CHAPTER 6

NEW SOLUTIONS IN FRACTURE MECHANICS

6.1. External circular crack under antisymmetric loading

Explicit expressions are derived for the field of stresses and displacements in a transversely isotropic space weakened by an external circular crack and subjected to two antisymmetrically applied concentrated forces. The presented results may be used as Green's functions for a general case of antisymmetric loading so that the complete solution can be presented in quadratures.

The external circular crack may be perceived as two elastic half-spaces connected in the plane $z=0$ by a circular domain which is called hereafter the crack neck. Ufliand (1967) was, probably, the first to consider the equilibrium of an *isotropic* elastic body weakened by an external circular crack and subjected to the action of two antisymmetric normal forces by an integral transform method. The same problem for the case of *transversely isotropic* body was solved in (Fabrikant, 1971d). All these solutions define the elastic field in the plane $z=0$ only. We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic space. One may argue that since the stresses exerted in the crack neck are known, we can substitute them into the Boussinesq point force solution (which is well known, for example, see Fabrikant, 1970) and obtain the complete solution in quadratures. Theoretically, yes, this can be done, but practically, this solution would be of little use since it would require double integration, with the integrand being singular. The computing time for this procedure would be quite significant, and its accuracy would be very doubtful. This is the main reason why, to the best of our knowledge, nobody tried so far to obtain a complete solution, even in the case of an isotropic body. On the other hand, knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems. For example, by using linear superposition of the solutions for symmetric and antisymmetric loading, we can obtain the solution to the problem of one-sided loading of a crack.

The complete solution has become possible due to the new results in potential theory given in Chapter 1. The expressions for the stresses in the crack neck are fed in the point force solution, with one important distinction: the integrals are computed in elementary functions and lead to remarkably simple and elementary expressions.

Theory. We consider a transversely isotropic elastic space weakened by an external circular crack of radius a in the plane z=0 (Fig. 6.1.1). Let two point

Fig. 6.1.1. The crack geometry

forces P be applied to the crack faces antisymmetrically in the Oz direction at the points with cylindrical coordinates $(r,\psi,0^+)$ and $(r,\psi,0^-)$. The problem, due to antisymmetric loading, can be reduced to that of a half-space $z\geq 0$, with the boundary conditions at the plane z=0

$$\begin{aligned}
&u = 0, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi;\\
&\sigma = 0, && \text{for } 0\leq\rho\leq a, && 0\leq\phi<2\pi;\\
&\sigma = P\delta(\rho-r,\phi-\psi)/\rho, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi;\\
&\tau = 0, && \text{for } a\leq\rho\leq\infty, && 0\leq\phi<2\pi.
\end{aligned} \tag{6.1.1}$$

Here $\sigma=-\sigma_z$ and $\tau=-\tau_z$ as they are defined in (5.1.12). It is known (Fabrikant, 1989a) that in the case of a transversely isotropic elastic half-space subjected to a general concentrated force with the components T_x, T_y and P, the complete solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1 - 1}\left[\frac{1}{2}\gamma_2(\overline{\Lambda}\chi_1 + \Lambda\overline{\chi}_1) + P\ln(R_1 + z_1)\right],$$

$$F_2 = \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1(\overline{\Lambda}\chi_2 + \Lambda\overline{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\frac{\gamma_3}{4\pi A_{44}}(\overline{\Lambda}\chi_3 - \Lambda\overline{\chi}_3). \tag{6.1.2}$$

Here (ρ_0,ϕ_0) is the point of the boundary where the concentrated force is applied;

$\chi_k(z) = \chi(z_k)$, $R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_k^2]^{1/2}$, for k=1,2,3;

$\chi(z) = T[z\ln(R_0 + z) - R_0]$, $T = T_x + \mathrm{i}T_y$,

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}. \tag{6.1.3}$$

Substitution of (2–3) in (5.1.6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3 + z_3)^2}\right]$$

$$+ \frac{H\gamma_2}{m_2 - 1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\overline{T}}{R_2(R_2 + z_2)^2}\right] + \frac{Pq}{R_2(R_2 + z_2)}\right\}$$

$$+ \frac{H\gamma_1}{m_1 - 1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\overline{T}}{R_1(R_1 + z_1)^2}\right] + \frac{Pq}{R_1(R_1 + z_1)}\right\}, \tag{6.1.4}$$

$$w = H\left\{\frac{1}{2}(T\overline{q} + \overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1 - 1)R_1(R_1 + z_1)} + \frac{\gamma_1 m_2}{(m_2 - 1)R_2(R_2 + z_2)}\right]\right.$$

$$\left. + P\left[\frac{m_1}{(m_1 - 1)R_1} + \frac{m_2}{(m_2 - 1)R_2}\right]\right\}. \tag{6.1.5}$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{6.1.6}$$

Expressions (4) and (5) simplify for the case when z=0

$$u = \frac{1}{2}G_1\frac{T}{R} + \frac{1}{2}G_2\frac{\bar{T}q^2}{R^3} - H\alpha\frac{P}{\bar{q}}, \qquad (6.1.7)$$

$$w = H\alpha\Re\left(\frac{T}{q}\right) + H\frac{P}{R}. \qquad (6.1.8)$$

Here $\Re$ is the sign indicating that the real part of the expression to follow is taken; H, α, G_1, and G_2 are defined by (5.1.9), and

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \qquad (6.1.9)$$

Expression (7) can be used for the integral equation formulation of the problem. The governing integral equation will take the form (Fabrikant, 1971d)

$$\frac{G_1}{2}\int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0 + \frac{G_2}{2}\int_0^{2\pi}\int_0^a \frac{q^2\bar{\tau}(\rho_0,\phi_0)}{R^3}\rho_0 d\rho_0 d\phi_0 = \frac{H\alpha P}{\rho e^{-i\phi} - r e^{-i\psi}} \qquad (6.1.10)$$

Here τ stands for τ_z as it was defined in (5.1.10). This equation has been solved in (Fabrikant, 1971d), and its exact solution reads

$$\tau(\rho,\phi) = -\frac{2PH\alpha}{\pi^2 G_1 r e^{-i\psi}(a^2-\rho^2)^{1/2}}\frac{1}{1-\bar{\zeta}}\left[1 + \left(\frac{\bar{\zeta}}{1-\bar{\zeta}}\right)^{1/2}\sin^{-1}(\bar{\zeta})^{1/2}\right]. \qquad (6.1.11)$$

Here $\bar{\zeta} = \rho e^{-i\phi}/(r e^{-i\psi})$. Expressions (2) can be used to obtain formulae for the potential functions in the case of a distributed loading. This will lead to computation of various integrals involving (11) and some functions of distance between points (see, for example, (4) and (5)). The simplest integral to compute is

$$I = \int_0^{2\pi}\int_0^a \frac{\tau(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0. \qquad (6.1.12)$$

Here τ is defined by (11), and R_0 is given by (3). Let us make use of the integral representation (1.2.21)

$$\frac{1}{R_0}=\frac{2}{\pi}\int_0^{\rho_0}\lambda\left(\frac{l_1^2(x)}{\rho\rho_0},\phi-\phi_0\right)\frac{[l_2^2(x)-x^2]^{1/2}\mathrm{d}x}{(r^2-x^2)^{1/2}[l_2^2(x)-l_1^2(x)]},$$

$$l_1(x)=\frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}-[(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x)=\frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\},$$

$$\lambda(k,\psi)=\frac{1-k^2}{1+k^2-2k\cos\psi}. \qquad (6.1.13)$$

and the series expansion for (11), namely,

$$\tau(\rho_0,\phi_0)=-\frac{2PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\frac{(\rho_0\mathrm{e}^{-i\phi_0})^n}{(r\mathrm{e}^{-i\psi})^{n+1}(a^2-\rho_0^2)^{1/2}}. \qquad (6.1.14)$$

Substitution of (13) and (14) in (12) yields, after integration with respect to ϕ_0

$$I=-\frac{8PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(r\mathrm{e}^{-i\psi})^{n+1}}\int_0^a\rho_0\mathrm{d}\rho_0\int_0^{\rho_0}\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)\mathrm{e}^{-i\phi}/\rho)^n\mathrm{d}x}{(\rho_0^2-x^2)^{1/2}(a^2-\rho_0^2)^{1/2}[l_2^2(x)-l_1^2(x)]}.$$

Changing the order of integration and consequent integration with respect to ρ_0 gives

$$I=-\frac{4PH\alpha}{\pi^{1/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(r\mathrm{e}^{-i\psi})^{n+1}}\int_0^a\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)\mathrm{e}^{-i\phi}/\rho)^n\mathrm{d}x}{l_2^2(x)-l_1^2(x)}. \qquad (6.1.15)$$

The summation in (15) can be performed, with the result

$$I=-\frac{4PH\alpha\rho\mathrm{e}^{i\phi}}{\pi G_1}\int_0^a\left[1+\frac{l_1(x)}{[b^2-l_1^2(x)]^{1/2}}\sin^{-1}\left(\frac{l_1(x)}{b}\right)\right]\frac{[l_2^2(x)-x^2]^{1/2}\mathrm{d}x}{[l_2^2(x)-l_1^2(x)][b^2-l_1^2(x)]}. \qquad (6.1.16)$$

By introducing a new variable $y=l_1(x)$, $x=y[1+z^2/(\rho^2-y^2)]^{1/2}$, the integral (16) will take the form

$$I = -\frac{4PH\alpha\rho e^{i\phi}}{\pi G_1}\int_0^{l_1}\left[1+\frac{y}{(b^2-y^2)^{1/2}}\sin^{-1}\left(\frac{y}{b}\right)\right]\frac{\mathrm{d}y}{(\rho^2-y^2)^{1/2}(b^2-y^2)}.$$

Throughout this section the abbreviations l_1 and l_2 denote $l_1(a)$ and $l_2(a)$ respectively. The last integral can be computed in an elementary manner, and the final result is

$$I = \frac{4PH\alpha}{\pi G_1\bar{q}}\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \tag{6.1.17}$$

Here q is defined by (6), and $b^2=\rho r e^{i(\phi-\psi)}$. In order to find the main potential functions (2), we need to compute the integral

$$I_1 = \int_0^{2\pi}\int_0^{a}\frac{\bar{q}\tau(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{R_0+z}. \tag{6.1.18}$$

This integral can be computed from (17) by means of application of the operator $\bar{\Lambda}$ to both sides of (17) and consequent integration of the result twice with respect to z. Application of $\bar{\Lambda}$ to (17) yields the following integral

$$\int_0^{2\pi}\int_0^{a}\frac{\bar{q}}{R_0^3}\tau(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0 = -\frac{4PH\alpha}{\pi G_1}\frac{l_1(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)(b^2-l_1^2)}\left[1+\frac{l_1}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \tag{6.1.19}$$

Integration of both sides of (19) with respect to z results in

$$\int_0^{2\pi}\int_0^{a}\frac{\bar{q}}{R_0(R_0+z)}\tau(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0 = -\frac{4PH\alpha}{\pi G_1}\left\{\frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\mathrm{d}x}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}\right\}. \tag{6.1.20}$$

Various formulae from Appendix 5 were used in the intermediary transformations. Yet another integration of (20) with respect to z gives

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0+z}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = \frac{4PH\alpha}{\pi G_1}\left\{\frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$\left.+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)dx}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\right\} \tag{6.1.21}$$

The last result allows us to define the potential functions (2) as follows:

$$F_1 = \frac{PH\gamma_1}{m_1-1}\left\{-\frac{2H\gamma_2\alpha}{\pi G_1}\left[f(z_1)+\bar{f}(z_1)\right]+\ln(R_1+z_1)\right\},$$

$$F_2 = \frac{PH\gamma_2}{m_2-1}\left\{-\frac{2H\gamma_1\alpha}{\pi G_1}\left[f(z_2)+\bar{f}(z_2)\right]+\ln(R_2+z_2)\right\},$$

$$F_3 = -\frac{i\gamma_3 PH\alpha}{\pi^2 A_{44}G_1}\left[f(z_3)-\bar{f}(z_3)\right]. \tag{6.1.22}$$

Here the notation was introduced

$$R_k = [\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_k^2]^{1/2},\quad z_k = z/\gamma_k,\quad \text{for}\quad k = 1,2,3;$$

$$f(z) = \frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)dx}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}. \tag{6.1.23}$$

Now the complete solution can be obtained by substitution of (22–23) into (5.1.6) and (5.1.12). The result is

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda\left[f(z_k)+\bar{f}(z_k)\right]+\frac{q\gamma_k}{R_k(R_k+z_k)}\right\}$$

$$+\frac{\gamma_3 PH\alpha}{\pi^2 A_{44}G_1}\Lambda\left[f(z_3)-\bar{f}(z_3)\right], \tag{6.1.24}$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1\gamma_k}\frac{\partial}{\partial z_k}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{1}{R_k}\right\}, \tag{6.1.25}$$

$$\sigma_1 = 2PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}\left[f(z_k)+\bar{f}(z_k)\right]-\frac{z}{R_k^3}\right\}, \tag{6.1.26}$$

$$\sigma_2 = 2PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda^2\left[f(z_k)+\bar{f}(z_k)\right]-\frac{\gamma_k q^2(2R_k+z_k)}{R_k^3(R_k+z_k)^2}\right\}$$

$$+\frac{2PH\alpha}{\pi^2 G_1\gamma_3}\Lambda^2\left[f(z_3)-\bar{f}(z_3)\right], \tag{6.1.27}$$

$$\sigma_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{z}{R_k^3}\right\}, \tag{6.1.28}$$

$$\tau_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi\gamma_k G_1}\Lambda\frac{\partial}{\partial z}\left[f(z_k)+\bar{f}(z_k)\right]+\frac{q}{R_k^3}\right\}$$

$$+\frac{PH\alpha}{\pi^2 G_1}\Lambda\frac{\partial}{\partial z_3}\left[f(z_3)-\bar{f}(z_3)\right]. \tag{6.1.29}$$

Here are the explicit expressions for various derivatives of f which will be needed

$$\frac{\partial f}{\partial z} = \frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\mathrm{d}x}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}, \tag{6.1.30}$$

$$\Lambda f = \frac{1}{\bar{q}}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{(\rho^2 - l_1^2)^{1/2}}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.31}$$

$$\Lambda \bar{f} = \frac{l_1 e^{i\phi}(\rho^2 - l_1^2)^{1/2}}{\rho(\bar{b}^2 - l_1^2)} - \frac{zre^{i\psi}}{\bar{b}^2 - a^2}\left\{\frac{1}{(\bar{b}^2 - a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right] + \frac{a}{\bar{b}^2} - \frac{(a^2 - l_1^2)^{1/2}}{\bar{b}^2 - l_1^2}\right\} + \frac{ze^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/\bar{b})\,dx}{[(a^2 - x^2)(\bar{b}^2 - x^2)]^{3/2}}, \tag{6.1.32}$$

$$\frac{\partial^2 f}{\partial z^2} = -\frac{l_1(\rho^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)(b^2 - l_1^2)}\left[1 + \frac{l_1}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.33}$$

$$\frac{\partial}{\partial z}\Lambda f = \frac{\rho e^{i\phi}(a^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)(b^2 - l_1^2)}\left[1 + \frac{l_1}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \tag{6.1.34}$$

$$\frac{\partial}{\partial z}\Lambda \bar{f} = \frac{(a^2 - l_1^2)^{1/2}}{\bar{b}^2 - l_1^2}\left[\frac{\rho e^{i\phi}}{l_2^2 - l_1^2} + \frac{re^{i\psi}}{\bar{b}^2 - a^2}\right] - \frac{l_1^3 e^{i\phi}(\rho^2 - l_1^2)\sin^{-1}(l_1/\bar{b})}{\rho(a^2 - l_1^2)^{1/2}(\bar{b}^2 - l_1^2)^{3/2}(l_2^2 - l_1^2)} - \frac{re^{i\psi}a}{\bar{b}^2(\bar{b}^2 - a^2)} - \frac{\rho_0 e^{i\phi_0}}{(\bar{b}^2 - a^2)^{3/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right] + \frac{e^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/\bar{b})\,dx}{[(a^2 - x^2)(\bar{b}^2 - x^2)]^{3/2}}, \tag{6.1.35}$$

$$\Lambda^2 f = -\frac{2}{\bar{q}^2}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{(\rho^2 - l_1^2)^{1/2}}{(b^2 - l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right] - \frac{e^{i\phi}(\rho^2 - l_1^2)^{1/2}}{\bar{q}(l_2^2 - l_1^2)}\left\{\frac{l_1}{\rho} + \frac{\rho(a^2 - l_1^2)}{l_1(b^2 - l_1^2)} + \frac{\sin^{-1}\left(\frac{l_1}{b}\right)}{(b^2 - l_1^2)^{1/2}}\left[\frac{l_2^2}{\rho} + \frac{\rho(a^2 - l_1^2)}{b^2 - l_1^2}\right]\right\}, \tag{6.1.36}$$

$$\Lambda^2 \bar{f} = \frac{zr^2 e^{2i\psi}}{(\bar{b}^2 - a^2)^2}\left\{\frac{3}{(\bar{b}^2 - a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2 - a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2 - l_1^2)^{1/2}}{(\bar{b}^2 - a^2)^{1/2}}\right)\right] + \frac{a(5\bar{b}^2 - 2a^2)}{\bar{b}^4}\right.$$

$$-\frac{(a^2-l_1^2)^{1/2}(5\bar{b}^2-2a^2-3l_1^2)}{(\bar{b}^2-l_1^2)^2}\Bigg\}-\frac{2re^{i(\phi+\psi)}(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)^2}\left[\frac{l_1}{\rho}+\frac{\rho(a^2-l_1^2)}{l_1(l_2^2-l_1^2)}\right]$$

$$+\frac{e^{2i\phi}l_1(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)(l_2^2-l_1^2)}\left[\frac{2l_1^2-\rho^2}{\rho^2}+\frac{2(a^2-l_1^2)}{\bar{b}^2-l_1^2}-\frac{l_1^3(\rho^2-l_1^2)\sin^{-1}(l_1/\bar{b})}{\rho^2(a^2-l_1^2)[\bar{b}^2-l_1^2]^{1/2}}\right]$$

$$+\frac{3ze^{2i\phi}}{\rho^2}\int_0^{l_1}\frac{x^6\sin^{-1}(x/\bar{b})\mathrm{d}x}{(a^2-x^2)^{5/2}(\bar{b}^2-x^2)^{3/2}}. \qquad (6.1.37)$$

Formulae (24–37) represent the main new results of this section. One can notice that some of the derivatives still contain uncomputed integrals, but the main advantage is that those integrals are single, rather than double, and that their integrands are non-singular which makes them easy to compute by any standard subroutine.

The main results are valid for isotropic bodies as well, provided that we substitute the elastic constants and compute the limits according to (5.1.14). These limits may be computed by using the L’Hôpital rule. The following scheme should be used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{f(z_1)}{m_1-1}+\frac{f(z_2)}{m_2-1}\right]=-f(z)-\frac{z}{2(1-\nu)}f'(z), \qquad (6.1.38)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1f(z_1)}{m_1-1}+\frac{m_2f(z_2)}{m_2-1}\right]=f(z)-\frac{z}{2(1-\nu)}f'(z), \qquad (6.1.39)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{[1-(1+m_1)(\gamma_3/\gamma_1)^2]f(z_1)}{m_1-1}+\frac{[1-(1+m_2)(\gamma_3/\gamma_2)^2]f(z_2)}{m_2-1}\right]$$

$$=\frac{2(1+\nu)f(z)+zf'(z)}{2(1-\nu)}, \qquad (6.1.40)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1f(z_1)}{m_1-1}+\frac{\gamma_2f(z_2)}{m_2-1}\right]=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \qquad (6.1.41)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{\gamma_1(m_1-1)}+\frac{m_2 f(z_2)}{\gamma_2(m_2-1)}\right]=\frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}. \tag{6.1.42}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1, \qquad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \tag{6.1.43}$$

and the symbol (′) indicates differentiation with respect to z. The field of displacements in the case of isotropy will take the form

$$u=\frac{1+\nu}{2\pi E}P\left\{\frac{zq}{R_0^3}-\frac{(1-2\nu)q}{R_0(R_0+z)}+\frac{2}{\pi}(1-2\nu)\Lambda f(z)-\frac{2\nu(1-2\nu)}{\pi(2-\nu)}\Lambda\bar{f}(z)\right.$$

$$\left.+\frac{1-2\nu}{\pi(2-\nu)}z\frac{\partial}{\partial z}\Lambda\left[f(z)+\bar{f}(z)\right]\right\}, \tag{6.1.44}$$

$$w=\frac{1+\nu}{2\pi E}P\left\{\frac{1-2\nu}{\pi(2-\nu)}\left[-(1-2\nu)[f'(z)+\bar{f}'(z)]+z[f''(z)+\bar{f}''(z)]\right]+\frac{2(1-\nu)}{R_0}+\frac{z^2}{R_0^3}\right\}. \tag{6.1.45}$$

The derivation of the field of stresses for the case of isotropy is left to the reader.

It is of interest to investigate the influence of crack neck on the field of displacements. This can be done by comparison of (44–45) with the case of an elastic half-space subjected to a normal concentrated load P which is given by the last two terms in (44–45). As we can see, the most difference will be achieved in the case of Poisson coefficient $\nu=0$, while in the other extreme, namely, $\nu=1/2$, both solutions coincide. The computations were made for the case $\nu=0$, $a=2$, $r=3$, $\psi=0$, $\phi=0$. The value of $u^*=(u/a)(2\pi E)/[P(1+\nu)]$ versus ρ/a for $z/a=0.0, 0.5, 1.0$ is plotted in Fig. 6.1.2. The negative value of ρ is understood as its value for $\phi=\pi$. A similar value of $w^*=(w/a)(2\pi E)/[P(1+\nu)]$ is plotted in Fig. 6.1.2. In both figures, the solid line curves correspond to formulae (44) and (45) respectively, while the broken line curves describe the field in an elastic half-space subjected to a normal load only. The results show that the field of normal displacements is practically unaffected even in this extreme case, while the field of tangential displacements differs significantly in the vicinity of the applied force and the crack neck. All the broken line curves

Fig. 6.1.2. The field of tangential displacements

Fig. 6.1.3. The field of normal displacements

in Fig. 6.1.2 go above the relevant solid line curves. A similar picture is observed in Fig. 6.1.3 for positive ρ, and it becomes reverse for negative ρ.

6.2. Penny-shaped crack under antisymmetric loading

Explicit expressions are derived for the field of stresses and displacements in a transversely isotropic space weakened by a penny-shaped crack and subjected to two antisymmetrically applied normal concentrated forces. The presented results may be used as Green's functions for a general case of antisymmetric loading so that the complete solution can be presented in quadratures. Several specific applications to fracture mechanics are considered.

We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic space. Though some axisymmetrical problems were considered before, we are unaware of any solution to the problem of a penny-shaped crack subjected to two antisymmetric normal forces applied at a *general* point. Knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems. For example, by using linear superposition of the solutions for symmetric and antisymmetric loading, we can obtain the solution to the problem of one-sided loading of a crack. Further application of the reciprocal theorem leads to the solution of the interaction problem between an external force and the crack.

Formulation of the antisymmetric circular crack problem. We consider a transversely isotropic elastic space weakened by a penny-shaped crack of radius a in the plane z=0 (Fig. 6.2.1). Let two point forces P be applied to the crack faces antisymmetrically in the Oz direction at the points with cylindrical coordinates $(r,\psi,0^+)$ and $(r,\psi,0^-)$. The problem, due to antisymmetric loading, can be reduced to that of a half-space $z\geq 0$, with the boundary conditions at the plane z=0

$$u = 0, \qquad \text{for } a\leq\rho\leq\infty, \qquad 0\leq\phi<2\pi;$$

$$\sigma = 0, \qquad \text{for } a\leq\rho\leq\infty, \qquad 0\leq\phi<2\pi;$$

$$\sigma = P\delta(\rho-r,\phi-\psi)/\rho, \qquad \text{for } 0\leq\rho\leq a, \qquad 0\leq\phi<2\pi;$$

$$\tau = 0, \qquad \text{for } 0\leq\rho\leq a, \qquad 0\leq\phi<2\pi. \tag{6.2.1}$$

It is known (Fabrikant, 1989a) that in the case of a transversely isotropic elastic half-space subjected to a general concentrated force with the components T_x, T_y and P, the complete solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1-1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1+z_1)\right],$$

Fig. 6.2.1. Loading of a penny-shaped crack

$$F_2 = \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1(\overline{\Lambda}\chi_2 + \Lambda\overline{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{4\pi A_{44}}(\overline{\Lambda}\chi_3 - \Lambda\overline{\chi}_3). \tag{6.2.2}$$

Here (ρ_0,ϕ_0) is the point of the boundary where the concentrated force is applied;

$$\chi_k(z) = \chi(z_k), \quad R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_k^2]^{1/2}, \quad \text{for} \quad k=1,2,3;$$

$$\chi(z) = T[z\ln(R_0 + z) - R_0], \quad T = T_x + \mathrm{i}T_y,$$

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}. \tag{6.2.3}$$

Substitution of (2–3) in (5.1.6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3 + z_3)^2}\right]$$

$$+\frac{H\gamma_2}{m_2-1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2}+\frac{q^2\overline{T}}{R_2(R_2+z_2)^2}\right]+\frac{Pq}{R_2(R_2+z_2)}\right\}$$

$$+\frac{H\gamma_1}{m_1-1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1}+\frac{q^2\overline{T}}{R_1(R_1+z_1)^2}\right]+\frac{Pq}{R_1(R_1+z_1)}\right\}, \tag{6.2.4}$$

$$w=H\left\{\frac{1}{2}(T\overline{q}+\overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)}+\frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]\right.$$

$$\left.+P\left[\frac{m_1}{(m_1-1)R_1}+\frac{m_2}{(m_2-1)R_2}\right]\right\}. \tag{6.2.5}$$

Here

$$q=\rho e^{i\phi}-\rho_0 e^{i\phi_0}. \tag{6.2.6}$$

Expressions (4) and (5) simplify for the case when $z=0$:

$$u=\frac{1}{2}G_1\frac{T}{R}+\frac{1}{2}G_2\frac{\overline{T}q^2}{R^3}-H\alpha\frac{P}{\overline{q}}, \tag{6.2.7}$$

$$w=H\alpha\Re\left(\frac{T}{q}\right)+H\frac{P}{R}. \tag{6.2.8}$$

Here $\Re$ is the real part sign; H, α, G_1, and G_2 are defined by (5.1.9), and

$$R=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \tag{6.2.9}$$

The complete solution. Expression (7) can be used for the integral equation formulation of the problem. The governing integral equation will take the form (Fabrikant, 1989a)

$$\frac{G_1}{2}\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0+\frac{G_2}{2}\int_0^{2\pi}\int_a^{\infty}\frac{q^2\overline{\tau}(\rho_0,\phi_0)}{R^3}\rho_0 d\rho_0 d\phi_0=\frac{H\alpha P}{\rho e^{-i\phi}-re^{-i\psi}} \tag{6.2.10}$$

Here τ stands for $-\tau_z$ as it was defined in (5.1.10). A general solution to this equation can be found in (Fabrikant, 1989a), and its exact solution in this case is elementary. It reads

$$\tau(\rho,\phi) = \frac{PH\alpha e^{i\phi}}{\pi G_1\rho(\rho^2 - a^2)^{1/2}}\left[\frac{G_2}{G_1 - G_2} + \frac{1}{(1-\bar{\zeta})^{3/2}}\right]. \tag{6.2.11}$$

Here $\bar{\zeta} = re^{-i\psi}/(\rho e^{-i\phi})$. Expressions (2) can be used to obtain formulae for the potential functions in the case of a distributed loading. This will lead to computation of various integrals involving (11) and some functions of distance between points (see, for example, (4) and (5)). The simplest integral to compute is

$$\int_0^{2\pi}\int_a^{\infty} \frac{\tau(\rho_0,\phi_0)}{R_0}\rho_0 d\rho_0 d\phi_0. \tag{6.2.12}$$

Here τ is defined by (11), and R_0 is given by (3). We need to compute the following integral

$$I = \int_0^{2\pi}\int_a^{\infty} \frac{e^{i\phi_0} d\rho_0 d\phi_0}{R_0(\rho_0^2 - a^2)^{1/2}[1 - re^{-i\psi}/(\rho_0 e^{-i\phi_0})]^{3/2}} \tag{6.2.13}$$

Let us make use of the integral representation (see Example 13, Chapter 1)

$$\frac{1}{R_0} = \frac{2}{\pi}\int_{\rho_0}^{\infty} \lambda\left(\frac{\rho\rho_0}{l_2^2(x)}, \phi - \phi_0\right)\frac{[x^2 - l_1^2(x)]^{1/2} dx}{(x^2 - \rho_0^2)^{1/2}[l_2^2(x) - l_1^2(x)]}, \tag{6.2.14}$$

where

$$l_1(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} - [(\rho - x)^2 + z^2]^{1/2}\},$$

$$l_2(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} + [(\rho - x)^2 + z^2]^{1/2}\}, \tag{6.2.15}$$

$$\lambda(k,\psi) = \frac{1 - k^2}{1 + k^2 - 2k\cos\psi}, \tag{6.2.16}$$

and the series expansion

$$(1-\zeta)^{-3/2} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \zeta^n. \tag{6.2.17}$$

Substitution of (14) and (17) in (13) yields, after integration with respect to ϕ_0

$$I = \frac{8e^{i\psi}}{r\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \int_a^{\infty} \frac{\rho_0 d\rho_0}{(\rho_0^2 - a^2)^{1/2}} \int_{\rho_0}^{\infty} \left(\frac{\rho r e^{i(\phi-\psi)}}{l_2^2(x)}\right)^{n+1} \frac{[x^2 - l_1^2(x)]^{1/2} dx}{[l_2^2(x) - l_1^2(x)](x^2 - \rho_0^2)^{1/2}}. \tag{6.2.18}$$

Changing the order of integration in (18) and consequent integration with respect to ρ_0 and summation gives

$$I = 2\pi\rho e^{i\phi} \int_a^{\infty} \frac{l_2(x) dl_2(x)}{[l_2^2(x) - \rho^2]^{1/2} [l_2^2(x) - \rho r e^{i(\phi-\psi)}]^{3/2}} \tag{6.2.19}$$

Here we used (17) and the substitution

$$\frac{dl_2(x)}{dx} = \frac{x(l_2^2(x) - \rho^2)}{l_2(x)[l_2^2(x) - l_1^2(x)]} \tag{6.2.20}$$

The last integral (19) can be computed in elementary manner, and the final result is

$$I = \frac{2\pi}{\rho e^{-i\phi} - r e^{-i\psi}} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}}\right]. \tag{6.2.21}$$

Here $b^2 = \rho r e^{i(\phi-\psi)}$. Throughout this section the abbreviations l_1 and l_2 denote $l_1(a)$ and $l_2(a)$ respectively. The last result (21) allows us to compute

$$\int_0^{2\pi} \int_a^{\infty} \frac{\tau(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0}{R_0} = \frac{2PH\alpha}{G_1} \left\{ \frac{1}{\rho e^{-i\phi} - r e^{-i\psi}} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - b^2)^{1/2}}\right] \right.$$
$$\left. + \frac{G_2 e^{i\phi}}{(G_1 - G_2)\rho} \left[1 - \frac{(l_2^2 - \rho^2)^{1/2}}{l_2}\right] \right\}. \tag{6.2.22}$$

In order to find the main potential functions (2), we need to compute the integral

$$I_1 = \int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{R_0+z}. \tag{6.2.23}$$

This integral can be computed from (22) by means of application of the operator $\bar{\Lambda}$ to both sides of (22) and consequent integration of the result twice with respect to z. Application of $\bar{\Lambda}$ to (22) yields the following integral

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0^3}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left[\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}} + \frac{G_2}{G_1-G_2}\frac{(l_2^2-\rho^2)^{1/2}}{l_2(l_2^2-l_1^2)}\right]. \tag{6.2.24}$$

Integration of both sides of (24) with respect to z results in

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0(R_0+z)}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left\{\frac{1}{a^2-b^2}\left[aE\left(\sin^{-1}(\frac{a}{l_2}),\frac{b}{a}\right) - \frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right] + \frac{G_2}{(G_1-G_2)a}\sin^{-1}(\frac{a}{l_2})\right\}. \tag{6.2.25}$$

Here $E(\cdot,\cdot)$ stands for the incomplete elliptic integral of the second kind. Various formulae from Appendix 5 were used in the intermediary transformations. Yet another integration of (25) with respect to z gives

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}}{R_0+z}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 = -\frac{2PH\alpha}{G_1}\left[f(z) + \frac{G_2}{G_1-G_2}f_0(z)\right], \tag{6.2.26}$$

where

$$f(z) = -\frac{z}{a^2-b^2}\left[aE\left(\sin^{-1}(\frac{a}{l_2}),\frac{b}{a}\right) - \frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right] + \frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}} - \ln[(l_2^2-b^2)^{1/2} + (l_2^2-\rho^2)^{1/2}], \tag{6.2.27}$$

$$f_0(z) = -\frac{z}{a}\sin^{-1}(\frac{a}{l_2}) + \frac{(a^2-l_1^2)^{1/2}}{a} - \ln[l_2 + (l_2^2-\rho^2)^{1/2}]. \tag{6.2.28}$$

Strictly speaking, the integral in (26) is divergent, so that the right hand side in (26) represents the finite part of the divergent integral. This result was obtained by using the following convergent integral

$$\int_0^{2\pi}\int_a^{\infty} \frac{\bar{q}\mathrm{e}^{i\phi_0}\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{(R_0+z)(\rho_0^2-a^2)^{1/2}}\left[\left(1-\frac{r\mathrm{e}^{-i\psi}}{\rho_0\mathrm{e}^{-i\phi_0}}\right)^{-3/2}-1\right] = 2\pi[-f(z)+f_0(z)]. \tag{6.2.29}$$

We notice that in the limiting case of $r=0$ function f coincides with f_0. Formulae (26-28) allow us to define the potential functions (2) as follows:

$$F_1 = \frac{PH}{m_1-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Re\{f(z_1)\} + \frac{G_2}{G_1}f_0(z_1)\right] + \gamma_1\ln(D_1+z_1)\right\},$$

$$F_2 = \frac{PH}{m_2-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Re\{f(z_2)\} + \frac{G_2}{G_1}f_0(z_2)\right] + \gamma_2\ln(D_2+z_2)\right\},$$

$$F_3 = -\frac{PH\alpha\gamma_3}{\pi A_{44}G_1}\Im\{f(z_3)\}. \tag{6.2.30}$$

Here $f(\cdot)$ and $f_0(\cdot)$ are defined by (27) and (28), and the following notation was introduced:

$$D_k = [\rho^2 + r^2 - 2\rho r\cos(\phi-\psi) + z_k^2]^{1/2},\quad z_k = z/\gamma_k,\quad \text{for } k = 1,2,3; \tag{6.2.31}$$

Now the complete solution can be obtained by substitution of (30) into (5.1.6) and (5.1.12). The result is

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda\Re\{f(z_k)\} + \frac{G_2}{G_1}\Lambda f_0(z_k)\right] + \frac{(\rho\mathrm{e}^{i\phi}-r\mathrm{e}^{i\psi})\gamma_k}{D_k(D_k+z_k)}\right\}$$

$$-\frac{i\gamma_3 PH\alpha}{\pi A_{44}G_1}\Lambda\Im\{f(z_3)\}, \tag{6.2.32}$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\} - \frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right] + \frac{1}{D_k}\right\}, \tag{6.2.33}$$

$$\sigma_1 = 2PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\frac{\partial^2}{\partial z_k^2}\Re\{f(z_k)\}+\frac{G_2}{G_1}\frac{(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}\right]-\frac{z}{D_k^3}\right\}, \tag{6.2.34}$$

$$\sigma_2 = 2PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\Lambda^2\Re\{f(z_k)\}+\frac{G_2}{G_1}\Lambda^2 f_0(z_k)\right]\right.$$

$$\left.-\frac{\gamma_k(\rho e^{i\phi}-re^{i\psi})^2(2D_k+z_k)}{D_k^3(D_k+z_k)^2}\right\}-\frac{2PH\alpha}{\pi G_1\gamma_3}\Lambda^2\Im\{f(z_3)\}, \tag{6.2.35}$$

$$\sigma_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left\{\alpha\left[\left(1-\frac{G_2}{G_1}\right)\frac{l_{2k}^2[l_{2k}^2-\rho^2]^{1/2}}{l_{2k}^2-l_{1k}^2}\Re\{(l_{2k}^2-b^2)^{-3/2}\}\right.\right.$$

$$\left.\left.+\frac{G_2}{G_1}\frac{(a^2-l_{1k}^2)^{1/2}}{a(l_{2k}^2-l_{1k}^2)}\right]-\frac{z}{D_k^3}\right\}, \tag{6.2.36}$$

$$\tau_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1-\frac{G_2}{G_1}\right)\Lambda\frac{\partial}{\partial z_k}\Re\{f(z_k)\}+\frac{G_2}{G_1}\Lambda\frac{\partial}{\partial z_k}f_0(z_k)\right]\right.$$

$$\left.-\frac{\rho e^{i\phi}-re^{i\psi}}{D_k^3}\right\}-\frac{PH\alpha}{\pi G_1}\Lambda\frac{\partial}{\partial z_3}\Im\{f(z_3)\}. \tag{6.2.37}$$

We recall that throughout this section the notations $\Re$ and $\Im$ stand for the real and imaginary part respectively. Here are the explicit expressions for various derivatives of f which will be needed

$$\frac{\partial f(z)}{\partial z} = -\frac{1}{a^2-b^2}\left[aE\left(\sin^{-1}\left(\frac{a}{l_2}\right),\frac{b}{a}\right)-\frac{b^2(l_2^2-a^2)^{1/2}}{l_2(l_2^2-b^2)^{1/2}}\right], \tag{6.2.38}$$

$$\Lambda f(z) = -\frac{1}{\rho e^{-i\phi}-re^{-i\psi}}\left[1-\frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}}\right], \tag{6.2.39}$$

$$\Lambda\bar{f}(z)=\frac{e^{i\phi}}{\rho}\left[\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-\bar{b}^2)^{3/2}}-1\right]+\frac{z}{a^2-\bar{b}^2}\left\{\frac{ae^{i\phi}}{\rho}\left[F\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)-\frac{a^2+\bar{b}^2}{a^2-\bar{b}^2}E\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right]\right.$$

$$\left.+\frac{re^{i\psi}(l_2^2-a^2)^{1/2}}{l_2(l_2^2-\bar{b}^2)^{1/2}}\left[\frac{2a^2}{a^2-\bar{b}^2}+\frac{\bar{b}^2}{l_2^2-\bar{b}^2}\right]\right\}, \tag{6.2.40}$$

$$\frac{\partial^2 f(z)}{\partial z^2}=\frac{l_2^2(l_2^2-\rho^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}} \tag{6.2.41}$$

$$\frac{\partial}{\partial z}\Lambda f(z)=\frac{l_2\rho e^{i\phi}(l_2^2-a^2)^{1/2}}{(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}}, \tag{6.2.42}$$

$$\frac{\partial}{\partial z}\Lambda\bar{f}(z)=\frac{l_2(l_2^2-a^2)^{1/2}}{(l_2^2-\bar{b}^2)^{3/2}}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{re^{i\psi}}{a^2-\bar{b}^2}\right]+\frac{ae^{i\phi}}{\rho(a^2-\bar{b}^2)}\left\{F\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right.$$

$$\left.+\frac{a^2+\bar{b}^2}{a^2-\bar{b}^2}\left[\frac{\bar{b}^2(l_2^2-a^2)^{1/2}}{al_2(l_2^2-\bar{b}^2)^{1/2}}-E\left(\sin^{-1}(\frac{a}{l_2}),\frac{\bar{b}}{a}\right)\right]\right\}, \tag{6.2.43}$$

$$\Lambda^2 f(z)=\frac{2}{(\rho e^{-i\phi}-re^{-i\psi})^2}\left[1-\frac{(l_2^2-\rho^2)^{1/2}}{(l_2^2-b^2)^{1/2}}\right]+\frac{\rho e^{i\phi}(l_2^2-\rho^2)^{1/2}(a^2b^2-l_2^4)}{(\rho e^{-i\phi}-re^{-i\psi})l_2^2(l_2^2-l_1^2)(l_2^2-b^2)^{3/2}}. \tag{6.2.44}$$

The necessary derivatives of f_0 are:

$$\Lambda f_0(z)=-\frac{e^{i\phi}}{\rho}\left[1-\frac{(a^2-l_1^2)^{1/2}}{a}\right], \tag{6.2.45}$$

$$\frac{\partial}{\partial z}f_0(z)=-\frac{1}{a}\sin^{-1}(\frac{a}{l_2}), \tag{6.2.46}$$

$$\frac{\partial}{\partial z}\Lambda f_0(z)=\frac{\rho e^{i\phi}(l_2^2-a^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}, \tag{6.2.47}$$

$$\frac{\partial^2}{\partial z^2}f_0(z)=\frac{(a^2-l_1^2)^{1/2}}{a(l_2^2-l_1^2)}, \tag{6.2.48}$$

$$\Lambda^2 f_0(z) = \frac{2e^{2i\phi}}{\rho^2}\left[1 - \frac{(a^2 - l_1^2)^{1/2}}{a}\right] - \frac{e^{2i\phi}(a^2 - l_1^2)^{1/2}}{a(l_2^2 - l_1^2)}. \tag{6.2.49}$$

Again, we notice that all the derivatives of f in the limiting case of $r = 0$ coincide with those of f_0. Formulae (31–38) represent the main new results of this section.

One-sided loading of a penny-shaped crack. Consider the case when a concentrated normal force P is applied to the positive side of the crack at the point $(r, \psi, 0^+)$, while the other side is stress-free. The complete solution to this problem can be obtained by the superposition of the symmetric loading solution (see Fabrikant, 1989a) and the antisymmetric one presented here (32–37).

$$u = \frac{HP}{\pi}\left[\frac{\gamma_1}{m_1 - 1} f_1(z_1) + \frac{\gamma_2}{m_2 - 1} f_1(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{1}{m_k - 1}\left\{\alpha\left[\left(1 - \frac{G_2}{G_1}\right)\Lambda\Re\{f(z_k)\}\right.\right.$$

$$\left.\left. + \frac{G_2}{G_1}\Lambda f_0(z_k)\right] + \frac{(\rho e^{i\phi} - r e^{i\psi})\gamma_k}{D_k(D_k + z_k)}\right\} - \frac{i\gamma_3 PH\alpha}{2\pi A_{44} G_1}\Lambda\Im\{f(z_3)\}, \tag{6.2.50}$$

$$w = \frac{HP}{\pi}\left[\frac{m_1}{m_1 - 1} f_2(z_1) + \frac{m_2}{m_2 - 1} f_2(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k - 1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1 - \frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\}\right.\right.$$

$$\left.\left. - \frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right] + \frac{1}{D_k}\right\}, \tag{6.2.51}$$

$$\sigma_1 = \frac{P}{\pi^2(\gamma_1 - \gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1 + 1)\gamma_3^2} - \frac{1}{\gamma_1}\right] f_3(z_1) - \left[\frac{\gamma_2}{(m_2 + 1)\gamma_3^2} - \frac{1}{\gamma_2}\right] f_3(z_2)\right\}$$

$$+ PHA_{66}\sum_{k=1}^{2}\frac{1 - (1 + m_k)(\gamma_3/\gamma_k)^2}{m_k - 1}\left\{\alpha\left[\left(1 - \frac{G_2}{G_1}\right)\frac{\partial^2}{\partial z_k^2}\Re\{f(z_k)\} + \frac{G_2}{G_1}\frac{(a^2 - l_1^2)^{1/2}}{a(l_2^2 - l_1^2)}\right] - \frac{z}{D_k^3}\right\}, \tag{6.2.52}$$

$$\sigma_2 = \frac{2}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1 - 1} f_4(z_1) + \frac{\gamma_2}{m_2 - 1} f_4(z_2)\right] + PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k - 1}\left\{\alpha\left[\left(1 - \frac{G_2}{G_1}\right)\Lambda^2\Re\{f(z_k)\}\right.\right.$$

$$+\frac{G_2}{G_1}\Lambda^2 f_0(z_k)\Bigg]-\frac{\gamma_k(\rho e^{i\phi}-re^{i\psi})^2(2D_k+z_k)}{D_k^3(D_k+z_k)^2}\Bigg\}-\frac{PH\alpha}{\pi G_1\gamma_3}\Lambda^2\Im\{f(z_3)\}, \tag{6.2.53}$$

$$\sigma_z=\frac{P}{2\pi^2(\gamma_1-\gamma_2)}\Big[\gamma_1 f_3(z_1)-\gamma_2 f_3(z_2)\Big]+\frac{P}{4\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\Bigg\{\alpha\Bigg[\Bigg(1$$

$$-\frac{G_2}{G_1}\Bigg)\frac{l_{2k}^2[l_{2k}^2-\rho^2]^{1/2}}{l_{2k}^2-l_{1k}^2}\Re\{(l_{2k}^2-b^2)^{-3/2}\}+\frac{G_2}{G_1}\frac{(a^2-l_{1k}^2)^{1/2}}{a(l_{2k}^2-l_{1k}^2)}\Bigg]-\frac{z}{D_k^3}\Bigg\}, \tag{6.2.54}$$

$$\tau_z=\frac{P}{2\pi^2(\gamma_1-\gamma_2)}\Big[f_5(z_1)-f_5(z_2)\Big]+\frac{P}{4\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\Bigg\{\frac{\alpha}{\gamma_k}\Bigg[\Bigg(1-\frac{G_2}{G_1}\Bigg)\Lambda\frac{\partial}{\partial z_k}\Re\{f(z_k)\}$$

$$+\frac{G_2}{G_1}\Lambda\frac{\partial}{\partial z_k}f_0(z_k)\Bigg]-\frac{\rho e^{i\phi}-re^{i\psi}}{D_k^3}\Bigg\}-\frac{PH\alpha}{2\pi G_1}\Lambda\frac{\partial}{\partial z_3}\Im\{f(z_3)\}. \tag{6.2.55}$$

where

$$f_1(z)=\frac{1}{\rho e^{-i\phi}-re^{-i\psi}}\Bigg[\frac{(a^2-r^2)^{1/2}}{(a^2-\overline{b}^2)^{1/2}}\tan^{-1}\frac{(a^2-\overline{b}^2)^{1/2}}{(l_2^2-a^2)^{1/2}}-\frac{z}{D}\tan^{-1}\Bigg(\frac{h}{D}\Bigg)\Bigg], \tag{6.2.56}$$

$$f_2(z)=\frac{1}{D}\tan^{-1}\Bigg(\frac{h}{D}\Bigg), \tag{6.2.57}$$

$$f_3(z)=\Bigg\{-\frac{z}{D^3}\tan^{-1}\Bigg(\frac{h}{D}\Bigg)+\frac{h}{z(D^2+h^2)}\Bigg[\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{z^2}{D^2}\Bigg]\Bigg\}, \tag{6.2.58}$$

$$f_4(z)=\frac{(a^2-r^2)^{1/2}}{(\rho e^{-i\phi}-re^{-i\psi})(a^2-\overline{b}^2)^{1/2}}\Bigg(\frac{re^{i\psi}}{a^2-\overline{b}^2}$$

$$-\frac{2}{\rho e^{-i\phi}-re^{-i\psi}}\Bigg)\tan^{-1}\Bigg(\frac{(a^2-\overline{b}^2)^{1/2}}{(l_2^2-a^2)^{1/2}}\Bigg)+\frac{z(3D^2-z^2)}{(\rho e^{-i\phi}-re^{-i\psi})^2D^3}\tan^{-1}\Bigg(\frac{h}{D}\Bigg)$$

$$-\frac{(a^2-r^2)^{1/2}(l_2^2-a^2)^{1/2}re^{i\psi}}{(\rho e^{-i\phi}-re^{-i\psi})(a^2-\overline{b}^2)[l_2^2-\rho re^{i(\phi-\psi)}]}$$

$$+\frac{zh}{D^2+h^2}\left[\frac{(\rho e^{i\phi}-re^{i\psi})}{(\rho e^{-i\phi}-re^{-i\psi})D^2}-\frac{\rho^2 e^{2i\phi}}{(l_2^2-l_1^2)(l_2^2-\rho^2)}\right], \tag{6.2.59}$$

$$f_5(z)=-\left\{\frac{\rho e^{i\phi}-re^{i\psi}}{D^3}\tan^{-1}\left(\frac{h}{D}\right)+\frac{h}{D^2+h^2}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{\rho e^{i\phi}-re^{i\psi}}{D^2}\right]\right\}. \tag{6.2.60}$$

The following notation was introduced:

$$D=[\rho^2+r^2-2\rho r\cos(\phi-\psi)+z^2]^{1/2},\quad h=(a^2-r^2)^{1/2}(a^2-l_1^2)^{1/2}/a. \tag{6.2.61}$$

As one can see from the structure of equations (50–55), the solution is given by the sum of the symmetric loading solution and the antisymmetric one. The case of normal loading of the negative crack face can be obtained as the difference of the two particular solutions.

Interaction of an external force with a penny-shaped crack. Consider a transversely isotropic space weakened in the plane $z=0$ by a penny-shaped crack of radius a. Let the crack faces be stress-free, and a concentrated force be applied in the Oz direction at the point (ρ,ϕ,z). Solution to this problem can be obtained by application of the reciprocal theorem to the general results obtained here. Let a unit normal force N be applied to the positive face of the crack at the point $(r,\psi,0^+)$. Denote w_N the normal displacements at the point (ρ,ϕ,z) due to the unit force N, and denote w_P the normal displacements at the point $(r,\psi,0^+)$ due to the external force P. Application of the reciprocal theorem immediately gives that

$$w_P=Pw_N, \tag{6.2.62}$$

which translates into

$$w(r,\psi,0^+)=\frac{PH}{\pi}\left[\frac{m_1}{m_1-1}f_2(z_1)+\frac{m_2}{m_2-1}f_2(z_2)\right]+\frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1\right.\right.\right.$$

$$\left.\left.\left.-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\}-\frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right]+\frac{1}{D_k}\right\}. \tag{6.2.63}$$

The normal displacements of the negative face of the crack can be found in a similar manner, with the result

$$w(r,\psi,0^-) = -\frac{PH}{\pi}\left[\frac{m_1}{m_1-1}f_2(z_1) + \frac{m_2}{m_2-1}f_2(z_2)\right] + \frac{1}{2}PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k}\left[\left(1\right.\right.\right.$$

$$\left.\left.\left.-\frac{G_2}{G_1}\right)\frac{\partial}{\partial z_k}\Re\{f(z_k)\} - \frac{G_2}{G_1 a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)\right] + \frac{1}{D_k}\right\}. \qquad (6.2.64)$$

The crack opening displacement can be obtained as the difference between (63) and (64). Computation of the crack shape was made for a transversely isotropic body, with the following values assigned to the elastic constants: $A_{11}=A_{33}=2.7777$, $A_{44}=A_{66}=1$, $A_{13}=0.6944$. The dimensionless quantity $w^*=wa/(PH)$ versus ρ/a is presented in Fig. 6.2.2 for $r/a=1.5$, $\phi=0$, $\psi=0$, and $z/a=0,1,2$. Since the crack

Fig. 6.2.2. Crack shape due to an external force

opening displacement is zero for $z=0$, and it tends to zero for $z\to\infty$, there should be a location for an external force where it produces maximum crack opening.

Discussion. Some particular cases of interest are considered here as well as some applications to various problems in fracture mechanics.

The main results are valid for isotropic bodies provided that we substitute the elastic constants and compute the limits according to (5.1.13). These limits may be computed by using the L'Hôpital rule (see 6.1.38–6.1.43) for the scheme involved. The field of displacements in the case of isotropy will take the form

$$u=\frac{(1+\nu)P}{2\pi E}\left\{-\frac{(1-2\nu)}{2-\nu}\left[(2-\nu)\Lambda f(z)+\nu\Lambda[f_0(z)-\bar{f}(z)]\right.\right.$$

$$\left.\left.+z\Lambda\frac{\partial}{\partial z}\left(\Re\{f(z)\}+\frac{\nu}{2(1-\nu)}f_0(z)\right)\right]+\frac{\rho e^{i\phi}-re^{i\psi}}{D}\left[\frac{z}{D^2}-\frac{1-2\nu}{D+z}\right]\right\}, \qquad (6.2.65)$$

$$w=\frac{(1+\nu)P}{2\pi E}\left\{\frac{(1-2\nu)^2}{2-\nu}\left[\Re\{f'(z)\}+\frac{\nu}{2(1-\nu)}f_0'(z)\right]\right.$$

$$\left.-\frac{1-2\nu}{2-\nu}z\left[\Re\{f''(z)+\frac{\nu}{2(1-\nu)}f_0''(z)\right]+\frac{2(1-\nu)}{D}+\frac{z^2}{D^3}\right\}. \qquad (6.2.66)$$

A comparison can be made between the field of displacements due to a concentrated normal loading of a half-space with the field defined by (65) and (66). Both fields coincide for $\nu=1/2$, and their difference increases with ν decreasing. Computations were made for $\nu=0$, $r=0.5a$, and different values of z. The dimensionless parameter $u^*=2\pi Eu/[(1+\nu)P]$ versus ρ/a is presented in Fig. 6.2.3. The value of $w^*=2\pi Ew/[(1+\nu)P]$ versus ρ/a for $z/a=0,2$ is presented in Fig. 6.2.4. Solid line curves in both figures correspond to formulae (65) and (66) respectively, while the broken line curves give the results due to the concentrated loading of the half-space. The derivation of the field of stresses for the case of isotropy is left to the reader.

In the case of axial symmetry $r=0$, and the complete solution can be expressed in terms of elementary functions. The main potential functions will take the form

$$F_1=\frac{PH}{m_1-1}\left[\alpha f_0(z_1)+\gamma_1\ln[(\rho^2+z_1^2)^{1/2}+z_1]\right], \qquad (6.2.67)$$

$$F_2=\frac{PH}{m_2-1}\left[\alpha f_0(z_2)+\gamma_2\ln[(\rho^2+z_2^2)^{1/2}+z_2]\right], \qquad (6.2.68)$$

$$F_3=0. \qquad (6.2.69)$$

We recall that f_0 is defined by (28). The field of displacements is

Fig. 6.2.3. Comparison of tangential displacements

Fig. 6.2.4. Comparison of normal displacements

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\alpha\frac{e^{i\phi}}{\rho}\left[1-\frac{(a^2-l_{1k}^2)^{1/2}}{a}\right]+\frac{\rho e^{i\phi}\gamma_k}{(\rho^2+z_k^2)^{1/2}[(\rho^2+z_k^2)^{1/2}+z_k]}\right\}, \tag{6.2.70}$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\frac{\alpha}{\gamma_k a}\sin^{-1}\left(\frac{a}{l_{2k}}\right)+\frac{1}{(\rho^2+z_k^2)^{1/2}}\right\}. \tag{6.2.71}$$

Again, we leave the derivation of the field of stresses to the reader. It can be done easily by using the appropriate differentiation formulae from Appendix 5.

The stress intensity factors (SIF) play an important role in fracture mechanics. The solution presented can be used for computing the SIF in various particular cases. Define the three mode stress intensity factors as follows:

$$k_1 = \lim_{\rho\to a}\{(\rho-a)^{1/2}\sigma_z\}, \tag{6.2.72}$$

$$k_2 + ik_3 = \lim_{\rho\to a}\{(\rho-a)^{1/2}\tau_z e^{-i\phi}\}, \tag{6.2.73}$$

Since the normal stress in the plane $z=0$ vanishes in the case of antisymmetric loading, so does the mode I SIF k_1. The remaining SIF can be obtained from (11) as follows:

$$k_2(\phi)+ik_3(\phi) = \frac{PH\alpha}{\pi a\sqrt{2a}G_1}\left\{\frac{G_2}{G_1-G_2}+\frac{1}{[1-(r/a)e^{i(\phi-\psi)}]^{3/2}}\right\}. \tag{6.2.74}$$

A similar result can be obtained by using the general expression of SIF through the limiting values of displacements (Fabrikant, 1989a)

$$k_2+ik_3 = -\frac{a}{\pi(G_1^2-G_2^2)\sqrt{2a}}\lim_{\rho\to a}\left[\frac{G_1 e^{-i\phi}u(\rho,\phi)+G_2 e^{i\phi}\overline{u}(\rho,\phi)}{(a^2-\rho^2)^{1/2}}\right]. \tag{6.2.75}$$

Here u stands for the complex tangential displacements in the plane $z=0$. Indeed, one can get from (32)

$$u(\rho,\phi,0) = PH\alpha(a^2-\rho^2)^{1/2}\left\{\frac{G_2 e^{i\phi}}{G_1\rho}\left[\frac{a^2}{(a^2-\overline{b}^2)^{3/2}}-\frac{1}{a}\right]\right.$$

$$\left.-\frac{1}{(\rho e^{-i\phi}-re^{-i\psi})(a^2-b^2)^{1/2}}\right\}, \tag{6.2.76}$$

and the substitution of (76) in (75) leads to (74).

In the case of one-sided loading of a penny-shaped crack, the SIF are

$$k_1(\phi) = \frac{P}{2\pi^2\sqrt{2a}}\frac{(a^2-r^2)^{1/2}}{a^2+r^2-2ar\cos(\phi-\psi)}, \tag{6.2.77}$$

$$k_2(\phi)+ik_3(\phi) = \frac{PH\alpha}{2\pi a\sqrt{2a}G_1}\left\{\frac{G_2}{G_1-G_2}+\frac{1}{[1-(r/a)e^{i(\phi-\psi)}]^{3/2}}\right\}. \tag{6.2.78}$$

Formulae (77) and (78) allow us to consider some cases of one-sided distributed loading. Let the load of intensity p per unit length be uniformly distributed along a circle of radius r. Direct integration results in

$$k_1 = \frac{p}{\pi\sqrt{2a}(a^2-r^2)^{1/2}}, \tag{6.2.79}$$

$$k_2 = \frac{p\alpha}{2a\gamma_1\gamma_2\sqrt{2a}}, \quad k_3 = 0. \tag{6.2.80}$$

Notice that k_2 does not depend on the loading location. The case of uniform loading p distributed over an annulus $r_1 \le r \le r_2$ can be considered in a similar manner. The result is

$$k_1 = \frac{p[(a^2-r_1^2)^{1/2}-(a^2-r_2^2)^{1/2}]}{\pi\sqrt{2a}}, \tag{6.2.81}$$

$$k_2 = \frac{p\alpha[r_2^2-r_1^2]}{4a\gamma_1\gamma_2\sqrt{2a}}. \tag{6.2.82}$$

Introduction of the total load $P = \pi p(r_2^2 - r_1^2)$ transforms (81) and (82) into

$$k_1 = \frac{P}{\pi^2\sqrt{2a}[(a^2-r_1^2)^{1/2}+(a^2-r_2^2)^{1/2}]}, \tag{6.2.83}$$

$$k_2 = \frac{P\alpha}{4\pi a\gamma_1\gamma_2\sqrt{2a}}.$$

Again, the value of k_2 does not depend on the location of the loading.

6.3. Annular crack under general normal loading

The flat annular crack in an elastic space under the action of a uniform pressure was considered by a number of authors starting with the early work of Smetanin (1968) and ending with the most recent work of Clements and Ang (1988) where the reader can find additional references. All these publications treat only the simplest case of uniform loading. To our knowledge, no attempt has been made to solve the general non-axisymmetric problem. Such a solution is presented here for the case of a transversely isotropic body. The method is based on the results described in Fabrikant (1989a). The problem is reduced to a set of two two-dimensional integral equations of Fredholm type with elementary kernels. The equations can be easily uncoupled and solved numerically with high accuracy.

Formulation of the problem. Consider a transversely isotropic elastic space weakened in the plane $z=0$ by a flat annular crack $b\le\rho\le a$. An arbitrary pressure $p(\rho,\phi)$ is applied to the crack faces in opposite directions. Due to the symmetry, the problem can be reduced to a mixed boundary value problem for an elastic half-space, with the following boundary conditions prescribed on the plane $z=0$:

$$w=0 \quad \text{for} \quad 0<\rho<b \quad \text{or} \quad \text{for} \quad \rho>a;$$

$$\sigma_z=-p(\rho,\phi) \quad \text{for} \quad b\le\rho\le a. \tag{6.3.1}$$

Here w denotes the normal displacement, and σ_z is the normal stress. If a normal concentrated load P is applied to the boundary of the half-space at the point $(\rho_0,\phi_0,0)$, then the normal displacements at $z=0$ can be defined as (Fabrikant, 1989a)

$$w(\rho,\phi)=H\frac{P}{R} \tag{6.3.2}$$

Here H is an elastic constant which in the case of isotropy transforms into $(1-\nu^2)/(\pi E)$, and

$$R=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{6.3.3}$$

In the case where the normal pressure σ is prescribed all over the plane $z=0$, the relevant normal displacements can be related to it in two different ways, namely, (Fabrikant, 1989b):

$$w(\rho,\phi)=4H\int_0^\rho\frac{dx}{(\rho^2-x^2)^{1/2}}\int_x^\infty\frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{1/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \tag{6.3.4}$$

$$w(\rho,\phi) = 4H\int_{\rho}^{\infty}\frac{\mathrm{d}x}{(x^2-\rho^2)^{1/2}}\int_0^x\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{6.3.5}$$

Here the $\mathcal{L}$-operator is defined as

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi}\lambda(k,\phi-\phi_0)\,f(\phi_0)\,\mathrm{d}\phi_0,$$

with

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}.$$

We can not use (4) and (5) directly since σ is known only on the interval $b\le\rho\le a$. Let us introduce two new unknowns σ_1 and σ_2 which are assumed to be equal to the normal pressure in the intervals $0\le\rho<b$ and $\rho>a$ respectively. These unknowns can be found from the first boundary condition (1). In the case $\rho>a$ it is convenient to use (5) which can be rewritten as

$$0 = 4H\int_{\rho}^{\infty}\frac{\mathrm{d}x}{(x^2-\rho^2)^{1/2}}\left\{\int_0^b\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi)\right.$$

$$\left.+\int_b^a\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)p(\rho_0,\phi)+\int_a^x\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi)\right\},$$

which immediately leads to

$$\int_0^b\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_1(\rho_0,\phi)+\int_b^a\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho_0}{x}\right)p(\rho_0,\phi)$$

$$+\int_a^x\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{1/2}}\,\mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_2(\rho_0,\phi) = 0 \tag{6.3.6}$$

Application of the operator

$$\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{x\,\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\,\mathcal{L}(x)$$

to both sides of (6) leads to the first governing equation

$$\sigma_2(\rho,\phi)+\frac{2}{\pi}\frac{1}{(\rho^2-a^2)^{1/2}}\int_0^b \frac{(a^2-\rho_0^2)^{1/2}}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma_1(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0$$

$$=-\frac{2}{\pi}\frac{1}{(\rho^2-a^2)^{1/2}}\int_b^a \frac{(a^2-\rho_0^2)^{1/2}}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)p(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0 \quad \text{for} \quad \rho>a. \tag{6.3.7}$$

A similar utilization of (4) for $0\le\rho\le b$ leads to yet another equation

$$\sigma_1(\rho,\phi)+\frac{2}{\pi}\frac{1}{\sqrt{b^2-\rho^2}}\int_a^\infty \frac{\sqrt{\rho_0^2-b^2}}{\rho_0^2-\rho^2}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0$$

$$=-\frac{2}{\pi}\frac{1}{\sqrt{b^2-\rho^2}}\int_b^a \frac{\sqrt{\rho_0^2-b^2}}{\rho_0^2-\rho^2}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)p(\rho_0,\phi)\,\rho_0\,\mathrm{d}\rho_0 \quad \text{for} \quad 0\le\rho<b. \tag{6.3.8}$$

Though the kernels of integral equations (7) and (8) look different, a special transformation can make them identical. Indeed, let us substitute in (8) the new variables x and t which are related to the old ones ρ and ρ_0 as follows: $\rho=\sqrt{ab}\,t$ and $\rho_0=\sqrt{ab}/x$. The result reads

$$\sigma_1(\sqrt{ab}\,t,\phi)+\frac{2}{\pi}\frac{1}{\sqrt{k^2-t^2}}\int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,\sigma_2(\sqrt{ab}/x,\phi)\,\frac{\mathrm{d}x}{x^2}$$

$$=-\frac{2}{\pi}\frac{1}{\sqrt{k^2-t^2}}\int_k^{1/k} \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,p(\sqrt{ab}/x,\phi)\,\frac{\mathrm{d}x}{x^2} \tag{6.3.9}$$

In a similar manner, substitution of $\rho=\sqrt{ab}/t$ and $\rho_0=\sqrt{ab}\,x$ in (7) results in

$$\sigma_2(\sqrt{ab}/t,\phi)+\frac{2}{\pi}\frac{t^3}{\sqrt{k^2-t^2}}\int_0^k\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,\sigma_1(\sqrt{ab}\,x,\phi)\,x\,\mathrm{d}x$$

$$=-\frac{2}{\pi}\frac{t^3}{\sqrt{k^2-t^2}}\int_0^k\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,\mathcal{L}(xt)\,p(\sqrt{ab}\,x,\phi)\,x\,\mathrm{d}x. \tag{6.3.10}$$

Equations (9) and (10) can be rewritten as

$$s_1(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_2(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_1(t,\phi), \tag{6.3.11}$$

$$s_2(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_1(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_2(t,\phi). \tag{6.3.12}$$

Here the following notation has been introduced

$$s_1(t,\phi)=\sigma_1(\sqrt{ab}t,\phi),\quad s_2(t,\phi)=\sigma_2(\sqrt{ab}/t,\phi)/t^3,\quad R_{xt}^2=1+x^2t^2-2xt\cos(\phi-\phi_0), \tag{6.3.13}$$

$$g_1(t,\phi)=-\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_k^{1/k}\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,p(\sqrt{ab}/x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0$$

$$g_2(t,\phi)=-\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_k^{1/k}\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,p(\sqrt{ab}x,\phi_0)\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi_0. \tag{6.3.14}$$

Equations (11) and (12) can be easily uncoupled by simple summation and subtraction

$$s_+(t,\phi)+\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^k\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}\,s_+(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_+(t,\phi), \tag{6.3.15}$$

$$s_-(t,\phi)-\frac{1}{\pi^2\sqrt{k^2-t^2}}\int_0^{2\pi}\int_0^{k}\frac{\sqrt{1-k^2x^2}}{R_{xt}^2}s_-(x,\phi_0)\,x\,\mathrm{d}x\,\mathrm{d}\phi_0=g_-(t,\phi). \tag{6.3.16}$$

Here we denoted

$$s_{\pm}=s_1\pm s_2,\quad g_{\pm}=g_1\pm g_2. \tag{6.3.17}$$

Equations (15) and (16) seem to be new. They represent the main result of this section.

Examples. Equations (15) and (16) can be solved numerically for arbitrary g_1 and g_2. One should though eliminate the singularity by multiplying both sides of each equation by the term $\sqrt{k^2-t^2}$. In the case of axial symmetry, equations (15) and (16) simplify as follows:

$$s_+(t)+\frac{2}{\pi\sqrt{k^2-t^2}}\int_0^{k}\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}s_+(x)\,x\,\mathrm{d}x=g_+(t), \tag{6.3.18}$$

$$s_-(t)-\frac{2}{\pi\sqrt{k^2-t^2}}\int_0^{k}\frac{\sqrt{1-k^2x^2}}{1-x^2t^2}s_-(x)\,x\,\mathrm{d}x=g_-(t). \tag{6.3.19}$$

We rewrite equations (18) and (19) as

$$S_+(t)+\frac{2}{\pi}\int_0^{k}\frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}\,(1-x^2t^2)}S_+(x)\,x\,\mathrm{d}x=h_+(t), \tag{6.3.20}$$

$$S_-(t)-\frac{2}{\pi}\int_0^{k}\frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}\,(1-x^2t^2)}S_-(x)\,x\,\mathrm{d}x=h_-(t). \tag{6.3.21}$$

The relevant notation is

$$S_{\pm}(t)=s_{\pm}(t)\sqrt{k^2-t^2},\quad h_{\pm}(t)=g_{\pm}(t)\sqrt{k^2-t^2}. \tag{6.3.22}$$

Consider in more detail the case of a uniform loading $p = \text{const.}$ In this case formulae (14) yield

$$h_{\pm}(t) = -\frac{2}{\pi}p\left\{\frac{\sqrt{1-k^4}}{k} - \sqrt{k^2-t^2}\;\sin^{-1}\frac{\sqrt{1-k^4}}{\sqrt{1-k^2t^2}}\right.$$

$$\left.\pm\frac{1}{t^2}\left[\sqrt{1-k^4} - \frac{\sqrt{k^2-t^2}}{t}\sin^{-1}\frac{t\sqrt{1-k^4}}{k\sqrt{1-k^2t^2}}\right]\right\}. \quad (6.3.23)$$

Now we give a description of the numerical method used for solving (9) and (10). Consider the following integral equation:

$$S(t) + \int_0^k \mathcal{K}(t,x)\,S(x)\,\mathrm{d}x = h(t). \quad (6.3.24)$$

Here h is a known function, $\mathcal{K}$ is the kernel, and S is the as yet unknown function. In the numerical methods used in the literature it is assumed usually that the unknown function is piecewise constant. We consider the unknown function S to be piecewise *linear*. The notation S stands for either S_+ or S_-. We divide the interval $[0,k]$ into $n-1$ equal subintervals of length $\Delta = k/(n-1)$. The points of division are called x_l, $l = 1, 2, \ldots, n$. Let $S_l = S(x_l)$, for $l = 1, 2, \ldots, n$. This implies that at the l-th subinterval the function S can be expressed as follows:

$$S(x) = S_l + (S_{l+1} - S_l)\left(\frac{x}{\Delta} - l\right), \quad \text{for} \quad x_l < x < x_{l+1}. \quad (6.3.25)$$

Substitution of (25) in (20) or (21) leads to a set of n linear algebraic equations

$$S_l + S_1\left[\kappa_1(t_l) - \frac{\theta_1(t_l)}{\Delta}\right] + \sum_{i=2}^{n-1} S_i\left[i\kappa_i(t_l) - (i-2)\kappa_{i-1}(t_l) - \frac{\theta_i(t_l) - \theta_{i-1}(t_l)}{\Delta}\right]$$

$$+ S_n\left[\frac{\theta_{n-1}(t_l)}{\Delta} - (n-2)\kappa_{n-1}(t_l)\right] = h(t_l), \quad t_l = x_l, \quad \text{for} \quad l = 1, 2, \ldots, n.$$

$$(6.3.26)$$

Here

$$\kappa_i(t_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(t_l, x)\,\mathrm{d}x. \tag{6.3.27}$$

$$\theta_i(t_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(t_l, x)\,x\,\mathrm{d}x. \tag{6.3.28}$$

Since the piecewise linear function follows the real function more close than the piecewise constant one, we should expect the set of equations (26) to give a more accurate solution.

Actual computations were made with $n = 301$ and the set of values (b/a): 0.04, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99. The dimensionless quantities $S_1^* = S_1/p$ and $S_2^* = S_2/p$ are plotted in Fig. 6.3.1 and Fig. 6.3.2 respectively versus the quantity $\xi = 1 + 300(t/k)$. This choice allowed us to plot all curves on the same base. In

Fig. 6.3.1. Annular crack under uniform normal loading (solution for S_1)

order to avoid overlapping, not all curves are actually plotted in Fig. 6.3.1 and 6.3.2. The stress intensity factors were defined as

$$K_a = \lim_{\rho \to a} \{\sigma_2(\rho)\sqrt{\rho - a}\}, \quad K_b = \lim_{\rho \to b} \{\sigma_1(\rho)\sqrt{b - \rho}\}. \tag{6.3.29}$$

Fig. 6.3.2. Annular crack under uniform normal loading (solution for S_2)

They can be simply defined in terms of the functions S_1 and S_2, namely,

$$K_a = k^2\sqrt{a/2}\, S_2(k), \quad K_b = \sqrt{a/2}\, S_1(k). \tag{6.3.30}$$

We have normalized these quantities by the stress intensity factor K_0 for a penny-shaped circular crack of radius a subjected to the same pressure p

$$K_0 = \frac{\sqrt{2a}}{\pi} p,$$

with the result

$$K_a^* = \frac{K_a}{K_0} = \frac{\pi k^2}{2p} S_2(k), \quad K_b^* = \frac{K_b}{K_0} = \frac{\pi}{2p} S_1(k). \tag{6.3.31}$$

The absolute values of the computed results are presented in the Table below

b/a	0.04	0.1	0.2	0.4	0.6	0.8	0.9	0.95	0.99
$\lvert K_b^*\rvert$	3.2245	2.0742	1.4892	1.0339	0.7646	0.5113	0.3541	0.2473	0.1064
$\lvert K_a^*\rvert$	0.9835	0.9578	0.9118	0.8055	0.6713	0.4848	0.3460	0.2450	0.1065

The results above are in excellent agreement with those of Clements and Ang

(1988) who have normalized their results relative to a straight two-dimensional crack of length $a-b$. As one should expect, the accuracy deteriorates as k approaches unity. Indeed, the results for $(b/a)=0.99$ are qualitatively incorrect since they give $K_b<K_a$. If one assumes the results of Clements and Ang as exact, then the correct values for this case are $K_b^*=-0.1112$ and $K_a^*=-0.1108$, with the error of about 5%. The accuracy of the other data is expected to be much better than that. Our results should be multiplied by the factor $2\sqrt{2}/(\pi\sqrt{1-k^2})$ in order to be compared with those of Clements and Ang.

The solution accuracy was also verified by checking the condition of equilibrium of the plane $z=0$, namely,

$$P+P_1+P_2=0. \tag{6.3.32}$$

Here

$$P=\pi p(a^2-b^2),\quad P_1=2\pi\int_0^b \sigma_1(\rho)\,\rho\,\mathrm{d}\rho=2\pi ab\int_0^k \frac{S_1(t)\,t\,\mathrm{d}t}{\sqrt{k^2-t^2}},$$

$$P_2=2\pi\int_a^\infty \sigma_2(\rho)\,\rho\,\mathrm{d}\rho=2\pi ab\int_0^k \frac{S_2(t)\,\mathrm{d}t}{\sqrt{k^2-t^2}}. \tag{6.3.33}$$

The condition (32) was satisfied with high accuracy which also deteriorates for k close to unity. For example, the discrepancy in equilibrium conditions for $(b/a)=0.99$ was about 11%. In the case of a very narrow annulus we need to use either n greater than 301 or to use the asymptotic two-dimensional solution.

The case of non-axisymmetric loading can be considered in a similar manner. For example, when the loading pressure can be described by a first harmonic, namely,

$$p(\rho,\phi)=p_1\rho\cos\phi, \tag{6.3.34}$$

the governing integral equations will take the form

$$s_+(t)+\frac{2t}{\pi\sqrt{k^2-t^2}}\int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,s_+(x)\,x^2\,\mathrm{d}x=g_+(t), \tag{6.3.35}$$

$$s_-(t)-\frac{2t}{\pi\sqrt{k^2-t^2}}\int_0^k \frac{\sqrt{1-k^2x^2}}{1-x^2t^2}\,s_-(x)\,x^2\,\mathrm{d}x=g_-(t), \tag{6.3.36}$$

where

$$g_{\pm}(t) = -\frac{2p_1 ak}{\pi t}\left\{\frac{\sqrt{1-k^4}}{\sqrt{k^2-t^2}} - \frac{1}{t}\sin^{-1}\frac{t\sqrt{1-k^4}}{k\sqrt{1-k^2t^2}} \pm \left[\frac{k\cos^{-1}(k^2)}{\sqrt{k^2-t^2}} - \cos^{-1}\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right]\right\}. \tag{6.3.37}$$

Again, singularities can be eliminated by an appropriate change of variables. The resulting equations can be written in a compact form

$$\psi_{\pm}(t) \pm \frac{2}{\pi}\int_0^k \frac{\sqrt{1-k^2x^2}}{\sqrt{k^2-x^2}(1-x^2t^2)}\,\psi_{\pm}(x)\,x^3\,\mathrm{d}x = h_{\pm}(t), \tag{6.3.38}$$

where

$$\psi_{\pm}(t) = s_{\pm}(t)\sqrt{k^2-t^2}/t, \quad h_{\pm}(t) = g_{\pm}(t)\sqrt{k^2-t^2}/t. \tag{6.3.39}$$

Note that both functions ψ and h are finite at $t=0$. We may deduce

$$h_{\pm}(0) = -\frac{1}{\pi}\left\{\frac{2(1-k^4)^{3/2}}{3k} \pm \left[\cos^{-1}(k^2) - \sqrt{1-k^4}\right]\right\}.$$

Equations (38) can be solved numerically. We note that the annular crack problem can also be solved by the method described in the section 3.7.