CHAPTER 5

NEW SOLUTIONS IN CONTACT MECHANICS

This Chapter contains complete solutions to several contact problems which were obtained recently, and could not be included in (Fabrikant, 1989a). Those comprise complete elastic fields around axisymmetric and inclined bonded punch. These fundamental solutions allow us to solve various problems of interaction between punches and anchor loads. Two of such solutions are included. A new approach is presented to a general annular punch problem, with analytical, numerical and asymptotic solutions derived and compared.

5.1. Axisymmetric bonded punch problem

The bonded punch problem belongs to the class of the mixed-mixed problems of elasticity theory which are among the most complicated due to the coupling between the normal and tangential parameters. We should mention the works of Mossakovskii (1954) and Ufliand (1956) among the first published exact solutions for the case of an *isotropic* half-space, obtained by using various integral transforms. A more compact solution has been reported by Kapshivyi and Masliuk (1967), who used a special apparatus of p-analytical functions. The first *elementary* exact solution for a *transversely isotropic* elastic half-space was published in (Fabrikant, 1971c). Four different types of solution of the governing set of integral equations were reported in (Fabrikant, 1986b).

All these solutions define the elastic field in the plane z=0 only. We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic half-space. One may argue that since the stresses exerted at the punch base are known, we can substitute them into the Boussinesq point force solution (which is well known, for example, see Fabrikant, 1970) and obtain the complete solution in quadratures. Theoretically, yes, this can be done, but practically, this solution would be of little use since it would require triple integration, with one being singular, and a numerical differentiation. The computing time for this procedure would be quite significant, and its accuracy would be very doubtful. This is the main reason why, to the best of our knowledge, nobody tried so far to obtain a complete solution, even in the case of an isotropic body. On the other hand, knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems of interaction between a bonded punch and anchor loads or cracks.

The complete solution has become possible due to the new results presented in Chapter 1. The expressions for the stresses exerted by the punch are fed in the point force solution, with one important distinction: two of the three integrations and the differentiation are performed exactly, and lead to remarkably simple and elementary expressions involving only one non-singular integration. The case of a circular centrally loaded punch bonded to an elastic half-space is considered as an example. Numerical results are obtained in order to compare the field of normal and tangential displacements due to a bonded punch with similar results for a smooth punch.

Theory. Consider a transversely isotropic elastic body which is characterized by five elastic constants A_{ik} defining the following stress-strain relationships:

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x} + (A_{11} - 2A_{66})\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11} - 2A_{66})\frac{\partial u_x}{\partial x} + A_{11}\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x} + A_{13}\frac{\partial u_y}{\partial y} + A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau_{zx} = A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z}\right). \tag{5.1.1}$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{5.1.2}$$

Substitution of (1) in (2) yields:

$$A_{11}\frac{\partial^2 u_x}{\partial x^2}+A_{66}\frac{\partial^2 u_x}{\partial y^2}+A_{44}\frac{\partial^2 u_x}{\partial z^2}+(A_{11}-A_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(A_{13}+A_{44})\frac{\partial^2 w}{\partial x\partial z}=0\,,$$

$$A_{66}\frac{\partial^2 u_y}{\partial x^2}+A_{11}\frac{\partial^2 u_y}{\partial y^2}+A_{44}\frac{\partial^2 u_y}{\partial z^2}+(A_{11}-A_{66})\frac{\partial^2 u_x}{\partial x\partial y}+(A_{13}+A_{44})\frac{\partial^2 w}{\partial y\partial z}=0\,,$$

$$A_{44}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]+A_{33}\frac{\partial^2 w}{\partial z^2}+(A_{44}+A_{13})\left[\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}\right]=0\,. \tag{5.1.3}$$

Introduce complex tangential displacements $u=u_x+iu_y$, and $\overline{u}=u_x-iu_y$. This will allow us to reduce the number of equations in (3) by one, and to rewrite these equations in a more compact manner, namely,

$$\frac{1}{2}(A_{11}+A_{66})\Delta u+A_{44}\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}(A_{11}-A_{66})\Lambda^2\overline{u}+(A_{13}+A_{44})\Lambda\frac{\partial w}{\partial z}=0,$$

$$A_{44}\Delta w+A_{33}\frac{\partial^2 w}{\partial z^2}+\frac{1}{2}(A_{13}+A_{44})\frac{\partial}{\partial z}(\overline{\Lambda}u+\Lambda\overline{u})=0. \tag{5.1.4}$$

Here the following differential operators were used:

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,,\qquad \Lambda=\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\,, \tag{5.1.5}$$

and the overbar everywhere indicates the complex conjugate value. Note also that $\Delta=\Lambda\overline{\Lambda}$. One can verify that equations (4) can be satisfied by

$$u=\Lambda(F_1+F_2+iF_3),\qquad w=m_1\frac{\partial F_1}{\partial z}+m_2\frac{\partial F_2}{\partial z}, \tag{5.1.6}$$

where all three functions F_k satisfy the equation (Elliott, 1948):

$$\Delta F_k+\gamma_k^2\frac{\partial^2 F_k}{\partial z^2}=0,\qquad \text{for } k=1,2,3, \tag{5.1.7}$$

and the values of m_k and γ_k are related by the following expressions (Elliott, 1948):

$$\frac{A_{44}+m_k(A_{13}+A_{44})}{A_{11}}=\frac{m_kA_{33}}{m_kA_{44}+A_{13}+A_{44}}=\gamma_k^2, \quad \text{for } k=1,2;$$

$$\gamma_3=\left(A_{44}/A_{66}\right)^{1/2}. \tag{5.1.8}$$

Introducing the notation $z_k=z/\gamma_k$, for k=1,2,3, we may call function $F_k=F(x,y,z_k)$ harmonic. Note the property $m_1m_2=1$, which seems to have escaped the attention of previous researchers, and which will help us to simplify various expressions to follow. The other elastic constants which will be used throughout the section are:

$$G_1=\beta+\gamma_1\gamma_2H, \qquad G_2=\beta-\gamma_1\gamma_2H,$$

$$H=\frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)}, \quad \alpha=\frac{(A_{11}A_{33})^{1/2}-A_{13}}{A_{11}(\gamma_1+\gamma_2)}, \quad \beta=\frac{\gamma_3}{2\pi A_{44}}. \tag{5.1.9}$$

Introduce the following inplane stress components:

$$\sigma_1=\sigma_x+\sigma_y, \quad \sigma_2=\sigma_x-\sigma_y+2i\tau_{xy}, \quad \tau_z=\tau_{zx}+i\tau_{yz}. \tag{5.1.10}$$

This will simplify expressions (1), namely

$$\sigma_1=(A_{11}-A_{66})(\bar{\Lambda}u+\Lambda\bar{u})+2A_{13}\frac{\partial w}{\partial z}, \quad \sigma_2=2A_{66}\Lambda u,$$

$$\sigma_z=\frac{1}{2}A_{13}(\bar{\Lambda}u+\Lambda\bar{u})+A_{33}\frac{\partial w}{\partial z}, \quad \tau_z=A_{44}\left[\frac{\partial u}{\partial z}+\Lambda w\right]. \tag{5.1.11}$$

We have now only four components of stress, instead of six, as it was in (1). The substitution of (6) in (11) yields:

$$\sigma_1=2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2-(1+m_1)\gamma_3^2]F_1+[\gamma_2^2-(1+m_2)\gamma_3^2]F_2\},$$

$$\sigma_2=2A_{66}\Lambda^2(F_1+F_2+iF_3),$$

$$\sigma_z=A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)\gamma_1^2F_1+(1+m_2)\gamma_2^2F_2]$$

$$= -A_{44}\,\Delta[(1+m_1)F_1 + (1+m_2)F_2]\,,$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1 + (1+m_2)F_2 + iF_3]\,. \qquad (5.1.12)$$

Here we used the fact that each F_k satisfies equation (7), and the relation: $A_{11}\gamma_k^2 - A_{13}m_k = A_{44}(1+m_k)$, (for k=1,2) which is an immediate consequence of (8). Expressions (6) and (12) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x,y,z) = c_k F(x,y,z_k)\,, \qquad (5.1.13)$$

where $z_k = z/\gamma_k$, and c_k is an as yet unknown complex constant. As we shall see further, this is possible indeed. All the results obtained in this section are valid for isotropic solids, provided that we take

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \quad H = \frac{1-\nu^2}{\pi E}, \quad \alpha = \frac{1-2\nu}{2(1-\nu)},$$

$$\beta = \frac{1+\nu}{\pi E}, \quad G_1 = \frac{(2-\nu)(1+\nu)}{\pi E}, \quad G_2 = \frac{\nu(1+\nu)}{\pi E}, \qquad (5.1.14)$$

where E is the elastic modulus, and ν is Poisson coefficient.

Consider a transversely isotropic elastic half-space $z \geq 0$. Let a point force, with components T_x, T_y, and P in Cartesian coordinates be applied at the point N_0 located at the boundary z=0 of a transversely isotropic elastic half-space. We may assume, without loss of generality, that the polar cylindrical coordinates of N_0 are $(\rho_0,\phi_0,0)$. We need to find the field of stresses and displacements at the point $M(\rho,\phi,z)$. Introduce the complex tangential force $T = T_x + iT_y$. The general solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1 - 1}\left[\frac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)\right],$$

$$F_2 = \frac{H\gamma_2}{m_2 - 1}\left[\frac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2 + z_2)\right],$$

$$F_3 = \mathrm{i}\,\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \qquad (5.1.15)$$

Here

$$\chi_k(z) = \chi(z_k), \quad R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2}, \quad \text{for} \quad k=1,2,3;$$

$$\chi(z) = T[z\ln(R_0 + z) - R_0], \quad R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}. \tag{5.1.16}$$

Substitution of (15–16) in (6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\overline{T}}{R_3(R_3+z_3)^2}\right]$$

$$+\frac{H\gamma_2}{m_2-1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\overline{T}}{R_2(R_2+z_2)^2}\right] + \frac{Pq}{R_2(R_2+z_2)}\right\}$$

$$+\frac{H\gamma_1}{m_1-1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\overline{T}}{R_1(R_1+z_1)^2}\right] + \frac{Pq}{R_1(R_1+z_1)}\right\}, \tag{5.1.17}$$

$$w = H\left\{\frac{1}{2}(T\overline{q} + \overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]\right.$$

$$\left. + P\left[\frac{m_1}{(m_1-1)R_1} + \frac{m_2}{(m_2-1)R_2}\right]\right\}. \tag{5.1.18}$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{5.1.19}$$

Expressions (17) and (18) simplify for the case when $z=0$

$$u = \frac{1}{2}G_1\frac{T}{R} + \frac{1}{2}G_2\frac{\overline{T}q^2}{R^3} - H\alpha\frac{P}{\overline{q}}, \tag{5.1.20}$$

$$w = H\alpha\Re\left(\frac{T}{q}\right) + H\frac{P}{R}. \tag{5.1.21}$$

Here $\Re$ is the real par sign; H, α, G_1, and G_2 are defined by (9), and

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \tag{5.1.22}$$

Formulation of the problem and its solution. Expressions (20) and (21) can be used for the integral equation formulation of the mixed-mixed boundary value problems in an elastic half-space. The boundary conditions in the case of axial symmetry are

$$u = u(\rho), \quad \text{for} \quad 0 \le \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$w = w(\rho), \quad \text{for} \quad 0 \le \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$\sigma = \sigma(\rho), \quad \text{for} \quad a \le \rho \le \infty, \quad 0 \le \phi < 2\pi;$$

$$\tau = \tau(\rho), \quad \text{for} \quad a \le \rho \le \infty, \quad 0 \le \phi < 2\pi. \tag{5.1.23}$$

The set of governing integral equations will take the form

$$2H\left\{-\pi\alpha\int_\rho^a \tau(\rho_0)\mathrm{d}\rho_0 + 2\int_0^\rho \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}}\right\} = \omega_1(\rho), \tag{5.1.24}$$

$$\frac{2H}{\rho}\left\{2\gamma_1\gamma_2\int_0^\rho \frac{x^2\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\tau(\rho_0)\,\mathrm{d}\rho_0}{(\rho_0^2 - x^2)^{1/2}} - \pi\alpha\int_0^\rho \sigma(\rho_0)\rho_0\mathrm{d}\rho_0\right\} = \omega_2(\rho). \tag{5.1.25}$$

The functions ω_1 and ω_2 are known from the boundary conditions, and are defined by

$$\omega_1(\rho) = w(\rho) + 2H\left\{\pi\alpha\int_a^\infty \tau(\rho_0)\mathrm{d}\rho_0 - 2\int_a^\infty \frac{\mathrm{d}x}{(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\sigma(\rho_0)\rho_0\mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}\right\}, \tag{5.1.26}$$

$$\omega_2(\rho) = u(\rho) - 4H\gamma_1\gamma_2\rho\int_a^\infty \frac{\mathrm{d}x}{x^2(x^2 - \rho^2)^{1/2}}\int_a^x \frac{\tau(\rho_0)\,\rho_0^2\mathrm{d}\rho_0}{(x^2 - \rho_0^2)^{1/2}}. \tag{5.1.27}$$

The solution to the problem may be presented in the form (Fabrikant, 1986b)

$$\sigma(\rho) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f_1(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}, \quad \tau_\rho(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f_2(t)\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}. \tag{5.1.28}$$

Here σ is the normal traction exerted by the punch, $\tau_\rho = \tau_{\rho z} + i\tau_{\theta z}$, and the stress functions f_1 and f_2 are defined by

$$f(y) = f_1(y) + i\,f_2(y) = \cosh^2(\pi\theta)\left[-\xi(y) + \frac{i}{\pi}\tanh(\pi\theta)\left(\frac{a+y}{a-y}\right)^{i\theta}\int_{-a}^{a}\left(\frac{a+r}{a-r}\right)^{-i\theta}\frac{\xi(r)\mathrm{d}r}{r-y}\right], \tag{5.1.29}$$

where

$$\xi(x) = \frac{1}{\pi^2 H}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\omega_1(\rho)\rho\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}} + \frac{\mathrm{i}}{\sqrt{\gamma_1\gamma_2}}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\omega_2(\rho)\rho^2\mathrm{d}\rho}{(x^2-\rho^2)^{1/2}}\right\}, \tag{5.1.30}$$

with ω_1 and ω_2 defined by (26) and (27). The general solution simplifies in the case of a bonded punch since $\omega_1 = w$ and $\omega_2 = u$.

Now we need to substitute formulae (28) in (15), modified for the case of distributed loading, and to compute the integrals involved. Here are some details of the derivation. Substitution of the first expression (28) in (15) leads to the integral

$$I_1 = \int_0^{2\pi}\int_0^a \ln(R_0 + z)\mathrm{d}\rho_0\mathrm{d}\phi_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{f_1(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{5.1.31}$$

By interchanging the order of integration in (31), we obtain

$$I_1 = -\int_0^a f_1(x)\,\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x \frac{\ln(R_0+z)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.1.32}$$

The double integral in (32) can be computed by using (A1–A7), with the result

$$I_1 = -2\pi\int_0^a f_1(x)\ln\{l_2(x) + [l_2^2(x) - \rho^2]^{1/2}\}\,\mathrm{d}x.$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} - [(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} + [(\rho-x)^2+z^2]^{1/2}\}. \tag{5.1.33}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2.

When substituting the second expression of (28) in (15), we have to remember the relationship between $\tau=\tau_{zx}+i\tau_{yz}$ and $\tau_\rho=\tau_{\rho z}+i\tau_{\theta z}$, namely, $\tau=\tau_\rho e^{i\phi}$. The substitution leads to the integral

$$I_2=\Lambda\int_0^{2\pi}\int_0^a[z\ln(R_0+z)-R_0]e^{-i\phi_0}\rho_0 d\rho_0 d\phi_0\frac{d}{d\rho_0}\int_{\rho_0}^a\frac{f_2(x)\,dx}{(x^2-\rho_0^2)^{1/2}}.$$

Again, interchanging the order of integration, we obtain

$$I_2=-\Lambda\int_0^a f_2(x)\frac{dx}{x}\frac{d}{dx}\int_0^{2\pi}\int_0^x[z\ln(R_0+z)-R_0]e^{-i\phi_0}\frac{\rho_0^2 d\rho_0 d\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.1.34}$$

The double integral in (34) can be computed according to (A17–A22), and the final result is rather simple

$$I_2=-2\pi\int_0^a f_2(x)\sin^{-1}\frac{x}{l_2(x)}\,dx. \tag{5.1.35}$$

Now the potential functions (15) can be expressed through the stress functions as follows:

$$F_1=-\frac{2\pi H}{m_1-1}\left\{\gamma_1\int_0^a f_1(x)\ln[l_{21}(x)+(l_{21}^2(x)-\rho^2)^{1/2}]\,dx+\sqrt{\gamma_1\gamma_2}\int_0^a f_2(x)\sin^{-1}\left(\frac{x}{l_{21}(x)}\right)dx\right\},$$

$$F_2=-\frac{2\pi H}{m_2-1}\left\{\gamma_2\int_0^a f_1(x)\ln[l_{22}(x)+(l_{22}^2(x)-\rho^2)^{1/2}]\,dx+\sqrt{\gamma_1\gamma_2}\int_0^a f_2(x)\sin^{-1}\left(\frac{x}{l_{22}(x)}\right)dx\right\},$$

$$F_3=0. \tag{5.1.36}$$

These remarkably simple and elementary expressions for the potential functions allow us to obtain the field of displacements and stresses by substitution of (36) in (6) and (12) respectively. The differentiations involved can be performed according to (A27–A45), and the complete solution is

$$u(\rho,\phi,z) = 2\pi H e^{i\phi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{\gamma_k}{\rho}\int_0^a\left[1-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}\right]f_1(x)\,\mathrm{d}x\right.$$

$$\left.+\sqrt{\gamma_1\gamma_2}\int_0^a\frac{l_{1k}(x)[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}(x)[l_{2k}^2(x)-l_{1k}^2(x)]}f_2(x)\,\mathrm{d}x\right\}, \qquad (5.1.37)$$

$$w(\rho,z) = 2\pi H\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{-\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}f_1(x)\,\mathrm{d}x\right.$$

$$\left.+\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]}f_2(x)\,\mathrm{d}x\right\}. \qquad (5.1.38)$$

The explicit presence of ϕ in (37) is due to the fact that the notation u stands not for the radial component of the tangential displacement but for a complex representation of its x- and y-components. The field of stresses can be obtained by substitution of (36) in (12), with the result

$$\sigma_1 = 4\pi H A_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[1-\frac{(1+m_k)\gamma_3^2}{\gamma_k^2}\right]\left[z\int_0^a\frac{l_{2k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[l_{2k}^2(x)-x^2]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\right]\right.$$

$$\left.+z_k\sqrt{\gamma_1\gamma_2}\int_0^a\frac{l_{1k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[x^2-l_{1k}^2(x)]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_2(x)\,\mathrm{d}x\right\}, \qquad (5.1.39)$$

$$\sigma_2 = 4\pi H A_{66}\sum_{k=1}^{2}\frac{e^{2i\phi}}{m_k-1}\left\{\gamma_k\int_0^a\left[\frac{2}{\rho^2}\right.\right.$$

$$\left.+\frac{x[x^2-l_1^2(x)]^{1/2}\{\rho^2[6x^2-l_{2k}^2(x)-3l_{1k}^2(x)]-2l_{2k}^4(x)\}}{l_{1k}^2(x)[l_{2k}^2(x)-l_{1k}^2(x)]^3}\right]f_1(x)\,\mathrm{d}x$$

$$\left.-\sqrt{\gamma_1\gamma_2}\int_0^a\frac{x[l_{2k}^2(x)-x^2]^{1/2}\{2l_{1k}^4(x)+\rho^2[l_{1k}^2(x)+3l_{2k}^2(x)-6x^2]\}}{l_{2k}^2(x)[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_2(x)\,\mathrm{d}x\right\},$$

(5.1.40)

$$\sigma_z = \frac{z_k}{\gamma_1 - \gamma_2} \sum_{k=1}^{2} (-1)^{k+1} \Bigg[\gamma_k \int_0^a \frac{l_{2k}^4(x) - x^2(2x^2 + 2z_k^2 - \rho^2)}{[l_2^2(x) - x^2]^{1/2}[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_1(x)\,\mathrm{d}x$$

$$+ \sqrt{\gamma_1\gamma_2} \int_0^a \frac{l_{1k}^4(x) - x^2(2x^2 + 2z_k^2 - \rho^2)}{[x^2 - l_{1k}^2(x)]^{1/2}[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_2(x)\,\mathrm{d}x \Bigg], \qquad (5.1.41)$$

$$\tau_z = \frac{\rho e^{i\phi}}{\gamma_1 - \gamma_2} \sum_{k=1}^{2} (-1)^{k+1} \Bigg\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}[l_{2k}^2(x) + 3l_{1k}^2(x) - 4x^2]}{[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_1(x)\,\mathrm{d}x$$

$$- \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}[3l_{2k}^2(x) + l_{1k}^2(x) - 4x^2]}{[l_{2k}^2(x) - l_{1k}^2(x)]^3} f_2(x)\,\mathrm{d}x \Bigg\}. \qquad (5.1.42)$$

Formulae (37–42) give the *complete* solution to the bonded punch problem which is the main result of this section. The complete solution for an *isotropic* half-space is readily available as the limiting case (14) of (37–42). Here is the field of displacements:

$$u = \frac{1+\nu}{E} e^{i\phi} \Bigg\{ \int_0^a \Bigg[(1-2\nu)\left(\frac{1}{\rho} - \frac{l_2(x)[x^2 - l_1^2(x)]^{1/2}}{l_1(x)[l_2^2(x) - l_1^2(x)]} \right)$$

$$+ \frac{z\rho[l_2^2(x) - x^2]^{1/2}[4x^2 - 3l_1^2(x) - l_2^2(x)]}{[l_2^2(x) - l_1^2(x)]^3} \Bigg] f_1(x)\,\mathrm{d}x$$

$$- \int_0^a \Bigg[2(1-\nu) \frac{l_1(x)[l_2^2(x) - x^2]^{1/2}}{l_2(x)[l_2^2(x) - l_1^2(x)]}$$

$$+ \frac{z\rho[x^2 - l_1^2(x)]^{1/2}[4x^2 - l_1^2(x) - 3l_2^2(x)]}{[l_2^2(x) - l_1^2(x)]^3} \Bigg] f_2(x)\,\mathrm{d}x \Bigg\}, \qquad (5.1.43)$$

$$w = \frac{1+\nu}{E} \Bigg\{ -\int_0^a \Bigg[2(1-\nu) \frac{[l_2^2(x) - x^2]^{1/2}}{l_2^2(x) - l_1^2(x)}$$

$$- \frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg] f_1(x)\,\mathrm{d}x$$

$$+ \int_0^a \Bigg[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)} - \frac{z^2[l_1^4(x)-x^2(2x^2+2z^2-\rho^2)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg] f_2(x)\,\mathrm{d}x\Bigg\}. \tag{5.1.44}$$

The limits were computed according to the L'Hôpital rule. The following scheme was used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{f(z_1)}{m_1-1}+\frac{f(z_2)}{m_2-1}\right] = -f(z) - \frac{z}{2(1-\nu)}f'(z), \tag{5.1.45}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{m_1-1}+\frac{m_2 f(z_2)}{m_2-1}\right] = f(z) - \frac{z}{2(1-\nu)}f'(z), \tag{5.1.46}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1}+\frac{\gamma_2 f(z_2)}{m_2-1}\right] = -\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{5.1.47}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{\gamma_1(m_1-1)}+\frac{m_2 f(z_2)}{\gamma_2(m_2-1)}\right] = \frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}. \tag{5.1.48}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1 = 1, \quad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial \gamma_1}\right] = 2(1-\nu), \tag{5.1.49}$$

and the symbol (′) indicates differentiation with respect to z. The derivation of the field of stresses for the case of isotropy is left to the reader.

Applications. Consider the case of a flat circular centrally loaded punch bonded to a transversely isotropic elastic half-space $z \geq 0$. The stress functions in this case are (Fabrikant, 1986b)

$$f_1(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta\, Y_c(x), \quad f_2(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta\, Y_s(x),$$

$$Y_c(x) = \cos\left(\theta \ln\frac{a+x}{a-x}\right), \qquad Y_s(x) = \sin\left(\theta \ln\frac{a+x}{a-x}\right), \tag{5.1.50}$$

where w_0 is the punch settlement, and

$$\theta = \frac{1}{2\pi}\ln\frac{\sqrt{\gamma_1\gamma_2}+\alpha}{\sqrt{\gamma_1\gamma_2}-\alpha}.$$

The complete solution can be obtained by substitution of (50) in (37–42). In the case of isotropy, the value of θ is defined by $\theta=(1/2\pi)\ln(3-4\nu)$. We have performed some computations in order to compare the field of displacements around a flat bonded punch with similar results for a smooth punch. The importance of such a comparison lies in the fact that the smooth and bonded punches represent two extreme cases of interaction between a punch and an elastic half-space, and usually give upper and lower bounds for various parameters of practical interest. The computations were made for the case of isotropy. The field of displacements around a bonded punch is

$$u = \frac{w_0 e^{i\phi}\cosh\pi\theta}{\pi(1-\nu)}\left\{-\int_0^a \left[(1-2\nu)\left(\frac{1}{\rho} - \frac{l_2(x)[x^2-l_1^2(x)]^{1/2}}{l_1(x)[l_2^2(x)-l_1^2(x)]}\right)\right.\right.$$
$$\left. + \frac{z\rho[l_2^2(x)-x^2]^{1/2}[4x^2-3l_1^2(x)-l_2^2(x)]}{[l_2^2(x)-l_1^2(x)]^3}\right] Y_c(x)\,\mathrm{d}x$$
$$+ \int_0^a \left[2(1-\nu)\frac{l_1(x)[l_2^2(x)-x^2]^{1/2}}{l_2(x)[l_2^2(x)-l_1^2(x)]}\right.$$
$$\left.\left. + \frac{z\rho[x^2-l_1^2(x)]^{1/2}[4x^2-l_1^2(x)-3l_2^2(x)]}{[l_2^2(x)-l_1^2(x)]^3}\right] Y_s(x)\,\mathrm{d}x\right\}, \tag{5.1.51}$$

$$w = \frac{w_0\cosh\pi\theta}{\pi(1-\nu)}\left\{\int_0^a \left[2(1-\nu)\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)} - \frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right] Y_c(x)\,\mathrm{d}x\right.$$
$$\left. - \int_0^a \left[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)} - \frac{z^2[l_1^4(x)-x^2(2x^2+2z^2-\rho^2)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right] Y_s(x)\,\mathrm{d}x\right\}.$$

(5.1.52)

The field of displacements in the case of a smooth punch takes the form (Fabrikant, 1989b)

$$u = \frac{w_0 e^{i\phi}}{\pi(1-\nu)}\left\{-(1-2\nu)\left[\frac{a-(a^2-l_1^2)^{1/2}}{\rho}\right]+\frac{z l_1 (l_2^2-a^2)^{1/2}}{l_2(l_2^2-l_1^2)}\right\}, \qquad (5.1.53)$$

$$w = \frac{2w_0}{\pi}\left[\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{z(a^2-l_1^2)^{1/2}}{2(1-\nu)(l_2^2-l_1^2)}\right]. \qquad (5.1.54)$$

The solutions (51–52) and (53–54) depend essentially on one parameter, namely, Poisson coefficient. In the case $\nu=1/2$, both solutions coincide. It would be a good exercise for the reader to prove this by direct integration of (51–52) which should yield (53–54). The greatest difference between solutions is attained for the Poisson coefficient $\nu=0$. This value was taken in numerical computations. The results are shown in Fig. 5.1.1 (the ratio u/w_0 versus ρ/a) and Fig. 5.1.2 (the ratio w/w_0 versus ρ/a) for $z/a=0, 0.1, 0.5, 1.0$. We took $a=1$ in all computations. The solid line curves correspond to the case of a bonded punch,

Fig. 5.1.1. The field of radial displacements

while the broken line curves give similar results for a smooth punch. As we could expect, the field of radial displacements under the bonded punch differs quite significantly from that of a smooth punch. On the contrary, the field of normal displacements differs very little from the case of a smooth punch, with

Fig. 5.1.2. The field of normal displacements

the maximum deviation not exceeding $0.06w_0$ at $z=0$, $\rho=1.1a$.

The complete solution (37–42) may be used for solving more complicated problems of interactions between bonded punches and anchor loads.

5.2. Inclined bonded circular punch

A complete solution is given to the problem of an inclined circular punch bonded to a transversely isotropic elastic half-space. Explicit expressions are derived for the field of stresses and displacements around such punch subjected to a shifting load and a tilting moment. The complete solution is initially expressed in terms of three potential functions. The displacements are defined through first derivatives of the potential functions, while the stresses are given by second derivatives. The complete solution for an isotropic body is obtained as a limiting case of the general solution. Specific computations are performed in order to compare the elastic field in the vicinity of a bonded punch with similar parameters for a smooth punch. Its is found that the influence of bonding on normal displacement is relatively small, while its influence on tangential displacements may be quite significant.

The problem of a bonded circular punch subjected to a shifting force and a tilting moment was first considered in (Fabrikant, 1971c). All known solutions define the elastic field in the plane z=0 only. We give below a complete solution.

The boundary conditions in the case of a flat circular bonded punch subjected to a shifting force and a tilting moment are

$$u = u_0 = \text{const.}, \qquad \text{for } 0 \le \rho \le a, \qquad 0 \le \phi < 2\pi;$$

$$w = -\delta\rho\cos\phi, \qquad \text{for } 0 \le \rho \le a, \qquad 0 \le \phi < 2\pi;$$

$$\sigma = 0, \qquad \text{for } a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi;$$

$$\tau = 0, \qquad \text{for } a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi. \tag{5.2.1}$$

The set of governing integral equations will take the form (Fabrikant, 1971c)

$$\frac{2G_1}{\rho^2}\int_0^\rho \frac{x^4\,dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\tau_2(\rho_0)d\rho_0}{\rho_0(\rho_0^2 - x^2)^{1/2}} - \frac{2\pi H\alpha}{\rho^2}\int_0^\rho \sigma_1(\rho_0)\rho_0^2 d\rho_0$$

$$+\frac{2G_2}{\rho^2}\int_0^\rho \frac{\rho^2 - 2x^2}{(\rho^2 - x^2)^{1/2}}dx\int_x^a \frac{\bar{\tau}_0(\rho_0)\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} = 0, \tag{5.2.2}$$

$$2G_2\int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{(\rho_0^2 - 2x^2)\,\bar{\tau}_2(\rho_0)d\rho_0}{\rho_0(\rho_0^2 - x^2)^{1/2}} + 2\pi H\alpha\int_\rho^a \sigma_{-1}(\rho_0)d\rho_0$$

$$+\,2G_1\int_0^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\tau_0(\rho_0)\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} = u_0, \tag{5.2.3}$$

$$2\pi H\alpha\,\Re\left\{\frac{e^{-i\phi}}{\rho}\int_0^\rho \tau_0(\rho_0)\rho_0 d\rho_0 - \rho e^{i\phi}\int_\rho^a \tau_2(\rho_0)\frac{d\rho_0}{\rho_0}\right\}$$

$$+\frac{4H}{\rho}\int_0^\rho \frac{x^2\,dx}{(\rho^2 - x^2)^{1/2}}\int_x^a \frac{\sigma_1(\rho_0)e^{i\phi} + \sigma_{-1}(\rho_0)e^{-i\phi}}{(\rho_0^2 - x^2)^{1/2}}d\rho_0 = -\delta\rho\cos\phi. \tag{5.2.4}$$

The structure of equations (2–4) is such that we may assume all the unknown functions τ_0, τ_2, σ_1, and σ_{-1} to be real. The solution may be represented in the form

$$\sigma_1(\rho) = \sigma_{-1}(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^\rho \frac{f(t)\mathrm{d}t}{(\rho^2 - t^2)^{1/2}},$$

$$\tau_0(\rho) = \bar{\tau}_0(\rho) = -\frac{C}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a \frac{f(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}} + \frac{D}{(a^2 - \rho^2)^{1/2}},$$

$$\tau_2(\rho) = \bar{\tau}_2(\rho) = -C\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\frac{1}{\rho^2}\int_\rho^a \frac{f(t)t\mathrm{d}t}{(t^2 - \rho^2)^{1/2}}\right\} - D\frac{2a^2 - \rho^2}{\rho^2(a^2 - \rho^2)^{1/2}}. \tag{5.2.5}$$

Here C and D are the constants to be determined, and f is the stress function. They are defined as follows (Fabrikant, 1989a):

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H\sinh(\pi\theta)}\left[tY_s(t) - \theta aY_c(t)\right] + AY_c(t). \tag{5.2.6}$$

$$C = \frac{\alpha}{\gamma_1\gamma_2}, \quad D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}A,$$

$$Y_{c,s}(t) = \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\left[\theta\ln\left(\frac{a+t}{a-t}\right)\right], \quad \tanh(\pi\theta) = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \tag{5.2.7}$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1 + G_2)}{\tanh(\pi\theta)(G_1 - G_2)}\right)\right]^{-1}. \tag{5.2.8}$$

The displacements of the punch are related to the applied loading as

$$u_0 = \frac{1}{8a}\left[\pi(G_1 + G_2) + \frac{(1 + 4\theta^2)\tanh(\pi\theta)}{\theta(1 + \theta^2)}(G_1 - G_2)\right]T - \frac{3H\alpha}{4a^2(1 + \theta^2)}M,$$

$$\delta = \frac{3H\alpha}{4a^2(1 + \theta^2)}\left[-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right]. \tag{5.2.9}$$

In order to proceed further, we need to express the normal stress distribution in a form slightly different from the first expression (5), namely,

$$\sigma_1(\rho) = \sigma_{-1}(\rho) = \frac{d}{d\rho}\int_\rho^a \frac{f_1(t)dt}{(t^2-\rho^2)^{1/2}}. \tag{5.2.10}$$

Here f_1 is a new stress function which can be related to f by an easily verifiable expression

$$f_1(x) = -\frac{2}{\pi}x\int_0^a \frac{(a^2-t^2)^{1/2}f(t)dt}{(a^2-x^2)^{1/2}(t^2-x^2)}. \tag{5.2.11}$$

Substitution of (6) in (11) yields, after integration (see Appendix A3.1 in Fabrikant, 1989a)

$$f_1(t) = \frac{\delta\cosh(\pi\theta)}{\pi^2 H}\left[tY_c(t) + \theta a Y_s(t)\right] + \tanh(\pi\theta)AY_s(t). \tag{5.2.12}$$

We have dropped here the term const$\cdot t/(a^2 - t^2)^{1/2}$ since its addition to or subtraction from f_1 does not change the value of normal stress defined by (10).

Now we need to substitute the last two formulae (5) and (10) in (5.1.15), modified for the case of distributed loading, and to compute the integrals involved. Here are some details of the derivation. Substitution of expression (10) in (5.1.15) leads to the integral

$$I_1 = \int_0^{2\pi}\int_0^a \ln(R_0+z)e^{i\phi_0}\rho_0 d\rho_0 d\phi_0 \frac{d}{d\rho_0}\int_{\rho_0}^a \frac{f_1(x)\,dx}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.13}$$

By interchanging the order of integration in (13), we obtain

$$I_1 = -\int_0^a \frac{1}{x}f_1(x)\,dx\frac{d}{dx}\int_0^{2\pi}\int_0^x \frac{e^{i\phi_0}\ln(R_0+z)\rho_0^2 d\rho_0 d\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.14}$$

The double integral in (14) can be computed by using (A17–A20), with the result

$$I_1 = 2\pi\frac{e^{i\phi}}{\rho}\int_0^a \{x - [x^2 - l_1^2(x)]^{1/2}\}f_1(x)\,dx. \tag{5.2.15}$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} - [(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2} + [(\rho-x)^2+z^2]^{1/2}\}. \tag{5.2.16}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2,3.

Substitution of the second expression of (5) in (5.1.15) leads to the integral

$$I_2 = -\Lambda\int_0^{2\pi}\int_0^a [z\ln(R_0+z) - R_0]\mathrm{d}\rho_0\mathrm{d}\phi_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a \frac{f(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.17}$$

Again, interchanging the order of integration, we obtain

$$I_2 = \Lambda\int_0^a f(x)\,\mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x [z\ln(R_0+z) - R_0]\frac{\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(x^2-\rho_0^2)^{1/2}}. \tag{5.2.18}$$

The double integral in (18) can be computed according to (A1–A7), and the final result is rather simple

$$I_2 = 2\pi\frac{e^{i\phi}}{\rho}\int_0^a \{z - [l_2^2(x) - x^2]^{1/2}\}f(x)\,\mathrm{d}x. \tag{5.2.19}$$

Substitution of the third expression of (5) in (5.1.15) yields the integral

$$I_3 = \overline{\Lambda}\int_0^{2\pi}\int_0^a [z\ln(R_0+z) - R_0]\rho_0^2 e^{2i\phi_0}\,\mathrm{d}\rho_0\mathrm{d}\phi_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}\left\{\frac{1}{\rho_0^2}\int_{\rho_0}^a \frac{f(x)x\,\mathrm{d}x}{(x^2-\rho_0^2)^{1/2}}\right\}.$$

Interchanging the order of integration, we obtain

$$I_3 = \overline{\Lambda}\int_0^a f(x)\,\mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi}\int_0^x e^{2i\phi_0}[z\ln(R_0+z) - R_0]\frac{(\rho_0^2-2x^2)\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho_0(x^2-\rho_0^2)^{1/2}}. \tag{5.2.20}$$

Again, the double integral in (20) can be computed according to (A8–A14), and the final result is

$$I_3 = 2\pi\frac{e^{i\phi}}{\rho}\int_0^a \{[l_2^2(x) - x^2]^{1/2} - (\rho^2 + z^2)^{1/2}\} f(x)\, dx. \tag{5.2.21}$$

Now the potential functions can be expressed through the stress functions as follows:

$$F_1 = \frac{4\pi H}{m_1 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_1 - [l_{21}^2(x) - x^2]^{1/2}\} f(x)\, dx\right.$$

$$\left. + \gamma_1\int_0^a \{x - [x^2 - l_{11}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_2 = \frac{4\pi H}{m_2 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_2 - [l_{22}^2(x) - x^2]^{1/2}\} f(x)\, dx\right.$$

$$\left. + \gamma_2\int_0^a \{x - [x^2 - l_{12}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_3 = D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\int_0^a \{z_3 - [l_{23}^2(x) - x^2]^{1/2}\}\, dx$$

$$= D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\left[z_3 a - \frac{(l_{23}^2 - a^2)^{1/2}(2a^2 - l_{13}^2)}{2a} - \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right] \tag{5.2.22}$$

These remarkably simple and elementary expressions for the potential functions allow us to obtain the field of displacements and stresses by substitution of (22) in (5.1.6) and (5.1.12) respectively. The differentiations involved can be performed according to (A27–A45), and the complete solution is

$$u(\rho,\phi,z) = 4\pi H\sum_{k=1}^{2}\frac{1}{m_k - 1}\left\{-\frac{e^{2i\phi}}{\rho^2}\left[\alpha\int_0^a \{z_k - [l_{2k}^2(x) - x^2]^{1/2}\}\, f(x)\, dx\right.\right.$$

$$+\gamma_k\int_0^a \{x-[x^2-l_{1k}^2(x)]^{1/2}\}f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{e^{2i\phi}+1}{2}\left[-\alpha\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f(x)\,\mathrm{d}x+\gamma_k\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\Bigg\}$$

$$-D\frac{2\gamma_3}{A_{44}}\left\{\frac{e^{2i\phi}}{\rho^2}\left[z_3a-(l_{23}^2-a^2)^{1/2}\left(a-\frac{l_{13}^2}{2a}\right)\right]-\frac{1}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right\}, \tag{5.2.23}$$

$$w(\rho,z)=4\pi H\frac{\cos\phi}{\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\alpha}{\gamma_k}\int_0^a\left(1-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)f(x)\,\mathrm{d}x\right.\right.$$

$$\left.\left.-\int_0^a \frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\right]\right\}. \tag{5.2.24}$$

We recall that the stress functions f and f_1 are defined by (6–8) and (11) respectively. The field of stresses can be obtained by substitution of (22) in (5.1.12), with the result

$$\sigma_1=8\pi HA_{66}\rho\cos\phi\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[1\right.\right.$$

$$-\frac{(1+m_k)\gamma_3^2}{\gamma_k^2}\left]\alpha\int_0^a \frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x\right.$$

$$\left.\left.-\gamma_k\int_0^a \frac{[x^2-l_{1k}^2(x)]^{1/2}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\right]\right\}, \tag{5.2.25}$$

$$\sigma_2=8\pi HA_{66}\sum_{k=1}^{2}\frac{e^{i\phi}}{m_k-1}\left\{\frac{4e^{2i\phi}}{\rho^3}\left[\alpha\int_0^a\left(z_k-[l_{2k}^2(x)-x^2]^{1/2}+\frac{\rho^2[l_{2k}^2(x)-x^2]^{1/2}}{2[l_{2k}^2(x)-l_{1k}^2(x)]}\right)f(x)\,\mathrm{d}x\right.\right.$$

$$+\gamma_k\int_0^a\left(x-[x^2-l_{1k}^2(x)]^{1/2}-\frac{\rho^2[x^2-l_{1k}^2(x)]^{1/2}}{2[l_{2k}^2(x)-l_{1k}^2(x)]}\right)f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{\mathrm{e}^{2i\phi}+1}{2}\rho\Bigg[-\alpha\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$+\gamma_k\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]f_1(x)\,\mathrm{d}x\Bigg]\Bigg\}$$

$$+4D\frac{\mathrm{e}^{i\phi}}{\gamma_3}\left\{\frac{4\mathrm{e}^{2i\phi}}{\rho^3}\left[z_3a-(l_{23}^2-a^2)^{1/2}\left(a-\frac{l_{13}^2}{2a}\right)\right]-\frac{1-\mathrm{e}^{2i\phi}}{2}\frac{l_{13}(l_{23}^2-a^2)^{1/2}}{l_{23}(l_{23}^2-l_{13}^2)}\right\}. \tag{5.2.26}$$

$$\sigma_z=\frac{2\rho\cos\phi}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{k+1}\Bigg\{\alpha\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}[4x^2-3l_{1k}^2(x)-l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$-\gamma_k\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}[4x^2-l_{1k}^2(x)-3l_{2k}^2(x)]}{[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\Bigg\}, \tag{5.2.27}$$

$$\tau_z=\frac{2}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{\mathrm{k}}\Bigg\{-\frac{\mathrm{e}^{2i\phi}}{\rho^2}\Bigg[\frac{\alpha}{\gamma_k}\int_0^a\frac{l_{2\mathrm{k}}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f(x)\,\mathrm{d}x$$

$$+\int_0^a\frac{l_{1\mathrm{k}}(x)[\rho^2-l_{1\mathrm{k}}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,\mathrm{d}x\Bigg]$$

$$+\frac{\mathrm{e}^{2i\phi}+1}{2}\Bigg[\frac{\alpha z_k}{\gamma_k}\int_0^a\frac{x^2(2x^2+2z_k^2-\rho^2)-l_{2k}^4(x)}{[l_{2k}^2(x)-x^2]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f(x)\,\mathrm{d}x$$

$$-z_k\int_0^a\frac{l_{2k}^4(x)-x^2(2x^2+2z_k^2-\rho^2)}{[x^2-l_{1k}^2(x)]^{1/2}[l_{2k}^2(x)-l_{1k}^2(x)]^3}f_1(x)\,\mathrm{d}x\Bigg]\Bigg\}$$

$$+\,2D(a^2-l_{13}^2)^{1/2}\left[\frac{e^{2i\phi}}{\rho^2}-\frac{1-e^{2i\phi}}{2(l_{23}^2-l_{13}^2)}\right]. \tag{5.2.28}$$

Formulae (23–28) give the *complete* solution to the bonded punch problem which is the main result of this section. The complete solution for an *isotropic* half-space is readily available as the limiting case (5.1.14) of (23–28). Here is the field of displacements:

$$u=\frac{1}{\mu}\left\{\frac{e^{2i\phi}}{\rho^2}(1-2\nu)\int_0^a\left[z-[l_2^2(x)-x^2]^{1/2}+\frac{z}{2(1-\nu)}\left(1-\frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right)\right]f(x)\,\mathrm{d}x\right.$$

$$+\frac{e^{2i\phi}}{\rho^2}\int_0^a\left[(1-2\nu)[x-[x^2-l_1^2(x)]^{1/2}]-\frac{z\,l_1(x)[\rho^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\right]f_1(x)\,\mathrm{d}x$$

$$+\frac{e^{2i\phi}+1}{2}(1-2\nu)\int_0^a\left[\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}+\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]f(x)\,\mathrm{d}x$$

$$\left.-\frac{e^{2i\phi}+1}{2}\int_0^a\left[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}+\frac{z^2[l_1^4(x)-x^2(2x^2+2z^2-\rho^2)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]f_1(x)\,\mathrm{d}x\right\}$$

$$-\frac{2D}{\mu}\left\{\frac{e^{2i\phi}}{\rho^2}\left[za-(l_2^2-a^2)^{1/2}\left(a-\frac{l_1^2}{2a}\right)\right]-\frac{1}{2}\sin^{-1}\left(\frac{a}{l_2}\right)\right\}, \tag{5.2.29}$$

$$w=\frac{\cos\phi}{\mu\rho}\left\{\frac{(1-2\nu)}{2(1-\nu)}\int_0^a\left[(1-2\nu)\left(1-\frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right)-\frac{z\rho^2[l_2^2(x)-x^2]^{1/2}}{[l_2^2(x)-l_1^2(x)]^3}[4x^2\right.\right.$$

$$\left.-\,3l_1^2(x)-l_2^2(x)]\right]f(x)\,\mathrm{d}x-\int_0^a\left[2(1-\nu)\frac{l_1(x)[\rho^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.$$

$$\left.\left.-\frac{z\rho^2[x^2-l_1^2(x)]^{1/2}}{[l_2^2(x)-l_1^2(x)]^3}[4x^2-l_1^2(x)-3l_2^2(x)]\right]f_1(x)\,\mathrm{d}x\right\}. \tag{5.2.30}$$

Here μ stands for the shear modulus, and ν is the Poisson coefficient. In the case of isotropy, the value of θ is defined by $\theta=(1/2\pi)\ln(3-4\nu)$. Formulae (6–8) and (12) in the case of isotropy take the form

$$f(x)=\frac{4}{\pi}\mu\cosh(\pi\theta)\left\{\frac{\delta}{(1-2\nu)}\left[-xY_{\mathrm{s}}(x)+\theta aY_{\mathrm{c}}(x)\right]+\frac{u_0+\delta a\theta}{1-2\nu+2\pi\theta}Y_{\mathrm{c}}(x)\right\},$$

$$f_1(x)=\frac{4}{\pi}\mu\sinh(\pi\theta)\left\{\frac{\delta}{(1-2\nu)}\left[xY_{\mathrm{c}}(x)+\theta aY_{\mathrm{s}}(x)\right]+\frac{u_0+\delta a\theta}{1-2\nu+2\pi\theta}Y_{\mathrm{s}}(x)\right\},$$

$$A=\frac{4\mu\cosh(\pi\theta)(u_0+\delta a\theta)}{\pi(1-2\nu+2\pi\theta)},\quad D=\frac{4\mu\theta(u_0+\delta a\theta)}{1-2\nu+2\pi\theta}$$

The limits were computed according to the L'Hôpital rule, and the symbol (´) indicates differentiation with respect to z. The derivation of the field of stresses for the case of isotropy is left to the reader.

We have performed some computations in order to compare the field of displacements around a flat inclined bonded punch with similar results for a smooth punch. The importance of such a comparison lies in the fact that the smooth and bonded punches represent two extreme cases of interaction between a punch and an elastic half-space, and usually give upper and lower bounds for various parameters of practical interest. The field of displacements in the case of a smooth punch takes the form (Fabrikant, 1989a)

$$u=-\frac{2\delta\rho\cos\phi}{\pi}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\left\{z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right.$$
$$\left.+\,\mathrm{e}^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right\}, \tag{5.2.31}$$

$$w=-\frac{2}{\pi}\delta\rho\cos\phi\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left[\sin^{-1}\left(\frac{a}{l_{2k}}\right)-\frac{a(l_{2k}^2-a^2)^{1/2}}{l_{2k}^2}\right]. \tag{5.2.32}$$

The solutions (23–24) and (31–32) in the case of θ=0 coincide. It would be a good exercise for the reader to prove this by a direct integration of (23–24) which should yield (31–32). In the case of isotropy, this situation corresponds to the value of Poisson coefficient $\nu=1/2$.

The computations were made for the case of isotropy, with the following numerical values assigned to the parameters: $u_0=0$, $\delta=1$, and $a=1.3$. Formula (32) in this case takes the form

$$w = -\frac{2}{\pi}\delta\rho\cos\phi\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{a}{l_2^2}(l_2^2-a^2)^{1/2} + \frac{za^2(a^2-l_1^2)^{1/2}}{(1-\nu)l_2^2(l_2^2-l_1^2)}\right]. \quad (5.2.33)$$

The greatest difference between solutions (30) and (33) is attained for the Poisson coefficient $\nu=0$. This value was taken in numerical computations. The results for $\delta=1$, $a=1.3$ (w versus ρ/a) are presented in Fig. 5.2.1. As before, the solid

Fig. 5.2.1. The influence of bonding on normal displacements

line curves correspond to the case of a bonded punch, while the broken line curves give the relevant results for a smooth punch. It was found that the field of normal displacements due to a bonded punch differs very little from the case of a smooth punch, with the maximum deviation not exceeding 3.5% at $z=0$, $\rho=1.1a$. On the contrary, it is evident, that the field of tangential displacements of a bonded punch might differ quite significantly from that of a smooth punch.

5.3. Interaction of a normal load with a bonded punch

We consider a transversely isotropic elastic half-space $z\geq 0$. A flat circular punch of radius a is bonded to its boundary $z=0$, with the punch centre coinciding with the coordinate system origin $\rho=0$. Let a point force N (Fig. 5.3.1) be applied in the Oz direction at the point with the polar cylindrical coordinates (ρ,ϕ,z). We may assume, without loss of generality, that $\phi=0$. We

Fig. 5.3.1. Geometry of the problem

need to find the punch settlement w_N, its tangential displacement u_N, and the angle of inclination δ_N which are due to the point load N. The reader is reminded that the punch settlement is understood as the normal displacement of the punch centre; the angle of inclination is the angle between the punch base and the plane $z=0$.

First of all, we need to solve two auxiliary problems, namely, the one of a centrally loaded bonded punch, and the second one is the problem of an inclined bonded punch. We consider below each problem separately, after which, the reciprocal theorem is used to obtain the solution to the main problem.

Problem 1. We consider the mixed-mixed problem characterized by the following boundary conditions:

$$u = 0, \quad \text{for } 0\leq\rho\leq a, \quad 0\leq\phi<2\pi;$$

$$w = w_0, \quad \text{for } 0\leq\rho\leq a, \quad 0\leq\phi<2\pi;$$

$$\sigma = 0, \quad \text{for } a\leq\rho\leq\infty, \quad 0\leq\phi<2\pi;$$

$$\tau = 0, \qquad \text{for } a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi. \tag{5.3.1}$$

The solution to the problem may be presented in the form (Fabrikant, 1986b)

$$\sigma(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{f_1(t)t\,dt}{(t^2-\rho^2)^{1/2}}, \quad \tau_\rho(\rho) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{d}{d\rho}\int_\rho^a \frac{f_2(t)\,dt}{(t^2-\rho^2)^{1/2}}. \tag{5.3.2}$$

Here σ is the normal traction exerted by the punch, $\tau_\rho = \tau_{\rho z} + i\tau_{\theta z}$, and the stress functions f_1 and f_2 are defined in this particular case by

$$f_1(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta\cos\left(\theta\ln\frac{a+x}{a-x}\right),$$

$$f_2(x) = -\frac{w_0}{\pi^2 H}\cosh\pi\theta\sin\left(\theta\ln\frac{a+x}{a-x}\right),$$

$$\theta = \frac{1}{2\pi}\ln\frac{\sqrt{\gamma_1\gamma_2}+\alpha}{\sqrt{\gamma_1\gamma_2}-\alpha}. \tag{5.3.3}$$

where w_0 is the punch settlement. In the case of isotropy, the value of θ is defined by $\theta=(1/2\pi)\ln(3-4\nu)$. Repeating the transformations from section 5.1, leading to (5.1.36), we come to the following expressions for the potential functions

$$F_1 = \frac{2w_0\cosh(\pi\theta)}{\pi(m_1-1)}\left\{\gamma_1\int_0^a Y_c(x)\ln[l_{21}(x)+(l_{21}^2(x)-\rho^2)^{1/2}]\,dx\right.$$

$$\left. + \sqrt{\gamma_1\gamma_2}\int_0^a Y_s(x)\sin^{-1}\left(\frac{x}{l_{21}(x)}\right)dx\right\},$$

$$F_2 = \frac{2w_0\cosh(\pi\theta)}{\pi(m_2-1)}\left\{\gamma_2\int_0^a Y_c(x)\ln[l_{22}(x)+(l_{22}^2(x)-\rho^2)^{1/2}]\,dx\right.$$

$$\left. + \sqrt{\gamma_1\gamma_2}\int_0^a Y_s(x)\sin^{-1}\left(\frac{x}{l_{22}(x)}\right)dx\right\},$$

$$F_3 = 0. \tag{5.3.4}$$

where w_0 is the punch settlement, θ is defined by (3), and

$$Y_c(x) = \cos\left(\theta \ln\frac{a+x}{a-x}\right), \quad Y_s(x) = \sin\left(\theta \ln\frac{a+x}{a-x}\right). \tag{5.3.5}$$

We need only the expression for the normal displacement

$$w(\rho,z) = \frac{2}{\pi} w_0 \cosh(\pi\theta) \sum_{k=1}^{2} \frac{m_k}{m_k - 1} \left\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_c(x)\,\mathrm{d}x \right.$$

$$\left. - \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_s(x)\,\mathrm{d}x \right\}. \tag{5.3.6}$$

Taking into consideration the relationship between the punch settlement w_0 and the applied to the punch force P (Fabrikant, 1989a)

$$w_0 = \frac{PH\tanh(\pi\theta)}{2a\theta}, \tag{5.3.7}$$

expression (6) can be rewritten as

$$w(\rho,z) = \frac{\mathrm{PH}\sinh(\pi\theta)}{\pi a\theta} \sum_{k=1}^{2} \frac{m_k}{m_k - 1} \left\{ \int_0^a \frac{[l_{2k}^2(x) - x^2]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_c(x)\,\mathrm{d}x \right.$$

$$\left. - \frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k} \int_0^a \frac{[x^2 - l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x) - l_{1k}^2(x)} Y_s(x)\,\mathrm{d}x \right\}. \tag{5.3.8}$$

Problem 2. The boundary conditions in the case of a flat circular bonded punch subjected to a shifting force T and a tilting moment M are

$u = u_0 = \text{const.}$, for $0 \le \rho \le a$, $0 \le \phi < 2\pi$;

$w = -\delta\rho\cos\phi$, for $0 \le \rho \le a$, $0 \le \phi < 2\pi$;

$\sigma = 0$, for $a \le \rho \le \infty$, $0 \le \phi < 2\pi$;

$$\tau = 0, \qquad \text{for } a \le \rho \le \infty, \qquad 0 \le \phi < 2\pi. \tag{5.3.9}$$

Here u_0 is the tangential displacement of the punch, and δ is the angle of inclination. Again, from the results in section 5.2 we get the potential functions

$$F_1 = \frac{4\pi H}{m_1 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_1 - [l_{21}^2(x) - x^2]^{1/2}\} f(x)\,dx\right.$$

$$\left. + \gamma_1 \int_0^a \{x - [x^2 - l_{11}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_2 = \frac{4\pi H}{m_2 - 1}\frac{\cos\phi}{\rho}\left\{\alpha\int_0^a \{z_2 - [l_{22}^2(x) - x^2]^{1/2}\} f(x)\,dx\right.$$

$$\left. + \gamma_2 \int_0^a \{x - [x^2 - l_{12}^2(x)]^{1/2}\} f_1(x)\right\},$$

$$F_3 = D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\int_0^a \{z_3 - [l_{23}^2(x) - x^2]^{1/2}\}\,dx$$

$$= D\frac{2\gamma_3}{A_{44}}\frac{\sin\phi}{\rho}\left[z_3 a - \frac{(l_{23}^2 - a^2)^{1/2}(2a^2 - l_{13}^2)}{2a} - \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_{23}}\right)\right] \tag{5.3.10}$$

Here f and f_1 are the stress functions and D is a constant. They are defined (Fabrikant, 1989a) as follows:

$$f(t) = -\frac{\delta\cosh^2(\pi\theta)}{\pi^2 H \sinh(\pi\theta)}\left[tY_s(t) - \theta a Y_c(t)\right] + AY_c(t). \tag{5.3.11}$$

$$f_1(t) = \frac{\delta\cosh(\pi\theta)}{\pi^2 H}\left[tY_c(t) + \theta a Y_s(t)\right] + \tanh(\pi\theta) A Y_s(t). \tag{5.3.12}$$

$$D = \frac{\pi\theta\alpha}{\gamma_1\gamma_2\sinh(\pi\theta)}A, \qquad \tanh(\pi\theta) = \frac{\alpha}{\sqrt{\gamma_1\gamma_2}} \tag{5.3.13}$$

$$A = \left(u_0 + \frac{\delta a\theta\alpha}{\tanh(\pi\theta)}\right)\left[\frac{\pi^2 H\alpha}{\cosh(\pi\theta)}\left(1 + \frac{\pi\theta(G_1+G_2)}{\tanh(\pi\theta)(G_1-G_2)}\right)\right]^{-1}. \quad (5.3.14)$$

Again, we need only the expression for the normal displacement

$$w(\rho,z) = 4\pi H\frac{\cos\phi}{\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\alpha}{\gamma_k}\int_0^a\left(1-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)f(x)\,dx\right.\right.$$

$$\left.\left.-\int_0^a\frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}f_1(x)\,dx\right]\right\}. \quad (5.3.15)$$

The displacements of the punch are related to the applied loading as (Fabrikant, 1989a)

$$u_0 = \frac{1}{8a}\left[\pi(G_1+G_2) + \frac{(1+4\theta^2)\tanh(\pi\theta)}{\theta(1+\theta^2)}(G_1-G_2)\right]T - \frac{3H\alpha}{4a^2(1+\theta^2)}M,$$

$$\delta = \frac{3H\alpha}{4a^2(1+\theta^2)}\left[-T + \frac{M}{a\theta\sqrt{\gamma_1\gamma_2}}\right]. \quad (5.3.16)$$

The main problem. Now we may apply the reciprocal theorem in order to obtain the punch displacements due to a normal concentrated force N applied at the point $(\rho,0,z)$. The normal displacement of the punch is readily available from (8) as

$$w_N = \frac{\mathrm{NH}\sinh(\pi\theta)}{\pi a\theta}\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{\int_0^a\frac{[l_{2k}^2(x)-x^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_c(x)\,dx\right.$$

$$\left.-\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\frac{[x^2-l_{1k}^2(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}Y_s(x)\,dx\right\}. \quad (5.3.17)$$

In order to find the tangential displacement of the punch, we have to apply a unit tangential force T in the positive Ox direction. From (11–14) and (16), one can find the stress functions as

$$f(x) = \frac{T\sqrt{\gamma_1\gamma_2}\cosh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left[3\frac{x}{a}Y_s(x) + \frac{1-2\theta^2}{\theta}Y_c(x)\right],$$

$$f_1(x) = \frac{T\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{4\pi^2 a(1+\theta^2)}\left[-3\frac{x}{a}Y_c(x) + \frac{1-2\theta^2}{\theta}Y_s(x)\right], \tag{5.3.18}$$

and the tangential displacements can be defined from (15) in the form

$$u_N = \frac{NH\sqrt{\gamma_1\gamma_2}\sinh(\pi\theta)}{\pi a(1+\theta^2)\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a\left(1\right.\right.\right.$$

$$\left.-\frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\right)\left(3\frac{x}{a}Y_s(x) + \frac{1-2\theta^2}{\theta}Y_c(x)\right)dx$$

$$\left.\left.-\int_0^a \frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(-3\frac{x}{a}Y_c(x) + \frac{1-2\theta^2}{\theta}Y_s(x)\right)dx\right]\right\}, \tag{5.3.19}$$

We need to apply to the punch a unit tilting moment M in order to find the angular displacement δ. The stress functions in this case are

$$f(x) = -\frac{3M\cosh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}\left(xY_s(x) - \theta a Y_c(x)\right),$$

$$f_1(x) = \frac{3M\sinh(\pi\theta)}{4\pi^2 a^3\theta(1+\theta^2)}\left(xY_c(x) + \theta a Y_s(x)\right), \tag{5.3.20}$$

and the angular displacement will take the form

$$\delta_N = \frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\frac{\sqrt{\gamma_1\gamma_2}}{\gamma_k}\int_0^a \frac{l_{2k}(x)[l_{2k}^2(x)-\rho^2]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(xY_s(x)-\theta a Y_c(x)\right)dx\right.\right.$$

$$\left.\left.-\int_0^a \frac{l_{1k}(x)[\rho^2-l_{1k}(x)]^{1/2}}{l_{2k}^2(x)-l_{1k}^2(x)}\left(xY_c(x)+\theta a Y_s(x)\right)dx\right]\right\}, \tag{5.3.21}$$

Formulae (17,19,21) are the main new results of this section.

It is of interest to compare the influence of a concentrated load on a bonded punch to that of a smooth punch. These two cases represent two extremes in the interaction between a punch and an elastic half-space, so the results give the upper and lower bounds for the parameters involved. The normal and angular displacements of a smooth punch due to a point load N applied at the point $(\rho,0,z)$ are (Fabrikant, 1989a)

$$w_N = \frac{NH}{a}\sum_{k=1}^{2}\frac{m_k}{m_k-1}\sin^{-1}\left(\frac{a}{l_{2k}}\right), \tag{5.3.22}$$

$$\delta_N = -\frac{3NH}{2a^3}\sum_{k=1}^{2}\left\{\frac{m_k}{m_k-1}\left[\rho\sin^{-1}\left(\frac{a}{l_{2k}}\right)-\frac{l_{1k}(l_{2k}^2-a^2)^{1/2}}{l_{2k}}\right]\right\}. \tag{5.3.23}$$

One should note that the solutions (17,21) and (22–23) coincide in the case of $\theta=0$. In the case of isotropy, this corresponds to the Poisson coefficient $\nu=1/2$. The greatest difference between the solutions for a bonded and a smooth punch is attained for the Poisson coefficient $\nu=0$. This value was used in numerical computations. Formulae (22–23) in the case of isotropy will take the form

$$w_N = \frac{NH}{a}\left[\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{z(a^2-l_1^2)^{1/2}}{2(1-\nu)(l_2^2-l_1^2)}\right], \tag{5.3.24}$$

$$\delta_N = \frac{3NH}{2a^3}\,\rho\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2}+\frac{za^2(a^2-l_1^2)^{1/2}}{(1-\nu)l_2^2(l_2^2-l_1^2)}\right]. \tag{5.3.25}$$

The limiting cases of (17) and (21) in the case of isotropy are

$$w_N = \frac{NH\sinh(\pi\theta)}{\pi a\theta}\left\{\int_0^a\left[\frac{[l_2^2(x)-x^2]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.\right.$$

$$\left.-\frac{z^2[x^2(2x^2+2z^2-\rho^2)-l_2^4(x)]}{2(1-\nu)[l_2^2(x)-x^2]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\right]Y_c(x)\,dx$$

$$-\frac{1}{2(1-\nu)}\int_0^a\left[(1-2\nu)\frac{[x^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}\right.$$

$$+\frac{z^2[x^2(2x^2+2z^2)-\rho^2)-l_1^4(x)]}{[x^2-l_1^2(x)]^{1/2}[l_2^2(x)-l_1^2(x)]^3}\Bigg]Y_s(x)\,\mathrm{d}x\Bigg]\Bigg\}, \tag{5.3.26}$$

$$\delta_N=-\frac{3NH\sinh(\pi\theta)}{\pi a^3\theta(1+\theta^2)\rho}\Bigg\{\frac{1}{2(1-\nu)}\int_0^a\Bigg[(1-2\nu)\frac{l_2(x)[l_2^2(x)-\rho^2]^{1/2}}{l_2^2(x)-l_1^2(x)}$$

$$+\frac{z\rho^2[l_2^2(x)-x^2]^{1/2}}{[l_2^2(x)-l_1^2(x)]^3}[4x^2-3l_1^2(x)-l_2^2(x)]\Bigg][a\theta Y_c(x)-xY_s(x)]\,\mathrm{d}x$$

$$+\int_0^a\Bigg[\frac{l_1(x)[\rho^2-l_1^2(x)]^{1/2}}{l_2^2(x)-l_1^2(x)}-\frac{z\rho^2[x^2-l_1^2(x)]^{1/2}}{2(1-\nu)[l_2^2(x)-l_1^2(x)]^3}[4x^2$$

$$-l_1^2(x)-3l_2^2(x)]\Bigg][xY_c(x)+a\theta Y_s(x)]\,\mathrm{d}x\Bigg\}. \tag{5.3.27}$$

The results of computation are presented in Fig. 5.3.2 (dimensionless parameter w_N/w^0 versus ρ/a) and Fig. 5.3.3 (the ratio δ_N/δ^0) for various $z/a=0, 0.1, 0.5, 1$. As before, the solid line curves give the results for a bonded punch, and the broken line curves give similar results for a smooth punch. The quantity $w^0=\pi NH/(2a)$ corresponds to the settlement of a smooth punch subjected to a central loading equal to N. The plot shows that the settlement of a smooth punch is always greater (up to about $0.1w^0$) than that of a bonded punch.

The parameter $\delta^0=3\pi NH/(4a^2)$ gives the maximum angle of inclination of a smooth punch. For the angle of inclination, the difference between the results for a smooth punch and a bonded punch does not exceed $0.08\delta^0$. All the computations were made for the Poisson coefficient $\nu=0$. Since in real materials $\nu>0$, the difference between the smooth and bonded punch solutions will be smaller than that indicated above.

5.4. Tangential loading underneath a smooth punch

The problem of a smooth circular punch penetrating a transversely isotropic elastic half-space and interacting with an arbitrarily located tangential concentrated load is considered. A closed form exact solution is obtained for the stress distribution under the punch as well as for the linear and angular displacements of the punch.

Fig. 5.3.2. Influence of bonding on the punch settlement

Fig. 5.3.3. Influence of bonding on angular inclination

The problem of interaction between a punch and an anchor load is of particular interest to geomechanics where the internal forces can be visualized as forces transmitted by anchoring regions located in the vicinity of structural foundation. Only some axisymmetric cases have been considered so far. The general case of interaction between a smooth circular punch and an arbitrarily located *tangenial* load is considered here for the first time. To the best of our knowledge, this problem was not considered before even for the case of isotropy.

Theory. We consider a transversely isotropic elastic half-space $z \geq 0$. A smooth punch of arbitrary base shape (Fig. 5.4.1) penetrates its boundary $z = 0$.

Fig. 5.4.1. Geometry of the problem

The domain of contact is a circle of radius a, with its centre coinciding with the coordinate system origin $\rho = 0$. Let a horizontal point force, with the components T_x and T_y, be applied at the point Q with the polar cylindrical coordinates (ρ, ϕ, z). We shall use its complex representation, namely, $T = T_x + iT_y$. We may assume, without loss of generality, that the system of external forces applied to the punch is such that eliminates the normal and angular displacements which otherwise would have been produced by the anchor load T. We need to find the normal tractions under the punch σ, the normal force N and the tilting moment M applied to the punch in order to compensate its displacements which otherwise would have been produced by the force T.

A general solution in terms of three harmonic functions to the mixed boundary value problem for a transversely isotropic elastic half-space was given in (Fabrikant, 1988c). One can deduce from the results of (Fabrikant, 1989a) that the normal displacements on the plane $z=0$ due to the force T can be defined as

$$w(\rho_0,\phi_0,0) = \frac{1}{2}H(T\bar{q} + \bar{T}q)\left[\frac{\gamma_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_2}{(m_2-1)R_2(R_2+z_2)}\right], \tag{5.4.1}$$

where parameters H, γ_1, γ_2, m_1, and m_2 are defined in section 1, and

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}, \quad z_k = z/\gamma_k,$$

$$R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_k^2]^{1/2}, \quad \text{for} \quad k=1,2. \tag{5.4.2}$$

The governing integral equation which relates the normal stress σ exerted by the punch to the displacement w, defined by (1), is as follows:

$$\int_0^{2\pi}\int_0^a \frac{\sigma(r,\psi)r\mathrm{d}r\mathrm{d}\psi}{[\rho_0^2 + r^2 - 2r\rho_0\cos(\phi_0-\psi)]^{1/2}}$$

$$= -\frac{1}{2}(T\bar{q} + \bar{T}q)\left[\frac{\gamma_1}{(m_1-1)R_1(R_1+z_1)} + \frac{\gamma_2}{(m_2-1)R_2(R_2+z_2)}\right], \tag{5.4.3}$$

The general solution of the equation of the type (3) is given first in (Fabrikant, 1971a), but it is somewhat difficult to use directly due to the complexity of the right-hand side of (3). We can use a shortcut instead. In addition to the point $Q(\rho,\phi,z)$, let us introduce two more points, namely, $Q_0(\rho_0,\phi_0,0)$ and $S(r,\psi,0)$. The following identity can be easily verified:

$$\int_0^{2\pi}\int_0^a \frac{z}{R^3(Q,S)}\left[\frac{R(Q,S)}{h} + \tan^{-1}\left(\frac{h}{R(Q,S)}\right)\right]\frac{r\mathrm{d}r\mathrm{d}\psi}{R(Q_0,S)} = \frac{\pi^2}{R(Q,Q_0)}. \tag{5.4.4}$$

Here $R(\cdot,\cdot)$ stands for the distance between two points, and

$$h = (a^2 - r^2)^{1/2}(a^2 - l_1^2)^{1/2}/a.$$

The following notation is used throughout this section:

$$l_1(x,\rho,z) \equiv l_1(x) = \frac{1}{2}\{[(\rho + x)^2 + z^2]^{1/2} - [(\rho - x)^2 + z^2]^{1/2}\},$$

$$l_2(x,\rho,z) \equiv l_2(x) = \frac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\}. \tag{5.4.5}$$

The abbreviations l_1 and l_2 everywhere stand for $l_1(a)$ and $l_2(a)$ respectively. The notations $l_{1k}(x)$ and $l_{2k}(x)$ are understood as $l_1(x,\rho,z_k)$ and $l_2(x,\rho,z_k)$ respectively, for k=1,2.

We can observe that the right-hand side of (3) can be obtained from the right-hand side of (4) by using integration with respect to z and the consequent application of the differentiation operator Λ defined by

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} . \tag{5.4.6}$$

A similar procedure, applied to the left-hand side of (4), will give us the required solution. The relevant integration and differentiation is performed in (Fabrikant, 1989a), and the result is

$$\sigma(r,\psi) = -\Re\left\{\frac{\bar{T}}{\pi^2}\left[\frac{\gamma_1 U(z_1)}{m_1-1}+\frac{\gamma_2 U(z_2)}{m_2-1}\right]\right\}, \tag{5.4.7}$$

where $\Re$ is the real part sign, the overbar everywhere indicates the complex conjugate value, and the following notation was introduced:

$$U(z) \equiv U(\rho,\phi,z;r,\psi) = \frac{q_1}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]-\frac{1}{h\bar{q}_1}\left\{1-\frac{(a^2-l_1^2)^{1/2}}{a}\right.$$
$$\left.-\frac{(a^2-l_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\left[\tan^{-1}\left(\frac{a(\bar{\zeta}-1)^{1/2}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}(\bar{\zeta}-1)^{1/2}\right]\right\}, \tag{5.4.8}$$

with

$$\zeta = \frac{\rho}{r}e^{i(\phi-\psi)}, \quad q_1 = \rho e^{i\phi} - re^{i\psi}, \quad R_0 = R(Q,S). \tag{5.4.9}$$

The resultant force N can be obtained by integration of σ

$$N = \int_0^{2\pi}\int_0^a \sigma(r,\psi)r\,dr\,d\psi. \tag{5.4.10}$$

Though expression (8) looks complicated, the integration in (10) can be performed (see Fabrikant, 1989a, for detail), and the result is rather simple:

$$N = -\Re\left\{\frac{2\overline{T}e^{i\phi}}{\pi\rho}\sum_{k=1}^{2}\left[\frac{\gamma_k[a-(a^2-l_{1k}^2)^{1/2}]}{m_k-1}\right]\right\}. \tag{5.4.11}$$

We define the tilting moment M in the complex form as follows:

$$M = M_x + iM_y = -i\int_0^{2\pi}\int_0^a \sigma(r,\psi)r^2 e^{i\psi}\mathrm{d}r\mathrm{d}\psi. \tag{5.4.12}$$

Again, the integration can be performed in elementary functions (see Fabrikant, 1989a), and the final result is

$$M = -\frac{2}{\pi}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[T\left(z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$
$$\left.\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.13}$$

Taking into consideration the relationship between the tilting moment and the complex angle of inclination

$$\delta = \delta_x + i\delta_y = \frac{3\pi H}{4a^3}M, \tag{5.4.14}$$

we may deduce that when no tilting moment is applied, the punch will tilt, and the angle will be

$$\delta = \frac{3H}{2a^3}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[T\left(z_k\sin^{-1}\left(\frac{a}{l_{2k}}\right)-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$
$$\left.\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.15}$$

Formulae (7–9), (11), (13), and (15) are the main new results of this section.

Applications. The simplicity of the results obtained allows us to consider various cases of distributed internal loadings. Let a uniformly distributed shear tractions τ_0 be applied over the segment $\rho_1 \le \rho \le \rho_2$, $\phi_1 \le \phi \le \phi_2$, at the distance z from the surface. Integrating (11) by the method described in (Fabrikant, 1989a), we obtain

$$N = \Re\left\{\frac{2i}{\pi}\tau_0(e^{i\phi_2} - e^{i\phi_1})\sum_{k=1}^{2}\left[\frac{\gamma_k}{m_k - 1}[V(\rho_2) - V(\rho_1)]\right]\right\}, \tag{5.4.16}$$

where

$$V(\rho) = \int [a - (a^2 - l_1^2)^{1/2}]d\rho = a\rho - \frac{1}{2}\rho(a^2 - l_1^2)^{1/2}$$

$$-\frac{1}{2}(a^2 - z^2)\sin^{-1}\left(\frac{l_1}{(a^2 + z^2)^{1/2}}\right) - az\ln\frac{l_2 + (\rho^2 - l_1^2)^{1/2}}{(a^2 + z^2)^{1/2}}. \tag{5.4.17}$$

In the case of a uniform loading over a complete annulus, formula (16) gives N=0 as it was expected. The value of the tilting moment can be obtained by integration of (13), and we present the result for the case of uniform shear loading of a circular annulus $\rho_1 \le \rho \le \rho_2$

$$M = -\frac{2i}{\pi}\tau_0\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\left[W(\rho_2) - W(\rho_1)\right]\right\}. \tag{5.4.18}$$

Here

$$W(\rho) = \int\left[z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2 - l_1^2)^{1/2}\right]\rho\, d\rho = \frac{1}{2}z\rho^2\sin^{-1}\left(\frac{a}{l_2}\right)$$

$$+\frac{z^2(2a^2 - l_1^2)}{2(a^2 - l_1^2)^{1/2}} - \frac{1}{3}(a^2 - l_1^2)^{1/2}(3l_2^2 + l_1^2 - 4a^2). \tag{5.4.19}$$

In the case of a torsional loading of a circular annulus, both the normal force and the tilting moment vanish.

The general solution is also valid in the case of isotropy, provided that we compute the limiting case

$$m_1 = m_2 = 1, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1, \quad H = \frac{1 - \nu^2}{\pi E},$$

where E is the elastic modulus, and ν is Poisson coefficient.

According to the L'Hôpital rule, the following scheme should be used

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1}+\frac{\gamma_2 f(z_2)}{m_2-1}\right]=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \quad (5.4.20)$$

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1, \qquad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \quad (5.4.21)$$

and the symbol (´) indicates differentiation with respect to z. Formulae (7), (11), (13), and (15) in the case of isotropy will take the form

$$\sigma(r,\psi)=\Re\left\{\frac{\overline{T}}{2\pi^2(1-\nu)}\,[(1-2\nu)U(z)+zU'(z)]\right\}, \quad (5.4.22)$$

with $U(z)$ defined by (8–9)

$$U'(z)=\frac{\partial U}{\partial z}=-\frac{3zq}{R_0^5}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]+\frac{z}{h(R_0^2+h^2)}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{q}{R_0^2}\right]; \quad (5.4.23)$$

$$N=\Re\left\{\frac{\overline{T}e^{i\phi}}{\pi\rho(1-\nu)}\left[(1-2\nu)[a-(a^2-l_1^2)^{1/2}]-\frac{zl_1(\rho^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right]\right\}, \quad (5.4.24)$$

$$M=\frac{\mathrm{i}}{\pi(1-\nu)}\left\{(1-2\nu)\left[T\left(z\sin^{-1}(\frac{a}{l_2})-(a^2-l_1^2)^{1/2}\right)\right.\right.$$

$$\left.-\overline{T}e^{2i\phi}\frac{2a^3-(l_1^2+2a^2)(a^2-l_1^2)^{1/2}}{3\rho^2}\right]+z\left[T\left(\sin^{-1}(\frac{a}{l_2})-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)\right.$$

$$\left.\left.+\overline{T}e^{2i\phi}\frac{al_1^2(l_2^2-a^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}\right]\right\}, \quad (5.4.25)$$

$$\delta=\frac{3\mathrm{i}}{8\pi\mu a^3}\left\{(1-2\nu)\left[T\left(z\sin^{-1}(\frac{a}{l_2})-(a^2-l_1^2)^{1/2}\right)\right.\right.$$

$$-\overline{T}e^{2i\phi}\frac{2a^3-(l_1^2+2a^2)(a^2-l_1^2)^{1/2}}{3\rho^2}\Bigg]+z\Bigg[T\left(\sin^{-1}(\frac{a}{l_2})-\frac{a(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)$$

$$+\overline{T}e^{2i\phi}\frac{a l_1^2(l_2^2-a^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}\Bigg]\Bigg\}, \qquad (5.4.26)$$

In order to illustrate the stress distribution under the punch, computations were made due to (22–23), for $\nu=0.3$, $\phi=0$, $z/a=0.25$, $\rho/a=0.5$. The cases of a unit radial and a unit transversal loading are considered separately (Fig. 5.4.2 and Fig. 5.4.3). The curves given by the solid dots, the dashed line, the line of

Fig. 5.4.2. Stress distribution due to a unit radial force

circles, and the solid line correspond to the values of $\psi=0$, $\pi/4$, $\pi/2$, $3\pi/4$ respectively. The negative values of r correspond to the argument $\psi+\pi$. It is important to emphasize that the presented results are valid only when the contact is maintained all over the circle $r\leq a$. Since the normal stress is negative at the part of the domain of contact, this implies that there should be additional external loading applied to the punch, so that the results stay valid.

Discussion. There exists an alternative way to solve the problem of interaction between an external load and a circular punch. Indeed, we could use the Green's functions derived in (Fabrikant, 1989a), combined with the reciprocal

Fig. 5.4.3. Stress distribution due to a unit transversal force

theorem. The tangential displacements around a circular punch are defined by

$$u(\rho,\phi,z) = \frac{1}{\pi^2}\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k - 1}\int_0^{2\pi}\int_0^{a} U(\rho,\phi,z_k;\, r,\psi)\,\omega(r,\psi)\, r\mathrm{d}r\mathrm{d}\psi\right\}. \qquad (5.4.27)$$

Here U is defined by (8), and ω denotes the normal displacements under the punch. We consider two systems in equilibrium, namely, a tangential force is applied at the point (ρ,ϕ,z), and a normal stress σ is applied at the domain of contact in order to eliminate the normal displacements; the second system represents a transversely isotropic half-space, with a normal displacement prescribed at the point $(r,\psi,0)$ in the form $\omega=\delta(r,\psi)$, where δ is the Dirac delta-function. Application of the reciprocal theorem yields

$$\Re\{u\bar{T}\} + \iint_S \sigma\,\omega\,\mathrm{d}S = 0. \qquad (5.4.28)$$

Substitution of (27) in (28) and subsequent use of the properties of the delta-function lead immediately to (7).

The tangential displacement around a flat circular punch, subjected to a unit normal displacement, is (Fabrikant, 1989a)

$$u = \frac{2e^{i\phi}}{\pi\rho}\sum_{k=1}^{2}\left[\frac{\gamma_k[a-(a^2-l_{1k}^2)^{1/2}]}{m_k-1}\right]. \tag{5.4.29}$$

Again, application of the reciprocal theorem yields (11). In order to derive (26), we should use the expression for tangential displacement around an flat punch inclined by the angle $\delta=\delta_x+i\delta_y$ (Fabrikant, 1989a)

$$u = \frac{2}{\pi}i\sum_{k=1}^{2}\left\{\frac{\gamma_k}{m_k-1}\left[\delta\left(z_k\sin^{-1}(\frac{a}{l_{2k}})-(a^2-l_{1k}^2)^{1/2}\right)\right.\right.$$

$$\left.\left.-\bar{\delta}e^{2i\phi}\frac{2a^3-(l_{1k}^2+2a^2)(a^2-l_{1k}^2)^{1/2}}{3\rho^2}\right]\right\}. \tag{5.4.30}$$

The derivation of (13) from (30) is straightforward.

5.5. The general annular punch problem

Quite a few papers have been published on the subject. One can find many references related to contact problem in (Barber, 1983), other references related to the equivalent electrostatic problem can be found in Love (1976). Why is there any need for yet another work on the subject? The main reason is that the majority of publications is devoted to the simplest flat centrally loaded annular punch problem. A very small number of publications treat non-flat but still axisymmetric problems (Barber 1976, 1983). Though some results related to consideration of specific harmonics have been published (Williams, 1963; Cooke, 1963), no general solution to the problem has been attempted as yet. Such a solution is presented here. The problem is reduced to a two-dimensional Fredholm integral equation with an elementary kernel which can be solved numerically. Flat inclined and centrally loaded annular punches are considered as examples. Asymptotic formulae are derived for the case of a very narrow annulus.

Theory. Consider a rigid annular punch $b\le\rho\le a$ penetrating a transversely isotropic elastic half space $z>0$. Neglecting the shear stress under the punch base, the boundary conditions for the problem can be formulated as follows:

$$w(\rho,\phi) = \delta - s(\rho,\phi), \quad \text{for} \quad b<\rho<a, \quad 0\le\phi<2\pi;$$

$$\sigma_z = 0, \quad \text{for} \quad \rho < b \quad \text{or} \quad \rho > a, \quad 0 \le \phi < 2\pi;$$

$$\tau_{yz} = \tau_{zx} = 0, \quad \text{for} \quad 0 \le \rho < \infty, \quad 0 \le \phi < 2\pi. \tag{5.5.1}$$

Here δ is the maximum punch penetration and s describes the shape of the punch base. It is well known that the problem can be reduced to the governing integral equation

$$H \int_0^{2\pi} \int_b^a \frac{\sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}} = w(\rho, \phi). \tag{5.5.2}$$

Here H is the elastic constant (see 5.1.9), w is the known function (1), and $\sigma = -\sigma_z$ is the yet unknown function. The following integral representation for the reciprocal of the distance between two points can be found in (1.2.22)

$$\frac{1}{R} = \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}} = \frac{2}{\pi} \int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)\, dx}{(\rho^2 - x^2)^{1/2} (\rho_0^2 - x^2)^{1/2}}. \tag{5.5.3}$$

Here

$$\lambda(k,\psi) = \frac{1 - k^2}{1 - 2k \cos\psi + k^2}. \tag{5.5.4}$$

Substitution of (3) in (2) leads to the governing integral equation

$$4 \int_b^\rho \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0, \phi)$$

$$+ 4 \int_0^b \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_b^a \frac{\rho_0 d\rho_0}{(\rho_0^2 - x^2)^{1/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma(\rho_0, \phi) = \frac{w(\rho, \phi)}{H}. \tag{5.5.5}$$

The $\mathcal{L}$-operator for $k \le 1$ was introduced as follows:

$$\mathcal{L}(k) f(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \lambda(k, \phi - \phi_0)\, f(\rho, \phi_0)\, d\phi_0$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|} e^{in\phi} \int_0^{2\pi} e^{-in\phi_0} f(\rho,\phi_0)\, d\phi_0 = \sum_{n=-\infty}^{\infty} k^{|n|} f_n(\rho) e^{in\phi}. \tag{5.5.6}$$

Here f_n is the n-th Fourier coefficient of the function f. Application of the operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_b^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)$$

to both sides of (5) yields

$$2\pi\int_r^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-r^2}}\,\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma(\rho_0,\phi) + \frac{4r}{\sqrt{r^2-b^2}}\int_0^b \frac{\sqrt{b^2-x^2}\,dx}{r^2-x^2}\int_b^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\left(\frac{x^2}{r\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= \frac{1}{H}\,\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_b^r \frac{\rho\, d\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho) w(\rho,\phi). \tag{5.5.7}$$

We introduce a new unknown function

$$\chi(r,\phi) = \int_r^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-r^2}}\,\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{5.5.8}$$

The inverse of (8) is readily available, and is

$$\sigma(\rho,\phi) = -\frac{2}{\pi}\frac{\mathcal{L}(\rho)}{\rho}\frac{d}{d\rho}\int_\rho^a \frac{r\, dr}{\sqrt{r^2-\rho^2}}\,\mathcal{L}\left(\frac{1}{r}\right)\chi(r,\phi). \tag{5.5.9}$$

Substitution of (8) in (7) gives

$$2\pi\chi(r,\phi) + \frac{8}{\pi}\frac{r}{\sqrt{r^2-b^2}}\int_0^b \frac{b^2-x^2}{r^2-x^2}\,dx \int_b^a \frac{y\, dy}{\sqrt{y^2-b^2}(y^2-x^2)}\,\mathcal{L}\left(\frac{x^2}{yr}\right)\chi(y,\phi)$$

$$= \frac{1}{H}\,\mathcal{L}\!\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)w(\rho,\phi). \tag{5.5.10}$$

One can interchange the order of integration in the second term of (10) and perform the integration with respect to x. The result is

$$\chi(r,\phi) + \frac{1}{\pi^3}\int_0^{2\pi}\int_b^a \frac{K(y,r,\phi-\phi_0)-K(r,y,\phi-\phi_0)}{y^2-r^2}\,\chi(y,\phi_0)\,\mathrm{d}y\,\mathrm{d}\phi_0$$

$$= \frac{1}{2\pi H}\,\mathcal{L}\!\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)w(\rho,\phi). \tag{5.5.11}$$

The kernel of (11) can be expressed in terms of elementary functions as follows:

$$K(y,r,\phi-\phi_0) = ry\left(\frac{r^2-b^2}{y^2-b^2}\right)^{1/2}\left\{-\lambda\!\left(\frac{r}{y},\,\phi-\phi_0\right)\frac{1}{r}\ln\!\left(\frac{r+b}{r-b}\right)\right.$$

$$\left. + 2\Re\left[\frac{1}{\xi\left(1-\frac{r}{y}\mathrm{e}^{-i(\phi-\phi_0)}\right)}\ln\frac{\xi+b}{\xi-b}\right]\right\}, \tag{5.5.12}$$

where

$$\xi = \sqrt{yr\mathrm{e}^{i(\phi-\phi_0)}}. \tag{5.5.13}$$

Here $\Re$ denotes the real part of the expression to follow. Thus, the general problem of annular punch has been reduced to a Fredholm integral equation (11) with an elementary kernel which can be solved numerically. It is noteworthy that the governing equation for each specific harmonic will also have an elementary kernel. For example, the equation corresponding to the zero harmonic is

$$\chi_0(r) + \frac{2}{\pi^2}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{y^2-r^2}\,\chi_0(y)\,\mathrm{d}y = \frac{1}{2\pi H}\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{w_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \tag{5.5.14}$$

with

$$K_0(y,r) = r\left(\frac{y^2-b^2}{r^2-b^2}\right)^{1/2} \ln\frac{y+b}{y-b}. \tag{5.5.15}$$

There have been so many variations of the governing integral equation published for the case of axial symmetry, that there is no doubt that equation (14) coincides with some known result, though we have difficulty to pinpoint exactly which one. The governing integral equation for the first harmonic will take the form

$$\chi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r)-K_1(r,y)}{y^2-r^2}\chi_1(y)\,\mathrm{d}y = \frac{1}{2\pi H}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{w_1(\rho)\,\rho^2\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \tag{5.5.16}$$

with

$$K_1(y,r) = \left(\frac{y^2-b^2}{r^2-b^2}\right)^{1/2}\left[y\ln\frac{y+b}{y-b} - 2b\right]. \tag{5.5.17}$$

There is no need to compute the stress distribution σ if one is interested in the integral characteristics only. Indeed, both the resultant force P and the tilting moment M can be expressed through the new unknown function χ as follows:

$$P = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\chi(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{\rho^2-b^2}} = 4\int_b^a \frac{\chi_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}, \tag{5.5.18}$$

$$M = -\frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{(2\rho^2-b^2)\chi(\rho,\phi)\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{\rho^2-b^2}} = -2\int_b^a \frac{(2\rho^2-b^2)\chi_1(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.19}$$

We note also that the kernels in (14) and (16) are finite at the point $y=r$. The following limits can be computed

$$\lim_{y\to r}\frac{K_0(y,r)-K_0(r,y)}{y^2-r^2} = \frac{1}{r^2-b^2}\left[\frac{r^2+b^2}{2r}\ln\frac{r+b}{r-b} - b\right],$$

$$\lim_{y\to r}\frac{K_1(y,r)-K_1(r,y)}{y^2-r^2} = \frac{1}{r^2-b^2}\left[\frac{3r^2-b^2}{2r}\ln\frac{r+b}{r-b} - 3b\right]. \tag{5.5.20}$$

Equations (11), (12), (18), and (19) are the main new results of this section.

Description of the numerical procedure. Consider the following integral equation:

$$h(r)\,f(r) + \int_b^a \mathcal{K}(r,x)\,f(x)\,\mathrm{d}x = g(r). \tag{5.5.21}$$

Here h and g are known functions, $\mathcal{K}$ is the kernel, and f is the as yet unknown function. The procedure which is usually used may be described as follows. We divide the interval $[b,a]$ into $n-1$ equal subintervals of length $\Delta=(a-b)/(n-1)$. The points of division are called x_k, $k=1,2,\ ...,n$. Assume the unknown function f to be piecewise *constant* on each of the subintervals and equal to f_k on the subinterval number k. Introduce a set of points $r_k=(x_k+x_{k+1})/2$, for $k=1,2,\ ...,n-1$. These assumptions allow us to reduce the integral equation (21) to a set of $n-1$ linear algebraic equations

$$h(r_k)\,f(r_k) + \sum_{i=1}^{n-1} \kappa_i(r_k)\,f_i = g(r_k), \quad \text{for} \quad k=1,2,...,n-1. \tag{5.5.22}$$

Here

$$\kappa_i(r_k) = \int_{x_i}^{x_{i+1}} \mathcal{K}(r_k,x)\,\mathrm{d}x. \tag{5.5.23}$$

The second method to be used here is somewhat different from that above. We consider the unknown function f to be piecewise *linear*. Assuming $f_k=f(x_k)$, for $k=1,2,\ ...,n$, this implies that at the k-th subinterval the function f can be expressed as follows:

$$f(x) = f_k + (f_{k+1}-f_k)\left(\frac{x-b}{\Delta}-k\right), \quad \text{for} \quad x_k<x<x_{k+1}. \tag{5.5.24}$$

Substitution of (24) in (21) leads to a set of n linear algebraic equations

$$h(r_l)f_l + f_1\left[\kappa_1(r_l) - \frac{\theta_1(r_l)}{\Delta}\right] + \sum_{i=2}^{n-1} f_i\left[i\kappa_i(r_l) - (i-2)\kappa_{i-1}(r_l) - \frac{\theta_i(r_l)-\theta_{i-1}(r_l)}{\Delta}\right]$$

$$+ f_n\left[\frac{\theta_{n-1}(r_l)}{\Delta} - (n-2)\kappa_{n-1}(r_l)\right] = g(r_l), \quad r_l=x_l, \quad \text{for} \quad l=1,2,...,n. \tag{5.5.25}$$

Here

$$\theta_i(r_l) = \int_{x_i}^{x_{i+1}} \mathcal{K}(r_l, x)\,(x-b)\,\mathrm{d}x. \tag{5.5.26}$$

Since the piecewise linear function follows the real function more close than the piecewise constant one, we should expect the set of equations (25) to give a more accurate solution than (22). One can also assume the function f to be piecewise *quadratic*. This exercise is left to the reader. Several examples are considered below.

Flat centrally loaded annular punch. In this case w_0=const., and the governing integral equation (14) will take the form

$$\chi_0(r) + \frac{2}{\pi^2}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{y^2-r^2}\,\chi_0(y)\,\mathrm{d}y = \frac{w_0}{2\pi H}\frac{r}{\sqrt{r^2-b^2}}. \tag{5.5.27}$$

It is well known that the stress distribution σ has square root singularities at $\rho = a$ and at $\rho = b$. We can then conclude from (8) that function χ_0 will have a logarithmic singularity at the point $\rho = b$. In order to obtain an effective numerical solution of (27) we have to eliminate singularities whenever possible. We introduce a new unknown function

$$f(r) = \frac{\chi_0(r)}{\ln\dfrac{r+b}{r-b}}, \tag{5.5.28}$$

which will have no singularities and will be limited on the $[b, a]$. Substitution of (28) in (27) allows us to rewrite it as follows:

$$\frac{\sqrt{r^2-b^2}}{r}\ln\frac{r+b}{r-b}f(r) + \frac{2}{\pi^2}\frac{\sqrt{r^2-b^2}}{r}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{y^2-r^2}f(y)\ln\left(\frac{y+b}{y-b}\right)\mathrm{d}y = \frac{w_0}{2\pi H}. \tag{5.5.29}$$

Note that in the limiting case of $r \to b$ equation (29) yields

$$\int_b^a \ln^2\left(\frac{y+b}{y-b}\right)\frac{f(y)\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{\pi w_0}{4H}. \tag{5.5.30}$$

The problem was solved numerically by using both methods from the previous section. The value of the total force P was computed in the first method according to the formula (18) as follows:

$$P = 4\sum_{i=1}^{n-1} f_i \int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.31}$$

The resultant force in the case of the second numerical method was computed as

$$P = 4\sum_{i=1}^{n-1}\left\{\left[\left(i+\frac{b}{\Delta}\right)f_i - \left(i+\frac{b}{\Delta}-1\right)f_{i+1}\right]\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}\right.$$

$$\left. + \frac{f_{i+1}-f_i}{\Delta}\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{\rho^2\,\mathrm{d}\rho}{\sqrt{\rho^2-b^2}}\right\}. \tag{5.5.32}$$

The integrals in (31) and (32) can be computed exactly in terms of elementary functions or it can be computed numerically.

Numerical computations were performed according to both methods for different values of n and various ratios b/a. The dimensionless quantity $f^* = Hf/w_0$ is plotted in Fig. 5.5.2. versus $\rho^* = (\rho - b)/\Delta + 1$. The argument of each plot was scaled in such a way that it would stretch over the same interval. The dimensionless resultant force $P^* = P/P_0$ is presented in the Table 5.5.1. The quantity $P_0 = 2w_0 a/(\pi H)$ corresponds to the resultant force producing normal displacement w_0 when applied to a circular punch of radius a (see Fabrikant, 1989a, p. 342). The column denoted as exact was computed independently according to the formula derived in (Love, 1976). This formula in our notation reads

$$P^* = 1 - \sum_{n=1}^{\infty} k\int_0^k \left\{\int_0^k K_L^n(u,t)\frac{\mathrm{d}t}{t}\right\}^2 \mathrm{d}u. \tag{5.5.33}$$

Here $k = \sqrt{b/a}$, and K_L^n is the n-th iteration of the kernel

$$K_L(u,t) = \frac{2}{\pi}\frac{ut}{1-u^2t^2}. \tag{5.5.34}$$

Fig. 5.5.2. Solution for a centrally loaded annular punch

Let us point out some interesting features of the numerical results in Table 5.1.1. First of all, two different methods lead to different results, but the discrepancy between them decreases as n increases, and in such a way that their average changes very little being very close to the exact value. The second conclusion is that each of the methods gives either upper or lower bound for the computed quantity. This feature is extremely important since it allows us to estimate the error of computation. As we expected, the second method is everywhere more accurate than the first one.

An asymptotic solution for a very narrow annulus can be found by using the analogy with a two-dimensional contact problem. The stress distribution can be taken in the form

$$\sigma(\rho) = \frac{\sigma_0}{\sqrt{c^2 - (\rho - r_0)^2}}. \tag{5.5.35}$$

Here σ_0 is the as yet unknown constant,

$$c = (a - b)/2, \quad r_0 = (a + b)/2. \tag{5.5.36}$$

Substitution of (35) in (8) yields

Table 5.5.1.

n	b/a	Method 1 P^*	Method 2 P^*	Average P^*	Exact P^*
10	0.20000	1.0325	0.9855	1.0090	0.9989
	0.40000	1.0117	0.9807	0.9962	0.9907
	0.60000	0.9776	0.9587	0.9682	0.9651
	0.80000	0.9027	0.8950	0.8988	0.8976
	0.90000	0.8224	0.8204	0.8214	0.8210
	0.95000	0.7473	0.7483	0.7478	0.7478
	0.99500	0.5598	0.5632	0.5615	0.5618
	0.99950	0.4440	0.4471	0.4456	0.4458
	0.99995	0.3676	0.3702	0.3689	0.3691
20	0.20000	1.0119	0.9924	1.0022	0.9989
	0.40000	0.9993	0.9859	0.9926	0.9907
	0.60000	0.9704	0.9620	0.9662	0.9651
	0.80000	0.8998	0.8964	0.8981	0.8976
	0.90000	0.8216	0.8207	0.8212	0.8210
	0.95000	0.7475	0.7481	0.7478	0.7478
	0.99500	0.5608	0.5625	0.5617	0.5618
	0.99950	0.4449	0.4465	0.4457	0.4458
	0.99995	0.3684	0.3697	0.3690	0.3691
30	0.20000	1.0066	0.9946	1.0006	0.9989
	0.40000	0.9959	0.9876	0.9918	0.9907
	0.60000	0.9684	0.9631	0.9657	0.9651
	0.80000	0.8990	0.8968	0.8979	0.8976
	0.90000	0.8214	0.8208	0.8211	0.8210
	0.95000	0.7476	0.7480	0.7478	0.7478
	0.99500	0.5612	0.5623	0.5617	0.5618
	0.99950	0.4452	0.4463	0.4457	0.4458
	0.99995	0.3686	0.3695	0.3691	0.3691
40	0.20000	1.0043	0.9958	1.0000	0.9989
	0.40000	0.9944	0.9884	0.9914	0.9907
	0.60000	0.9674	0.9636	0.9655	0.9651
	0.80000	0.8986	0.8970	0.8978	0.8976
	0.90000	0.8213	0.8209	0.8211	0.8210
	0.95000	0.7477	0.7479	0.7478	0.7478
	0.99500	0.5613	0.5622	0.5618	0.5618
	0.99950	0.4454	0.4462	0.4458	0.4458
	0.99995	0.3687	0.3694	0.3691	0.3691

$$\chi(t) \approx \sigma_0 \frac{\sqrt{r_0}}{\sqrt{2}} \int_t^c \frac{\mathrm{d}x}{\sqrt{c^2 - x^2}\sqrt{x - t}}. \tag{5.5.37}$$

Here the following new variables were introduced

$$\rho_0 = r_0 + x, \quad r = r_0 + t, \tag{5.5.38}$$

and the small quantities of the order of c/r_0, x/r_0, and t/r_0 were neglected. The same procedure applied to (18) and (30) yields respectively

$$P = 2\sigma_0 \sqrt{2r_0} \int_{-c}^{c} \frac{\chi(t)\,\mathrm{d}t}{\sqrt{t + c}}, \tag{5.5.39}$$

$$w_0 = \frac{2\sqrt{2}\sigma_0}{\pi H \sqrt{r_0}} \int_{-c}^{c} \ln\left(\frac{2r_0}{t + c}\right) \frac{\chi(t)\,\mathrm{d}t}{\sqrt{t + c}}, \tag{5.5.40}$$

Substitution of (37) in (39) and (40), interchange of the order of integration and subsequent integration yield

$$P = 2\pi^2 r_0 \sigma_0, \tag{5.5.41}$$

$$w_0 = 2\pi H \sigma_0 \ln\left(\frac{16 r_0}{c}\right). \tag{5.5.42}$$

Here the following integral was used

$$\int_{-c}^{x} \ln\left(\frac{2r_0}{t + c}\right) \frac{\mathrm{d}t}{\sqrt{t + c}\sqrt{x - t}} = \pi \ln\left(\frac{8r_0}{x + c}\right). \tag{5.5.43}$$

We may now deduce from (41) and (42) that

$$P = \frac{\pi w_0 r_0}{H \ln\left(\frac{16 r_0}{c}\right)}, \tag{5.5.44}$$

$$\sigma_0 = \frac{P}{2\pi^2 r_0} = \frac{w_0}{2\pi H \ln\left(\frac{16 r_0}{c}\right)}, \tag{5.5.45}$$

$$P^* = \frac{\pi^2}{2\ln\left(\frac{16(a+b)}{a-b}\right)}. \tag{5.5.46}$$

The last result is in agreement with that of Smythe (1951) and Collins (1963).

Flat inclined annular punch. Assume that the punch is tilted about axis Oy in the positive direction, and that the angle of rotation is α. The normal displacements under the punch can be expressed as

$$w(\rho,\phi) = -\alpha\rho\cos\phi. \tag{5.5.47}$$

Substitution of (47) in (16) leads to the governing integral equation

$$\chi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r) - K_1(r,y)}{y^2 - r^2}\chi_1(y)\,\mathrm{d}y = -\frac{\alpha}{2\pi H}\frac{2r^2 - b^2}{\sqrt{r^2 - b^2}}. \tag{5.5.48}$$

It is reminded that the kernel K_1 is defined by (17). We may conclude once again that since the stress distribution is singular at the edges $\rho = b$ and $\rho = a$, the function χ_1 will have a logarithmic singularity at the point $\rho = b$. Introducing a new unknown function q as

$$q(r) = \frac{\chi_1(r)}{\ln\frac{r+b}{r-b}}, \tag{5.5.49}$$

we may rewrite (48) in the form

$$\frac{\sqrt{r^2-b^2}}{2r^2-b^2}\ln\frac{r+b}{r-b}\,q(r) + \frac{2}{\pi^2}\frac{\sqrt{r^2-b^2}}{2r^2-b^2}\int_b^a \frac{K_1(y,r)-K_1(r,y)}{y^2-r^2}\,q(y)\ln\left(\frac{y+b}{y-b}\right)\mathrm{d}y = -\frac{\alpha}{2\pi H}. \tag{5.5.50}$$

The problem was solved numerically by using both methods from the previous section. The value of the tilting moment M was computed in the first method according to the formula (19) as follows:

$$M = -2\sum_{i=1}^{n-1} q_i \int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}}. \tag{5.5.51}$$

The following formula was used in the second numerical method

$$M = -2\sum_{i=1}^{n-1}\left\{\left[q_i\left(i+\frac{b}{\Delta}\right)-q_{i+1}\left(i+\frac{b}{\Delta}-1\right)\right]\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}}\right.$$

$$\left.+\frac{q_{i+1}-q_i}{\Delta}\int_{x_i}^{x_{i+1}} \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,\rho\,d\rho}{\sqrt{\rho^2-b^2}}\right\}. \tag{5.5.52}$$

The integrals in (51) and (52) can be computed in terms of elementary functions, namely,

$$\int \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,d\rho}{\sqrt{\rho^2-b^2}} = \sqrt{\rho^2-b^2}\left[2b+\rho\ln\left(\frac{\rho+b}{\rho-b}\right)\right],$$

$$\int \ln\left(\frac{\rho+b}{\rho-b}\right)\frac{(2\rho^2-b^2)\,\rho\,d\rho}{\sqrt{\rho^2-b^2}} = \frac{1}{3}(2\rho^2+b^2)\sqrt{\rho^2-b^2}\,\ln\left(\frac{\rho+b}{\rho-b}\right)$$

$$+\frac{2}{3}b\left(\rho\sqrt{\rho^2-b^2} + 2b^2\ln(\rho+\sqrt{\rho^2-b^2}\right).$$

Numerical computations were performed according to both methods for different values of n and various ratios b/a. The dimensionless quantity $q^* = H|f|/(a\alpha)$ is plotted in Fig. 5.5.3. versus $\rho^* = (\rho - b)/\Delta + 1$. The conventions are the same as in the Fig. 5.5.2. The dimensionless tilting moment $M^* = M/M_0$ is presented in the Table 5.5.2. The quantity $M_0 = 4a^3\alpha/(3\pi H)$ corresponds to the tilting moment producing angular displacement α when applied to a circular punch of radius a. Here we no longer have that peculiar property that each method gives either upper or lower bound for the solution. It does not hold for $b/a = 0.2$, though it appears to be valid for $b/a \geq 0.4$.

An asymptotic solution for a very narrow $[(a-b)/a] \ll 1$ annulus can be attempted as above. Assume

Fig. 5.5.3. Solution for an inclined annular punch

$$\sigma(\rho,\phi)=\frac{\sigma_1\cos\phi}{\sqrt{c^2-(\rho-r_0)^2}}. \tag{5.5.53}$$

Here, as before, $2c$ is the annulus thickness and r_0 is its average radius as defined in (36). Substitution of (53) in (8) yields

$$\chi_1(t)\approx\sigma_1\frac{\sqrt{r_0}}{\sqrt{2}}\int_t^c\frac{\mathrm{d}x}{\sqrt{c^2-x^2}\sqrt{x-t}}. \tag{5.5.54}$$

Here the new variables were introduced in the manner similar to (38). In the limiting case of $r\to b$ we can deduce from (50) that

$$\alpha=-\frac{4H}{\pi b^2}\int_b^a\left[y\ln\left(\frac{y+b}{y-b}\right)-2b\right]\frac{\chi_1(y)\,\mathrm{d}y}{\sqrt{y^2-b^2}}\approx-\frac{2\sqrt{2}H}{\pi b^2\sqrt{r_0}}\int_{-c}^c\left[r_0\ln\left(\frac{2r_0}{t+c}\right)-2b\right]\frac{\chi_1(t)\,\mathrm{d}t}{\sqrt{t+c}}. \tag{5.5.55}$$

Substitution of (54) in (55) yields after interchanging the order of integration and subsequent integration

$$\alpha=-\frac{2\pi H\sigma_1}{r_0}\left[\ln\left(\frac{16r_0}{c}\right)-2\right]. \tag{5.5.56}$$

Table 5.5.2.

n	b/a	Method 1 M^*	Method 2 M^*	Average M^*	Exact M^*
10	0.2000	1.0028	1.0017	1.0023	0.99996
	0.4000	1.0079	0.9959	1.0019	0.99878
	0.6000	1.0012	0.9839	0.9925	0.98930
	0.8000	0.9491	0.9368	0.9429	0.94084
	0.9000	0.8658	0.8609	0.8633	0.86243
	0.9400	0.7991	0.7982	0.7986	0.79830
	0.9800	0.6677	0.6707	0.6692	0.66942
	0.9900	0.5989	0.6027	0.6008	0.60115
	0.9990	0.4392	0.4429	0.4411	0.44138
	0.9999	0.3449	0.3479	0.3464	0.34661
20	0.2000	1.0012	1.0000	1.0006	0.99996
	0.4000	1.0024	0.9971	0.9997	0.99878
	0.6000	0.9941	0.9866	0.9904	0.98930
	0.8000	0.9443	0.9389	0.9416	0.94084
	0.9000	0.8638	0.8617	0.8627	0.86243
	0.9400	0.7986	0.7982	0.7984	0.79830
	0.9800	0.6685	0.6701	0.6693	0.66942
	0.9900	0.6000	0.6020	0.6010	0.60115
	0.9990	0.4403	0.4422	0.4412	0.44138
	0.9999	0.3458	0.3473	0.3465	0.34661
30	0.2000	1.0007	0.9998	1.0003	0.99996
	0.4000	1.0010	0.9976	0.9993	0.99878
	0.6000	0.9923	0.9875	0.9899	0.98930
	0.8000	0.9430	0.9396	0.9413	0.94084
	0.9000	0.8633	0.8620	0.8626	0.86243
	0.9400	0.7984	0.7983	0.7983	0.79830
	0.9800	0.6688	0.6699	0.6693	0.66942
	0.9900	0.6004	0.6017	0.6010	0.60115
	0.9990	0.4407	0.4419	0.4413	0.44138
	0.9999	0.3461	0.3471	0.3466	0.34661
40	0.2000	1.0005	0.9998	1.0002	0.99996
	0.4000	1.0003	0.9979	0.9991	0.99878
	0.6000	0.9914	0.9880	0.9897	0.98930
	0.8000	0.9424	0.9399	0.9411	0.94084
	0.9000	0.8630	0.8621	0.8625	0.86243
	0.9400	0.7984	0.7983	0.7983	0.79830
	0.9800	0.6690	0.6697	0.6694	0.66942
	0.9900	0.6006	0.6016	0.6011	0.60115
	0.9990	0.4408	0.4418	0.4413	0.44138
	0.9999	0.3462	0.3470	0.3466	0.34661

A similar procedure performed on (19) gives

$$M = -\pi^2 r_0^2 \sigma_1. \tag{5.5.57}$$

We may now deduce from (56) and (57) that

$$\sigma_1 = -\frac{M}{\pi^2 r_0^2}, \tag{5.5.58}$$

$$\alpha = \frac{2HM}{\pi r_0^3}\left[\ln\left(\frac{16(a+b}{a-b}\right) - 2\right]. \tag{5.5.59}$$

Taking into consideration that for a circular punch of radius a we have

$$M_0 = \frac{4a^3\alpha}{3\pi H}, \tag{5.5.60}$$

the following expression for the dimensionless moment can be written

$$M^* = \frac{M}{M_0} = \frac{3\pi^2(a+b)^3}{64a^3\left[\ln\left(\frac{16(a+b}{a-b}\right) - 2\right]} \tag{5.5.61}$$

We are unaware of a similar result published elsewhere.

Discussion. An attempt can be made to obtain an approximate analytical solution. We can multiply both sides of (27) by $4r\,\mathrm{d}r/\sqrt{r^2-b^2}$ and integrate with respect to r from $b+\varepsilon$ to a. The result is

$$4\int_{b+\varepsilon}^{a} \frac{\chi_0(r)\,r\,\mathrm{d}r}{\sqrt{r^2-b^2}} + \frac{4}{\pi^2}\int_b^a \frac{1}{\sqrt{y^2-b^2}}\left\{\ln\left(\frac{y+b}{y-b}\right)\left[y\ln\frac{(y-b)(a+y)}{(y+b)(a-y)} - b\ln\frac{(a+b)\varepsilon}{(a-b)2b}\right.\right.$$

$$\left.\left. - y\int_b^a \ln\left(\frac{x+b}{x-b}\right)\frac{x\,\mathrm{d}x}{y^2-x^2}\right\}\chi_0(y)\,\mathrm{d}y = \frac{2w_0}{\pi H}\left[a - b - \frac{b}{2}\ln\frac{(a+b)\varepsilon}{(a-b)2b}\right]. \tag{5.5.62}$$

By using identity (30) the limiting case of $\varepsilon \to 0$ can be computed

$$P + \frac{4}{\pi^2}\int_b^a T(y)\,\frac{\chi_0(y)\,y\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{2w_0(a-b)}{\pi H}, \tag{5.5.63}$$

with

$$T(y) = \ln\left(\frac{y+b}{y-b}\right)\ln\left(\frac{(a+y)(y-b)}{(a-y)(y+b)}\right) - \int_b^a \ln\left(\frac{x+b}{x-b}\right)\frac{x\,\mathrm{d}x}{y^2-x^2}. \tag{5.5.64}$$

Taking into consideration that χ_0 does not change sign in the interval $[b, a]$, we can use the mean value theorem and to rewrite (63) as

$$P\left[1 + \frac{1}{\pi^2}T(Y)\right] = \frac{2w_0(a-b)}{\pi H},$$

with an immediate consequence

$$P = \frac{2w_0(a-b)}{\pi H\left(1 + \frac{1}{\pi^2}T(Y)\right)}. \tag{5.5.65}$$

We know about the value of Y only that it is located somewhere in the interval $[b, a]$. This condition allows infinite variation of T, thus making (65) of little practical value. On the other hand, formula (65) is exact in two limiting cases, namely for $b \to 0$ and $b \to a$. this means that an additional investigation can reveal an optimal value of Y, making (65) useful.

Yet another solution can be deduced from (18) and (30) which can be rewritten as

$$\int_b^a \ln\left(\frac{y+b}{y-b}\right)\frac{\chi_0(y)\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{\pi w_0}{4H}. \tag{5.5.66}$$

Taking into consideration that χ_0 does not change sign in the $[b, a]$, we can apply the mean value theorem to (66), with the result

$$\frac{1}{Y}\ln\left(\frac{Y+b}{Y-b}\right)\int_b^a \frac{\chi_0(y)\,y\,\mathrm{d}y}{\sqrt{y^2-b^2}} = \frac{\pi w_0}{4H}. \tag{5.5.67}$$

Comparison of (67) with (18) yields

$$P = \frac{\pi w_0 Y}{H \ln\left(\dfrac{Y+b}{Y-b}\right)}. \tag{5.5.68}$$

Again, the main problem with (68) is the fact that we know about the value of Y only that it is located somewhere between b and a, which allows infinite variation. This does not preclude, however, from finding some optimal value for Y which would make (68) useful. This investigation is beyond the scope of this book.

The complete solution, namely, the explicit expressions for the field of stresses and displacements in the elastic half-space due to the annular punch indentation, can be derived in terms of the function χ. Indeed, in order to obtain the field of normal displacements, one has to compute the integral

$$I(\rho,\phi,z) = \int_0^{2\pi}\int_b^a \frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}}. \tag{5.5.69}$$

Substitution of (9) in (69) yields, after interchanging the order of integration and subsequent integration

$$I(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{[l_2^2(x)-x^2]^{1/2}\,\chi(x,\phi_0)\,dx\,d\phi_0}{\rho^2+x^2-2\rho x\cos(\phi-\phi_0)+z^2}$$

$$+\frac{8}{\pi}\int_b^a\left\{\int_0^{l_1(b)} \mathcal{L}\left(\frac{x^2}{\rho y}\right)\frac{\sqrt{b^2-g^2(x)}\,dx}{[y^2-g^2(x)]\sqrt{\rho^2-x^2}}\right\}\frac{\chi(y,\phi)\,y\,dy}{\sqrt{y^2-b^2}}. \tag{5.5.70}$$

Here

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2}-\sqrt{(\rho-x)^2+z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2}+\sqrt{(\rho-x)^2+z^2}\},$$

$$g(x) = x\left[1+\frac{z^2}{\rho^2-x^2}\right]^{1/2}. \tag{5.5.71}$$

Note that function g is inverse to both l_1 and l_2. The method of integration is described in (Fabrikant, 1989a). The reader may now try to find a complete solution. One can also apply to the problem of an annular punch the results of sections 3.6 and 3.8.

Appendix 5

We present here some mathematical results used in various transformations throughout this book. The main properties of l_1 and l_2 are:

$$l_1 l_2 = a\rho, \quad l_1^2 + l_2^2 = a^2 + \rho^2 + z^2, \tag{A5.1}$$

$$(l_2^2 - \rho^2)^{1/2}(l_2^2 - a^2)^{1/2} = z l_2, \quad (a^2 - l_1^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z l_1,$$

$$(a^2 - l_1^2)^{1/2}(l_2^2 - a^2)^{1/2} = za, \quad (l_2^2 - \rho^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z\rho. \tag{A5.2}$$

$$\frac{\partial l_1}{\partial z} = -\frac{z l_1}{l_2^2 - l_1^2}, \quad \frac{\partial l_2}{\partial z} = \frac{z l_2}{l_2^2 - l_1^2},$$

$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2 - l_1^2} = \frac{\rho(a^2 - l_1^2)}{l_1(l_2^2 - l_1^2)}, \quad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2 - l_1^2} = \frac{\rho(l_2^2 - a^2)}{l_2(l_2^2 - l_1^2)}. \tag{A5.3}$$

Here are some derivatives used

$$\frac{\partial}{\partial z}(l_2^2 - a^2)^{1/2} = \frac{l_2(l_2^2 - \rho^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A5.4}$$

$$\Lambda(l_2^2 - a^2)^{1/2} = \frac{\rho e^{i\phi}(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A5.5}$$

$$\Lambda\frac{\partial}{\partial z}(l_2^2 - a^2)^{1/2} = \frac{z[a^2(2a^2 + 2z^2 - \rho^2) - l_2^4]}{(l_2^2 - a^2)^{1/2}(l_2^2 - l_1^2)^3}\rho e^{i\phi}, \tag{A5.6}$$

$$\frac{\partial^2}{\partial z^2}(l_2^2 - a^2)^{1/2} = \frac{a^2(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - l_1^2)^3}(l_2^2 + 3l_1^2 - 4\rho^2), \tag{A5.7}$$

$$\frac{\partial}{\partial z}(a^2 - l_1^2)^{1/2} = \frac{l_1(\rho^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A5.8}$$

$$\Lambda(a^2-l_1^2)^{1/2} = -\frac{\rho e^{i\phi}(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.9}$$

$$\Lambda\frac{\partial}{\partial z}(a^2-l_1^2)^{1/2} = -\frac{\rho e^{i\phi} z[l_1^4-a^2(2a^2+2z^2-\rho^2)]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3}, \tag{A5.10}$$

$$\frac{\partial^2}{\partial z^2}(a^2-l_1^2)^{1/2} = \frac{a^2(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)^3}(4\rho^2-l_1^2-3l_2^2). \tag{A5.11}$$

$$\frac{\partial}{\partial a}(a^2-l_1^2)^{1/2} = \frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2} = \frac{\partial}{\partial z}(l_2^2-a^2)^{1/2}, \tag{A5.12}$$

$$\frac{\partial}{\partial a}\sin^{-1}\left(\frac{a}{l_2}\right) = \frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}, \quad \frac{\partial}{\partial a}\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.13}$$

$$\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}, \quad \frac{\partial}{\partial z}\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}, \tag{A5.14}$$

$$\Lambda\sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{l_1(l_2^2-a^2)^{1/2}}{l_2[l_2^2-l_1^2]}\, e^{i\phi}, \tag{A5.15}$$

$$\Lambda^2\sin^{-1}\left(\frac{a}{l_2}\right) = \frac{a e^{2i\phi}(l_2^2-a^2)^{1/2}}{l_2^2[l_2^2-l_1^2]^3}\,[3\rho^2 l_2^2+\rho^2 l_1^2-6a^2\rho^2+2l_1^4] \tag{A5.16}$$

$$\Lambda\ln[l_2+(l_2^2-\rho^2)^{1/2}] = \frac{e^{i\phi}}{\rho}\left[1-\frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}\right], \tag{A5.17}$$

$$\Lambda^2\ln[l_2+(l_2^2-\rho^2)^{1/2}] = -\frac{2e^{2i\phi}}{\rho^2} - \frac{a e^{2i\phi}(a^2-l_1^2)^{1/2}}{l_1^2(l_2^2-l_1^2)^3}\,[6a^2\rho^2-2l_2^4-\rho^2 l_2^2-3\rho^2 l_1^2], \tag{A5.18}$$

$$\frac{\partial^2}{\partial z^2}\sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{\partial}{\partial z}\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right) = \frac{z[a^2(2a^2+2z^2-\rho^2)-l_1^4]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3}, \tag{A5.19}$$

$$\frac{\partial^2}{\partial z^2}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] = \frac{\partial}{\partial z}\left(\frac{(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{z[a^2(2a^2 + 2z^2 - \rho^2) - l_2^4]}{(l_2^2 - a^2)^{1/2}(l_2^2 - l_1^2)^3}, \quad \text{(A5.20)}$$

$$\Lambda\frac{\partial}{\partial z}\sin^{-1}(\frac{a}{l_2}) = -\Lambda\left(\frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{\rho e^{i\phi}(a^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)^3}[3l_2^2 + l_1^2 - 4a^2], \quad \text{(A5.21)}$$

$$\Lambda\frac{\partial}{\partial z}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] = \Lambda\left(\frac{(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}\right) = \frac{\rho e^{i\phi}(l_2^2 - a^2)^{1/2}}{(l_2^2 - l_1^2)^3}[4a^2 - l_2^2 - 3l_1^2]. \quad \text{(A5.22)}$$

The following indefinite integrals are used:

$$\int (l_2^2 - a^2)^{1/2}\mathrm{d}a = (l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a}\right) + \frac{\rho^2}{2}\sin^{-1}(\frac{a}{l_2}), \quad \text{(A5.23)}$$

$$\int (a^2 - l_1^2)^{1/2}\mathrm{d}a = -(a^2 - l_1^2)^{1/2}\left(\frac{l_2^2}{2a} - a\right) - \frac{\rho^2}{2}\ln\frac{a + (a^2 - l_1^2)^{1/2}}{l_1}, \quad \text{(A5.24)}$$

$$\int (a^2 - l_1^2)^{1/2}a\mathrm{d}a = \frac{1}{6}(a^2 - l_1^2)^{1/2}(2a^2 - 3\rho^2 + l_1^2) + \frac{1}{2}z\rho^2\sin^{-1}(\frac{a}{l_2}), \quad \text{(A5.25)}$$

$$\int (l_2^2 - a^2)^{1/2}\mathrm{d}z = (a^2 - l_1^2)^{1/2}\,\frac{l_2^2 - 2a^2}{2a} + \frac{\rho^2}{2}\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \quad \text{(A5.26)}$$

$$\int (l_2^2 - a^2)^{1/2}\,l_1^2\mathrm{d}z = -a(a^2 - l_1^2)^{1/2}\,\frac{l_1^2 + 2a^2}{3} + a^2\rho^2\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \quad \text{(A5.27)}$$

$$\int (a^2 - l_1^2)^{1/2}\mathrm{d}z = \frac{2a^2 - l_1^2}{2a}(l_2^2 - a^2)^{1/2} + \frac{\rho^2}{2}\sin^{-1}(\frac{a}{l_2}), \quad \text{(A5.28)}$$

$$\int (a^2 - l_1^2)^{1/2}\,l_1^2\mathrm{d}z = -\frac{l_1^2(2l_1^2 + 3\rho^2)}{8a}(l_2^2 - a^2)^{1/2} + \rho^2(\frac{3}{8}\rho^2 - a^2)\sin^{-1}(\frac{a}{l_2}), \quad \text{(A5.29)}$$

$$\int l_2^2(a^2 - l_1^2)^{1/2}\mathrm{d}z = \frac{1}{3}a(l_2^2 - a^2)^{1/2}(2a^2 + l_2^2) + a^2\rho^2\sin^{-1}(\frac{a}{l_2}), \quad \text{(A5.30)}$$

$$\int (l_2^2 - a^2)^{1/2} \frac{l_1^2}{l_2^2} \mathrm{d}z = a(a^2 - l_1^2)^{1/2} \left[1 - \frac{8a^2}{15\rho^2} - \frac{4a^2 + 3l_1^2}{15 l_2^2} \right], \qquad \text{(A5.31)}$$

$$\int \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2} \mathrm{d}z = -\sin^{-1}\left(\frac{a}{l_2}\right), \qquad \text{(A5.32)}$$

$$\int \frac{(a^2 - l_1^2)^{1/2}}{l_2^2 (l_2^2 - l_1^2)} \mathrm{d}z = \frac{1}{2a^2} \left[\frac{a(l_2^2 - a^2)^{1/2}}{l_2^2} - \sin^{-1}\left(\frac{a}{l_2}\right) \right], \qquad \text{(A5.33)}$$

$$\int \sin^{-1}\left(\frac{a}{l_2}\right) \mathrm{d}z = z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2 - l_1^2)^{1/2} + a\ln[l_2 + (l_2^2 - \rho^2)^{1/2}], \qquad \text{(A5.34)}$$

$$\int z\sin^{-1}\left(\frac{a}{l_2}\right) \mathrm{d}z = \frac{1}{4}(2a^2 + 2z^2 + \rho^2)\sin^{-1}\left(\frac{a}{l_2}\right) + (l_2^2 - a^2)^{1/2} \frac{2a^2 + l_1^2}{4a}, \qquad \text{(A5.35)}$$

$$\int z^2\sin^{-1}\left(\frac{a}{l_2}\right) \mathrm{d}z = \frac{1}{3} z^3 \sin^{-1}\left(\frac{a}{l_2}\right) + \frac{1}{18}(a^2 - l_1^2)^{1/2}(3l_2^2 + 6\rho^2 + 8a^2 - 2l_1^2)$$

$$- \frac{1}{6} a\,(3\rho^2 + 2a^2) \ln[l_2 + (l_2^2 - \rho^2)^{1/2}]. \qquad \text{(A5.36)}$$

The following integrals may be computed by the method described in (Fabrikant, 1989a)

$$I = \int_0^{2\pi} \int_0^a \frac{z\ln(R_0 + z) - R_0}{(a^2 - \rho_0^2)^{1/2}} \rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0 = \pi \left\{ \left(z^2 - a^2 - \frac{\rho^2}{2} \right) \sin^{-1}\left(\frac{a}{l_2}\right) \right.$$

$$\left. - \frac{3(2a^2 - l_1^2)}{2a} (l_2^2 - a^2)^{1/2} + 2az\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \right\}, \qquad \text{(A5.37)}$$

We recall that R_0 is defined by (5.1.16), and the parameters l_1 and l_2 are given by (5.1.33). Here are some partial derivatives used in this Chapter

$$\frac{\partial I}{\partial z} = \int_0^{2\pi} \int_0^a \frac{\ln(R_0 + z)}{(a^2 - \rho_0^2)^{1/2}} \rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0$$

$$= 2\pi\left\{ z\sin^{-1}\left(\frac{a}{l_2}\right) - (a^2 - l_1^2)^{1/2} + a\ln[l_2 + (l_2^2 - \rho^2)^{1/2}]\right\}, \quad \text{(A5.38)}$$

$$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{z\ln(R_0 + z) - R_0}{(a^2 - \rho_0^2)^{1/2}}\,\rho_0 d\rho_0 d\phi_0$$

$$= 2\pi\left\{ z\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] - a\sin^{-1}\left(\frac{a}{l_2}\right) - (l_2^2 - a^2)^{1/2}\right\}, \quad \text{(A5.39)}$$

$$\Lambda I = -\int_0^{2\pi}\int_0^a \frac{q\rho_0 d\rho_0 d\phi_0}{(R_0 + z)(a^2 - \rho_0^2)^{1/2}}$$

$$= -2\pi\frac{e^{i\phi}}{\rho}\left[(l_2^2 - a^2)^{1/2}\left(a - \frac{l_1^2}{2a}\right) - za + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right)\right], \quad \text{(A5.40)}$$

$$\frac{\partial^2 I}{\partial z^2} = \int_0^{2\pi}\int_0^a \frac{\rho_0 d\rho_0 d\phi_0}{R_0(a^2 - \rho_0^2)^{1/2}} = 2\pi\sin^{-1}\left(\frac{a}{l_2}\right) \quad \text{(A5.41)}$$

$$\frac{\partial^2 I}{\partial z\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{\ln(R_0 + z)}{(a^2 - \rho_0^2)^{1/2}}\,\rho_0 d\rho_0 d\phi_0 = 2\pi\ln[l_2 + (l_2^2 - \rho^2)^{1/2}] \quad \text{(A5.42)}$$

$$\Lambda\frac{\partial I}{\partial a} = -\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a \frac{q\rho_0 d\rho_0 d\phi_0}{(R_0 + z)(a^2 - \rho_0^2)^{1/2}} = 2\pi\frac{e^{i\phi}}{\rho}[z - (l_2^2 - a^2)^{1/2}]. \quad \text{(A5.43)}$$

Yet another important integral is

$$J = \int_0^{2\pi}\int_0^a e^{2i\phi_0}[z\ln(R_0 + z) - R_0]\,\frac{\rho_0^2 - 2a^2}{\rho_0(a^2 - \rho_0^2)^{1/2}}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left\{(l_2^2-a^2)^{1/2}\left[\frac{a(l_2^2+2l_1^2-2a^2)}{3}-\frac{l_1^2(2l_1^2+3\rho^2)}{24a}\right]\right.$$

$$\left.+\frac{\rho^4}{8}\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{a(\rho^2+z^2)^{3/2}}{3}+\frac{za^3}{3}\right\}. \qquad \text{(A5.44)}$$

Several partial derivatives may be computed as follows:

$$\frac{\partial J}{\partial z}=\int_0^{2\pi}\int_0^{a} e^{2i\phi_0}\ln(R_0+z)\,\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,d\rho_0 d\phi_0 =$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left[(a^2-l_1^2)^{1/2}\left(l_2^2+\frac{1}{3}l_1^2-\frac{4}{3}a^2\right)-az(\rho^2+z^2)^{1/2}+\frac{1}{3}a^3\right], \qquad \text{(A5.45)}$$

$$\overline{\Lambda}J=-\int_0^{2\pi}\int_0^{a}\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,\frac{\overline{q}e^{2i\phi_0}}{R_0+z}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{i\phi}}{\rho}\left[(l_2^2-a^2)^{1/2}\left(a-\frac{l_1^2}{2a}\right)+\frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right)-a(\rho^2+z^2)^{1/2}\right], \qquad \text{(A5.46)}$$

$$\frac{\partial J}{\partial a}=\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^{a} e^{2i\phi_0}[z\ln(R_0+z)-R_0]\,\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left\{\frac{1}{3}(l_2^2-a^2)^{1/2}(l_2^2+3l_1^2-4a^2)-\frac{1}{3}(\rho^2+z^2)^{3/2}+a^2z\right\}, \qquad \text{(A5.47)}$$

$$\frac{\partial}{\partial a}(\overline{\Lambda}J)=-\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^{a}\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,\frac{\overline{q}e^{2i\phi_0}}{R_0+z}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{i\phi}}{\rho}\left[(l_2^2-a^2)^{1/2}-(\rho^2+z^2)^{1/2}\right], \qquad \text{(A5.48)}$$

$$\frac{\partial^2 J}{\partial a \partial z} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^a e^{2i\phi_0}\ln(R_0+z)\frac{\rho_0^2-2a^2}{\rho_0(a^2-\rho_0^2)^{1/2}}\,d\rho_0 d\phi_0$$

$$= 2\pi\frac{e^{2i\phi}}{\rho^2}\left\{(a^2-l_1^2)^{1/2}\left(\frac{l_2^2}{a}-2a\right)-z(\rho^2+z^2)^{1/2}+a^2\right\}, \qquad \text{(A5.49)}$$

$$\frac{\partial^2 J}{\partial z^2} = \int_0^{2\pi}\int_0^a \frac{\rho_0^2-2a^2}{R_0\rho_0(a^2-\rho_0^2)^{1/2}}e^{2i\phi_0}\,d\rho_0 d\phi_0 =$$

$$= 2\pi\frac{e^{2i\phi}}{a\rho^2}\left[(l_2^2-a^2)^{1/2}(2a^2-l_1^2)-\frac{a^2(2z^2+\rho^2)}{(\rho^2+z^2)^{1/2}}\right] \qquad \text{(A5.50)}$$

It is reminded that q is defined by (5.1.19). The integral in (A5.50) can be presented as a linear combination of two integrals, namely,

$$I_1 = \int_0^{2\pi}\int_0^a \frac{e^{2i\phi_0}\rho_0 d\rho_0 d\phi_0}{R_0(a^2-\rho_0^2)^{1/2}} = 2\pi\frac{e^{2i\phi}}{a\rho^2}\left[(l_2^2-a^2)^{1/2}\left(\frac{l_2^2-a^2}{3}-\rho^2-z^2\right)+\frac{2}{3}(\rho^2+z^2)^{3/2}\right], \qquad \text{(A5.51)}$$

and

$$I_2 = \int_0^{2\pi}\int_0^a \frac{e^{2i\phi_0}\,d\rho_0 d\phi_0}{R_0\rho_0(a^2-\rho_0^2)^{1/2}} = \frac{\pi e^{2i\phi}}{3a^3\rho^2}\left[\frac{3a^2(2z^2+\rho^2)+2(\rho^2+z^2)^2}{(\rho^2+z^2)^{1/2}}-2(l_2^2-a^2)^{1/2}(2a^2+l_2^2)\right].$$

(A5.52)

The third basic integral is

$$N = \int_0^{2\pi}\int_0^a [z\ln(R_0+z)-R_0]\frac{e^{i\phi_0}\rho_0^2 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\rho e^{i\phi}\left[\left(\frac{a^2}{2}+\frac{z^2}{2}-\frac{\rho^2}{8}\right)\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{2za^3}{3\rho^2}\right.$$

$$\left.+\left(\frac{5l_1^2}{8a}-\frac{a}{2}+\frac{2a^3}{3\rho^2}-\frac{al_1^2}{3\rho^2}-\frac{l_1^4}{12a\rho^2}\right)(l_2^2-a^2)^{1/2}\right]. \qquad \text{(A5.53)}$$

Here follow some of the derivatives:

$$\frac{\partial N}{\partial z} = \int_0^{2\pi}\int_0^{a} \ln(R_0+z)\,\frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\rho e^{i\phi}\left[z\sin^{-1}(\frac{a}{l_2}) - (a^2-l_1^2)^{1/2}\left(1-\frac{l_1^2+2a^2}{3\rho^2}\right) - \frac{2a^3}{3\rho^2}\right],$$

$$\frac{\partial^2 N}{\partial z^2} = \int_0^{2\pi}\int_0^{a} \frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{R_0(a^2-\rho_0^2)^{1/2}} = \pi\rho e^{i\phi}\left[\sin^{-1}(\frac{a}{l_2}) - \frac{a(l_2^2-a^2)^{1/2}}{l_2^2}\right], \quad \text{(A5.54)}$$

$$\frac{\partial^2 N}{\partial z\partial a} = \frac{\partial}{\partial a}\int_0^{2\pi}\int_0^{a} \ln(R_0+z)\,\frac{e^{i\phi_0}\,\rho_0^2 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}} = -2\pi\frac{a}{\rho}e^{i\phi}\left[a-(a^2-l_1^2)^{1/2}\right], \quad \text{(A5.55)}$$

$$\overline{\Lambda}N = -\int_0^{2\pi}\int_0^{a} \frac{e^{i\phi_0}\,\overline{q}\rho_0^2 d\rho_0 d\phi_0}{(R_0+z)\,(a^2-\rho_0^2)^{1/2}}$$

$$= \pi\left[\left(a^2+z^2-\frac{\rho^2}{2}\right)\sin^{-1}(\frac{a}{l_2}) - \left(a-\frac{3l_1^2}{2a}\right)(l_2^2-a^2)^{1/2}\right], \quad \text{(A5.56)}$$

$$\frac{\partial}{\partial a}\overline{\Lambda}N = -\frac{\partial}{\partial a}\int_0^{2\pi}\int_0^{a} \frac{e^{i\phi_0}\,\overline{q}\rho_0^2 d\rho_0 d\phi_0}{(R_0+z)\,(a^2-\rho_0^2)^{1/2}} = 2\pi a\sin^{-1}(\frac{a}{l_2}). \quad \text{(A5.57)}$$