CHAPTER 4

APPLICATIONS IN DIFFUSION AND ACOUSTICS

4.1. Diffusion through perforated membranes

The diffusion process through a thin membrane, perforated by several holes of arbitrary shape, is considered. A general theorem is established which relates the total flux through each hole with the concentration distribution of some chemical species, prescribed in the hole, by a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located elliptical holes. Several specific examples are considered.

Advances in bioengineering have generated wide interest in the diffusion mechanism of biological membranes. An exact solution to the problem of steady-state diffusion through an isolated circular hole had been found many years ago (Rayleigh, 1948), along with a very good approximate solution for the case of a thick membrane. There seems to be no exact solutions for the case of several holes. Some quantitative considerations on this subject were made by Lord Rayleigh (1948) who indicated that the interaction between the holes should result in *decreasing* of the flux through each hole while the total flux should increase. The limiting case, when the number of holes tends to infinity, results in the total flux tending to infinity while the flux through each hole tends to zero. In spite of these well established facts, several papers have been published (S. Prager and H.L. Frish, J. Chem. Phys. **62**, 89 (1975); G.M. Malone, Suk Youn Suh, and S. Prager, in *Lecture Notes in Mathematics* **1035**, (1983), pp. 370-390.) in which the result is quite opposite: the holes' interaction *increases* the flux through each hole! Since the authors of these papers never explain the reason for such a discrepancy, one should think that those papers are just incorrect.

Here, a general theorem is established which is valid for arbitrary holes, and relates the flux through each hole, with the chemical species concentration prescribed inside the hole, to a system of linear algebraic equations. Since the coefficients of these equations depend on the concentration outside a hole of arbitrary shape, the theorem can be used effectively for elliptical holes only, as there seems to be no publication giving the concentration outside a nonelliptical hole. The exact values of the coefficients are not known but, since their variation is quite limited, the established theorem allows one to obtain the upper and the lower bounds for the parameters sought. There is no need to know the flux intensity distribution in the hole which presents a significant advantage for the user. Several examples are considered to show that these bounds may be so close that they provide, in fact, a reasonably accurate solution to the problem.

Theory. Consider a thin membrane conceived as the plane $z=0$, which separates two half-spaces, namely, $z>0$ and $z<0$. Assume that at $z\to\infty$ the concentration of some chemical species is $w^{+}=const.$, with the corresponding concentration $w^{-}=const.$ at $z\to-\infty$. The membrane is perforated by N holes of arbitrary shape and location. The analysis is limited to the steady-state diffusion through these holes.

For the sake of mathematical convenience, we can consider an equivalent problem with the limiting values $w=0$ for $z\to\infty$ and $w=w^{-}-w^{+}$ for $z\to-\infty$. This will enable us to use the Newton potential representation for the solution. The mathematical formulation of the problem reads: find a function W harmonic in the upper half-space ($\Delta W=0$) subject to the boundary conditions

$$W=w_n(M) \quad \text{for} \quad M\in S_n,$$

$$\sigma_n(M)=0 \quad \text{for} \quad M\notin S_n, \quad n=1,2,...,N. \qquad (4.1.1)$$

where S_n denotes the area of the nth hole, and σ stands for the concentration gradient normal to the plane $z = 0$

$$\sigma=-\frac{\partial W}{\partial z}.$$

In our case $w_n=(w^{-}-w^{+})/2=\text{const.}$ but we are deliberately considering a more general case in order to show the applicability of the method. Using the Newton potential representation, we can write

$$W(Q)=D\sum_{n=1}^{N}\iint_{S_n}\frac{\sigma_n(T_n)}{R(T_n,Q)}\,dS_n. \qquad (4.1.2)$$

where D is the mass diffusivity coefficient, and $R(T_n,Q)$ denotes the distance between a point $T_n\in S_n$ and a field point Q. Substitution of the boundary conditions (1) in (2) leads to a set of N integral equations. The exact solutions of these equations are not known at the present time even for the case of several circles. Here, we are to show that we do not really need to know these solutions. We can single out, without loss of generality, the first hole, and consider the related integral equation

$$w_1(Q_1) = D \iint\limits_{S_1} \frac{\sigma_1(T_1)}{R(T_1,Q_1)}]\, \mathrm{d}S_1 + D \sum_{n=2}^{N} \iint\limits_{S_n} \frac{\sigma_n(T_n)}{R(T_n,Q_1)}\, \mathrm{d}S_n. \tag{4.1.3}$$

Suppose that the function σ_0 is known, satisfying the following integral equation inside S_1

$$\iint\limits_{S_1} \frac{\sigma_0(Q_1)}{R(T_1,Q_1)}\, \mathrm{d}S_1 = 1. \tag{4.1.4}$$

Multiplication of both sides of (3) by $\sigma_0(Q_1)$ and integration over the area S_1 yields

$$\iint\limits_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D \iint\limits_{S_1} \sigma_0(Q_1)\, \mathrm{d}S_1 \iint\limits_{S_1} \frac{\sigma_1(T_1)}{R(T_1,Q_1)}]\, \mathrm{d}S_1$$

$$+ D \sum_{n=2}^{N} \iint\limits_{S_1} \sigma_0(Q_1)\, \mathrm{d}S_1 \iint\limits_{S_n} \frac{\sigma_n(T_n)}{R(T_n,Q_1)}\, \mathrm{d}S_n. \tag{4.1.5}$$

By changing the order of integration in (5) and taking into consideration that σ_0 satisfies (4), the following result can be obtained

$$\iint\limits_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D \left[P_1 + \sum_{n=2}^{N} \iint\limits_{S_n} w_{1n}(T_n)\, \sigma_n(T_n)\, \mathrm{d}S_n \right]. \tag{4.1.6}$$

where P_1 is the total flux through the first hole

$$P_1 = \iint\limits_{S_1} \sigma_1(Q_1)\, \mathrm{d}S_1\,,$$

and

$$w_{1n}(T_n) = \iint\limits_{S_1} \frac{\sigma_0(Q_1)}{R(T_n,Q_1)}\, \mathrm{d}S_1. \tag{4.1.7}$$

which is proportional to the concentration in the domain S_n due to a unit concentration of the chemical species in S_1. By evoking the mean value theorem which is valid when σ_n does not change sign, we come to the linear algebraic equation

$$\iint_{S_1} \sigma_0(Q_1)\, w_1(Q_1)\, \mathrm{d}S_1 = D\left[P_1 + \sum_{n=2}^{N} w_{1n}(C_n)\, P_n \right]. \tag{4.1.8}$$

The exact location of the point C_n is not known but the fact that $C_n \in S_n$ allows only limited variation within S_n, and in many cases provides sufficiently close upper and lower bounds for the parameters sought. By using the same logic, $N-1$ additional linear algebraic equations can be derived for the remaining holes. This set of equations provides the necessary relationships between the individual fluxes through the holes and the concentration gradients prescribed in the holes.

The set of linear algebraic equations of the type (8), is the main result of this section. In order to use (8), one needs to know the concentration distribution outside every hole in the system due to a unit concentration prescribed inside it which, at the moment, is available for the elliptical holes only.

Application to elliptical holes. Consider the interaction of a set of N elliptical holes arbitrarily located in an infinite thin membrane. Let a_n and b_n be the major and the minor semiaxes of the nth ellipse; X_n and Y_n be its center, and θ_n be the angle between the axis Ox and the major semiaxis a_n; P_n denotes the flux passing through the nth hole.

Here are some results implicitly given by Lur'e (1964). The function σ_0 has the form

$$\sigma_0 = \frac{1}{2\pi b_1 K(k_1)}\left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2}, \tag{4.1.9}$$

where $K(k_1)$ stands for the complete elliptic integral of the first kind, and k_1 is the eccentricity of the first ellipse

$$k_1 = \sqrt{1-(b_1/a_1)^2}. \tag{4.1.10}$$

Introduce the notation

$$R = \sqrt{(x-x_0)^2 + (y-y_0)^2},$$

$$Z_0 = \left[1 - \frac{x_0^2}{a_1^2} - \frac{y_0^2}{b_1^2} \right]^{1/2}.$$

The following integrals are valid

$$\iint_{S_1} \frac{dx_0 dy_0}{RZ_0} = \begin{cases} 2\pi b_1 K(k_1), & \text{for } (x,y) \in S_1; \\ 2\pi b_1 F(\phi_1, k_1), & \text{for } (x,y) \notin S_1; \end{cases} \tag{4.1.11}$$

where

$$\phi_1 = \sin^{-1}\left[\frac{1}{\rho_1} \right], \tag{4.1.12}$$

$$\rho_1 = [L + (L^2 - k_1^2 x^2/a_1^2)^{1/2}]^{1/2}, \qquad L = \frac{1}{2}[k_1^2 + (x^2+y^2)/a_1^2]. \tag{4.1.13}$$

The remaining integral is elementary

$$\iint_{S_1} \left[1 - \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} \right]^{-1/2} dx\, dy = 2\pi a_1 b_1. \tag{4.1.14}$$

The boundary conditions (1) in this case will take the form

$$w_n = (w^- - w^+)/2 = \delta. \tag{4.1.15}$$

Substitution of (15) into (8) yields

$$\frac{a_1}{K(k_1)}\delta = D\left[P_1 + \sum_{n=2}^{N} \frac{F(\phi_{1n}, k_1)}{K(k_1)} P_n \right], \tag{4.1.16}$$

where $K(k_1)$ and $F(\phi_{1n}, k_1)$ stand for the complete and incomplete elliptic integrals of the first kind ; ρ_{1n} and ϕ_{1n} are defined according to (12) and (13); formulae (9), (11), and (14) were used in the derivation of (16); Equation (16) represents the first of the set of N linear algebraic equations, which allows one to obtain the exact upper and lower bounds for the fluxes P_n, $n=1,2, \ldots ,N$ without having solved the system of integral equations (3). It is also important to note that each equation in the set is valid in the local system of the coordinates related to the center of the ellipse.

Two equal elliptical holes. Consider the case $N = 2$, $a_1 = a_2 = \mathrm{a}$, $b_1 = b_2 = \mathrm{b}$, $X_1 = Y_1 = 0$, $X_2 = \mathrm{h}$, $Y_2 = 0$, $\theta_1 = \theta_2 = 0$. Since $P_1 = P_2 = P$ then, due to the symmetry of the system, the set of equations, equivalent to (16), reduces to just one equation, namely

$$\frac{a}{K(k)}\delta = D\left[P + \frac{F(\phi, k)}{K(k)} P \right], \tag{4.1.17}$$

with an immediate result

$$P = \frac{P_0}{1 + \dfrac{F(\phi, k)}{K(k)}}. \tag{4.1.18}$$

where $P_0 = \delta a / DK(k)]$ indicates the flux through a solitary hole. Equation (18) shows that the interaction between the holes decreases the flux through each hole. The upper and the lower bounds for P can be obtained from (18) by taking

$$\phi = \sin^{-1}\left[\frac{a}{h-a}\right], \quad \text{and} \quad \phi = \sin^{-1}\left[\frac{a}{h+a}\right]. \tag{4.1.19}$$

respectively. We shall also consider the central estimation for P defined by

$$\phi = \sin^{-1}\left[\frac{a}{h}\right]. \tag{4.1.20}$$

Fig. 4.1.1 plots the ratio P/P_0 versus h/a for $a = 2$, $b = 1$. The solid line gives the upper bound, the broken line gives the lower bound, and the small circles plot the central estimation. Numerical computations show that the maximal error of the central estimation is less than 9% for $h/a > 3.5$, it is less than 5% for $h/a > 5$, it is less than 2% for $h/a > 8$, and it is less than 1% for $h/a > 12$. Since there is no accurate solution available for this case, it is difficult to say how great the real error of the central estimation is, but there is a reason to believe that it is much less than the one indicated above. The reason for such a belief comes from a comparison of the central estimation for two equal circular holes with the numerical solution by Kobayashi (1939) If one takes the Kobayashi’s solution as exact then the maximum error of the central estimation does not exceed 0.4% in the whole range of $2 \le h/a < \infty$. Even if one assumes the accuracy in the case of two elliptic holes being ten times worse than the accuracy of the central estimation for two circular holes, this still would give the maximum error

Fig. 4.1.1. The ratio P/P_0 for two equal elliptical holes

at 4% which is not bad. Having this in mind, we shall evaluate the central estimation only in the examples to follow.

Four equal elliptical holes. Consider a rectangle with sides l and h. Locate the centers of four equal elliptical holes at its apices, with their axes being along the sides of the rectangle (Fig. 4.1.2). The holes are numbered in the clockwise direction. Due to the symmetry of the system, it is sufficient to consider just one equation of the set (16). The result is

$$\frac{a}{K(k)}\,\delta = DP\left[\,1+\frac{F(\phi_{12},k)}{K(k)}+\frac{F(\phi_{13},k)}{K(k)}+\frac{F(\phi_{14},k)}{K(k)}\,\right], \tag{4.1.21}$$

where

$$\phi_{1n} = \sin^{-1}\frac{1}{\rho_{1n}}, \qquad \text{for} \quad n = 2,3,4\,; \tag{4.1.22}$$

$$\rho_{12} = [1+(h^2-b^2)/a^2]^{1/2}, \qquad \rho_{14} = \frac{l}{a},$$

$$\rho_{13} = [L+\sqrt{L^2-k^2l^2/a^2}]^{1/2}, \quad L = \frac{1}{2}[k^2+(l^2+h^2)/a^2]. \tag{4.1.23}$$

We can note certain relation between our results and those of Grinberg (1948) who established some relevant theorems in electrostatics. It is also of interest to indicate certain limiting cases. In the case of a circular hole the eccentricity $k_1 \to 0$, and

Fig. 4.1.2. Four equal elliptical holes

$$\frac{F(\phi_{1n}, k_1)}{K(k_1)} \rightarrow \frac{2}{\pi} \sin^{-1}\left(\frac{a_1}{r_{1n}}\right). \tag{4.1.24}$$

and equation (16) will take the form

$$\frac{2}{\pi} a_1 \delta_1 = D\left[P_1 + \frac{2}{\pi} \sum_{n=2}^{N} P_n \sin^{-1} \frac{a_1}{r_{1n}} \right], \tag{4.1.25}$$

where a_1 is the radius of the hole number one and r_{1n} is the distance from the center of the first hole to the nth hole. Formula (25) is in agreement with the result by Fabrikant (1985)

4.2. Interaction between circular pores

By using a special integral representation for the Green's function, a system of Fredholm integral equations is derived with respect to the flux density. Equations are non-singular and allow an accurate numerical solution. The total flux can be found from a system of linear algebraic equations.

We consider a thin membrane $z=0$, perforated by n circular holes. The center of i-th hole is located at the point (x_i, y_i), and its radius is denoted a_i. We can formulate the problem of diffusion as that of finding a harmonic function W subject to the boundary conditions

$$W(\rho,\phi,0) = w_i(\rho,\phi), \quad \text{for} \quad i=1,2,...,n, \text{ and } (\rho,\phi)\in S_i;$$

$$\frac{\partial W}{\partial z} \equiv -\sigma = 0 \quad \text{for } z=0 \text{ and } (\rho,\phi)\notin S_i.$$

Here S_i is the area of i-th pore, σ is the flux density. The mass diffusivity was assumed to be unity. We can single out, without loss of generality, the pore number 1, and locate the origin of polar coordinates at its center. The governing integral equation for the first pore can be written as

$$w_1(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{a_1} \frac{\sigma_1(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}$$

$$+\frac{1}{2\pi}\sum_{k=2}^{n}\iint_{S_k} \frac{\sigma_k(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}. \tag{4.2.1}$$

Substituting the integral representation (1.2.22) in (1), we obtain

$$w_1(\rho,\phi) = \frac{2}{\pi}\int_0^{\rho} \frac{dx}{(\rho^2-x^2)^{1/2}} \int_x^{a_1} \frac{\rho_0\,d\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma_1(\rho_0,\phi)$$

$$+\frac{1}{\pi^2}\sum_{k=2}^{n}\int_0^{\rho}\frac{dx}{(\rho^2-x^2)^{1/2}}\iint_{S_k}\frac{\rho_0\,d\rho_0}{(\rho_0^2-x^2)^{1/2}}\,\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)\sigma_k(\rho_0,\phi_0)\,d\phi_0. \tag{4.2.2}$$

We apply the integral operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_0^{r}\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)$$

to both sides of (2). The result is

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w_1(\rho,\phi) = \int_r^{a_1} \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}\,\mathcal{L}\left(\frac{r}{\rho_0}\right)\sigma_1(\rho_0,\phi)$$

$$+\frac{1}{2\pi}\sum_{k=2}^{n}\int\int_{S_k} \frac{\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\sqrt{\rho_0^2-r^2}}\,\lambda\left(\frac{r}{\rho_0},\phi-\phi_0\right)\sigma_k(\rho_0,\phi_0). \tag{4.2.3}$$

The next operator to apply is

$$\mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1} \frac{r\,\mathrm{d}r}{\sqrt{r^2-t^2}}\,\mathcal{L}\left(\frac{1}{r}\right).$$

The final result is

$$-\frac{2}{\pi}\,\mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1} \frac{r\,\mathrm{d}r}{\sqrt{r^2-t^2}}\,\mathcal{L}\left(\frac{1}{r^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\,\mathcal{L}(\rho)\,w_1(\rho,\phi) = \sigma_1(t,\phi)$$

$$+\frac{1}{\pi^2\sqrt{a_1^2-t^2}}\sum_{k=2}^{n}\int\int_{S_k} \frac{\sqrt{\rho_0^2-a_1^2}\,\sigma_k(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{t^2+\rho_0^2-2t\rho_0\cos(\phi-\phi_0)}. \tag{4.2.4}$$

Similar equations can be obtained for the remaining $n-1$ pores. The integral equations are non-singular, and can be solved by any regular numerical method. In the case when $w_1 = c_0$ is a constant, the governing integral equation will take the form

$$\sigma_1(\rho,\phi) = \frac{2c_0}{\pi\sqrt{a_1^2-\rho^2}} - \frac{1}{\pi^2\sqrt{a_1^2-\rho^2}}\sum_{k=2}^{n}\int\int_{S_k} \frac{\sqrt{\rho_0^2-a_1^2}\,\sigma_k(\rho_0,\phi_0)\,\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\,\rho_0\cos(\phi-\phi_0)}. \tag{4.2.5}$$

If we need to obtain the total flux only, we do not have to solve integral equations (5). Indeed, we can multiply both sides of (5) by $\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$ and integrate over the first pore. The result is

$$P_1 = 4a_1c_0 - \frac{2}{\pi}\sum_{k=2}^{n}\int\int_{S_k} \sigma_k(\rho_0, \phi_0)\sin^{-1}\left(\frac{a_1}{\rho_0}\right)\rho_0\, d\rho_0\, d\phi_0. \tag{4.2.6}$$

Here P_1 is the flux through the first pore. Note that the relationship (6) is exact. We can now apply the mean value theorem which is valid when σ_k does not change sign, and get the set of linear algebraic equations with respect to the total fluxes

$$P_k + \frac{2}{\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} P_i \sin^{-1}\left(\frac{a_k}{b_{ki}}\right) = 4a_kc_0, \quad \text{for} \quad k = 1, 2, ..., n. \tag{4.2.7}$$

Here

$$r_{ki} - a_i \leq b_{ki} \leq r_{ki} + a_i, \quad r_{ki} = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}. \tag{4.2.8}$$

Of course, the exact value of b_{ki} is not known, but the set of equations (7) can be used to obtain the upper and the lower bounds for the flux by simple variation of b_{ki} in the admissible range (8). In general $b_{ki} \neq b_{ik}$, except for the case $a_k = a_i$. Due to the reciprocal theorem, it can be shown that for $a_k < a_i$ the admissible range for b_{ki} becomes more narrow which might improve the flux estimation. In certain cases the flux estimation for unequal holes is sharper than that for the equal ones. Since the case of equal pores might be the least accurate, it is analyzed in the examples to follow.

Two circular pores. Let a_1 and a_2 be the radii of the holes, the distance between their centers being b. The system (7) in this case takes the form

$$P_1 + \frac{2}{\pi}\sin^{-1}\left(\frac{a_1}{b_{12}}\right) = 4\, a_1\, c_0,$$

$$\frac{2}{\pi}P_1 \sin^{-1}\left(\frac{a_2}{b_{21}}\right) + P_2 = 4\, a_2\, c_0.$$

The solution is

$$P_1 = 4c_0 \frac{a_1 - \frac{2}{\pi} a_2 \sin^{-1}\left(\frac{a_1}{b_{12}}\right)}{1 - \frac{4}{\pi^2}\sin^{-1}\left(\frac{a_1}{b_{12}}\right)\sin^{-1}\left(\frac{a_2}{b_{21}}\right)},$$

$$P_2 = 4c_0 \frac{a_2 - \frac{2}{\pi} a_1 \sin^{-1}\left(\frac{a_2}{b_{21}}\right)}{1 - \frac{4}{\pi^2}\sin^{-1}\left(\frac{a_1}{b_{12}}\right)\sin^{-1}\left(\frac{a_2}{b_{21}}\right)}.$$

When $a_1 = a_2 = a$, the solution simplifies to

$$P_1 = P_2 = \frac{4c_0 a}{1 + \frac{2}{\pi}\sin^{-1}\left(\frac{a}{b_{12}}\right)} = \frac{P_0}{1 + \frac{2}{\pi}\sin^{-1}\left(\frac{a}{b_{12}}\right)}. \tag{4.2.9}$$

Here P_0 is the flux through an isolated hole and the denominator in (9) shows the degree of the flux reduction due to the second hole. When the distance $b \to \infty$, the two holes do not interact. The smallest possible value is given by $b = 2a$, and the flux through each hole will be $0.75P_0$. Computations were performed in Fabrikant (1985) which showed a very good accuracy when b_{12} was taken equal to b. This is called the central estimation. The different estimations for the dimensionless flux $P^* = P/P_0$ is given in Table 4.2.1.

General configuration. We consider the case of N equal holes of radius a located at the apexes of a regular polygon. The hole of radius a_0 is located at the geometric center of the polygon. Let the distance between the center of any apex be h. From geometric consideration, the distance between the n-th and k-th apex is ely,

$$P_0 + \frac{2}{\pi} N P \sin^{-1}\left(\frac{a_0}{h}\right) = 4a_0 c_0,$$

$$\frac{2}{\pi} P_0 \sin^{-1}\left(\frac{a}{h}\right) + P\left[1 + \frac{2}{\pi}\sum_{n=1}^{N-1} \sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right] = 4ac_0.$$

The solution is

Table 4.2.1. Comparison of our results with Kobayashi's (1939)

Distance between the centres	Upper bound for the flux F^*	Lower bound for the flux F^*	Central estimation of F^*	Kobayashi's result	Our numerical result
2.0	0.8221338	0.5000000	0.7500000	0.75272	0.75239
2.2	0.8317164	0.6145749	0.7689962	0.77014	0.76996
2.4	0.8402998	0.6637918	0.7851738	0.78545	0.78537
2.6	0.8480356	0.6993975	0.7991486	0.79898	0.79893
2.8	0.8550458	0.7272786	0.8113602	0.81096	0.81093
3.0	0.8614294	0.7499999	0.8221338	0.82162	0.82161
3.5	0.8751494	0.7924057	0.8442654	0.84370	0.84370
4.0	0.8863767	0.8221338	0.8614294	0.86093	0.86092
5.0	0.9036683	0.8614294	0.8863767	0.88602	0.88602
7.0	0.9261093	0.9036683	0.9163737	0.91619	0.91620
10.0	0.9452202	0.9338098	0.9400541	0.93999	0.93998
∞	1.0	1.0	1.0	1.0	1.0

$$P_0 = 4c_0 \frac{a_0\left[1 + \frac{2}{\pi}\sum_{n=1}^{N-1} \sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right] - \frac{2}{\pi} a N \sin^{-1}\left(\frac{a_0}{h}\right)}{\left[1 + \frac{2}{\pi}\sum_{n=1}^{N-1} \sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right] - \frac{4}{\pi^2} N \sin^{-1}\left(\frac{a}{h}\right)\sin^{-1}\left(\frac{a_0}{h}\right)},$$

$$P = 4c_0 \frac{a - \frac{2}{\pi} a_0 \sin^{-1}\left(\frac{a}{h}\right)}{\left[1 + \frac{2}{\pi}\sum_{n=1}^{N-1} \sin^{-1}\left(\frac{a}{2h\sin(\pi n/N)}\right)\right] - \frac{4}{\pi^2} N \sin^{-1}\left(\frac{a}{h}\right)\sin^{-1}\left(\frac{a_0}{h}\right)}.$$

The same method can be used for membranes of finite thickness, and it is presented in the next section.

4.3. Pore length effect

In the previous sections the membrane was assumed to be infinitely thin. The diffusion of chemical species through a pore of finite length, including entrance and exit effects, has important applications to transport and filtration processes in biological membranes or membrane-like structures. We consider now a system comprising of a thick layer penetrated by discrete uniform pores, which are approximated as identical right cylinders of radius a and length $2l$ (Fig. 4.3.1). The layer is sandwiched between two stagnant fluids of infinite

Fig. 4.3.1. Geometry of the problem

extent. The solute concentration far from the pore in either fluid is held constant (say, at w^+ and w^- respectively). The problem is to find a steady-state concentration in the combined space.

The method of solution proposed by Kelman (1965) involves the use of oblate spheroidal coordinates in the half-space and the polar cylindrical coordinated inside the pore. In order to satisfy the matching conditions at the pore opening, one has to expand each term of an infinite sum representing the concentration, into yet another infinite sum in terms of the other system of coordinates. All this makes the method very cumbersome, and its accuracy doubtful.

An alternative approach is presented here. It uses the closed form representation for the concentration in region I (half-space) by utilizing the Green's function for a half-space. The Fourier-Bessel series representation is used in region II (pore). Some simple relations involving concentration and its normal derivatives can be established. Such relations will allow us to arrive at the general form for permeability P, expressions for the local flux and concentration profile. Due to symmetry, it is sufficient to focus attention on half the system, i.e. a pore of radius a and length $2l$ with an infinite region above the pore.

Let ρ, ϕ, and z denote cylindrical coordinates measured from the pore opening. The problem to solve now reads

$$\Delta w = 0 \quad \text{for} \quad -l < z < 0 \quad \text{and} \quad 0 < \rho < a; \quad \text{or} \quad z > 0 \quad \text{and} \quad \rho > 0,$$

subject to the boundary conditions

$$\frac{\partial w}{\partial \rho} = 0 \quad \text{for} \quad \rho = 0, \quad z > -l, \text{ or} \quad \rho = a, \quad -l < z < 0,$$

$$\frac{\partial w}{\partial z} = 0 \quad \text{for} \quad z = 0, \quad \rho > a,$$

$$w = \frac{w^+ + w^-}{2} \quad \text{for} \quad z = -l, \quad 0 \le \rho < a,$$

$$w \to w^+ \quad \text{for} \quad \sqrt{\rho^2 + z^2} \to \infty, \quad z > 0.$$

The solution in the half-space $z \ge 0$ can be presented as

$$w(\rho, z) = w^+ + \frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0)\, \rho_0 d\rho_0 d\phi_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2}}. \tag{4.3.1}$$

As before,

$$\sigma(\rho) = -\frac{\partial w}{\partial z} \quad \text{at} \quad z = 0.$$

The solution in the pore region may be presented as

$$w(\rho, z) = \frac{w^+ + w^-}{2} + A_0(l + z) + \sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l + z}{a}\right) J_0\left(x_n \frac{\rho}{a}\right). \tag{4.3.2}$$

Here x_n are positive roots of equation $J_1(x) = 0$, and A_n are the as yet unknown constants. Notice that both (1) and (2) satisfy the Laplace equation and the boundary conditions in their respective domains of validity, i.e. the half space and the cylinder respectively. The unknown constants are to be determined from the continuity conditions

$$w(\rho, 0^+) = w(\rho, 0^-); \quad \frac{\partial w}{\partial z}\Big|_{z=0^+} = \frac{\partial w}{\partial z}\Big|_{z=0^-}. \tag{4.3.3}$$

Differentiation of (2) yields

$$\sigma(\rho) = -\left[A_0 + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} \cosh\left(x_n \frac{l}{a}\right) J_0\left(x_n \frac{\rho}{a}\right)\right]. \tag{4.3.4}$$

By using integral representation (1.2.22) in the case of axial symmetry, equation (1) can be rewritten as

$$w^*(\rho) = \frac{2}{\pi}\int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_x^a \frac{\sigma(\rho_0)\,\rho_0\,d\rho_0}{(\rho_0^2 - x^2)^{1/2}}. \tag{4.3.5}$$

Here

$$w^* = w(\rho, 0) - w^+. \tag{4.3.6}$$

Application of the operator

$$\frac{d}{dr}\int_0^r \frac{\rho\,d\rho}{\sqrt{r^2 - \rho^2}} \tag{4.3.7}$$

to both sides of (5) yields

$$\frac{d}{dr}\int_0^r \frac{w^*(\rho)\,\rho\,d\rho}{\sqrt{r^2 - \rho^2}} = \int_r^a \frac{\sigma(\rho_0)\,\rho_0\,d\rho_0}{\sqrt{\rho_0^2 - r^2}}. \tag{4.3.8}$$

Substitution (2), (4), and (6) in (8) results in

$$A_0\,l - \frac{w^+ - w^-}{2} + \sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\cos\left(x_n \frac{r}{a}\right) = -\left[A_0\,(a^2 - r^2)^{1/2}\right.$$

$$\left. + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} \cosh\left(x_n \frac{l}{a}\right)\int_r^a \frac{J_0(x_n\rho_0/a)\,\rho_0\,d\rho_0}{\sqrt{\rho_0^2 - r^2}}\right]. \tag{4.3.9}$$

Here the following integral was used (Gradshtein and Ryzhik, 1965):

$$\int_0^r \frac{J_0(x_n\rho/a)\,\rho\,d\rho}{\sqrt{r^2 - \rho^2}} = \frac{a}{x_n}\sin\left(x_n \frac{r}{a}\right). \tag{4.3.10}$$

The remaining integrals are elementary. Integration of both sides of (9) with respect to r from 0 to a gives

$$A_0\left(l+\frac{\pi}{4}a\right)+\sum_{n=1}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\frac{\sin x_n}{x_n}=\frac{1}{2}(w^+-w^-). \tag{4.3.11}$$

Multiplication of both sides of (9) by $\cos(x_k r/a)$ and subsequent integration with respect to r from 0 to a results in

$$A_0 l\,\frac{\sin x_k}{x_k}+\frac{1}{2}\sum_{\substack{n=1\\ n\neq k}}^{\infty} A_n \sinh\left(x_n \frac{l}{a}\right)\left[\frac{\sin(x_n-x_k)}{x_n-x_k}+\frac{\sin(x_n+x_k)}{x_n+x_k}\right]$$

$$+\frac{1}{2}A_k \sinh\left(x_k \frac{l}{a}\right)\left[1+\frac{\sin(2x_k)}{2x_k}+\coth\left(x_k\frac{l}{a}\right)\frac{\pi x_k}{2}J_0^2(x_k)\right]=\frac{1}{2}(w^+-w^-)\,\frac{\sin x_k}{x_k}. \tag{4.3.12}$$

Here are some details of the transformations performed.

$$\int_0^a \cos\left(x_k\frac{r}{a}\right)\mathrm{d}r\int_r^a \frac{J_0(x_n\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-r^2}}=\int_0^a J_0(x_n\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0\int_0^{\rho_0}\frac{\cos\left(x_k\frac{r}{a}\right)\mathrm{d}r}{\sqrt{\rho_0^2-r^2}}$$

$$=\frac{\pi}{2}\int_0^a J_0(x_n\rho_0/a)\,J_0(x_k\rho_0/a)\,\rho_0\,\mathrm{d}\rho_0=\begin{cases}\frac{\pi}{4}[aJ_0(x_k)]^2 & \text{for } n=k\\ 0 & \text{for } n\neq k\end{cases} \tag{4.3.13}$$

$$\int_0^a (a^2-r^2)^{1/2}\cos\left(x_k\frac{r}{a}\right)\mathrm{d}r=a^2 J_1(x_k)=0. \tag{4.3.14}$$

Now the constants A_0, A_1, ..., may be found from the infinite system of algebraic equations (11) and (12). By introducing the notation

$$c_0=\frac{w^+-w^-}{2},\quad \lambda=\frac{l}{a},\quad B_0=A_0 l,\quad B_n=A_n\sinh\left(x_n\frac{l}{a}\right),\quad \xi_n=\frac{\sin x_n}{x_n},\quad n=1,2,....,\tag{4.3.15}$$

the system (11)–(12) can be rewritten in a more compact form, namely,

$$B_0\left(1+\frac{\pi}{4\lambda}\right)+\sum_{n=1}^{\infty} B_n\,\xi_n=c_0, \tag{4.3.16}$$

$$B_0 \xi_k + \sum_{n=1}^{\infty} B_n q_{nk} = c_0 \xi_k, \quad \text{for} \quad k = 1, 2, ..., \tag{4.3.17}$$

Here

$$q_{nk} = \frac{1}{2}\left[\frac{\sin(x_n - x_k)}{x_n - x_k} + \frac{\sin(x_n + x_k)}{x_n + x_k} \right], \quad \text{for} \quad n \neq k;$$

and

$$q_{nk} = \frac{1}{2}\left[1 + \frac{\sin(2x_k)}{2x_k} + \coth(x_k \lambda) \frac{\pi x_k}{2} J_0^2(x_k) \right], \quad \text{for} \quad n = k. \tag{4.3.18}$$

Notice that the matrix of the system (16)–(17) is symmetric, and that only diagonal elements depend on the value of the ratio $(l/a) = \lambda$, and the diagonal elements are dominating. The off-diagonal elements depend only on the values of the roots x_k, and they are decreasing with the distance from diagonal. These features make the system (16)–(17) very well-behaved, and guarantee high accuracy for any truncated system. Actual computations were made with $N = 100$. The most interesting constant is A_0 since it is proportional to the flux. The ratio F^* of total flux F through a pore of length l to the flux F_0 through an infinitely thin membrane is given in Fig. 4.3.2.

Some simple approximate formulae can be obtained as follows. We consider a truncated system of just two equations

$$B_0\left(1 + \frac{\pi}{4\lambda}\right) + B_1 \xi_1 = c_0,$$

$$B_0 \xi_1 + B_1 q_{11} = c_0 \xi_1. \tag{4.3.19}$$

The solution is

$$B_0 = A_0 l = c_0 \frac{q_{11} - \xi_1^2}{q_{11}\left(1 + \frac{\pi}{4\lambda}\right) - \xi_1^2}, \tag{4.3.20}$$

$$B_1 = A_1 \sinh(\lambda x_1) = c_0 \frac{\frac{\pi}{4\lambda}\xi_1}{q_{11}\left(1 + \frac{\pi}{4\lambda}\right) - \xi_1^2}. \tag{4.3.21}$$

Since $x_1 = 3.8317$, and $J_0(x_1) = -0.4028$, their substitution in (20) gives

Fig. 4.3.2. Dimensionless flux through a pore of length l

$$A_0 = \frac{c_0}{\left(l + \frac{\pi}{4}a\right)\eta - \frac{0.0276\, l}{0.5365 + 0.4883\coth(\lambda x_1)}}. \tag{4.3.22}$$

Here

$$\eta = \frac{0.5641 + 0.4883\coth(\lambda x_1)}{0.5365 + 0.4883\coth(\lambda x_1)}. \tag{4.3.23}$$

A very simple analysis shows that the value of η is almost constant, being unity for $\lambda \to 0$ and $\eta = 1.0269$ for $\lambda \to \infty$. Hence, a simple formula may be suggested for A_0, namely,

$$A_0 = \frac{4c_0}{\pi a}\frac{1}{1 + \frac{4}{\pi}\lambda}. \tag{4.3.24}$$

Formula (24) is exact when $\lambda = 0$, and its overall performance is good. The maximum error is about 2%, and it is achieved near $\lambda = 0.5$. Formula of Kelman (1965) has its maximum error of 5% for $\lambda = 0$, though for $\lambda > 0.5$ its error is less than 1%. Kelman's formula may be recommended for large λ when high accuracy is necessary, otherwise (24) may be used in the whole range $0 \le \lambda < \infty$. A very accurate formula (the maximum error less than 0.15%) can be obtained by a curve-fitting technique, namely,

$$F^* \equiv \frac{F}{F_0} = \frac{1}{1 + \frac{4}{\pi}\lambda + \frac{1}{21.4479 + 0.2564 \coth(0.3439\lambda)}}.$$

Concentration profiles. Finding of σ from (4) is difficult due to a bad convergence. This can be illustrated by the limiting case $\lambda=0$. The exact solution in this case is known:

$$w^*(\rho,z) = -\frac{2}{\pi} c_0 \sin^{-1}\!\left(\frac{a}{l_2}\right), \quad l_2 = \frac{1}{2}\{\sqrt{(a+\rho)^2+z^2} + \sqrt{(a-\rho)^2+z^2}\},$$

$$\frac{\partial w}{\partial z}\Big|_{z=0} = \frac{2c_0}{\pi(a^2-\rho^2)^{1/2}}. \tag{4.3.25}$$

On the other hand, the exact solution of (16)–(17) in this case is

$$A_0 = 4\frac{c_0}{\pi a}, \quad A_k = \frac{4c_0 \sin x_k}{\pi x_k^2 J_0^2(x_k)}, \quad \text{for} \quad k = 1, 2, ..., \tag{4.3.26}$$

By using integrals (10) and (13), we can show the validity of an expansion

$$A_0 + \sum_{n=1}^{\infty} A_n \frac{x_n}{a} J_0\!\left(x_n \frac{\rho}{a}\right) = \frac{4c_0}{\pi a}\left\{1 + \sum_{n=1}^{\infty} \frac{\sin x_n}{x_n J_0^2(x_n)} J_0\!\left(x_n \frac{\rho}{a}\right)\right\} = \frac{2c_0}{\pi(a^2-\rho^2)^{1/2}}. \tag{4.3.27}$$

Substitution of numerical values of x_n into (27) shows that the term $\sin x_n/[x_n J_0^2(x_n)]$ is approximately equal $(-1)^k$, and slightly increasing with k. This makes the expansion (27) practically non-convergent. This is true for any λ, so that expressions (4) and (1) become unfit for use.

An alternative approach is based on the solution of (5) which has the form

$$\sigma(\rho) = -\frac{2}{\pi\rho}\frac{d}{d\rho}\int_\rho^a \frac{t\,dt}{(t^2-\rho^2)^{1/2}}\frac{d}{dt}\int_0^t \frac{w^*(\rho_0)\,\rho_0\,d\rho_0}{\sqrt{t^2-\rho_0^2}}. \tag{4.3.28}$$

The last expression can be rewritten as

$$\sigma(\rho) = \frac{2}{\pi}\left[\frac{f(a)}{(a^2-\rho^2)^{1/2}} - \int_a^\rho \frac{df(t)}{\sqrt{t^2-\rho^2}}\right], \tag{4.3.29}$$

with

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{w^*(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{t^2-\rho_0^2}} = A_0\,l - c_0 + \sum_{n=1}^{\infty} A_n \sinh(x_n\lambda)\cos\left(x_n\frac{t}{a}\right). \tag{4.3.30}$$

Substitution of (30) in (29) yields

$$\sigma(\rho) = \frac{2}{\pi}\left[\frac{A_0\,l - c_0 + \sum_{n=1}^{\infty} A_n \sinh(x_n\lambda)\cos(x_n)}{(a^2-\rho^2)^{1/2}} + \sum_{n=1}^{\infty} A_n \sinh(x_n\lambda)\frac{x_n}{a}\int_\rho^a \frac{\sin(x_n t/a)\,\mathrm{d}t}{\sqrt{t^2-\rho^2}}\right]. \tag{4.3.31}$$

The convergence of (31) is better than that of (5), especially close to $\rho = a$, where the first term becomes dominating. The results of computation of the dimensionless local flux $\sigma^* = |\sigma|\pi a/(2c_0)$ is given in Fig. 4.3.3.

Fig. 4.3.3. Local flux at the pore entrance

The value of w^* in the half-space $z\geq 0$ may be expressed as

$$w^*(\rho,z)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\sigma(\rho_0)\,\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}}. \tag{4.3.32}$$

Now we make use of the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\rho_0\,\mathrm{d}\rho_0}{(a^2-\rho_0^2)^{1/2}\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}}=\sin^{-1}(\frac{a}{l_2}). \tag{4.3.33}$$

Substitution of (29) in (32) yields, after integration according to (33),

$$w^*(\rho,z)=\frac{2}{\pi}\left\{f(a)\sin^{-1}(\frac{a}{l_2})-\int_0^{a}\sin^{-1}\left(\frac{t}{l_2(t)}\right)\mathrm{d}f(t)\right\}. \tag{4.3.34}$$

Integration by parts in (34) gives the final result

$$w^*(\rho,z)=\frac{1}{\pi}\int_0^{a}\frac{\sqrt{l_2^2(t)-t^2}}{l_2^2(t)-l_1^2(t)}f(t)\,\mathrm{d}t. \tag{4.3.35}$$

The terms l_1 and l_2 were defined on many occasions, see, for example, (3.6.4). Now the complete numerical procedure may be outlined as follows. Solution of the truncated system (16)–(17) gives the values of A_k, $k=0,\ \ ,\ \ ,...,N-1$. Here N denotes the order of truncation. The next step is evaluation of the function $f(t)$ according to (30). Its substitution in (35) gives the value of $w^*(\rho,z)$ in the half-space, while formula (2) gives the relevant values in the pore. It is recommended that the derivative $\frac{\partial w^*}{\partial z}$ be evaluated by numerical differentiation procedure. When ρ is close to a and $z=0$, formula (31) is appropriate to use. Notice also that for $z=0$, formula (35) changes to

$$w^*(\rho,0)=\frac{2}{\pi}\int_0^{\min(\rho,a)}\frac{f(t)\,\mathrm{d}t}{\sqrt{\rho^2-t^2}}. \tag{4.3.36}$$

Because of singularity when $\rho\leq a$, formula (36) is not good for numerical integration, and should be transformed to

$$w^*(\rho, 0) = \frac{2}{\pi}\int_0^{\pi/2} f(\rho \sin\phi)\, d\phi.$$

Fig. 4.3.4. Local concentration profile at the pore entrance

Fig. 4.3.5. Isoconcentration profiles: (a) $\lambda = 0$, (b) $\lambda = 0.5$, (c) $\lambda = 1.$, (d) $\lambda = 5.$

We note also that $w^*(0,0) = f(0)$. The results of computations are given in Fig. 4.3.4 and Fig. 4.3.5.

4.4. Sound transmission through an aperture in a rigid screen

Van Bladel has reduced the problem of low-frequency scattering through an aperture in a rigid screen to a sequence of static integral equations. Analytical solutions are known at the moment for a circular and an elliptic aperture only. A new analytical method is proposed here which is valid for the nonelliptical apertures. Specific approximate formulae are derived for evaluating the average value of the quadratic term in the low-frequency expansion for an aperture of general shape. Specific examples are considered. All the formulae are checked against the solutions known in the literature, and a good accuracy is confirmed.

The diffraction of a plane wave by an aperture in a rigid screen is an important acoustical problem. Though significant efforts were spent on the investigation of circular and elliptical apertures (see, for example, Van Bladel, 1967), very little is known about the apertures of general shape, except for some numerical solutions (De Smedt, 1981). Here we reproduce some essential results from (Van Bladel, 1967), which are necessary for better understanding of the new approach presented in this section.

Consider a flat rigid screen with a general aperture S. Let the incident field be a plane wave $P^i = \mathrm{e}^{-jkR}$, where $R = \mathbf{u}_i \cdot \mathbf{r}$, $\mathbf{u}_i$ is the incidence vector and $\mathbf{r}$ is the field point vector, the wave number $k = 2\pi/\lambda$, and λ is the wavelength. The governing integral equation takes the form

$$-\frac{1}{2\pi}\int\int_S \frac{\partial P}{\partial z_M}\frac{e^{-jkR(M,N)}}{R(M,N)}\,\mathrm{d}S_M = P^i(N)$$

where P is the acoustic pressure in the aperture. In the low-frequency case, the characteristic length of the aperture is much smaller than the wavelength, and the following expansions become valid:

$$P(\mathbf{r}) = P_0(\mathbf{r}) + jkP_1(\mathbf{r}) + \frac{1}{2}(jk)^2 P_2(\mathbf{r}) + \dots\ ,$$

$$-\frac{\partial P}{\partial z} = \alpha + jk\beta + \frac{1}{2}(jk)^2\gamma + \dots .$$

Van Bladel (1967) has proven that the diffraction problem can be reduced to the solution of a sequence of integral equations of the following type:

$$w(N) = \iint_S \frac{\sigma(M)}{R(M,N)}\,dS\,, \tag{4.4.1}$$

where S is a two-dimensional domain, $R(M,N)$ stands for the distance between the points M and N, w is a known function, and σ is the unknown function. If we denote σ_0, σ_x, σ_y, σ_{xx}, *etc.*, as solutions of (1), corresponding, respectively, to the function w taking on values $2\pi\sqrt{A}$, $2\pi x$, $2\pi x^2/\sqrt{A}$, *etc.*, where A is the area of the aperture, then the various parameters can be defined quite simply through these solutions. For example,

$$\alpha = \sigma_0/\sqrt{A},$$

$$\beta = -\mathbf{u}_i\cdot(\sigma_x\mathbf{u}_x + \sigma_y\mathbf{u}_y) + \frac{\sigma_0}{2\pi A}\iint_S \sigma_0\,dS, \quad \ldots$$

The problem of sound penetration through an aperture was solved numerically for several specific shapes (De Smedt 1979). We shall use his results for the verification of the accuracy of the new method. The zeroth-order term in the low-frequency expansion was found analytically for an arbitrary aperture in (Fabrikant 1986c). The apparatus used there is essentially the same as that in section 3.3, so it is not repeated here. The first (linear) term can be found from section 3.4 where the mathematically equivalent problem of magnetic polarizability of small apertures of arbitrary shape was considered. Here, a similar method is used for the analysis of the quadratic term in the low-frequency expansion. The relevant theory is given further, with applications to specific aperture shapes (polygon, rectangle, rhombus, cross) to follow. The possibility of using the variational approach is discussed in the last part.

Theory. We outline the idea of the analytical treatment of such problems which allows the derivation of simple yet accurate formulae for various aperture shapes. The approach is based on the integral representation for the reciprocal distance between two points established in Chapter 1.

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right)dx}{(\rho^2-x^2)^{1/2}(\rho_0^2-x^2)^{1/2}}, \tag{4.4.2}$$

where

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}. \tag{4.4.3}$$

Substitution of (2) into (1) gives, after changing the order of integration

$$w(\rho,\phi) = \frac{2}{\pi}\int_0^{\rho} \frac{\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \mathrm{d}\phi_0 \int_x^{a(\phi_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)}{(\rho_0{}^2 - x^2)^{1/2}}\, \sigma(\rho_0, \phi_0)\, \rho_0 \mathrm{d}\rho_0. \qquad (4.4.4)$$

Despite the fact that (4) is valid only inside a circle inscribed into the aperture, it will be shown further that expression (4) allows us to obtain approximate solutions of high accuracy for various aperture shapes.

Consider an aperture of general shape in a rigid screen. Let the boundary of the aperture S be given in the polar coordinates as

$$\rho = a(\phi),$$

where the function $a(\phi)$ is bounded and single-valued. Let the known function w take the form

$$w = g_x y^2 + g_{xy} xy + g_y x^2, \qquad (4.4.5)$$

where g_x, g_y and g_{xy} are known constants.

Assume the distribution of σ in the aperture as

$$\sigma(\rho,\phi) = \frac{a(\phi)\,[\alpha_0 + \rho^2(\alpha_x \sin^2\phi + \alpha_{xy}\sin\phi\cos\phi + \alpha_y \cos^2\phi)]}{\sqrt{a^2(\phi) - \rho^2}}, \qquad (4.4.6)$$

where α_0, α_x, α_y and α_{xy} are yet unknown constants. Now it is necessary to relate α_0, α_x, α_y and α_{xy} to the parameters g_x, g_y and g_{xy}. This can be done by substitution of (6) into (4) which yields after integration with respect to ρ_0

$$w(\rho,\phi) = \alpha_0 \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \mathrm{e}^{in(\phi - \phi_0)} F(\frac{2-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)})\, \mathrm{d}\phi_0$$

$$+ \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^3\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \mathrm{e}^{in(\phi - \phi_0)} F\left(\frac{4-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right)$$

$$\times (\alpha_x \sin^2\phi_0 + \alpha_{xy}\sin\phi_0\cos\phi_0 + \alpha_y\cos^2\phi_0)\, \mathrm{d}\phi_0\,. \qquad (4.4.7)$$

Here F stands for the Gauss hypergeometric function. Further evaluation of w can be done separately for each harmonic. Note that all the odd harmonics of w will be zero if $a(\phi)$ contains only the even harmonics. The zero*th* harmonic will take the form

$$w_0(\rho,\phi) = \frac{\pi}{4}\int_0^{2\pi} \Bigg[2\alpha_0 + \left(a^2(\phi_0) + \frac{1}{2}\rho^2 \right) \times (\alpha_x \sin^2\phi_0 + \alpha_{xy}\sin\phi_0\cos\phi_0 + \alpha_y\cos^2\phi_0) \Bigg] a(\phi_0)\, d\phi_0, \quad (4.4.8)$$

which can be simplified as

$$w_0(\rho,\phi) = \frac{\pi}{4}\left[2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y + \frac{1}{2}\rho^2 (\alpha_x J_x + \alpha_{xy} J_{xy} + \alpha_y J_y) \right], \quad (4.4.9)$$

where the following quantities were introduced

$$B_x = \int_0^{2\pi} a^3(\phi)\sin^2\phi\, d\phi, \qquad B_y = \int_0^{2\pi} a^3(\phi)\cos^2\phi\, d\phi,$$

$$B_{xy} = \int_0^{2\pi} a^3(\phi)\sin\phi\cos\phi\, d\phi. \quad (4.4.10)$$

Since their tensor properties are similar to those of the moments of inertia, we shall call B_x and B_y *the cubic moments of a two-dimensional domain about the axes Ox and Oy* respectively, B_{xy} will be called *the cubic product of a two-dimensional domain about the axes Ox and Oy*.

$$J_0 = \int_0^{2\pi} a(\phi)\, d\phi, \qquad J_x = \int_0^{2\pi} a(\phi)\sin^2\phi\, d\phi,$$

$$J_y = \int_0^{2\pi} a(\phi)\cos^2\phi\, d\phi, \qquad J_{xy} = \int_0^{2\pi} a(\phi)\sin\phi\cos\phi\, d\phi. \quad (4.4.11)$$

These quantities were introduced in section 3.4 We shall call J_x and J_y *the linear moments of a two-dimensional domain about the axes Ox and Oy* respectively, J_{xy} will be called *the linear product of a two-dimensional domain about the axes Ox and Oy*. J_0 will be called the linear polar moment. The following property is quite clear, $J_0 = J_x + J_y$.

Here is the expression for the second harmonic,

$$w_2(\rho,\phi) = \frac{3}{8}\pi\rho^2 \int_0^{2\pi} (\alpha_x \sin^2\phi_0 + \alpha_{xy}\sin\phi_0\cos\phi_0 + \alpha_y\cos^2\phi_0)\, a(\phi_0)\cos 2(\phi-\phi_0)\, d\phi_0,$$

which can be modified as

$$w_2(\rho,\phi) = \frac{3}{8}\pi\rho^2 \{-\alpha_x[(C_{xxxx} - C_{xxyy})\cos 2\phi + 2C_{xxxy}\sin 2\phi] + \alpha_y[(C_{yyyy} - C_{xxyy})\cos 2\phi$$
$$+ 2C_{xyyy}\sin 2\phi] + \alpha_{xy}[(C_{xyyy} - C_{xxxy})\cos 2\phi + 2C_{xxyy}\sin 2\phi]\}. \tag{4.4.12}$$

Here, the following geometrical characteristics of the domain of aperture were introduced:

$$C_{xxxx} = \int_0^{2\pi} a(\phi)\sin^4\phi\, d\phi, \qquad C_{xxxy} = \int_0^{2\pi} a(\phi)\sin^3\phi\cos\phi\, d\phi,$$
$$C_{xxyy} = \int_0^{2\pi} a(\phi)\sin^2\phi\cos^2\phi\, d\phi, \qquad C_{xyyy} = \int_0^{2\pi} a(\phi)\sin\phi\cos^3\phi\, d\phi$$
$$C_{yyyy} = \int_0^{2\pi} a(\phi)\cos^4\phi\, d\phi. \tag{4.4.13}$$

The C moments will be called *the linear moments of the fourth order*. Their relationships with the J moments are easy to establish, for example, $J_x = C_{xxxx} + C_{xxyy}$, $J_{xy} = C_{xyyy} + C_{xxxy}$, *etc*. It is important to note that the parameter α_0 did not enter (12), and the parameters α_x, α_{xy}, and α_y will not enter the expression for the fourth harmonic. Investigation of the harmonics higher than 2 shows that their amplitude decreases. In the case of an ellipse they vanish thus making the solution *exact*. Of course, the odd harmonics do not vanish for the apertures of general shape, but we can always choose the system of coordinate origin in such a way as to eliminate the first harmonic and to reduce (or eliminate) the higher odd harmonics. For example, let $a(\phi) = a_1 + a_2\sin\phi$. Here we can eliminate all the odd harmonics just by moving the system of coordinate origin in the positive y direction by a_2. It can be shown that we can eliminate the first harmonic for an aperture of general shape by locating the system of coordinate origin at the center of gravity. This is why it seems justified to assume $w \approx w_0 + w_2$, ignoring the first harmonic and calling the remaining harmonics (the third and higher) the solution error. Now, we have an approximate expression for w as

$$w = \frac{\pi}{4}\{2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y$$

$$+ x^2[-\alpha_x (C_{xxxx} - 2C_{xxyy}) + \alpha_{xy}(2C_{xyyy} - C_{xxxy}) + \alpha_y (2C_{yyyy} - C_{xxyy})]$$

$$+ y^2[\alpha_x (2C_{xxxx} - C_{xxyy}) + \alpha_{xy}(2C_{xxxy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy})]$$

$$+ 6xy[\alpha_x C_{xxxy} + \alpha_{xy} C_{xxyy} + \alpha_y C_{xyyy}]\}. \tag{4.4.14}$$

The comparison of (5) and (14) leads to the following set of equations:

$$\frac{\pi}{4}(2\alpha_0 J_0 + \alpha_x B_x + \alpha_{xy} B_{xy} + \alpha_y B_y) = 0,$$

$$\frac{\pi}{4}[\alpha_x (2C_{xxxx} - C_{xxyy}) + \alpha_{xy}(2C_{xxxy} - C_{xyyy}) - \alpha_y (C_{yyyy} - 2C_{xxyy})] = g_x,$$

$$\frac{\pi}{4}[-\alpha_x (C_{xxxx} - 2C_{xxyy}) + \alpha_{xy}(2C_{xyyy} - C_{xxxy}) + \alpha_y (2C_{yyyy} - C_{xxyy})] = g_y,$$

$$\frac{3\pi}{2}[\alpha_x C_{xxxy} + \alpha_{xy} C_{xxyy} + \alpha_y C_{xyyy}] = g_{xy}. \tag{4.4.15}$$

The last three equations of (15) can be solved with respect to α_x, α_y, and α_{xy}, after which the value of α_0 can be found from the first equation (15).

A significant simplification occurs when the aperture S has at least one axis of symmetry. In this case $C_{xxxy} = C_{xyyy} = B_{xy} = 0$. The last equation (15) becomes decoupled from the previous three. The solutions can be written explicitly,

$$\alpha_x = \frac{4[g_x(2C_{yyyy} - C_{xxyy}) + g_y(C_{yyyy} - 2C_{xxyy})]}{3\pi(C_{xxxx}C_{yyyy} - C_{xxyy}^2)},$$

$$\alpha_y = \frac{4[g_x(C_{xxxx} - 2C_{xxyy}) + g_y(2C_{xxxx} - C_{xxyy})]}{3\pi(C_{xxxx}C_{yyyy} - C_{xxyy}^2)},$$

$$\alpha_{xy} = \frac{2g_{xy}}{3\pi C_{xxyy}}. \tag{4.4.16}$$

Substitution of (16) into the first equation (15) gives

$$\alpha_0 = -\frac{2(g_x\beta_x + g_y\beta_y)}{3\pi J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)}, \tag{4.4.17}$$

where

$$\beta_x = B_x(2C_{yyyy} - C_{xxyy}) + B_y(C_{xxxx} - 2C_{xxyy}),$$

$$\beta_y = B_x(C_{yyyy} - 2C_{xxyy}) + B_y(2C_{xxxx} - C_{xxyy}).$$

Expressions (6), (7), (16), (17) give a complete and *exact* solution for an ellipse. We hope they will perform well for an aperture of general shape. We expect (6) to be reasonably accurate in the neighborhood of the coordinate origin while the error might become quite significant close to the boundary of the domain S, mainly due to the fact that the assumption of a square-root singularity in (6) is wrong, especially for a domain with sharp angles.

It is appropriate to discuss the following particular cases: $g_y = 2\pi/\sqrt{A}$, $g_x = g_{xy} = 0$; and the case $g_x = 2\pi/\sqrt{A}$, $g_y = g_{xy} = 0$. In every case let us compute the integral

$$p = \frac{1}{A}\int\int_S \sigma\, dS,$$

which is proportional to the average value of σ, and is dimensionless thus characterizing the shape of S and being independent of its size. We shall denote these parameters by p_y and p_x for each case respectively. These parameters correspond to the coefficients in the quadratic terms in the low-frequency expansion (Van Bladel, 1967). Formulae (7), (16), and (17) lead to the following expressions for the parameters p_y and p_x:

$$p_y = \frac{8\left(8J_0\left[I_x(C_{yyyy} - 2C_{xxyy}) + I_y(2C_{xxxx} - C_{xxyy})\right] - 3A\beta_y\right)}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)},$$

$$p_x = \frac{8\left(8J_0\left[I_x(2C_{yyyy} - C_{xxyy}) + I_y(C_{xxxx} - 2C_{xxyy})\right] - 3A\beta_x\right)}{9A^{3/2}J_0(C_{xxxx}C_{yyyy} - C_{xxyy}^2)}, \tag{4.4.18}$$

where I_x and I_y are the well-known moments of inertia of the domain of aperture. Some further simplifications take place when the domain S possesses a central symmetry which implies that all the moments about the axis Ox are equal to the similar moments about the axis Oy. In this case

$$p_y = p_x = \frac{8(8I_0 J_0 - 3AB_0)}{3A^{3/2} J_0^2}, \tag{4.4.19}$$

where the moments with the subindex 0 indicate corresponding polar moments. Formula (17) also simplifies as follows

$$\alpha_0 = \frac{2}{\pi J_0}\left[g_0 - \frac{B_0}{J_0}(g_x + g_y)\right]. \tag{4.4.20}$$

Formulae (18) are the main results of this section. The quadratic terms in the low-frequency expansion can now be found by a relatively simple computation of the geometrical characteristics (moments) of the domain of aperture.

Several aperture shapes are considered below. The general solution is given by the formulae (18). We present only the necessary computations of the moments involved. A sufficiently high degree of accuracy of formulae derived is confirmed by comparison with available numerical solutions.

Polygon. Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The system of coordinate origin is located at the polygon's center of gravity in order to eliminate the first harmonic from $a(\phi)$. Let us number the polygon sides in a counter-clockwise direction from 1 to n, with a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote b_k the distance from the center of gravity to the kth apex; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. The following expressions can be obtained for the moments of inertia:

$$I_x = \sum_{k=1}^{n} -m_k \cos 2\psi_k + g_k \sin 2\psi_k + 2h_k \cos^2\psi_k\,,$$

$$I_y = \sum_{k=1}^{n} m_k \cos 2\psi_k - g_k \sin 2\psi_k + 2h_k \sin^2\psi_k\,,$$

$$I_{xy} = \sum_{k=1}^{n} (m_k - h_k)\sin 2\psi_k + g_k \cos 2\psi_k, \tag{4.4.21}$$

where

$$m_k = \frac{2A_k^3}{a_k^2}, \quad g_k = A_k^2 \frac{b_{k+1}^2 - b_k^2}{2a_k^2}, \quad h_k = \frac{A_k[3(b_{k+1}^2 + b_k^2) - a_k^2]}{24}. \tag{4.4.22}$$

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^{n} -q_k \cos 2\psi_k + s_k \sin 2\psi_k + 2t_k \cos^2\psi_k,$$

$$J_y = \sum_{k=1}^{n} q_k \cos 2\psi_k - s_k \sin 2\psi_k + 2t_k \sin^2\psi_k, \tag{4.4.23}$$

$$J_{xy} = \sum_{k=1}^{n} (q_k - t_k)\sin 2\psi_k + s_k \cos 2\psi_k,$$

where

$$q_k = \frac{A_k}{a_k^2}\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right)[a_k^2 + (b_k - b_{k+1})^2],$$

$$s_k = 4\frac{A_k^2}{a_k^2}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right), \tag{4.4.24}$$

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}.$$

The C moments can be computed by the following formulae:

$$C_{xxxx} = \sum_{k=1}^{n} -q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k$$

$$+ 4s_k \sin\psi_k \cos^3\psi_k + 2t_k \cos^4\psi_k,$$

$$C_{xxxy} = \sum_{k=1}^{n} v_k \cos 4\psi_k + u_k \sin 4\psi_k + s_k \cos^2\psi_k$$
$$\times (1 - 4\sin^2\psi_k) + \tfrac{1}{2} q_k \sin 2\psi_k - 2t_k \sin\psi_k \cos^3\psi_k,$$

$$C_{xxyy} = \sum_{k=1}^{n} u_k \cos 4\psi_k - v_k \sin 4\psi_k - \tfrac{1}{2} s_k \sin 4\psi_k + 2t_k \sin^2\psi_k \cos^2\psi_k,$$

$$C_{xyyy} = \sum_{k=1}^{n} -v_k \cos 4\psi_k - u_k \sin 4\psi_k - s_k \sin^2\psi_k$$
$$\times (1 - 4\cos^2\psi_k) + \tfrac{1}{2} q_k \sin 2\psi_k - 2t_k \sin^3\psi_k \cos\psi_k,$$

$$C_{yyyy} = \sum_{k=1}^{n} q_k \cos 2\psi_k - u_k \cos 4\psi_k + v_k \sin 4\psi_k$$
$$- 4s_k \sin^3\psi_k \cos\psi_k + 2t_k \sin^4\psi_k,$$

where q_k, s_k and t_k are defined by (24) and

$$u_k = \frac{A_k}{12a_k^4}\left\{\left[\frac{b_{k+1}^2 - b_k^2 + a_k^2}{b_{k+1}}\right]^3 + \left[\frac{b_k^2 - b_{k+1}^2 + a_k^2}{b_k}\right]^3\right\}, \qquad (4.4.25)$$

$$v_k = \frac{16A_k^4}{3a_k^4}\left[\frac{1}{b_{k+1}^3} - \frac{1}{b_k^3}\right].$$

The following formulae can be derived for the cubic moments:

$$B_x = \sum_{k=1}^{n} -j_k \cos 2\psi_k + r_k \sin 2\psi_k + 2f_k \cos^2\psi_k,$$

$$B_y = \sum_{k=1}^{n} j_k \cos 2\psi_k - r_k \sin 2\psi_k + 2f_k \sin^2\psi_k, \qquad (4.4.26)$$

$$B_{xy} = \sum_{k=1}^{n} (j_k - f_k) \sin 2\psi_k + r_k \cos 2\psi_k,$$

where

$$j_k = \left(\frac{2A_k}{a_k}\right)^3 \ln\frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k},$$

$$r_k = \left(\frac{2A_k}{a_k}\right)^2 (b_{k+1} - b_k),$$

$$f_k = \tfrac{1}{4}(b_k + b_{k+1})A_k\left[1 + \left(\frac{b_{k+1} - b_k}{a_k}\right)^2\right] + \tfrac{1}{4}j_k.$$

Substitution of (21–26) into (18) gives the complete solution for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k - 1)/n$, $A_k = [a^2\cot(\pi/n)]/4 = [b^2\sin(2\pi/n)]/2$, $A = nA_k$, and formulae (21–26) simplify to

$$I_x = I_y = \frac{na^4}{64}\cot\frac{\pi}{n}\left[\cot^2\frac{\pi}{n} + \frac{1}{3}\right] = \frac{nb^4}{24}\sin\frac{2\pi}{n}\left[2 + \cos\frac{2\pi}{n}\right], \tag{4.4.27}$$

$$J_x = J_y = \tfrac{1}{4}na\cot\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} = \tfrac{1}{2}nb\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}, \tag{4.4.28}$$

$$B_x = B_y = \tfrac{1}{4}nb^3\left[\sin\frac{2\pi}{n} + \cos^3\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}\right], \tag{4.4.29}$$

$$C_{xxxx} = C_{yyyy} = \tfrac{3}{8}nb\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)},$$

$$C_{xxyy} = \tfrac{1}{8}nb\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}. \tag{4.4.30}$$

Note that formulae (30) are valid for any regular polygon except the square, due to the fact that the trigonometric series summation, namely,

$$\sum_{k=1}^{n}\sin^4\left[(k\text{-}1)\frac{2\pi}{n}\right] = \sum_{k=1}^{n}\cos^4\left[(k\text{-}1)\frac{2\pi}{n}\right] = \tfrac{3}{8}n,$$

is not valid for a square. The C moments for a square with the side equal $2l$ can be expressed as

$$C_{xxxx} = C_{yyyy} = l\left[4\ln(1+\sqrt{2}) - \frac{2\sqrt{2}}{3}\right],$$
$$C_{xxyy} = \frac{2\sqrt{2}}{3}l. \tag{4.4.31}$$

Formulae (6, 7, 16, and 17) simplify for a regular polygon

$$\alpha_x = \frac{4(5g_x + g_y)}{3\pi J_0},$$
$$\alpha_y = \frac{4(g_x + 5g_y)}{3\pi J_0}, \tag{4.4.32}$$
$$\alpha_{xy} = \frac{16g_{xy}}{3\pi J_0}.$$

Again, one should note that formulae (32) are not valid for a square. The formulae to follow are valid for an arbitrary polygon including the square.

$$\alpha_0 = -\frac{2B_0}{\pi J_0{}^2}(g_x + g_y).$$

The dimensionless coefficients p_y and p_x will take the form

$$p_y = p_x = \frac{8\sqrt{2}}{\left(n\sin\frac{2\pi}{n}\right)^{1/2} n\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}} \times\left[\frac{23+7\cos\frac{2\pi}{n}}{36} - \frac{\sin\frac{\pi}{n}}{\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}\right]. \tag{4.4.33}$$

Consider several particular values of n. For an equilateral triangle ($n = 3$) formula (33) gives $p_y = p_x = 0.3782$. We did not find any numerical data to compare with this result. In the case of a square $n = 4$, and $p_y = p_x = 0.2697$. The result due to De Smedt (1979) is 0.2645, with the discrepancy less than 2%. Since formula (33), in the limiting case $n \to \infty$, gives the exact result for a circle $p_y = p_x = 4/(3\pi^{3/2}) =$ 0.2394, we should expect that the error of (33) will decrease with n. The value of the coefficients for a regular hexagon is 0.2443, and again, we did not find anything in the literature to compare with this result. It is noteworthy that the value of the coefficients does not change significantly in the whole range $3 \leq n < \infty$.

Rectangle. Consider an aperture with a rectangular base, a_1 and a_2 being its semiaxes along the axis Ox and Oy respectively. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (24–29) in this case reduce to

$$I_x = (4/3)\, a_1 a_2^3, \qquad I_y = (4/3)\, a_1^3 a_2\,, \tag{4.4.34}$$

$$J_x = 4a_1 \sinh^{-1} \varepsilon\,, \qquad J_y = 4a_2 \sinh^{-1}(1/\varepsilon), \tag{4.4.35}$$

$$C_{xxxx} = 4a_1\Big(\sinh^{-1} \varepsilon - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}\Big),$$

$$C_{xxyy} = 4a_1 \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}, \tag{4.4.36}$$

$$C_{yyyy} = 4a_2\Big(\sinh^{-1} \frac{1}{\varepsilon} - \frac{1}{3\sqrt{1+\varepsilon^2}}\Big),$$

$$B_x = 2a_1^3\Big(\varepsilon\sqrt{1+\varepsilon^2} - \sinh^{-1} \varepsilon + 2\varepsilon^3 \sinh^{-1} \frac{1}{\varepsilon}\Big),$$

$$B_y = 2a_1^3\Big[\varepsilon\sqrt{1+\varepsilon^2} - \varepsilon^3 \sinh^{-1} \frac{1}{\varepsilon} + 2\sinh^{-1} \varepsilon \Big]. \tag{4.4.37}$$

We have found in the literature some numerical results which seem to be more or less accurate. The coefficients p_y and p_x were computed by De Smedt (1979) for a rectangle with various aspect ratio ε. Here, we present his results along with those given by the method of this section

ε=	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt p_y=	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
our result p_y=	3.2809	1.3959	0.7782	0.5100	0.3485	0.2697
Discrepancy in p_y %	-9.4	-1.7	2.0	2.5	0.2	-2.0
De Smedt p_x=	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
our result p_x=	0.0284	0.0577	0.0963	0.1431	0.2086	0.2697
Discrepancy in p_x %	24.6	9.7	1.9	-2.3	-3.2	-2.0

Our formulae seem to perform satisfactorily in a sufficiently wide range of aspect ratio. The distribution of σ due to (7) can be compared with the numerical data received in a personal communication from De Smedt. Computations were made for $\varepsilon = 0.5$, $g_y = 2\pi/\sqrt{A}$, $g_x = 0$. Here are the results along the axis Ox, compared to those communicated by De Smedt

x/a_1=	0.0000	0.0833	0.2500	0.3333	0.5000	0.6667	0.7500	0.9167
De Smedt σ=	-0.4715	-0.4673	-0.3933	-0.3249	-0.1238	0.2515	0.5456	2.0580
our result σ=	-0.4731	-0.4647	-0.3953	-0.3314	-0.1290	0.2273	0.5141	1.8556
Discrepancy %	-0.3	0.6	-0.5	-2.0	-4.2	9.6	-1.5	9.8

We compare the same values along the axis Oy.

y/a_2=	0.0000	0.1667	0.3333	.5000	.6667	.8333
De Smedt σ=	-0.4715	-0.4765	-0.4774	-0.4837	-0.5063	-0.5311
our result σ=	-0.4731	-0.4744	-0.4791	-0.4907	-0.5198	-0.6138
Discrepancy %	-0.3	0.5	-0.3	-1.4	-2.7	-15.6

As we predicted, the discrepancy becomes quite significant close to the boundary.

Rhombus. Let a_1 and a_2 be its semiaxes along Ox and Oy respectively. Denote its side $l = (a_1^2 + a_2^2)^{1/2}$, and introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (21–26) in this case yield

$$I_x = \frac{l^4\varepsilon^3}{3(1+\varepsilon^2)^2}, \qquad I_y = \frac{l^4\varepsilon}{3(1+\varepsilon^2)^2}, \qquad A = \frac{2l^2\varepsilon}{(1+\varepsilon^2)}. \tag{4.4.38}$$

$$J_x = \frac{4l\varepsilon}{(1+\varepsilon^2)}\left[\frac{1-\varepsilon}{\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$J_y = \frac{4l\varepsilon}{(1+\varepsilon^2)}\left[-\frac{1-\varepsilon}{\sqrt{1+\varepsilon^2}} + \frac{1}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]. \tag{4.4.39}$$

$$B_x = \frac{2l^3\varepsilon^3}{(1+\varepsilon^2)^3}\left[\frac{\varepsilon^3+4\varepsilon-3}{\sqrt{1+\varepsilon^2}} + \frac{2-\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$B_y = \frac{2l^3\varepsilon}{(1+\varepsilon^2)^3}\left[\frac{1+4\varepsilon^2-3\varepsilon^3}{\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2(2\varepsilon^2-1)}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right]. \tag{4.4.40}$$

$$C_{xxxx} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{2-\varepsilon+5\varepsilon^2-4\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^4}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$C_{yyyy} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{-4+5\varepsilon-\varepsilon^2+2\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{1}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}\right],$$

$$C_{xxyy} = \frac{4l\varepsilon}{(1+\varepsilon^2)^2}\left[\frac{1-2\varepsilon-2\varepsilon^2+\varepsilon^3}{3\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2}{(1+\varepsilon^2)}\ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}} \right].$$

Again, we have only the numerical results by De Smedt (1979) to compare with ours which are given below

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=$	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
our result $p_y=$	3.7425	1.6605	0.9192	0.5770	0.3661	0.2697
Discrepancy %	19.6	12.1	6.6	2.7	-0.2	-2.5
De Smedt $p_x=$	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
our result $p_x=$	0.1944	0.1435	0.1345	0.1532	0.2050	0.2697
Discrepancy %	-518.4	-161.3	-56.0	-20.6	-6.6	-2.5

Though our results are satisfactory for p_y, they are unacceptable for p_x when $\varepsilon \leq 0.5$. The reason for this is our assumption of a square-root singularity in (6) which is grossly incorrect for contours with sharp angles. An alternative approach which uses the variational principle and somewhat improves the accuracy, is discussed further.

Cross. Consider an aperture with a configuration obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$ $(a \geq b)$. Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments will take the form

$$A = 4a^2\varepsilon(2-\varepsilon)\,, \qquad I_x = I_y = \frac{4}{3}a^4\varepsilon(1+\varepsilon^2-\varepsilon^3),$$

$$J_x = J_y = 4a\left[\ln(\varepsilon+\sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon} \right], \tag{4.4.41}$$

$$B_x = B_y = 2a^3\left\{ 2\varepsilon\sqrt{1+\varepsilon^2} + \ln(\varepsilon+\sqrt{1+\varepsilon^2}) + \varepsilon^3\left[\ln\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon(1+\sqrt{2})} - \sqrt{2} \right] \right\}.$$

The comparison between the results of this section and those given by De Smedt (1979) are presented below

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt $p_y=p_x=$	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
our result $p_y=p_x=$	1.6943	0.6765	0.3716	0.2683	0.2517	0.2697
Discrepancy %	-75.1	-39.4	-13.6	-0.5	0.3	-2.0

Taking into consideration the shape complexity, we should consider the resulting agreement as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$. The discrepancy becomes unacceptably big for $\varepsilon \le 0.3$. It will be shown further that the variational approach slightly improves the results.

Variational approach. An alternative method can be suggested by using the variational approach (Noble 1960). The following functional assumes its stationary value at the exact solution of (1)

$$I(\sigma) = 2\iint_S \sigma(M)w(M)\mathrm{d}S_M - \iint_S \sigma(M)\left[\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \tag{4.4.42}$$

Take

$$\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N \approx w_0 + w_2, \tag{4.4.43}$$

where σ is defined by (6) and $w_0 + w_2$ is given by (14). Substitution of (5), (6), (14), and (43) into (42) makes it possible to consider the functional I as a function of α_0, α_x, α_y, and α_{xy}. The extremum conditions

$$\frac{\partial I}{\partial \alpha_0} = 0, \quad \frac{\partial I}{\partial \alpha_x} = 0, \quad \frac{\partial I}{\partial \alpha_y} = 0, \quad \frac{\partial I}{\partial \alpha_{xy}} = 0,$$

give four linear algebraic equations with respect to the unknown α_0, α_x, α_y, and α_{xy}. The complete solution is pretty cumbersome. Here, we present the set of equations for the coefficients α_0, α_x, and α_y which are valid only for the domains having at least one axis of symmetry.

$$\begin{aligned} c_{11}\alpha_0 + c_{12}\alpha_x + c_{13}\alpha_y &= \frac{16}{3}(I_x g_x + I_y g_y), \\ c_{12}\alpha_0 + c_{22}\alpha_x + c_{23}\alpha_y &= \frac{16}{15}(D_{xxxx} g_x + D_{xxyy} g_y), \\ c_{13}\alpha_0 + c_{23}\alpha_x + c_{33}\alpha_y &= \frac{16}{15}(D_{xxyy} g_x + D_{yyyy} g_y). \end{aligned} \tag{4.4.44}$$

Here,

$$c_{11} = 2\pi J_0 A,$$

$$c_{12} = \frac{1}{2}\pi\left[B_x A + \frac{4}{3}I_x(2J_0 + 2C_{xxxx} - C_{xxyy}) - \frac{4}{3}I_y(C_{xxxx} - 2C_{xxyy})\right],$$

$$c_{13} = \frac{1}{2}\pi\,[B_y A + \frac{4}{3} I_y (2J_0 + 2C_{yyyy} - C_{xxyy}) - \frac{4}{3} I_x (C_{yyyy} - 2C_{xxyy})],$$

$$c_{22} = \frac{4}{15}\pi\,[5B_x I_x + D_{xxxx}(2C_{xxxx} - C_{xxyy}) - D_{xxyy}(C_{xxxx} - 2C_{xxyy})],$$

$$c_{23} = \frac{2}{15}\pi\,[5(B_x I_y + B_y I_x) - D_{xxxx}(C_{yyyy} - 2C_{xxyy}) + 2D_{xxyy}(C_{xxxx} + C_{yyyy} - C_{xxyy}) - D_{yyyy}(C_{xxxx} - 2C_{xxyy})],$$

$$c_{33} = \frac{4}{15}\pi\,[5B_y I_y + D_{yyyy}(2C_{yyyy} - C_{xxyy}) - D_{xxyy}(C_{yyyy} - 2C_{xxyy})]. \qquad (4.4.45)$$

The D moments are introduced similar to (13) as

$$D_{xxxx} = \int_0^{2\pi} a^6(\phi)\sin^4\phi\, d\phi, \qquad D_{xxyy} = \int_0^{2\pi} a^6(\phi)\sin^2\phi\cos^2\phi\, d\phi,$$

$$D_{yyyy} = \int_0^{2\pi} a^6(\phi)\cos^4\phi\, d\phi. \qquad (4.4.46)$$

It is quite clear that the variational approach solution is more cumbersome than the one introduced in the first part. It remains to be seen whether it will be more accurate. One advantage should be noted: the matrix of (44) is symmetric (as it is required by the reciprocal theorem) while the matrix of (15) generally is not symmetric.

Let us compare the results for several particular configurations. First of all, consider a regular polygon. In the tables hereafter the word *simple* refers to the method introduced in (18), the word *variational* refers to the solution of the set of equations (44). In the case of a regular polygon we shall need the polar D moment only

$$D_0 = \tfrac{1}{5} b^6 n \sin\frac{2\pi}{n}\left[1 + \tfrac{4}{3}\cos^2\frac{\pi}{n} + \tfrac{8}{3}\cos^4\frac{\pi}{n}\right].$$

Here are the results of computations for a regular polygon with n sides

n=	3	4	5	6	7	9	100
simple $p_y = p_x$=	0.3782	0.2697	0.2502	0.2443	0.2420	0.2403	0.2394
variational $p_y = p_x$=	0.3409	0.2612	0.2472	0.2429	0.2412	0.2401	0.2394
Discrepancy %	9.9	3.2	1.2	0.6	0.3	0.1	0.0

Both methods seem to work well. If one considers the result by De Smedt (1979) for a square 0.2645 as exact then this might be an indication that the variational approach is somewhat more accurate. In the limiting case of $n \to \infty$ both methods give the exact result for a circle.

The D moments for the rectangle considered earlier will take the form

$$D_{xxxx} = \frac{24}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{24}{5} a_1^5 a_2, \quad D_{xxyy} = \frac{8}{3} a_1^3 a_2^3.$$

Here are the numerical results computed for a rectangle

ε=	0.1000	0.2000	0.3330	0.5000	0.7500	1.0000
De Smedt p_y=	2.9980	1.3730	0.7942	0.5229	0.3491	0.2645
variational p_y=	3.4239	1.4523	0.8023	0.5166	0.3437	0.2612
Discrepancy %	-14.2	-5.8	-1.0	1.2	1.5	1.2
De Smedt p_x=	0.0376	0.0639	0.0982	0.1399	0.2022	0.2645
variational p_x=	0.0316	0.0588	0.0939	0.1370	0.1997	0.2612
Discrepancy %	15.9	7.9	4.3	2.1	1.2	1.2

Again, the general impression is that the variational approach is more accurate but not everywhere, for example, the discrepancy in p_y for $\varepsilon = 0.1$ increased as compared to the simple method result given earlier.

The D moments for a rhombus will take the form

$$D_{xxxx} = \frac{4}{5} a_1 a_2^5, \quad D_{yyyy} = \frac{4}{5} a_1^5 a_2, \quad D_{xxyy} = \frac{2}{15} a_1^3 a_2^3.$$

We present below the numerical results for a rhombus due to the variational approach compared to those by De Smedt (1979)

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt p_y=	4.6520	1.8890	0.9844	0.5933	0.3655	0.2631
variational p_y=	-0.5952	3.3549	0.9465	0.5580	0.3534	0.2612
Discrepancy %	112.8	-77.6	3.8	6.0	3.3	0.7
De Smedt p_x=	0.0314	0.0549	0.0862	0.1270	0.1923	0.2631
variational p_x=	0.0090	0.1464	0.1110	0.1400	0.1971	0.2612
Discrepancy %	71.5	-166.7	-28.8	-10.2	-2.5	0.7

Though the discrepancy decreased for $\varepsilon > 0.33$, we should state that both methods fail for a domain with sharp angles, since the results are unacceptable for $\varepsilon <$ 0.33.

In the case of a cross-shaped aperture, the D moments can be expressed as follows:

$$D_{xxxx} = D_{yyyy} = \frac{24}{5} l^6 \varepsilon (1 + \varepsilon^4 - \varepsilon^5), \qquad D_{xxyy} = \frac{8}{3} l^6 \varepsilon^3 (2 - \varepsilon^3).$$

Here are the numerical results due to the variational approach compared to those by De Smedt (1979)

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	1.0000
De Smedt p_y=p_x=	0.9675	0.4854	0.3271	0.2671	0.2523	0.2645
variational p_y=p_x=	1.4346	0.5822	0.3397	0.2606	0.2482	0.2612
Discrepancy %	-48.3	-19.9	-3.9	2.4	1.6	1.2

Comparison of this table with a similar one given earlier leads to the same conclusion: the results become valid in a wider range of the aspect ratio ε, but the theory fails for very small ε. It is up to the user to decide whether a somewhat better accuracy of the variational approach is worth more cumbersome computations.

We have to caution the reader willing to use the reciprocal theorem and the solution in sections 3.3 and 3.4 in order to find further terms in the low-frequency expansion. The results might be good for the domains with the aspect ratio close to unity (like, for example, a square) but the accuracy deteriorates quickly as the aspect ratio moves away from unity. It is advisable in each particular case to use the method similar to the one in this section.

Formulae (18) give a simple and effective solution to the problem of evaluating the quadratic terms in the low-frequency expansion for the problem of sound penetration through an aperture in a rigid screen. Their high accuracy in a sufficiently wide range of aspect ratio is confirmed by numerous examples. The case of a domain with sharp angles seems to be outside this class. An investigation of the nature of singularity is absolutely indispensable for this type of problems. A similar method can be used for evaluating further terms of the low-frequency expansion.

4.5. Sound penetration through a general aperture in a soft screen

The term soft screen represents an abstraction opposite to that of a rigid screen. The diffraction of a plane wave by an aperture in a soft screen is an important acoustical problem. Again, very little is known about the apertures of general shape, except for some numerical solutions (De Meulenaere and Van Bladel, 1977; Okon and Harrington, 1981). Here we reproduce some essential results from (Van Bladel, 1968) which are necessary for better understanding of the problem formulation.

Consider a flat soft screen with a general aperture S, whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{4.5.1}$$

Let the incident field be a plane wave $P^i = e^{-jkR}$, where $R = \mathbf{u}_i \cdot \mathbf{r}$, $\mathbf{u}_i$ is the incidence vector and $\mathbf{r}$ is the field point vector and k is the wave number. The governing integral equation in the case of a soft screen takes the form

$$\frac{1}{2\pi} \iint_S P(M) \frac{\partial}{\partial z_M}\left[\frac{e^{-jkR(M,N)}}{R(M,N)} \right] \mathrm{d}S_M = P^i(N),$$

where P is the acoustic pressure in the aperture. In the low-frequency case the characteristic length of the aperture is much smaller that the wavelength, and the following expansion becomes valid

$$P(\mathbf{r}) = P_0(\mathbf{r}) + jkP_1(\mathbf{r}) + \tfrac{1}{2}(jk)^2 P_2(\mathbf{r}) + \ldots ,$$

Van Bladel (1968) has proven that the diffraction problem can be reduced to the solution of a sequence of integral equations of the following type

$$\sigma(N) = \Delta \iint_S \frac{w(M)}{R(M,N)} \mathrm{d}S, \tag{4.5.2}$$

where Δ is the two-dimensional Laplace operator, S is the aperture domain, $R(M,N)$ stands for the distance between the points M and N, w denotes the unknown function and σ is a known function, If we denote w°, w_x, w_y, w_{xx}, *etc.* as solutions of (2) corresponding respectively to the function σ taking on values $-2\pi/\sqrt{A}$, $-2\pi x/A$, $-2\pi y/A$, $-2\pi x^2/A^{3/2}$, *etc.*, where A is the area of the aperture, then the various parameters can be defined quite simply through these solutions. For example,

$$P_1 = jk\sqrt{A}\cos\theta_i\, w^{\circ},$$

where θ_i is the angle of incidence. The reader is referred to the original paper by Van Bladel (1968) for the rest of the theory. The most important seems to be the zeroth-order term w°. The analytical solution w° is known for a circle and an ellipse only. The case of non-elliptic aperture had to be treated numerically. The problem of sound penetration through an aperture is mathematically equivalent to the one of the electrical polarizability. Therefore, in order to avoid unnecessary repetition, the reader is referred for the rest of the theory to section 3.5.