      DO 9 K=2,N
      IF(ABS(YY(K)/ZZ(K)).LT.ABS(QR))GO TO 10
      QR=YY(K)/ZZ(K) $ MR=K
   10 IF(ABS(YY(K)).LT.ABS(QA))GO TO 9
      QA=YY(K) $ MY=K
    9 CONTINUE
      XA=XX(MY) $ XR=XX(MR)
      DO 11 K=1,N
   11 YY(K)=YY(K)**2
      CALL IQHSCU(XX,YY,N,CC,161,IER)
      CALL DCSQDU(XX,YY,N,CC,161,XX(1),XX(N),QS,IER)
      QS=SQRT(QS)
      RETURN
      END
C
```

```
C
C
      FUNCTION  SK(N,X,D)
C
C     CALCULATION  OF  THE  SINGULAR  INTEGRAL
C     CORRESPONDING  TO  THE  DERIVATIVE
C     OF  THE  TERM  X**N*(1–X)**(.5+D)*(1+X)**(.5–D)
C
      PI=3.1415926
      S=0.    $  L=N–1  $  B=1.5+D  $  C=3  $  Z=2  $  A1=2.–N
      SK=–PI/COS(PI*D)
      IF(N.EQ.0)GO  TO  2
      SK=SK*2*(X–D)
      IF(N.EQ.1)  GO  TO  2
      IF(X.EQ.0.)GO  TO  3
      DO  1  I=1,L
      K=I–1  $  A1=–K
    1 S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)*(N–1,–K)*
      *X**(N–2–K)
    4 SK=S–PI/COS(PI*D)*((N+1)*X–2.*N*D)*X**(N–1)
      GO  TO  2
    3 S=PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)
      GO  TO  4
    2 RETURN
      END

C
C
      FUNCTION  F21(A,B,C,Z)
C
C     COMPUTATION  OF  THE  GAUSS  GYPERGEOMETRIC  FUNCTION
C
      A1=A  $  B1=B  $  C1=C  $  G=S=R=1.
    1 R=R*A1*B1*Z/C1/G
      S=S+R  $  A1=A1+1.    $  B1=B1+1  $  C1=C1+1.    $  G=G+1.
      IF(R.EQ.0..AND.S.EQ.0.)GO  TO  2
      IF(R.NE.0..AND.S.EQ.0.)GO  TO  1
      IF(ABS(R/S).GT..000001)GO  TO  1
    2 F21=S
      RETURN
      END
C
C
      FUNCTION  FI(N,X,D)
C
C     CALCULATION  OF  THE  SINGULAR  INTEGRAL
C     CORRESPONDING  TO  THE  TERM
C     X**N*(1–X)**(.5+D)*(1+X)**(.5–D)
C
      PI=3.1415926
      S=–PI*(TAN(PI*D)*(1–X)**(.5+D)*(1+X)**(.5–D)+
      +(X–2.*D)/COS(PI*D))
      FI=S
      IF(N.EQ.0)GO  TO  2
```

```
      FI=FI*X+PI*(1.–4.*D*D)/2./COS(PI*D)
      B=1.5+D  $  C=3  $  Z=2
      IF(N.EQ.1)GO  TO  2
      S=S*X**N  $  A1=1.–N
      IF(X.EQ.0.)GO  TO  3
      DO  1  I=1,N
      K=I–1  $  A1=–K
 1    S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)*X**(N–1–K)
      GO  TO  4
 3    S=S+PI*(1.–4.*D*D)/2./COS(PI*D)*F21(A1,B,C,Z)
 4    FI=S
 2    RETURN
      END
```

7.3. Computer evaluation of two-dimensional singular integrals

An algorithm and a standard subroutine are developed for the evaluation of singular integrals over arbitrary two-dimensional domains. The integrand is a product of a Green's function with another function having a weak singularity at the boundary of the domain. Formulae are derived for an accurate estimation of the integral in the neighborhood of the singularities. The integral over the rest of the domain is evaluated by a library subroutine. The software developed is applied to a study of some contact problems of the theory of elasticity for non-classical domains.

There are numerous monographs on numerical integration. They discuss very thoroughly various methods of integration of non-singular functions, but deal very superficially with the problem of integration of singular ones. Though some theoretical results have been published, we are unaware of any standard subroutine available for the singular integrals evaluation. This lack of the standard software resulted in some cases in ignoring the singularities during computations which definitely undermined the accuracy of the numerical results. The development of such a procedure is the purpose of this section.

A method for the numerical integration of products of Green's functions with *non-singular* functions was reported in (Berger and Bernard, 1983). In many practical cases the second function is *singular* at the domain border. Their method is not applicable here. Another approach is required, and is discussed further. The approach is based on the formulae derived for the accurate estimation of the integral in the small neighborhood of the singularities. The integral over the remaining part of the domain is evaluated by a library subroutine. The software developed is checked against the case of an elliptical domain for which the exact solution is known, and an excellent accuracy is confirmed. In the last part, it is shown how the subroutines developed can be used for an approximate solution of some elastic contact problems for non-classical domains. The standard subroutines are presented in Appendix.

Theory. Consider a two-dimensional domain S. Let its boundary be given in polar coordinates as

$$\rho = a(\phi). \tag{7.3.1}$$

The following integral is encountered in various applications

$$I(N) = \iint_S \frac{\sigma(M)}{R(M,N)}\, dS, \tag{7.3.2}$$

where $R(M,N)$ is the distance between M and N which gives a singularity at N when $N \in S$; the function σ is also singular, and can be presented in polar coordinates in the form

$$\sigma(M) \equiv \sigma(\rho, \phi) = \frac{a(\phi)\, f(\rho, \phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \tag{7.3.3}$$

where the function f has no singularities in S. So, we have an integrand in (2) which has a singularity at $N \in S$ and a square root singularity along the border $\rho = a(\phi)$. Split the domain S into three subdomains (Fig. 7.3.1), namely, S_1 indicates a circular disk of radius ε having its centre at N; S_2 stands for a narrow closed strip $a(\phi) - \varepsilon < \rho < a(\phi)$; and S_3 denote the remaining part of S. It is assumed that $\varepsilon \ll a(\phi)$. The integral (2) over the subdomain S_3 is non-singular and can be evaluated by any standard subroutine. The problem lies in the evaluation of (2) over the subdomains S_1 and S_2. Here, we are to show that some formulae can be derived for the estimation of (2) over the subdomains S_1 and S_2 which are reasonably accurate for an almost arbitrary non-singular f. Two separate cases are to be considered, namely, *i*) when N is inside S (subdomains S_1 and S_2 do not intersect); *ii*) when N is at the border (singularities overlap) which is the most difficult. The case when $N \notin S$ does not require any special consideration.

Consider the first case. Let the polar coordinates of the points M and N be (ρ_0, ϕ_0) and (ρ, ϕ) respectively. The integrals to be evaluated are

$$I_1(\rho, \phi) = \iint_{S_1} \frac{a(\phi_0)\, f(\rho_0, \phi_0)}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2} [\, \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)\,]^{1/2}}\, dS, \tag{7.3.4}$$

Fig. 7.3.1. Subdomains of integration

$$I_2(\rho, \phi) = \iint\limits_{S_2} \frac{a(\phi_0)\, f(\rho_0,\phi_0)}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2}[\, \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)\,]^{1/2}}\, \mathrm{d}S. \qquad (7.3.5)$$

Introduction of a local system of polar coordinates centered at N eliminates the singularity and allows a very simple estimation of I_1

$$I_1(\rho, \phi) \equiv 2\pi\varepsilon \frac{a(\phi)\, f(\rho,\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}. \qquad (7.3.6)$$

For the evaluation of I_2, rewrite (5) as

$$I_2(\rho, \phi) = \int_0^{2\pi} \mathrm{d}\phi_0 \int_{a(\phi_0)-\varepsilon}^{a(\phi_0)} \frac{a(\phi_0)\, f(\rho_0,\phi_0)\, \rho_0 \mathrm{d}\rho_0}{[\, a^2(\phi_0) - \rho_0^2\,]^{1/2}[\, \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)\,]^{1/2}}. \qquad (7.3.7)$$

Since ε is small and function f has no singularities at the border, an approximate integration with respect to ρ_0 can be performed, with the result

$$I_2(\rho, \phi) \approx (2\varepsilon)^{1/2} \int_0^{2\pi} \frac{a(\phi_0)^{3/2} f(a(\phi_0), \phi_0)\, d\phi_0}{[\,\rho^2 + a^2(\phi_0) - 2\rho a(\phi_0)\cos(\phi - \phi_0)]^{1/2}}. \tag{7.3.8}$$

Thus, the singular two-dimensional integral (5) has been reduced to a single integral with a regular integrand and can be evaluated by any standard subroutine. The formulae (6) and (8) not only give the necessary estimations but also indicate the order of the error if the singularities are ignored: it is ε in the neighborhood of N and it is of the order of $\sqrt{\varepsilon}$ at the boundary.

Consider now the case when N is at the border. Introduce the notation

$$\frac{d}{d\phi} a(\phi + 0) = a'_+, \quad \frac{d}{d\phi} a(\phi - 0) = a'_-. \tag{7.3.9}$$

The most general is the case of an angular point at the border (Fig. 7.3.2).

Fig. 7.3.2. Geometry related to the derivation of (19) and (20)

The case of a smooth curve can be obtained from the general one by putting $a'_+ = a'_-$. Define the angles γ_+ and γ_- between the normal to the boundary and the polar radius as

$$\gamma_+ = -\frac{a'_+}{[\,a^2(\phi) + a'^2_+\,]^{1/2}}, \quad \gamma_- = \frac{a'_-}{[\,a^2(\phi) + a'^2_-\,]^{1/2}}. \tag{7.3.10}$$

The signs in (10) are chosen in such a way that the formula to be derived could be applied automatically to an arbitrary case, including the case of a smooth curve. It seems logical to represent the subarea S_1 as two rhombi, with the side equal ε and the angles at the apex N equal $\pi/2-\gamma_+$ and $\pi/2-\gamma_-$ respectively. Consider the estimation of the integral of the type (4) over a rhombus with the side ε and the acute angle $\pi/2-\gamma$. Since $\varepsilon \ll a(\phi)$, the following approximation is valid

$$a^2(\phi)-\rho^2=[a(\phi)+\rho][a(\phi)-\rho]\approx 2a(\phi)\,[a(\phi)-\rho]. \qquad (7.3.11)$$

Introducing a local system of polar coordinates (r,ψ) centered at N, the following relationship can be obtained

$$a(\phi)-\rho\approx r\sin\psi\,/\,\cos\gamma. \qquad (7.3.12)$$

In the new system of coordinates, the integral in question can be rewritten in the form

$$I\approx[\tfrac{1}{2}a(\phi)\cos\gamma]^{1/2}\mathrm{f}(a(\phi),\phi)\int_0^{\pi/2-\gamma}\mathrm{d}\psi\int_0^{b(\psi)}\frac{\mathrm{d}r}{(r\sin\psi)^{1/2}}$$

$$=[2a(\phi)\cos\gamma]^{1/2}\,\mathrm{f}(a(\phi),\phi)\int_0^{\pi/2-\gamma}\left[\frac{b(\psi)}{\sin\psi}\right]^{1/2}\mathrm{d}\psi, \qquad (7.3.13)$$

where $b(\psi)$ is the equation of rhombus in the local system of polar coordinates which has the form

$$b(\psi)=\begin{cases}\dfrac{\varepsilon\cos\gamma}{\cos(\psi+\gamma)} & \text{for}\quad 0<\psi<\dfrac{\pi}{4}-\dfrac{\gamma}{2};\\[2mm] \dfrac{\varepsilon\cos\gamma}{\sin\psi} & \text{for}\quad \dfrac{\pi}{4}-\dfrac{\gamma}{2}<\psi<\dfrac{\pi}{2}-\gamma.\end{cases} \qquad (7.3.14)$$

Due to (14), the integral (13) can be rewritten as

$$I\approx[2\varepsilon a(\phi)]^{1/2}f(a(\phi),\phi)\cos\gamma\left[\int_0^{\pi/4-\gamma/2}\frac{\mathrm{d}\psi}{[\,\sin\psi\cos(\psi+\gamma)\,]^{1/2}}+\int_{\pi/4-\gamma/2}^{\pi/2-\gamma}\frac{\mathrm{d}\psi}{\sin\psi}\right]. \qquad (7.3.15)$$

The substitution $\psi=(x-\gamma)/2$ reduces the first integral to

$$\int_{\gamma}^{\pi/2} \frac{dx}{[2(\sin x - \sin\gamma)]^{1/2}} = K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right), \qquad (7.3.16)$$

where K stands for the complete elliptic integral of the first kind. Formula (3.673) from (Gradshtein and Ryzhik, 1965) was used for the evaluation of (16). The second integral in (15) is elementary. Finally, evaluation of (15) yields

$$I \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi)\cos\gamma\, C(\gamma), \qquad (7.3.17)$$

where C is the coefficient which depends on the angle γ only

$$C(\gamma) = K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right) + \ln\left[\frac{1 + \cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}\right]. \qquad (7.3.18)$$

Since the subdomain S_1 consists of two rhombi, the estimation of (4) will take the form, according to (17) and (18),

$$I_1(\phi) \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi)\cos\gamma\,[\, C(\gamma_+) + C(\gamma_-)\,]. \qquad (7.3.19)$$

In the case of a smooth boundary $\gamma_+ = -\gamma_- = \gamma$, and formula (19) simplifies as follows

$$I_1(\phi) \approx [2\varepsilon a(\phi)]^{1/2} f(a(\phi),\phi)\cos\gamma\left[\, K\left(\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)\right) + K\left(\sin\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)\right)\right.$$

$$\left. + \ln\left(\frac{\cos(\gamma/2) + 1/\sqrt{2}}{\cos(\gamma/2) - 1/\sqrt{2}}\right)\right]. \qquad (7.3.20)$$

Formulae (6), (8), (19) and (20) give the necessary estimations, and we can proceed with the development of a standard subroutine for the singular integral evaluation.

The software description. The main subroutine is a real function SING(F,A,R,FI,E,ER,IE). The parameters are as follows. **F** is the nonsingular part of the real function to be integrated (should be provided by the user, expressed in the polar coordinates and should be specified in the calling program as EXTERNAL). **A** is a real function giving the boundary of the domain of integration (the same remark as previous). **R** and **FI** are the polar radius and the polar angle respectively of the field point (input), can be specified inside, outside or at the boundary of the domain of integration. **E** and **ER** are respectively the values of the desired (input) and the achieved (output) relative error. **IE** is the code of the error (output), according to the specifications of the standard library IMSL. The subroutines DCADRE and DBLIN from this library have been used for the evaluation of the regular part of the integral. The accuracy of the software developed was checked against the case when the domain of integration is an ellipse with the semiaxes ratio $b:a = 2:1$, and F = 1. The exact value of the integral in this case is known, and is $2\pi abK(k)$, where K stands for the complete elliptic integral of the first kind, and k is the ellipse eccentricity $k = (1 - b^2/a^2)^{1/2}$. All the computations were made with the relative error E = 0.00001. It was necessary to investigate the influence of the value of ε on the relative error of the integral computed. Of course, it is quite clear that the smaller the value of ε the more accurate is the result, but the problem is that we can not take ε arbitrary small for two reasons: increase of the computing time, and the IMSL subroutine DBLIN would not tolerate a very small ε and would, now and then, give us 'terminal error'. Here are some results. For $\varepsilon = 0.1$ the maximum relative error was about 1% for regular points, and it was about 7% for the points close to the boundary but not at the boundary. This phenomena is due to the rapid change in the denominator of (3) which reduces the accuracy of the estimation (5). The conclusion, one should make, is that in the case of evaluating the integral (2) very close to the boundary, it is advisable to evaluate the integral at the boundary and to make the necessary interpolation. Ignoring the singularity due to $1/R$ in (2) would lead to an error from 5% for the field points close to the center to 25% for the points close to the boundary. The integral over a narrow strip $a(\phi)-\varepsilon<\rho<a(\phi)$ is responsible for 25% to 35% of the value of the integral (2). Of course, the value 0.1 is too large for ε, and we took it in order to demonstrate that our procedure is sufficiently accurate even in this case, and also to demonstrate how large the error can be if the singularities are neglected. In the case $\varepsilon = 0.03$ the maximum error everywhere does not exceed 0.3%. The singularities are still responsible for up to 12% and 22% of the integral (2) respectively. Again, the portion due to the singularities increases with the movement of the field point closer to the boundary. In the case $\varepsilon = 0.001$ the result is accurate up to the fifth digit, and this value of ε was used in the subroutine SING. Even for such a small ε the singularities are still responsible for up to 1.2% and 12% respectively which means that the singularities should not be ignored. The listing of SING along with the accompanying subroutines is given in Appendix.

Application to contact problems. It is well known that the contact problem in the theory of elasticity is reduced to the solution of the integral equation

$$w(N) = H \int\int_S \frac{\sigma(M)}{R(M,N)}\,\mathrm{d}S \tag{7.3.21}$$

where w denotes the normal displacements under the punch (a known function), σ stands for the normal stress exerted by the punch (an unknown function), and H is a constant (see 5.1.9) which in the case of an isotropic elastic half-space takes on the value $H = (1-\nu^2)/\pi E$, ν and E being respectively the Poisson coefficient and the elasticity modulus.

The software developed allows us to obtain reasonably accurate solutions to various non-classical contact problems. Let us consider a flat-ended punch with an arbitrary planform S under the action of a centrally applied normal force P. Let the normal stress distribution under the punch be

$$\sigma = \frac{c\, a(\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \tag{7.3.22}$$

where, as before $a(\phi)$ is the equation of the contact region in polar coordinates, and c is a constant which can be easily defined from the condition that the integral of σ over S should give the total force P

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{a(\phi)} \frac{c\, a(\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}} \rho \mathrm{d}\rho = c \int_0^{2\pi} a^2(\phi)\, \mathrm{d}\phi = 2Ac = P, \tag{7.3.23}$$

where A is the area of S. One gets immediately from (23) that

$$\sigma = \frac{P\, a(\phi)}{2A\, [\, a^2(\phi) - \rho^2\,]^{1/2}} \tag{7.3.24}$$

For the case of a flat punch $w = const$. Now substituting (24) in (21), we can verify how close to a constant will be the displacements, produced by the stress distribution (24). Denote by w_0 the normal displacements at the center of the punch (ρ=0). One can get from (21)

$$w_0 = \frac{\pi H P A_1}{4A}, \tag{7.3.25}$$

where

$$A_1 = \int_0^{2\pi} a(\phi)\, \mathrm{d}\phi. \tag{7.3.26}$$

Computations were made for three different punch planforms: a square, an oval and a 'shamrock'. The expression of $a(\phi)$ for the square with the edge $2l$ was taken in the form

$$a(\phi) = \begin{cases} \dfrac{l}{\cos\phi} & \text{for } -\pi/4 < \phi < \pi/4 \\ \dfrac{l}{\cos(\phi - \pi/2)} & \text{for } \pi/4 < \phi < 3\pi/4 \\ \dfrac{l}{\cos(\phi - \pi)} & \text{for } 3\pi/4 < \phi < 5\pi/4 \\ \dfrac{l}{\cos(\phi - 3\pi/2)} & \text{for } 5\pi/4 < \phi < 7\pi/4 \end{cases} \tag{7.3.27}$$

for the oval

$$a(\phi) = [\, a_1^2\cos^2\phi + a_2^2\sin^2\phi \,]^{1/2}\,, \tag{7.3.28}$$

and for the 'shamrock'

$$a(\phi) = a_1 + a_2\cos 3x\,, \quad a_1 > a_2 > 0. \tag{7.3.29}$$

The case of a square punch has been considered before by several authors, the oval or the 'shamrock' punch seem never to have been considered before.

The results of computation of the dimensionless normal displacement $w^* = w/w_0$ against the dimensionless $x^* = x/l$ and $y^* = y/l$ are presented in Table 7.3.1.

Table 7.3.1. The normal displacements under a square punch.

y^* \ x^*	0.000	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	1.000
1.00	1.024	1.020	1.015	1.007	0.997	0.983	0.965	0.940	0.903	0.874	0.794
0.95	1.021	1.017	1.012	1.004	0.994	0.980	0.962	0.938	0.904	0.882	0.874
0.90	1.018	1.014	1.009	1.001	0.991	0.978	0.961	0.939	0.911	0.904	0.903
0.80	1.013	1.009	1.004	0.997	0.988	0.976	0.961	0.946	0.939	0.938	0.940
0.70	1.009	1.005	1.001	0.994	0.986	0.976	0.967	0.961	0.961	0.962	0.965
0.60	1.006	1.002	0.998	0.993	0.987	0.980	0.976	0.976	0.978	0.980	0.983
0.50	1.003	1.001	0.997	0.993	0.989	0.987	0.986	0.988	0.991	0.994	0.997
0.40	1.002	0.999	0.997	0.994	0.993	0.993	0.994	0.997	1.001	1.004	1.007
0.30	1.001	0.999	0.998	0.997	0.997	0.998	1.001	1.004	1.009	1.012	1.015
0.20	1.000	0.999	0.999	0.999	1.001	1.002	1.005	1.009	1.014	1.017	1.020
0.10	1.000	1.000	1.000	1.001	1.003	1.005	1.008	1.012	1.017	1.020	1.023
0.00	1.000	1.000	1.001	1.002	1.003	1.006	1.009	1.013	1.018	1.021	1.024

As one can see, at the major part of the square the displacements are very close to unity, the maximum error of 20% being achieved only at the apex, and the error decreases very rapidly with the distance from the apex. Taking into consideration that the sign of the error fluctuates, we may assume that the relationship between the total force P and the punch settlement δ, computed on the basis of (25), should be reasonably accurate. Introduce the following relationship

$$\delta = \frac{HP}{g\sqrt{A}}, \tag{7.3.30}$$

where g is a dimensionless coefficient

$$g = \frac{4\sqrt{A}}{\pi A_1}. \tag{7.3.31}$$

According to (26) and (27), one can easily compute the coefficient g for the square as

$$g = \frac{1}{\pi \ln(1+\sqrt{2})} = 0.3611,$$

which is very close to the value 0.3607 given by Maxwell for the capacitance of the square. Using the electrostatic analogy, one can easily deduce that our coefficient g is related to the capacity C of a flat lamina by

$$g = \frac{C}{\sqrt{A}}. \tag{7.3.32}$$

Of course, closeness to the result by Maxwell does not mean that our result is so accurate. The value of g which seems to be accurate was obtained in (Noble, 1960), and is 0.367, so that our result is in error by 1.6% which is not bad. The accuracy can be improved further by using the variational principle. According to Noble (1960), the following inequality is valid

$$g_e \geq g\left\{2 - \frac{1}{2A}\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{w^*(\rho,\phi)\,\rho d\rho}{[\,a^2(\phi) - \rho^2\,]^{1/2}}\right\}, \tag{7.3.33}$$

where g_e stands for the exact value of the coefficient, g is the approximate value defined by (31). Since w^* has already been computed in Table 7.3.1, the additional computation due to (33) can be performed very easily, the singularity can be removed by the change of variables $\rho = a(\phi)\sin\psi$. The final result, according to (33), is $g_e \geq 0.365$ which is only 0.55% away from the correct value.

For the oval, defined by (28), one can compute

$$A = \frac{\pi}{2}(a_1^2 + a_2^2), \quad A_1 = 4a_1 E(k), \tag{7.3.34}$$

where it is assumed that $a_1 > a_2$; E stands for the complete elliptic integral of the second kind, and $k = (1 - a_2^2/a_1^2)^{1/2}$. The value of g can be estimated as

$$g = \frac{(2 - k^2)^{1/2}}{\sqrt{2}\pi E(k)}. \tag{7.3.35}$$

For the numerical computations, we assumed $a_1 = 4$, $a_2 = 1$. in order to have an oval with a 'waist' which would be quite different from the elliptic shape. The results are presented in Table 7.3.2 as the dimensionless normal displacements $w^* = w/w_0$ against the polar angle ϕ and dimensionless radius $\rho^* = \rho/a(\phi)$.

Table 7.3.2. The normal displacements under the oval punch.

ϕ \ ρ^*	0	1/6	1/4	1/3	1/2	2/3	3/4	5/6	11/12	1.0
$\pi/2$	1	1.000	1.000	1.001	1.002	1.006	1.009	1.014	1.022	1.042
$11\pi/24$	1	1.000	1.000	1.001	1.002	1.005	1.008	1.010	1.010	1.004
$5\pi/12$	1	1.000	1.000	1.000	1.001	0.998	0.993	0.985	0.976	0.967
$3\pi/8$	1	1.000	1.000	0.999	0.994	0.981	0.974	0.966	0.958	0.951
$\pi/3$	1	1.000	0.999	0.996	0.987	0.973	0.967	0.961	0.956	0.951
$7\pi/24$	1	0.999	0.997	0.994	0.985	0.975	0.972	0.969	0.966	0.964
$\pi/4$	1	0.999	0.997	0.994	0.988	0.984	0.983	0.983	0.984	0.984
$5\pi/24$	1	0.999	0.998	0.996	0.995	0.997	0.999	1.001	1.004	1.007
$\pi/6$	1	1.000	0.999	0.999	1.003	1.010	1.015	1.019	1.024	1.029
$\pi/8$	1	1.000	1.002	1.004	1.012	1.023	1.029	1.036	1.042	1.049
$\pi/12$	1	1.001	1.004	1.007	1.019	1.033	1.040	1.048	1.056	1.064
$\pi/24$	1	1.002	1.005	1.010	1.023	1.039	1.048	1.056	1.065	1.074
0	1	1.002	1.005	1.011	1.025	1.041	1.050	1.059	1.068	1.077

As we can see, the positive error does not exceed 8% while the negative error is below 6%, the error being mainly restricted to a narrow strip close to the boundary. Computations, according to (33), give the lower bound $g_e \geq 0.3758$. Formula (35) gives $g = 0.3835$ which is about 2% above the lower bound. Since the negative error in Table 7.3.2 is below 6%, one can estimate the upper bound as 0.3946 which guarantees that formula (35) is accurate, at least, within 3%. We strongly believe that the real accuracy is much better than 3% but even 3% is not bad for such a simple formula. It is also clear that the value of error will decrease with a_1 approaching a_2 since formula (30) in the case of an ellipse is exact.

The following values can be computed for the ’shamrock’-shaped punch (29): $A=\pi(a_1^2+a_2^2/2)$, $A_1=2\pi a_1$, and the value of g will be defined as

$$g=\frac{2\,[\,\pi(a_1^2+a_2^2/2)\,]^{1/2}}{\pi^2 a_1} \tag{7.3.36}$$

The results of computations for $a_1=1.5$, $a_2=1$ are presented in Table 7.3.3. Here, the maximum positive error and the maximum negative error are about 5%. The value of g due to (36) is 0.3971 while the lower bound for g, according to (33), is 0.4006. Since the error fluctuates almost symmetrically around zero, there is a reason to believe that the exact value of g will be very close to the lower bound.

Table 7.3.3. The numerical results for the ’shamrock’-shaped punch.

ϕ \ ρ^*	0	1/6	1/4	1/3	5/12	1/2	7/12	2/3	3/4	10/12	11/12	1.00
$\pi/3$	1	1.000	1.001	1.001	1.002	1.004	1.006	1.009	1.013	1.019	1.029	1.050
$11\pi/36$	1	1.000	1.001	1.001	1.002	1.004	1.006	1.010	1.014	1.019	1.026	1.026
$5\pi/18$	1	1.000	1.001	1.002	1.003	1.005	1.007	1.009	1.011	1.009	1.003	0.996
$\pi/4$	1	1.000	1.001	1.002	1.003	1.004	1.003	1.000	0.994	0.986	0.979	0.971
$2\pi/9$	1	1.000	1.001	1.001	1.000	0.997	0.991	0.983	0.976	0.968	0.962	0.955
$7\pi/36$	1	1.000	1.000	0.998	0.993	0.986	0.979	0.971	0.965	0.959	0.954	0.949
$\pi/6$	1	0.999	0.997	0.992	0.986	0.979	0.972	0.967	0.962	0.958	0.955	0.953
$5\pi/36$	1	0.998	0.994	0.988	0.981	0.976	0.971	0.968	0.966	0.965	0.965	0.965
$\pi/9$	1	0.997	0.992	0.985	0.980	0.977	0.975	0.974	0.975	0.976	0.978	0.982
$\pi/12$	1	0.996	0.990	0.985	0.982	0.980	0.981	0.983	0.986	0.989	0.994	0.999
$\pi/18$	1	0.995	0.989	0.985	0.984	0.984	0.987	0.990	0.995	1.001	1.007	1.014
$\pi/36$	1	0.994	0.989	0.986	0.986	0.987	0.991	0.996	1.002	1.009	1.016	1.025
0	1	0.994	0.989	0.986	0.986	0.989	0.993	0.998	1.004	1.012	1.020	1.028

In order to verify the validity of the above speculations, we have performed some computations for a ’shamrock’-shaped punch by using an iterative procedure. The results are presented below.

ratio a_1/a_2	1.1	1.5	2.0
g (36)	0.4270	0.3971	0.3810
g numerical	0.4147	0.4013	0.3890
error of (36) (%)	3.0	1.0	2.0

As we can see, the numerical data do support all the above considerations. We note also that in the limiting case of $a_1/a_2 \to \infty$, formula (36) gives the exact result for a circle, as it should.

Discussion. The examples presented show that the subroutine can be successfully used for solving various problems for non-classical domains. The applications are not limited to contact problems since the integral of the type (2) is involved in fluid mechanics, heat transfer, electrostatics, wave propagation *etc.* In the elastic contact problems, the subroutine helped to establish a very simple relationship between the applied force and the punch settlement (30–31) and to confirm its accuracy.

The method of this section can be successfully used for the computer evaluation of singular integrals of various types. For example, recently we have found the following integral representation for the reciprocal distance

$$\frac{1}{[\,\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)\,]^{1/2}} = \frac{1}{\pi^2}\int_0^{2\pi} d\psi \int_0^{min(\rho_0,\rho)} \frac{(\rho^2-x^2)^{1/2}}{\rho^2+x^2-2x\rho\cos(\phi-\psi)}$$

$$\times \frac{(\rho_0^2-x^2)^{1/2}}{\rho_0{}^2+x^2-2x\rho_0\cos(\phi_0-\psi)}\,dx.$$

When $\rho<\rho_0$, the integrand in the last expression has a singularity at the boundary ($x=\rho$, $\psi=\phi$). Since the domain of integration in this case is a circle, the rhombi in Fig. 7.3.2 transform in two squares, and the following formula can be obtained for the integral estimation in the neighborhood of the singularity

$$I = \frac{1}{\pi^2}\left[\frac{2\varepsilon}{\rho}\right]^{1/2}[\,\pi(1+\sqrt{2})-2\sqrt{2}\ln(1+\sqrt{2})\,]\,\frac{(\rho_0{}^2-\rho^2)^{1/2}}{\rho^2+\rho_0{}^2-2\rho\rho_0\cos(\phi-\phi_0)}.$$

As it was noticed by Noble (1960), the existing methods generally provide the accuracy of about 5%. Of course, it would be very desirable to establish a Gauss type formula for the singular integral evaluation but so far no success was reported. Our subroutine can be used not only to verify the accuracy of an assumed solution but actually to find one. The first error in the suggested solution (22) is the assumption of a square root singularity at the edge which, in general, is incorrect. The second error is the assumption that c in (22) is a constant. It is of interest to establish the impact of each error on the accuracy of the solution. In order to do so, we assumed the solution in the form

$$\sigma(\rho,\phi) = \frac{c\,a^{1-u(\phi)}(\phi)}{[\,a^2(\phi)-\rho^2\,]^{[1-u(\phi)]/2}}. \qquad (7.3.37)$$

The punch settlement at $\rho = 0$ will be defined by

$$w_0 = c\,\frac{\sqrt{\pi}}{2}\int_0^{2\pi} a(\phi)\frac{\Gamma[(1+u(\phi))/2]}{\Gamma[1+u(\phi)/2]}\,d\phi. \tag{7.3.38}$$

Certain modifications are to be made in formulae (6), (8), (19) and (18) which will take the respective forms

$$I_1(\rho,\phi) \approx 2\pi\varepsilon\frac{a^{1-u(\phi)}(\phi)\, f(\rho,\phi)}{[\,a^2(\phi)-\rho^2\,]^{[1-u(\phi)]/2}}, \tag{7.3.39}$$

$$I_2(\rho,\phi) \approx \int_0^{2\pi}\frac{(2\varepsilon)^{[1+u(\phi_0)]/2}[a(\phi_0)]^{[3-u(\phi_0)]/2} f(a(\phi_0),\phi_0)\,d\phi_0}{[1+u(\phi_0)][\,\rho^2+a^2(\phi_0)-2\rho a(\phi_0)\cos(\phi-\phi_0)]^{1/2}}, \tag{7.3.40}$$

$$I_1(\phi) \approx (2\varepsilon)^{[1+u(\phi)]/2}[a(\phi)]^{[1-u(\phi)]/2} f(a(\phi),\phi)\cos\gamma\,[\,C(\gamma_+)+C(\gamma_-)], \tag{7.3.41}$$

$$C(\gamma) = \int_0^{\pi/4-\gamma/2}\frac{d\psi}{\sin^{[1-u(\phi)]/2}\psi\,\cos^{[1+u(\phi)]/2}(\psi+\gamma)} + \ln\frac{1+\cos\left(\dfrac{\pi}{4}-\dfrac{\gamma}{2}\right)}{\cos\left(\dfrac{\pi}{4}-\dfrac{\gamma}{2}\right)}.$$

The function $u(\phi)$ for the case of a square was taken $u(\phi) = 0.04-0.3(4\phi/\pi)^5$ for $0<\phi<\pi/4$. The pattern is repeated for $\phi>\pi/4$. The results of computations are presented in Table 7.3.4 which compares very favourably with Table 7.3.1 since now the normal displacements inside the square are practically uniform, except for a small portion close to the boundary where the maximum error is about 5%.

Table 7.3.4. Influence of the singularity on the displacements(square punch).

y^* \ x^*	0.000	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	1.000
1.000	0.999	0.997	0.995	0.995	0.996	1.001	1.013	1.030	1.050	1.053	0.978
0.950	1.006	1.004	1.002	1.000	0.999	1.000	1.002	1.004	1.000	0.995	1.053
0.900	1.006	1.004	1.002	1.000	0.999	0.998	0.997	0.995	0.991	1.000	1.050
0.800	1.006	1.004	1.002	1.000	0.997	0.995	0.992	0.989	0.995	1.004	1.030
0.700	1.005	1.003	1.001	0.998	0.996	0.993	0.991	0.992	0.997	1.002	1.013
0.600	1.003	1.002	1.000	0.997	0.995	0.993	0.993	0.995	0.998	1.000	1.001
0.500	1.002	1.001	0.999	0.997	0.996	0.995	0.996	0.997	0.999	0.999	0.996
0.400	1.001	1.000	0.999	0.998	0.997	0.997	0.998	1.000	1.000	1.000	0.995
0.300	1.001	1.000	0.999	0.999	0.999	1.000	1.001	1.002	1.002	1.002	0.995
0.200	1.000	1.000	1.000	1.000	1.001	1.002	1.003	1.004	1.004	1.004	0.997

0.100	1.000	1.000	1.000	1.001	1.002	1.003	1.004	1.005	1.006	1.005	0.999
0.000	1.000	1.000	1.001	1.001	1.002	1.003	1.005	1.006	1.006	1.006	0.999

Evaluation of (38) gives g = 0.3675, formula (33) gives the lower bound $g_e \geq 0.3666$, both results being very close to the value given by Noble. One can draw the conclusion: correct estimation of the singularities is the most important thing; the variation of c inside the square has a minor impact and is limited to, may be, several percent. The other idea which comes to mind is the possibility of the development of self-adaptive subroutine which would be able to find a solution with a prescribed degree of accuracy using the feedback principle. The first version of such a procedure has been developed. It takes (22) as initial stress distribution. The value of $c(\rho, \phi)$ is being adjusted at each point proportionally to the discrepancy between the computed potential and unity. The results of application of the procedure to a rectangle are given in Table 7.3.5.

Table 7.3.5. The normal displacements after 10 iterations (rectangle 1:10).

y^* \ x^*	0.00000	2.00000	3.00000	4.00000	5.00000	6.00000	7.00000	8.00000	9.00000	9.50000	10.00000
1.00000	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00001	1.00001	1.00003	1.00005	1.00000
0.95000	0.99996	1.00001	0.99992	0.99979	0.99947	0.99926	0.99823	0.99783	0.99538	0.99489	1.00000
0.90000	1.00001	0.99997	0.99983	0.99966	0.99923	0.99891	0.99776	0.99691	0.99504	0.99368	1.00000
0.80000	1.00002	0.99993	0.99976	0.99951	0.99906	0.99840	0.99756	0.99639	0.99615	0.99911	0.99999
0.70000	1.00002	0.99990	0.99973	0.99942	0.99894	0.99837	0.99749	0.99705	0.99892	1.00049	0.99999
0.60000	1.00001	0.99988	0.99971	0.99941	0.99897	0.99845	0.99810	0.99892	1.00046	1.00119	0.99999
0.50000	1.00001	0.99988	0.99974	0.99943	0.99912	0.99901	0.99951	1.00035	1.00103	1.00109	0.99999
0.40000	1.00001	0.99988	0.99986	0.99959	0.99963	1.00011	1.00066	1.00118	1.00119	1.00088	0.99999
0.30000	1.00000	0.99990	0.99990	1.00002	1.00047	1.00100	1.00145	1.00165	1.00115	1.00064	1.00000
0.20000	1.00000	0.99997	1.00019	1.00060	1.00111	1.00161	1.00193	1.00190	1.00105	1.00045	1.00000
0.10000	1.00000	1.00015	1.00051	1.00098	1.00149	1.00196	1.00219	1.00201	1.00097	1.00033	1.00000
0.00000	1.00000	1.00024	1.00062	1.00111	1.00162	1.00207	1.00227	1.00204	1.00094	1.00030	1.00000

Appendix

```
FUNCTION  SING(F,A,R,FI,E,ER,IE)
IMPLICIT  REAL*8  (A−H,O−Z)
EXTERNAL  F,A,AN,AK,FA,FB,RR1,RR2,ANK,AKK,AK1,AK2
COMMON/A/  R0,F0,EP,EP1
R0=R
F0=FI
PI=3.1415926536
EP1=.01
IF(R−A(FI).GT..00001)  GO  TO  2
AA=0.
P2=2.*PI
IF(R.LE.EP)  GO  TO  1
AL=ASIN(EP/R)
BB=FI−AL−EP1
BB1=FI−AL
CC1=FI+AL
```

```
      AA1=P2+FI−AL
      CC=FI+AL+EP1
      AA=P2+FI−AL−EP1
      IF(A(FI)−R.LE.EP)  GO  TO  3
      R1=DBLIN(FA,CC,AA,AN,AK,E,ER,IE)
      R21=DBLIN(FA,BB,BB1,AN,AK1,E,ER,IE)
      R22=DBLIN(FA,CC1,CC,AN,AK1,E,ER,IE)
      R23=DBLIN(FA,BB,BB1,AK2,AK,E,ER,IE)
      R24=DBLIN(FA,CC1,CC,AK2,AK,E,ER,IE)
      R25=DBLIN(FA,BB,BB1,AK1,AK2,E,ER,IE)
      R26=DBLIN(FA,CC1,CC,AK1,AK2,E,ER,IE)
      R2=R21+R22+R23+R24+R25+R26
      R3=DBLIN(FA,BB1,CC1,AN,RR1,E,ER,IE)
      R4=DBLIN(FA,BB1,CC1,RR2,AK,E,ER,IE)
      R5=P2*EP*F(FI,R)/SQRT(A(FI)**2−R*R)*A(FI)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
    5 FORMAT(1X,6F10.5)
      SING=R1+R2+R3+R4+R5+R6
      GO  TO  4
    1 R1=DBLIN(FA,AA,P2,ANK,AK,E,ER,EI)
      R5=P2*EP*F(FI,0.)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
      SING=R1+R5+R6
      GO  TO  4
    2 TYPE*,'OUTSIDE'
      R=A(FI)
      R1=DBLIN(FA,AA,P2,AN,AK,E,ER,IE)
      R6=DCADRE(FB,0.,P2,E,E,ER,IE)
      SING=R1+R6
    6 FORMAT(1X,7HOUTSIDE)
      GO  TO  4
    3 DA=(A(FI+EP)−A(FI))/EP
      GP=−ASIN(DA/SQRT(A(FI)**2+DA**2))
      DA=(A(FI)−A(FI−EP))/EP
      GM=ASIN(DA/SQRT(A(FI)**2+DA**2))
      AL1=ASIN(EP*COS(GP)/R)+FI
      AL2=FI−ASIN(EP*COS(GM)/R)
      BB=AL2−EP1
      BB1=AL2
      CC1=AL1
      AA1=P2+AL2
      CC=AL1+EP1
      AA=P2+AL2−EP1
      R1=DBLIN(FA,CC,AA,AN,AK,E,ER,IE)
      R5=(FG(GP)+FG(GM))
      R21=DBLIN(FA,BB,BB1,AN,AKK,E,ER,IE)
      R22=DBLIN(FA,CC1,CC,AN,AKK,E,ER,IE)
      BB2=BB+.99*EP1
      R23=DBLIN(FA,BB,BB2,AKK,AK,E,ER,IE)
      R231=DBLIN(FA,BB2,BB1,AKK,AK,E,ER,IE)
      R24=DBLIN(FA,CC1,CC,AKK,AK,E,ER,IE)
      R2=R21+R22+R23+R24+R231
      R3=DBLIN(FA,BB1,CC1,AN,AKK,E,ER,IE)
      R4=DBLIN(FA,BB1,CC1,AKK,AK,E,ER,IE)
```

```
      R6=DCADRE(FB,CC,AA,E,E,ER,IE)
      1+DCADRE(FB,AL1,CC,E,E,ER,IE)
      2+DCADRE(FB,BB,AL2,E,E,ER,IE)
      SING=R1+R2+R3+R4+R6+R5
    4 RETURN
      END
C     ******************************
      FUNCTION F2(XF)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      DIMENSION XX(12),YY(12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      XL=XX(12)
      YL=XF
      IF(ABS(YY(12)–YL).LT..0000001) YL=YY(12)
      IF(ABS(YY(1)–YL).LT..0000001) YL=YY(1)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
    5 FORMAT(1X,5F15.4)
      Z=1.+U(XF)
      F2=FL*A(XF)**2*SIN(EP)**Z/Z
      RETURN
      END
C     *******************************
      FUNCTION FU(FI,PS)
      IMPLICIT REAL*8 (A–H,O–Z)
      DIMENSION XX(12),YY(12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      XL=SIN(PS)
      YL=FI
      IF(ABS(XX(1)–XL).LT..0000001) XL=XX(1)
      IF(ABS(YY(12)–YL).LT..0000001) YL=YY(12)
      IF(ABS(XX(12)–XL).LT..0000001) XL=XX(12)
      IF(ABS(YY(1)–YL).LT..0000001) YL=YY(1)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
    5 FORMAT(1X,5F15.4)
      FU=FL*A(FI)**2*SIN(PS)*COS(PS)**U(FI)
      RETURN
      END
C     *************************************
      FUNCTION FU1(FI,PS)
      IMPLICIT REAL*8 (A–H,O–Z)
      DIMENSION XX(12),YY(12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      XL=SIN(PS)
      YL=FI
      IF(ABS(XX(1)–XL).LT..0000001) XL=XX(1)
      IF(ABS(YY(12)–YL).LT..0000001) YL=YY(12)
      IF(ABS(XX(12)–XL).LT..0000001) XL=XX(12)
      IF(ABS(YY(1)–YL).LT..0000001) YL=YY(1)
      CALL IBCEVL(XX,12,YY,12,C,12,XL,YL,FL,IE)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FLS,IE)
    5 FORMAT(1X,5F15.4)
      IF(COS(PS).LE.0.) TYPE*,'FU1',PS,COS(PS),FI
      FU1=FL*FLS*A(FI)**2*SIN(PS)*COS(PS)**U(FI)
```

```
      RETURN
      END
C     ***************************************
      FUNCTION FU2(XF)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      DIMENSION XX(12),YY(12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      XL=XX(12)
      YL=XF
      IF(ABS(YY(12)–YL).LT..0000001) YL=YY(12)
      IF(ABS(YY(1)–YL).LT..0000001) YL=YY(1)
      CALL IBCEVL(XX,12,YY,12,C,12,XL,YL,FL,IE)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FLS,IE)
    5 FORMAT(1X,5F15.4)
      Z=1.+U(XF)
      FU2=FL*FLS*A(XF)**2*SIN(EP)**Z/Z
      RETURN
      END
C     ******************************************
      FUNCTION F1(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      F1=A(X)*GAMMA((1.+U(X))/2.)/GAMMA(1.+U(X)/2.)
      RETURN
      END
C     *****************************************
      FUNCTION APE(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      APE=3.1415926536/2.–EP
      RETURN
      END
C     *******************************************
      FUNCTION AP2(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      AP2=3.1415926536/2.
      RETURN
      END
C     *******************************************
      FUNCTION F(F0,X)
      IMPLICIT REAL*8 (A–H,O–Z)
      DIMENSION XX(12),YY(12),CS(2,12,2,12),C(2,12,2,12)
      COMMON/E/ C,XX,YY,CS
      PI=3.1415926536
      FI=F0
    1 IF(FI.LT.0.) FI=FI+2.*PI
    2 IF(FI.GT.PI/3.) FI=FI–2.*PI/3.
      IF(FI.GT.PI/3.) GO TO 2
      XL=X/A(FI)
      YL=ABS(FI)
      IF(ABS(XX(1)–XL).LT..00001) XL=XX(1)
      IF(ABS(YY(12)–YL).LT..00001) YL=YY(12)
      IF(ABS(XX(12)–XL).LT..00001) XL=XX(12)
      IF(ABS(YY(1)–YL).LT..00001) YL=YY(1)
```

```
    5 FORMAT(1X,5F15.4)
      CALL IBCEVL(XX,12,YY,12,CS,12,XL,YL,FL,IE)
      IF(IE.NE.0) TYPE*,'F ER',IE,XL,YL,FL
      F=FL
      RETURN
      END
C     ****************************************
      FUNCTION U(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      DIMENSION UD(23),PSI(23),CRR(22,3)
      COMMON/D/ UD,PSI,CRR
      PI=3.1415926536
      Y=ABS(X)
    1 IF(Y.GT.PI/3.) Y=ABS(Y–2.*PI/3.)
      IF(Y.GT.PI/3.) GO TO 1
      CALL ICSEVU(PSI,UD,23,CRR,22,Y,RES,1,IER)
      IF(IER.NE.0) TYPE*,'U ER',X,Y,RES
    5 FORMAT(1X,3F15.4)
      U=RES
      RETURN
      END
C     ****************************************
      FUNCTION AKK(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      AKK=A(X)–EP–EP1
      RETURN
      END
C     ****************************************
      FUNCTION AK(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      AK=A(X)–EP
      RETURN
      END
C     ****************************************
      FUNCTION RR1(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      SQ=(EP**2–(R*SIN(FI–X))**2)
      IF(SQ.LT.0.) SQ=0.
      RR1=R*COS(FI–X)–SQRT(SQ)
      IF(RR1.GT.AK(X)) RR1=AK(X)
      RETURN
      END
C     ****************************************
      FUNCTION RR2(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      SQ=(EP**2–(R*SIN(FI–X))**2)
      IF(SQ.LT.0.) SQ=0.
      RR2=R*COS(FI–X)+SQRT(SQ)
      IF(RR2.GT.AK(X)) RR2=AK(X)
      RETURN
```

```
      END
C     ****************************************
      FUNCTION AN(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      AN=0.
      RETURN
      END
C     ***************************************
      FUNCTION A(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/B/ A1,A2,B
      A=A1+A2*COS(3.*X)
      RETURN
      END
C     ************************************************
      FUNCTION MMDELK(Z)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/MM/ ARG
      REAL*8 MMDELK
      MMDELK=1./SQRT(1.–(ARG*SIN(Z))**2)
      RETURN
      END
C     ************************************************
      FUNCTION FG(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      COMMON/MM/ ARG
      EXTERNAL MMDELK
      REAL*8 MMDELK
      E=.00001
      PI=3.1415926536
      Y=PI/4.–X/2.
      ARG=SIN(Y)
      C=COS(Y)
      P2=PI/2.
      FG=DCADRE(MMDELK,0.,P2,E,E,ER,IE)+LOG((1.+C)/C)
      FG=FG*COS(X)*F(FI,A(FI))*SQRT(2.*A(FI)*EP)
      RETURN
      END
C     ***************************************************
      FUNCTION AK1(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      Y=R–EP1
      IF(Y.GT.AK(X))Y=AK(X)
      AK1=Y
      RETURN
      END
C     ************************************************
      FUNCTION AK2(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      Y=R+EP1
      IF(Y.GT.AK(X))Y=AK(X)
```

```
      AK2=Y
      RETURN
      END
C     *************************************************
      FUNCTION FA(PS,X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      RR=SQRT(R*R+X*X–2.*R*X*COS(FI–PS))
      SQ=SQRT(A(PS)**2–X*X)
      FA=F(PS,X)*(A(PS)/SQ)**(1.–U(PS))/RR*X
      RETURN
      END
C     *************************************************
      FUNCTION ANK(X)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      ANK=EP
      RETURN
      END
C     ************************************************
      FUNCTION FB(PS)
      IMPLICIT REAL*8 (A–H,O–Z)
      COMMON/A/ R,FI,EP,EP1
      AA=A(PS)
      RR=SQRT(R*R+AA*AA–2.*R*AA*COS(FI–PS))
      Z=U(PS)
      FB=AA**(1.–Z)*(2.*EP*AA)**((1.+Z)/2.)/(1.+Z)*F(PS,AA)/RR
      RETURN
      END
```