CHAPTER 3

APPLICATIONS IN ELECTROMAGNETICS

The general results of Chapter 1 are applied here to investigation of interaction of several charged coaxial and arbitrarily located disks. New type of governing integral equation is derived for the Dirichlet and Neumann problems for a circular annulus domain. Simple yet accurate formulae are derived for the capacity of flat laminae. Similar results were obtained for the electrical and magnetic polarizability of small apertures of general shape.

3.1. Interaction of several coaxial disks

The problem of charged coaxial circular disks has been attracting the attention of scientists during the last century. Kirchhoff, Ignatowsky, Love, Nomura, Cooke and others made a significant contribution to its solution. A comprehensive literature review can be found in (Ufliand, 1977) and (Leppington and Levine, 1970). The latest published important result seems to be (Kuz'min, 1971) where an axisymmetric problem of several charged coaxial disks is considered by the dual integral equation method. The solution is obtained for the case when the distance between disks is long in comparison with their radii.

A general solution to the non-axisymmetric problem is given here by a new approach. A mathematical outline of the method is given first. Formulation of the problem and its solution follows. The charge density distribution can be found from a system of Fredholm integral equation. The total charge and some other integral characteristics can be estimated without solving the system. The collocation method is used for the numerical solution of the integral equation. A way is found to assess the error of the computation. A high degree of accuracy of the solution allowed us to correct some numerical results published by Cooke (1958). Several examples are considered.

Preliminaries. Some mathematical considerations are presented here to simplify understanding of the method. Introduce the following quantities:

$$l_{1,2}(x,\rho,z)\equiv l_{1,2}(x)=\tfrac{1}{2}[\sqrt{(x+\rho)^2+z^2}\mp\sqrt{(x-\rho)^2+z^2}\,], \tag{3.1.1}$$

$$l_{10,20}(x) \equiv l_{1,2}(x, \rho_0, z_0). \tag{3.1.2}$$

The following integral is valid

$$\int T(x)\, T^0(x)\, \lambda\left(\frac{l_1(x)\, l_{10}(x)}{l_2(x)\, l_{20}(x)}, \phi - \phi_0\right) \mathrm{d}x = -\frac{1}{2R_1} \tan^{-1} \frac{\eta_1(x)}{R_1} - \frac{1}{2R_2} \tan^{-1} \frac{\eta_2}{R_2}. \tag{3.1.3}$$

Here

$$T(x) = \frac{1}{\sqrt{\rho^2 - l_1^2(x)}} \frac{\partial l_1(x)}{\partial x} = \frac{\sqrt{l_2^2(x) - x^2}}{l_2^2(x) - l_1^2(x)}, \tag{3.1.4}$$

$$T^0(x) = \frac{1}{(\rho_0^2 - (l_{10}^2(x))^{1/2}} \frac{\partial l_{10}}{\partial x} = \frac{\sqrt{l_{20}^2(x) - x^2}}{l_{20}^2(x) - l_{10}^2(x)}, \tag{3.1.5}$$

$$\lambda(k, \phi - \phi_0) = \frac{1 - k^2}{1 + k^2 - 2k \cos(\phi - \phi_0)}, \tag{3.1.6}$$

$$R_{1,2}^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + (z \mp z_0)^2, \tag{3.1.7}$$

$$\eta_{1,2}(x) = S(x) \pm z z_0 / S(x), \tag{3.1.8}$$

$$S(x) = (l_2^2(x) - x^2)^{1/2} (l_{20}^2(x) - x^2)^{1/2} / x. \tag{3.1.9}$$

Formula (3) can be verified using the following relationships

$$\lambda\left(\frac{l_1(x) l_{10}(x)}{l_2(x) l_{20}(x)}, \phi - \phi_0\right)$$

$$= \frac{1}{2x^2}\left[l_2^2(x)\, l_{20}^2(x) - l_1^2(x)\, l_{10}^2(x) \right]\left[\frac{1}{R_1^2 + \eta_1^2(x)} + \frac{1}{R_2^2 + \eta_2^2(x)} \right], \tag{3.1.10}$$

$$\frac{\partial}{\partial x} S(x) = -\frac{S(x)[l_2^2(x)\, l_{20}^2(x) - l_1^2(x) l_{10}^2(x)]}{x[l_2^2(x) - l_1^2(x)][(l_{20}^2(x) - (l_{10}^2(x)]}, \tag{3.1.11}$$

$$l_2(x) l_2(x) = x\rho, \qquad l_1^2(x) + l_2^2(x) = x^2 + \rho^2 + z^2. \tag{3.1.12}$$

Several particular cases of (3) will also be used in this section. For example, setting $z = 0$ in (3) one gets

$$\int \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}} T^0(x)\, \lambda\left(\frac{l_{10}^2(x)}{\rho\rho_0}, \phi-\phi_0\right) = -\frac{1}{R}\tan^{-1}\frac{\sqrt{\rho^2-x^2}\,[l_{20}^2(x)-x^2]^{1/2}}{xR}. \tag{3.1.13}$$

where

$$R^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_0^2.$$

Another variation of (13) can be obtained by a substitution $y = l_1(x)$. The result is

$$\int \frac{\mathrm{d}y}{(\rho^2+g^2(y))^{1/2}(\rho_0^2-y^2)^{1/2}}\, \lambda\left(\frac{y^2}{\rho\rho_0}, \phi-\phi_0\right) = -\frac{1}{R}\tan^{-1}\frac{(\rho^2+g^2(y))^{1/2}(\rho_0^2-y^2)^{1/2}}{yR}. \tag{3.1.14}$$

Here

$$g(y) = y\sqrt{1+z_0^2/(\rho_0^2-y^2)}. \tag{3.1.15}$$

Formulation of the problem. We consider a system of n charged circular coaxial disks. We place a set of the cylindrical coordinate axes so that the axis Oz passes through the centers of the disks. Let a_i be the radius of the i-th disk, z_i be the z-coordinate of its center. The problem is to find the electrostatic field potential of the system of charged disks, i.e. to find a harmonic function $V(\rho,\phi,z)$ subject to the following boundary conditions:

$$V(\rho,\phi,z_k) = v_k(\rho,\phi) \quad \text{for } 0 \le \rho \le a_k, \quad 0 \le \phi < 2\pi, \quad k = 1,2,\ldots,n. \tag{3.1.16}$$

The potential can be represented through the simple layer distribution as follows:

$$V(\rho,\phi,z) = \sum_{i=1}^{n} \int_0^{2\pi}\int_0^{a_i} \frac{q_i(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z-z_i)^2}}. \tag{3.1.17}$$

Here q_i are yet unknown charge densities which can be found from a system of integral equations obtained by substitution of the boundary conditions (16) in (17). Now some transformations of (17) are necessary. Introduce the notations

$$l_{ik,1}(x,\rho_0,h_{ik}) \equiv l_{ik,1}(x) = l_1(x,\rho_0,h_{ik}), \tag{3.1.18}$$

$$h_{ik} = |z_i - z_k|. \tag{3.1.19}$$

Expressions (13) and (14) allows the following integral representations for the reciprocal distance

$$\frac{1}{R_{ik}}=\frac{2}{\pi}\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}T_{ik}(x,\rho_0)\lambda\left(\frac{l_{ik,1}^2(x)}{\rho\rho_0},\phi-\phi_0\right), \tag{3.1.20}$$

$$\frac{1}{R_{ik}}=\frac{2}{\pi}\int_0^{l_{ik,1}(\rho)}\frac{\mathrm{d}x}{(\rho^2-g_{ik}^2(x))^{1/2}\sqrt{\rho_0^2-x^2}}\,\lambda(\frac{x^2}{\rho\rho_0},\,\phi-\phi_0), \tag{3.1.21}$$

where $g_{ik}(x)$ according to (15) can be defined as

$$g_{ik}(x)=x\sqrt{1+h_{ik}^2/(\rho_0^2-x^2)}, \tag{3.1.22}$$

and

$$R_{ik}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+h_{ik}^2, \tag{3.1.23}$$

$$T_{ik}(x,\rho_0)=\frac{1}{\sqrt{\rho_0^2-l_{ik,1}^2(x)}}\frac{\partial}{\partial x}l_{ik,1}(x). \tag{3.1.24}$$

Now we can single out the k-th disk, without loss of generality, and do all the transformations of the integral equation related to the boundary conditions at the surface of the k-th disk, having in mind that the integral equations related to the other disks can be transformed in a similar manner. Substituting the boundary condition (16) in (17) and using (20) and (21), one obtains

$$4\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^{a_k}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)q_k(\rho_0,\phi)$$
$$+\sum_{i=1,i\neq k}^{n}4\int_0^{a_i}\rho_0\mathrm{d}\rho_0\left\{\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}T_{ik}(x,\rho_0)\mathcal{L}\left(\frac{l_{ik,1}^2(x)}{\rho\rho_0}\right)\right\}q_i(\rho_0,\phi)=v_k(\rho,\phi). \tag{3.1.25}$$

Equation (25) can be transformed to the Fredholm type by the following procedure. Apply the operator

$$\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{r}\frac{\rho\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}(\rho)$$

to both sides of (25). The result is

$$2\pi\int_r^{a_k} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - r^2}} \mathcal{L}\left(\frac{r}{\rho_0}\right) q_k(\rho_0,\phi) + 2\pi \sum_{i=1, i\neq k}^{n} \int_0^{a_i} T_{ik}(r,\rho_0)\, \mathcal{L}\left(\frac{l_{ik,1}^2(r)}{r\rho_0}\right) q_i(\rho_0,\phi)\, \rho_0\, d\rho_0$$

$$= \mathcal{L}\left(\frac{1}{r}\right) \frac{d}{dr} \int_0^r \frac{\rho\, d\rho}{\sqrt{r^2 - \rho^2}} \mathcal{L}(\rho) v_k(\rho,\phi). \tag{3.1.26}$$

The next operator to apply is

$$\frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_t^{a_k} \frac{r dr}{(r^2 - t^2)^{1/2}} \mathcal{L}(1/r)$$

which yields

$$-\pi^2 q_k(t,\phi) + 2\pi \sum_{i=1, i\neq k}^{n} \int_0^{a_i} \rho_0\, d\rho_0$$

$$\times \left[\frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_t^{a_k} \frac{r dr}{(r^2 - t^2)^{1/2}} T_{ik}(r,\rho_0)\, \mathcal{L}\left(\frac{\rho_0}{l_{ik,2}^2(r)}\right)\right] q_i(\rho_0,\phi) = M_k v_k(t,\phi), \tag{3.1.27}$$

where

$$l_{ik,2}(r) = l_2(r,\rho_0,h_{ik}), \tag{3.1.28}$$

and the M-operator is understood as

$$M_k v_k(t,\phi) = \frac{\mathcal{L}(t)}{t}\frac{d}{dt}\int_r^{a_k} \frac{r dr}{(r^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{r^2}\right)\frac{d}{dt}\int_0^r \frac{\rho}{(r^2 - \rho^2)^{1/2}} \mathcal{L}(\rho) v_k(\rho,\phi). \tag{3.1.29}$$

Changing the order of integration in (27) and integrating with respect to r, one gets the following system of Fredholm integral equations

$$q_k(\rho,\phi) = -\frac{1}{\pi^2} \sum_{i=1, i\neq k}^{n} \int_0^{2\pi}\int_0^{a_i} K_{ik}(\rho,\rho_0,\phi,\phi_0) q_i(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 -$$

$$-\frac{1}{\pi^2}(M_k v_k)(\rho,\phi), \quad \text{for} \quad k = 1,2,...,n. \tag{3.1.30}$$

Here, the kernel K_{ik} can be expressed in terms of elementary functions:

$$K_{ik}(\rho,\rho_0,\phi,\phi_0) = \frac{h_{ik}}{R_{ik}^3}\left[\frac{R_{ik}}{\xi_{ik}(a_k,\rho)} + \tan^{-1}\frac{\xi_{ik}(a_k,\rho)}{R_{ik}}\right], \tag{3.1.31}$$

$$\xi_{ik}(a_k,\rho) = \frac{(a_k^2-\rho^2)^{1/2}[a_k^2 - l_{ik,1}^2(a_k)]^{1/2}}{a_k}. \tag{3.1.32}$$

R_{ik} is defined by (23), and h_{ik} by (19). Notice that in the case v_k = const, $M_k v_k(\rho) = v_k(a_k^2 - \rho^2)^{-1/2}$. It is advisable to multiply both sides of (30) by $(a_k^2 - \rho^2)^{1/2}$ in order to eliminate a weak singularity at $\rho \to a_k$.

The system (30) has a unique solution which can be obtained by successive approximations. To prove this, introduce a new function $\sigma_k = (a_k^2 - \rho^2)^{1/2} q_k$. According to the Banach's theorem, it is sufficient to prove that the integral operator

$$W(\sigma_i) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2-\rho^2)^{1/2}K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{\sigma_i(\rho_0,\phi_0)}{(a_i^2-\rho_0^2)^{1/2}}\rho_0 d\rho_0 d\phi_0$$

is a contraction operator. Determine the distance in the class of continuous functions by

$$\delta(\sigma_i,\tilde{\sigma}_i) = \max|\sigma_i(\rho_0,\phi_0) - \tilde{\sigma}_i(\rho_0,\phi_0)|.$$

Assess the value of

$$|W(\sigma_i) - W(\tilde{\sigma}_i)|$$

$$\le \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2-\rho^2)^{1/2}K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{|\sigma_i(\rho_0,\phi_0)-\tilde{\sigma}_i(\rho_0,\phi_0)|}{(a_i^2-\rho_0^2)^{1/2}}\rho_0 d\rho_0 d\phi_0$$

$$\le \frac{\delta(\sigma_i,\tilde{\sigma}_i)}{\pi^2}\int_0^{2\pi}\int_0^{a_i}(a_k^2-\rho^2)^{1/2}K_{ik}(\rho,\rho_0,\phi,\phi_0)\frac{\rho_0 d\rho_0 d\phi_0}{(a_i^2-\rho_0^2)^{1/2}} \le \varepsilon\delta(\sigma_i,\tilde{\sigma}_i), \tag{3.1.33}$$

where

$$\varepsilon = \frac{1}{\pi}\left[\tan^{-1}\frac{a_i + a_k}{h_{ik}} + \tan^{-1}\frac{a_i}{h_{ik}}\right] < 1.$$

The last expression proves that W is a contraction operator thus proving the theorem. The following integral representation was used for the evaluation of the integral in (33)

$$T_{ik}(r,\rho_0) = \frac{h_{ik}}{\pi}\int_0^{\rho_0} \frac{d\tau}{(\rho_0^2 - \tau^2)^{1/2}}\left[\frac{1}{(\tau + r)^2 + h_{ik}^2} + \frac{1}{(\tau - r)^2 + h_{ik}^2}\right]. \tag{3.1.34}$$

After the system (30) is solved, the electrostatic field potential can be defined by

$$V(\rho,\phi,z) = 4\sum_{k=1}^{n}\int_0^{b_k} \frac{dx}{\sqrt{\rho^2 - x^2}}\int_{g_k(x)}^{a_k} \frac{\rho_0 d\rho_0}{(\rho_0^2 - g_k^2(x))^{1/2}}\, \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) q_k(\rho_0,\phi), \tag{3.1.35}$$

where

$$b_k = l_1(a_k, \rho, z - z_k), \tag{3.1.36}$$

$$g_k(x) = x\sqrt{1 + (z - z_k)^2/(\rho^2 - x^2)}. \tag{3.1.37}$$

The main advantage of (35) over the equivalent expression (17) is its convenience for various mathematical transformations, including the exact evaluation of the integrals as it will be shown further.

The number of unknown densities in (35) can be reduced by substitution of q_k defined by (27) in (35). Using (3), one gets, after simplification

$$V(\rho,\phi,z) = \sum_{i=1, i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i} G_{ik}(\rho,z,\rho_0,\phi - \phi_0) q_i(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0 + P_k v_k(\rho,\phi), \tag{3.1.38}$$

where G_{ik} can be expressed in elementary functions as follows,

$$G_{ik}(\rho,z,\rho_0,\phi-\phi_0)=\frac{1}{2R_{i,1}}\left[1+\frac{2}{\pi}\tan^{-1}\frac{\eta_{ik,1}}{R_{i,1}}\right]-\frac{1}{2R_{i,2}}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\eta_{ik,2}}{R_{i,2}}\right], \tag{3.1.39}$$

$$R_{i,1}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z-z_i)^2, \tag{3.1.40}$$

$$R_{i,2}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z-z_k+z_i-z_k)^2, \tag{3.1.41}$$

$$\eta_{ik,1}=S_{ik}(a_k)+(z-z_k)(z_i-z_k)/S_{ik}(a_k), \tag{3.1.42}$$

$$\eta_{ik,2}=S_{ik}(a_k)-(z-z_k)(z_i-z_k)/S_{ik}(a_k), \tag{3.1.43}$$

$$S_{ik}(a_k)=\frac{\sqrt{l_2^2(a_k,\rho,z-z_k)-a_k^2}\sqrt{l_2^2(a_k,\rho_0,z_i-z_k)-a_k^2}}{a_k}, \tag{3.1.44}$$

and the operator P_k reads

$$P_k v_k(\rho,\phi)=\frac{2}{\pi}\int_0^{a_k} T_k(x,\rho)\,\mathcal{L}\left(\frac{\rho}{l_{k,2}^2(x)}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}(\rho_0)\,v_k(\rho_0,\phi). \tag{3.1.45}$$

Here,

$$T_k(x,\rho)=\frac{1}{(\rho^2-l_1^2(x,\rho,z-z_k))^{1/2}}\frac{\partial}{\partial x}l_1(x,\rho,z-z_k), \tag{3.1.46}$$

$$l_{k,2}(x)=l_2(x,\rho,z-z_k). \tag{3.1.47}$$

In the case $v_k=\text{constant}$

$$P_k v_k=\frac{2}{\pi}v_k\sin^{-1}\frac{a_k}{l_{k,2}(a_k)}. \tag{3.1.48}$$

The solution of some practical problems very often do not require the knowledge of the charge distributions q_k, and only the integral characteristics are of interest, like the total charge, the moments, etc. An assessment of these values can be made without solving the set of equations (30). Indeed, multiplying both sides of (26) by r^m and integrating over the surface of the k-th disk, yields

$$\frac{1}{2}(\pi)^{1/2}\frac{\Gamma[\frac{1}{2}(m+1)]}{\Gamma(1+\frac{1}{2}m)}\int_0^{2\pi}\int_0^{a_k} q_k(\rho_0,\phi_0)\,\rho_0^{m+1}\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0$$

$$+\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i}\left\{\int_0^{a_k} T_{ik}(r,\rho_0)\,r^m\mathrm{d}r\right\}q_i(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{a_k} r^m\,\mathrm{d}r\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{r}\frac{v_k(\rho,\phi)\rho\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}. \tag{3.1.49}$$

It is easy to show that all the intermediary integrals with respect to r in (49) can be evaluated in terms of elementary functions. Expression (49) simplifies for $m = 0$ as follows:

$$Q_k+\frac{2}{\pi}\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i} q_i(\rho,\phi)\sin^{-1}\frac{a_k}{l_2(a_k,\rho,h_{ik})}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = B_k, \tag{3.1.50}$$

where Q_k is the total charge at the k-th disk, and

$$B_k=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_k}\frac{v_k(\rho,\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_k^2-\rho^2}}. \tag{3.1.51}$$

For the case $m = 1$, expression (49) reduces to

$$\int_0^{2\pi}\int_0^{a_k} q_k(\rho,\phi)\rho^2\mathrm{d}\rho\mathrm{d}\phi+\sum_{i=1,i\neq k}^{n}\int_0^{2\pi}\int_0^{a_i}\Bigg[\sqrt{\rho^2+h_{ik}^2}\;-\sqrt{c_{ik}^2-a_k^2}$$

$$+h_{ik}\ln\frac{c_{ik}(\sqrt{c_i^2-a_k^2}+h_{ik}}{\sqrt{c_{ik}^2-a_k^2}(\sqrt{\rho^2+h_{ik}^2}+h_{ik})}\Bigg]q_i(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a_k}\left[\frac{a_k}{\sqrt{a_k^2-\rho^2}}-\cosh^{-1}\frac{a_k}{\rho}\right]v_k(\rho,\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi, \tag{3.1.52}$$

where

$$c_{ik} = l_2(a_k, \rho, h_{ik}).$$

Evoking the mean value theorem which is valid when q_i does not change sign, (50) can be rewritten as:

$$Q_k + \frac{2}{\pi}\sum_{i=1, i\neq k}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k, \rho_{ik}, h_{ik})} = B_k \qquad \text{for } k = 1, 2, ..., n. \tag{3.1.53}$$

where, according to theorem, $0 \le \rho_{ik} \le a_i$. Despite the fact that the exact value of ρ_{ik} is unknown, the set of linear algebraic equations (53) is very useful for various purposes. For example, one can assess the values of Q_i in the form $Q_{i,\min} \le Q_i \le Q_{i,\max}$ by variation of all the admissible values of ρ_{ik}. In the case of constant potentials v_k, $k = 1,2,...,n$, the set of equations (53) takes the form

$$\frac{\pi}{2a_k}Q_k + \frac{1}{a_k}\sum_{i=1, i\neq k}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k, \rho_{ik}, h_{ik})} = v_k \qquad \text{for } k = 1, 2, ..., n. \tag{3.1.54}$$

Remembering that the matrix of the capacitances c_{ij} is symmetrical, one can introduce several ways to make the estimation of the values of interest more sharp. For example, let Q_i' be the set of solutions of (54) for $v_{\mathrm{j}} = 1$, and $v_k = 0$, $k = 1,....,n$ $(k \neq j)$. Let Q_i'' $(i = 1,...,n)$ be the set of solutions of (54) for $v_l = 1$ and all the other $v_k = 0$. Now it is easy to deduce that the following equality should hold

$$c_{lj} = Q_l' = Q_{\mathrm{j}}'' = c_{jl}. \tag{3.1.55}$$

As both quantities are assessed in the form

$$Q_{l,\max}' > Q_l' > Q_{l,\min}', \qquad Q_{j,\max}'' > Q_j'' > Q_{j,\min}'', \tag{3.1.56}$$

then the final assessment of c_{lj} will be obtained as an intersection of the intervals (56), which might give a sharper estimation. Here is an illustrative example. Consider a set of two disks with $a_1 = 2$, $a_2 = 1$, $h = 0.6$, $v_1 = v_2 = 1$. The direct estimation of the total charges gives $1.60828 < Q_1 < 2.54544$, $-0.66970 < Q_2 < 0.49519$ while the usage of the described technique allows a much sharper estimation, namely $1.67554 < Q_1 < 1.7801$, $0.27799 < Q_2 < 0.39838$. It is obvious that in the case of equal disks, both intervals in (56) are the same, and their intersection will not improve the estimation. One can expect that in some cases the sharpness of the estimation of certain values for equal disks will be inferior to those of unequal ones.

The set of equations (50) can also be used to verify the accuracy of an approximate solution or a numerical procedure. For example, numerical results for the case of two equal coaxial disks are presented in (Cooke, 1958). Let us check just one point, corresponding to the value $h_{ik}/a_k = 20$. The values, proportional to the total charge Q, given in (Cooke, 1958) for the case of equal potentials and for opposite potentials are $A_0^{(1)} = \frac{1}{2}\pi Q = 0.9683$ and $A_0^{(2)} = 1.0319$ respectively. The estimation of these values by solving (54) is $0.969213 > A_0^{(1)} > 0.969176$, $1.032849 > A_0^{(2)} > 1.032807$ which means that both values given in (Cooke, 1958) are outside the admissible range, and therefore are inaccurate. The correct values are $A_0^{(1)} = 0.969201$ and $A_0^{(2)} = 1.032821$. The numerical procedure to obtain these and other results is discussed further. One can notice also that since $h_{kk} = 0$ and $l_2(a_k,\rho_{kk},0) = a_k$, then the set (53) can be rewritten as:

$$\frac{2}{\pi}\sum_{i=1}^{n} Q_i \sin^{-1}\frac{a_k}{l_2(a_k,\rho_{ik},h_{ik})} = B_k, \qquad \text{for } k = 1, 2, ..., n. \tag{3.1.57}$$

Numerical solution. One needs accurate numerical results in order to estimate the error of various approximate formulae. The collocation method, used here, proved to give an adequate accuracy in a wide range of distances between disks. A case of coaxial disks maintained at constant potential was considered. The set of equations (26) for $v_k = \text{const.}$ takes the form:

$$2\pi\int_r^{a_k}\frac{q_k(\rho_0)\rho_0 d\rho_0}{\sqrt{\rho_0^2 - r^2}} + 2\pi\sum_{i=1,i\neq k}^{n}\int_0^{a_i} T_{ik}(r,\rho_0)q_i(\rho_0)\rho_0 d\rho_0 = v_k. \tag{3.1.58}$$

Let the solution of (58) be presented in the form:

$$q_k(\rho) = (a_k^2 - \rho^2)^{-1/2}\sum_{m=0}^{N} C_{km}\rho^{2m}, \tag{3.1.59}$$

where C_{km} are yet unknown constants. Substituting (59) in (58) and using (34), one gets:

$$\sum_{m=0}^{N} C_{km} f_{km}(r) + \sum_{i=1, i\neq k}^{n} \frac{h_{ik}}{\pi} \int_0^{a_i} \left[\frac{1}{(t+r)^2 + h_{ik}^2} + \frac{1}{(t-r)^2 + h_{ik}^2} \right] \sum_{m=0}^{N} C_{im} f_{im}(t)\, \mathrm{d}t = v_k, \tag{3.1.60}$$

where

$$f_{km}(r) = \int_r^{a_k} \frac{\rho_0^{2m+1}\, \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - r^2}\sqrt{a_k^2 - \rho_0^2}} = \tfrac{1}{2}\pi a_k^{2m} F(-m, \tfrac{1}{2}; 1; 1 - r^2/a_k^2). \tag{3.1.61}$$

is a polynomial in r^2 and F is the Gauss hypergeometric function. All the integrals in (60) can be evaluated exactly and expressed in elementary functions, using the following formulae:

$$\int_0^a \left[\frac{1}{(t+r)^2 + h^2} + \frac{1}{(t-r)^2 + h^2} \right] t^{2n}\, \mathrm{d}t$$

$$= -nr^{2n-1} F\left(1-n, \tfrac{1}{2}-n; \tfrac{2}{3}; -\frac{h^2}{r^2}\right) \ln\frac{(a+r)^2 + h^2}{(a-r)^2 + h^2}$$

$$+ \frac{r^{2n}}{h} F\left(-n, \tfrac{1}{2}-n; \tfrac{1}{2}; -\frac{h^2}{r^2}\right)\left(\tan^{-1}\frac{a+r}{h} + \tan^{-1}\frac{a-r}{h}\right)$$

$$+ \sum_{m=0}^{n-1} \frac{2a^{2n-2m-1} r^{2m}}{2n-2m-1} (2m+1) F\left(-m, \tfrac{1}{2}-m; \tfrac{3}{2}; -\frac{h^2}{r^2}\right), \tag{3.1.62}$$

$$\int_0^a \left[\frac{1}{(t+r)^2 + h^2} + \frac{1}{(t-r)^2 + h^2} \right] t^{2n+1}\, \mathrm{d}t$$

$$= r^{2n}\left\{ (2n+1) F\left(-n, \tfrac{1}{2}-n; \tfrac{3}{2}; -\frac{h^2}{r^2}\right) \ln\frac{l_2^2 - l_1^2}{r^2 + h^2} \right.$$

$$\left. + \frac{r^{2n+1}}{h} F\left(-n, \tfrac{1}{2}-n; \tfrac{1}{2}; -\frac{h^2}{r^2}\right)\left(2\tan^{-1}\frac{r}{h} + \tan^{-1}\frac{a-r}{h} - \tan^{-1}\frac{a+r}{h}\right)\right\}$$

$$+\sum_{m=0}^{n-1} \frac{2m+1}{n-m} a^{2n-2m} r^{2m} F\left(-m, \tfrac{1}{2}-m; \tfrac{3}{2}; -\frac{h^2}{r^2}\right), \tag{3.1.63}$$

where l_1 and l_2 are understood as $l_{1,2}(a,r,h)$ defined by (1).

Now one has to specify $N+1$ points of collocation at each of the intervals $0 \le r \le a_k$, $k = 1,2,...,n$, and request that the set of equations (60) be satisfied at these points. This leads to the set of $n(N+1)$ linear algebraic equations:

$$\sum_{m=0}^{N} C_{km} f_{km}(r_j) + \sum_{i=1, i\neq k}^{n} \frac{h_{ik}}{\pi} \sum_{m=0}^{N} C_{im} \int_0^{a_i} \left[\frac{1}{(t+r_j)^2 + h_{ik}^2} + \frac{1}{(t-r_j)^2 + h_{ik}^2} \right] f_{im}(t)\, \mathrm{d}t = v_k,$$

$$\text{for } j = 0,1,...,N \quad \text{and} \quad k = 1,2,...,n. \tag{3.1.64}$$

from which the constants C_{km} are to be defined. After the system (64) is solved, all the other parameters of interest can be obtained rather easily, for example, the total charge at each disk can be defined by

$$Q_k = \sum_{m=0}^{N} \frac{1}{2} (\pi)^{1/2} \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} a_k^{2m+1} C_{km}. \tag{3.1.65}$$

The potential value at an arbitrary point in space can also be expressed in terms of elementary functions. Indeed, substitution of (59) in (35) gives, after the first integration,

$$V(\rho,\phi,z) = 4 \sum_{k=1}^{n} \int_0^{b_k} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \sum_{m=0}^{N} C_{km} f_{km}(g_k(x)). \tag{3.1.66}$$

Since f_{km} here is a polynomial, according to (61), then the remaining integrals can always be evaluated exactly in terms of elementary functions. The accuracy of the solution was assessed by the error function $E_k(r)$ defined by:

$$E_k(r) = \sum_{m=0}^{N} C_{km} f_{km}(r)$$

$$+\sum_{i=1,i\neq k}^{n}\frac{h_{ik}}{\pi}\sum_{m=0}^{N}C_{im}\int_{0}^{a_i}\left[\frac{1}{(t+r)^2+h_{ik}^2}+\frac{1}{(t-r)^2+h_{ik}^2}\right]f_{im}(t)\,\mathrm{d}t-v_k$$

$$\text{for}\quad 0\le r\le a_k,\qquad k=1,2,...,n. \tag{3.1.67}$$

It is obvious that $E_k = 0$ indicates that the solution is exact. The value of

$$\Delta = \max|E_k(r)| \tag{3.1.68}$$

was used as a measure of accuracy of the solution, which means that, out of two solutions, the one with the smallest Δ was considered more accurate. The typical behavior of the error function E_k is presented in Fig. 3.1.1 and Fig. 3.1.2 for the case of two equal coaxial disks of radius 1 held at unit opposite potentials and a gap between them $h = 0.1$. The solid line in both

Fig. 3.1.1. Error function for equidistant points of collocations.

figures plots the error function for the case of three points of collocations, the dashed line gives the same plot for five points of collocations, and the error function for 11 points of collocations is given by circles. Fig. 3.1.1 corresponds to the case of equidistant points while in Fig. 3.1.2 the points of collocations were taken at $r_j = a_k\sin(\pi j/2N)$, $j = 0,1,...,N$. Comparison of figures leads to a conclusion that the accuracy for the case of equidistant points of collocation is

Fig. 3.1.2. Error function for non-equidistant points of collocations.

inferior to the second choice. Our investigation also showed that further increase in the number of points of collocation generally does not improve the accuracy of the solution, and in many cases the accuracy deteriorates. The error of evaluation of the total charge Q_k was taken as a product $Q_k\Delta$. The real error is unknown but it will definitely be less than $Q_k\Delta$ due to the fluctuation of the error function with the area under the positive half-wave being almost equal to the negative area.

The results of the numerical procedure with 11 points of collocation $r_j = \sin(\pi j/20)$, $j = 0, 1, ..., 10$, for the case of two equal coaxial disks of radius 1, held at equal and at opposite potentials with a variable gap h between them, are given in Table 3.1.1. The value of $Q^* = Q_k\pi/2v_k a_k$ along with the absolute error assessment is given in Table 3.1.1. Table 3.1.1 corrects some inaccuracies in similar results published in (Cooke, 1958).

Figure 3.1.3 plots the charge density distribution for different values of the gap h. Several examples are considered below.

Two equal coaxial disks. Denote $a_1 = a_2 = a$; $h_{12} = h_{21} = h$. Two fundamental cases are considered: $v_1 = v_2 = v$ and $v_1 = -v_2 = v$. The solution of the set of equations (54) has the form

Table 3.1.1. Values of the total charge.

h/a	Equal potentials		Opposite potentials	
	Q^*	Error	Q^*	Error
0	0.5	0	∞	0
0.001	0.500877	0.002	789.29	2
0.01	0.505320	0.0015	80.457	0.3
0.05	0.520553	0.0010	17.22936	0.03
0.1	0.535883	0.0002	9.233071	0.001
0.2	0.561362	0.00005	5.175753	0.0001
0.4	0.602499	10^{-7}	3.102305	10^{-5}
0.6	0.636407	10^{-8}	2.395441	10^{-6}
0.8	0.665610	10^{-9}	2.037267	10^{-7}
1.0	0.691207	10^{-10}	1.820785	10^{-8}
1.2	0.713812	10^{-11}	1.676043	10^{-9}
1.5	0.743019	10^{-11}	1.531444	10^{-10}
2.0	0.781752	10^{-11}	1.388027	10^{-11}
2.5	0.811259	10^{-11}	1.303422	10^{-11}
3.0	0.834216	10^{-12}	1.248107	10^{-11}
5.0	0.889579	10^{-13}	1.141723	10^{-12}
10.0	0.940518	10^{-15}	1.067514	10^{-13}
20.0	0.969201	10^{-15}	1.032821	10^{-15}
100.0	0.993674	10^{-16}	1.006406	10^{-15}
∞	1.	0	1.	0

Fig. 3.1.3. Charge density distribution (two disks at unit opposite potentials).

$$Q_1^{\pm} = \frac{2}{\pi} \frac{av}{1 \pm \frac{2}{\pi} \sin^{-1}[a/l_2(a,\rho^{\pm},h)]}. \tag{3.1.69}$$

The plus sign in (69) corresponds to the case of equal potentials, and the minus to opposite ones. Notice that formula (69) gives exact results for $h \to 0$ and $h \to \infty$. Considering Q_1^+ and Q_1^- as functions of ρ^+ and ρ^- respectively, one can analyze numerically the overall performance of (69). The simplest approximate formula can be obtained by averaging of the maximum and the minimum admissible values of $Q_1^{\pm}$, namely

$$Q_1^+ = \tfrac{1}{2}[Q_1^+(a) + Q_1^+(0)], \qquad Q_1^- = \tfrac{1}{2}[Q_1^-(a) + Q_1^-(0)]. \tag{3.1.70}$$

Comparison with results of Table 1 shows that the maximum error of the first formula (70) is about 3%, while the second one yields about 12%. Better accuracy can be obtained by assuming

$$Q_1^+ = \tfrac{1}{2}[Q_1^+(0.67a) + Q_1^+(0.98a)], \qquad Q_1^- = \tfrac{1}{2}[Q_1^-(0.562a) + Q_1^-(0.983a)]. \tag{3.1.71}$$

The maximum error is less than 0.1% for the first formula of (71), and is about 0.62% for the second one.

If the accuracy achieved is still not satisfactory, one has to analyze ρ^+ and ρ^- as functions of h. The physical meaning of $\rho^{\pm}$ can be explained as a substitution of a disk by an infinitely thin annulus of radius $\rho^{\pm}$, having the same total charge and an equivalent influence on the total charge of the second disk. Plotting of both curves $\rho^+ = \rho^+(h)$ and $\rho^- = \rho^-(h)$ can help also to verify the accuracy of a numerical procedure. Elementary logic suggests that both should be smooth curves merging as $h \to \infty$.

Using (69) and Table 1, the curves $\rho^+(h)$ and $\rho^-(h)$ were plotted in Fig. 3.1.4 by the dashed line and the solid line respectively. The limiting values were established as follows:

$$\rho^+(0) = a, \quad \rho^-(0) = \sqrt{3/4}\,a, \quad \rho^+(\infty) = \rho^-(\infty) = \sqrt{2/3}\,a. \tag{3.1.72}$$

Similar computations were made by using the data from (Cooke, 1958). The results for ρ^+ and ρ^- are presented in Fig. 3.1.4 by non-solid circles and solid circles respectively. Looking at Fig. 3.1.4, we note immediately that there are some troubles with the accuracy of the data given by Cooke. We note also that the results of Cooke for $h=5$ and $h=10$ are not incorrect, they just do not

Fig. 3.1.4. Test of the accuracy of numerical results

have sufficient number of decimal places in the data, and this indicates how sensitive the parameter $\rho^{\pm}$ is for large h. No reasonable values for $\rho^{\pm}$ can be obtained from Cooke's data for $h=20$. As was shown earlier, these data are beyond the admissible interval.

One can approximate the function $\rho^{-}(h)$ as

$$\rho^{-} = (-0.02347\,\mathrm{e}^{-0.036h} + 0.073\,\mathrm{e}^{-8.7h} + \sqrt{2/3})a. \tag{3.1.73}$$

Substitution of (73) in (69) makes it highly accurate in the whole range $0<h<\infty$, with the maximum error not exceeding 0.3%.

Three equal coaxial disks at constant potentials. We consider the case of equal disks because it is the least favorable case in terms of accuracy, as it was stated in previous section. Put $a_1=a_2=a_3=a$, $h_{12}=h_{23}=h$, $h_{13}=2h$. The set of equations to be solved is

$$\frac{\pi}{2a}Q_1 + Q_2\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{21},h)} + Q_3\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{31},2h)} = v_1,$$

$$Q_1\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{21},h)} + \frac{\pi}{2a}Q_2 + Q_3\frac{1}{a}\sin^{-1}\frac{a}{l_2(a,\rho_{23},h)} = v_2,$$

$$Q_1 \frac{1}{a} \sin^{-1} \frac{a}{l_2(a,\rho_{31},2h)} + Q_2 \frac{1}{a} \sin^{-1} \frac{a}{l_2(a,\rho_{23},h)} + \frac{\pi}{2a} Q_3 = v_3. \tag{3.1.74}$$

Solution of (74) for all the combinations of ρ_{21}, ρ_{31} and ρ_{23} equal 0 or a gives the upper and the lower bound for the values of total charge Q_1, Q_2 and Q_3. The problem of three equal, equally spaced disks was considered in (Kuz'min,1971). The following approximate solution was given there in the assumption that $h > a$.

$$\begin{aligned} Q_1 &= v_1(c_{11} + c_{12} + c_{13}) - v_2 c_{12} - v_3 c_{13}, \\ Q_2 &= -v_1 c_{12} + v_2(c_{22} + c_{21} + c_{23}) - v_3 c_{23}, \\ Q_3 &= -v_1 c_{31} - v_2 c_{32} + v_3(c_{33} + c_{31} + c_{32}), \end{aligned} \tag{3.1.75}$$

where

$$\begin{aligned} c_{11} &= c_{33} = 0.6366a(1 - 0.9549\varepsilon + 1.1145\varepsilon^2 - 0.6514\varepsilon^3 - 0.02069\varepsilon^4), \\ c_{22} &= 0.6366a(1 - 1.2732\varepsilon + 1.2159\varepsilon^2 - 0.5702\varepsilon^3 + 0.01874\varepsilon^4), \\ c_{12} &= c_{21} = c_{32} = c_{23} = 0.4053a\varepsilon(1 - 0.3183\varepsilon + 0.2452\varepsilon^2 - 0.2830\varepsilon^3), \\ c_{31} &= c_{13} = 0.2026a\varepsilon(1 - 1.2732\varepsilon + 0.7452\varepsilon^2 + 0.2786\varepsilon^3), \quad \varepsilon = a/h. \end{aligned} \tag{3.1.76}$$

It is of interest to compare the results of solution of (74) with those of (75) and (76). Necessary calculations were performed for two particular cases: $v_1 = v_2 = v_3 = v$ and $v_1 = -v_2 = v$, $v_3 = 0$. The results of evaluation of the dimensionless $Q_1^* = \pi Q_1/(2va)$ are presented in Fig. 3.1.5 and Fig. 3.1.6 respectively. The solid line in both figures gives the upper bound, the dashed line gives the lower bound, (computed from (74)) and the results of (75) are plotted by circles. We can see that formulae (75) and (76) give good accuracy for $h/a > 2$; the results sharply deviate from the admissible region for $h/a < 1.5$.

Discussion. It is of interest to establish a relationship between some of the results of this section and those previously reported in literature. Expression (39) corresponds to the Green's function for a conducting disk under the influence of a charged point, found by Hobson (1900). His expression in our notation takes the form

$$G_{ik} = \frac{1}{2R_{i,1}}\left[1 \pm \frac{2}{\pi}\tan^{-1}\frac{\sqrt{\eta_{ik,1}^2}}{R_{i,1}}\right] - \frac{1}{2R_{i,2}}\left[1 \mp \frac{2}{\pi}\tan^{-1}\frac{\sqrt{\eta_{ik,2}^2}}{R_{i,2}}\right],$$

where the ambiguous signs are assigned according to pretty complicated rules

Fig. 3.1.5. Total charge at the first disk (three-disk system $v_1 = v_2 = v_3 = v$)

Fig. 3.1.6. Total charge at the first disk (three-disk system $v_1 = -v_2 = v$, $v_3 = 0$)

depending on the position of the points. The geometrical form, in which Hobson presented his result, did not let him to notice the square root of a complete square which led to the ambiguous signs. Our expressions (39) is simpler and free of this mishap. Expression (31) corresponds to another source function, also found by Hobson in geometrical form.

Certain relationship can be established between the set of equations (26) and the Love type integral equations derived in (Kuz'min, 1971). Introduce a new unknown function

$$\chi_k(r,\phi) = 2\pi \int_r^{a_k} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - r^2}} \mathcal{L}\left(\frac{r}{\rho_0}\right) q_k(\rho_0,\phi), \quad k=1, 2,..., n. \tag{3.1.77}$$

Inversion of (77) gives

$$q_k(\rho_0,\phi) = -\frac{\mathcal{L}(\rho_0)}{\pi^2 \rho_0} \frac{\mathrm{d}}{\mathrm{d}\rho_0} \int_{\rho_0}^{a_k} \frac{x \mathrm{d}x}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{1}{x}\right) \chi_k(x,\phi). \tag{3.1.78}$$

Substitution of (77) and (78) in (26) yields, after the change of the order of integration

$$\chi_k(r,\phi) + \frac{2}{\pi} \sum_{i=1, i\neq k}^{n} \int_0^{a_i} \left\{ \mathcal{L}\left(\frac{1}{x}\right) \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x T_{ik}(r,\rho_0) \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{l_{ik,1}^2(r)}{r}\right) \right\} \chi_i(x,\phi)\, \mathrm{d}x$$

$$= \mathcal{L}\left(\frac{1}{r}\right) \frac{\mathrm{d}}{\mathrm{d}r} \int_0^r \frac{\rho \mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}} \mathcal{L}(\rho) v_k(\rho,\phi). \tag{3.1.79}$$

The internal integral with respect to ρ_0 in (79) can be evaluated and expressed in terms of elementary function for each particular harmonic. For example, in the case of axial symmetry one gets from (79)

$$\chi_{k0}(r) + \frac{1}{\pi} \sum_{i=1, i\neq k}^{n} h_{ik} \int_0^{a_i} \left[\frac{1}{(x+r)^2 + h_{ik}^2} + \frac{1}{(x-r)^2 + h_{ik}^2} \right] \chi_{i0}(x)\, \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}r} \int_0^r \frac{v_{k0}(\rho) \rho \mathrm{d}\rho}{(r^2 - \rho^2)^{1/2}}. \tag{3.1.80}$$

The set of equations (80) corresponds to the one derived in (Kuz'min, 1971). Here, χ_{k0} and v_{k0} are the zero harmonics of the functions χ_k and v_k respectively. In general, for the m-th harmonic the following integral equation can be obtained from (79)

$$\chi_{km}(r)+\frac{2}{\pi}\sum_{i=1,i\neq k}^{n} h_{ik}\int_0^{a_i} H_{ikm}(r,x)\chi_{im}(x)\,\mathrm{d}x=\frac{1}{r^m}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{v_{km}(\rho)\rho^{m+1}\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}},$$

for $k=0,1,...,n$. (3.1.81)

The kernel H_{ikm} can be expressed in elementary functions:

$$H_{ikm}(r,x)=\left(\frac{r}{x}\right)^m\left\{\frac{2x^2}{(l_2^2-l_1^2)^2}-\sum_{k=1}^{m-1}\frac{(-1)^k}{\Gamma(k)}\left(\frac{l_1}{r}\right)^{2k}\left[\frac{x^2-l_2^2}{l_2^2}\frac{\partial^{k-1}}{\partial\tau^{k-1}}((1-\tau)^{k-1}\mu(\tau))\right.\right.$$

$$\left.\left.+\frac{x^2(l_1^2+l_2^2-2r^2)}{l_2^2(l_2^2-l_1^2)}\frac{\partial^{k-1}}{\partial\tau^{k-1}}\left((1-\tau)^k\frac{\partial\mu(\tau)}{\partial\tau}\right)\right]\frac{1}{l_2^2-l_1^2}\right\}$$

for $m\geq 1$. (3.1.82)

where l_1 and l_2 are understood as $l_{1,2}(r,x,h_{ik})$, and

$$\mu(\tau)=\frac{1}{\sqrt{\tau}}\ln\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}},\qquad \tau=\frac{l_1^2(r,x,h_{ik})}{l_2^2(r,x,h_{ik})}. \tag{3.1.83}$$

Here are explicit expressions of the kernel for the first six harmonics

$$H_{ik,1}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}-\frac{1}{(r+x)^2+h_{ik}^2}\right]=\frac{2l_1l_2}{(l_2^2-l_1^2)^2},$$

$$H_{ik,2}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}+\frac{1}{(r+x)^2+h_{ik}^2}-\frac{1}{2rx}\ln\frac{(r+x)^2+h_{ik}^2}{(r-x)^2+h_{ik}^2}\right]$$

$$=\frac{l_2^2+l_1^2}{(l_2^2-l_1^2)^2}-\frac{1}{2l_1l_2}\ln\frac{l_2+l_1}{l_2-l_1},$$

$$H_{ik,3}(r,x)=\frac{1}{2}\left[\frac{1}{(r-x)^2+h_{ik}^2}-\frac{1}{(r+x)^2+h_{ik}^2}+\frac{3}{rx}\right.$$

$$\left.-\frac{3(r^2+x^2+h_{ik}^2)}{4r^2x^2}\ln\frac{(r+x)^2+h_{ik}^2}{(r-x)^2+h_{ik}^2}\right]$$

$$= \frac{2l_1 l_2}{(l_2^2 - l_1^2)^2} + \frac{3}{l_1 l_2} - \frac{3(l_2^2 + l_1^2)}{4l_1^2 l_2^2} \ln \frac{l_2 + l_1}{l_2 - l_1},$$

$$H_{ik,4}(r,x) = \frac{l_2^2 + l_1^2}{(l_2^2 - l_1^2)^2} + \frac{15(l_2^2 + l_1^2)}{8l_1^2 l_2^2} - \frac{3(5l_2^4 + 6l_1^2 l_2^2 + 5l_1^4)}{16 l_1^3 l_2^3} \ln \frac{l_2 + l_1}{l_2 - l_1},$$

$$H_{ik,5}(r,x) = \frac{105 l_2^8 - 40\, l_1^2 l_2^6 - 34 l_1^4 l_2^4 - 40 l_1^6 l_2^2 + 105 l_1^8}{48 l_1^3 l_2^3 (l_2^2 - l_1^2)^2}$$

$$- \frac{5(l_2^2 + l_1^2)(7l_2^4 + 2l_1^2 l_2^2 + 7l_1^4)}{32 l_1^4 l_2^4} \ln \frac{l_2 + l_1}{l_2 - l_1},$$

$$H_{ik,6}(r,x) = \frac{(l_2^2 + l_1^2)(315 l_2^8 - 420 l_1^2 l_2^6 + 338 l_1^4 l_2^4 - 420 l_1^6 l_2^2 + 315 l_1^8)}{128 l_1^4 l_2^4 (l_2^2 - l_1^2)^2}$$

$$- \frac{15(21 l_2^8 + 28 l_1^2 l_2^6 + 30 l_1^4 l_2^4 + 28 l_1^6 l_2^2 + 21 l_1^8)}{256 l_1^5 l_2^5} \ln \frac{l_2 + l_1}{l_2 - l_1}, \tag{3.1.84}$$

We recall once again that in this section, unlike elsewhere in the book, the abbreviations l_1 and l_2 denote $l_1(r,x,h_{ik})$ and $l_2(r,x,h_{ik})$ respectively. The kernels defined by (82) contain no singularities therefore the regular methods of solution of Fredholm integral equations are applicable here. Formulae (81)-(84) seem not to have been reported in literature before.

3.2. Potential of arbitrarily located disks

The electrostatic field of several non-parallel circular disks is considered. A set of governing integral equations is derived by a new method. It is shown that some integral characteristics can be found without solving the integral equations. The upper and the lower bounds for the total charge are found from a set of linear algebraic equations whose coefficients are defined by simple geometric characteristics of the system. Example considered shows sufficient sharpness of the estimations.

There are just a few papers where the problem of two *coplanar* disks is considered; among them we know of only one (Kobayashi 1939) where some numerical results of sufficient accuracy are given. A solution to the problem of two *non-parallel* disks, whose centers are located in one plane orthogonal to the planes of both disks, can be found in Ufliand (1977). It was solved by the Mehler-Fok transform with consequent use of the small parameter method. To the best of our knowledge there are no publications considering the electrostatic problem of two or more *arbitrarily located* disks, mainly due to the fact that existing methods are not capable of solving these problems.

We consider a system of n charged arbitrarily located circular disks. Let a_i be the radius of the i-th disk, and S_i be its surface. We can single out, without loss of generality, disk number one and place the origin of the set of cylindrical coordinates (ρ, ϕ, z) at its center so that the Oz axis is orthogonal to the disk's plane. Let the position vector $\mathbf{r}_i$ indicate the center of the i-th disk, and the unit vector $\mathbf{n}_i$, orthogonal to the disk's plane, indicate its orientation. The problem is to find the electrostatic potential due to the system of charged disks, i.e. to find a harmonic function $V(\rho,\phi,z)$, satisfying the following boundary conditions:

$$V(\rho,\phi,z) = v_i(\rho,\phi,z) \quad \text{for} \quad (\rho,\phi,z) \subset S_i; \quad i = 1,2,...,n. \tag{3.2.1}$$

The potential can be represented by a simple layer distribution as follows

$$V = \sum_{i=1}^{n} \int\int_{S_i} \frac{q_i}{R_i} \mathrm{d}S_i. \tag{3.2.2}$$

Here q_i are the as yet unknown charge densities, and R_i stands for the distance between a point of integration inside S_i and an arbitrary point in space.

Now make use of the following integral representation for the reciprocal distance (see (1.2.19))

$$\frac{1}{\sqrt{\rho^2 + \rho_i^2 - 2\rho\rho_i \cos(\phi - \phi_i) + z_i^2}} = \frac{2}{\pi} \int_0^{c_i(\rho)} \frac{\lambda\left(\frac{x^2}{\rho\rho_i}, \phi - \phi_i\right) \mathrm{d}x}{\sqrt{\rho_i^2 - x^2}\sqrt{\rho^2 - g_i^2(x)}}, \tag{3.2.3}$$

where

$$g_i^2(x) = x^2\left[1 + \frac{z_i^2}{\rho_i^2 - x^2}\right], \tag{3.2.4}$$

$$\lambda(k,\psi) = \frac{1 - k^2}{1 + k^2 - 2k\cos\psi}, \tag{3.2.5}$$

and

$$c_i(\rho) = \frac{1}{2}\{[(\rho + \rho_i)^2 + z_i^2]^{1/2} - [(\rho - \rho_i)^2 + z_i^2]^{1/2}\}. \tag{3.2.6}$$

An obvious simplification of (3) is valid when $z_i = 0$, namely

$$\frac{1}{\sqrt{\rho^2+\rho_i^2-2\rho\rho_i\cos(\phi-\phi_i)}}=\frac{2}{\pi}\int_0^{\min(\rho,\rho_i)}\frac{\lambda\left(\frac{x^2}{\rho\rho_i}\phi-\phi_i\right)\mathrm{d}x}{\sqrt{\rho_i^2-x^2}\sqrt{\rho^2-x^2}}, \tag{3.2.7}$$

Substituting the boundary conditions (1) for the first disk into (2) and using (3) and (7) yields the following integral equation

$$4\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^{a_1}\frac{r\,\mathrm{d}r}{\sqrt{r^2-x^2}}\,\mathcal{L}\left(\frac{x^2}{\rho r}\right)q_1(r,\phi)$$

$$+\frac{2}{\pi}\sum_{i=2}^{n}\int\int_{S_i}\left\{\int_0^{c_i(\rho)}\frac{\lambda\left(\frac{x^2}{\rho\rho_i}\phi-\phi_i\right)\mathrm{d}x}{\sqrt{\rho_i^2-x^2}\sqrt{\rho^2-g_i^2(x)}}\right\}q_i\,\mathrm{d}S_i=v_1(\rho,\phi). \tag{3.2.8}$$

Let us apply the operator

$$\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)$$

to both sides of (8). The result of application of the operator above is

$$2\pi\int_y^{a_1}\frac{r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\,\mathcal{L}\left(\frac{y}{r}\right)q_1(r,\phi)+\sum_{i=2}^{n}\int\int_{S_i}\frac{\lambda[c_i^2(y)/y\rho_i,\,\phi-\phi_i]}{\sqrt{\rho_i^2-c_i^2(y)}}\frac{\mathrm{d}c_i(y)}{\mathrm{d}y}\,q_i\,\mathrm{d}S_i$$

$$=\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)\,v_1(\rho,\phi). \tag{3.2.9}$$

Here the following rule for the interchange of the order of integration was used

$$\int_0^{y}\mathrm{d}\rho\int_0^{c_i(\rho)}\mathrm{d}x=\int_0^{c_i(y)}\mathrm{d}x\int_{g_i(x)}^{y}\mathrm{d}\rho. \tag{3.2.10}$$

One can easily notice that each function $g(x)$ is inverse to the relevant function $c(\rho)$. The next operator to apply is

$$\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y}\right)$$

with the result

$$q_1(t,\phi)-\frac{1}{\pi^2}\sum_{i=2}^{n}\int\!\!\int_{S_i}\left[\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\frac{\lambda[c_i^2(y)/(y^2\rho_i),\,\phi-\phi_i]}{\sqrt{\rho_i^2-c_i^2(y)}}\,\frac{\mathrm{d}c_i(y)}{\mathrm{d}y}\right]q_i\,\mathrm{d}S_i$$

$$=-\frac{1}{\pi^2}\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y^2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)v_1(\rho,\phi). \tag{3.2.11}$$

Integration with respect to y can be performed in (11) (see (1.3.32)) to give

$$q_1(t,\phi)=-\frac{1}{\pi^2}\sum_{i=2}^{n}\int\!\!\int_{S_i}K_{1i}(t,\phi,\rho_i,\phi_i,z_i)\,q_i\,\mathrm{d}S_i+\frac{1}{\pi^2}M_1v_1(t,\phi), \tag{3.2.12}$$

where the kernel can be expressed in elementary functions

$$K_{1i}(t,\phi,\rho_i,\phi_i,z_i)=\frac{|z_i|}{R_{1i}^3}\left[\frac{R_{1i}}{\xi_{1i}}+\tan^{-1}\frac{\xi_{1i}}{R_{1i}}\right],$$

$$\xi_{1i}=\sqrt{a_1^2-t^2}\ \ \sqrt{a_1^2-c_i^2(a_1)}/a_1,$$

$$R_{1i}=[t^2+\rho_i^2-2\,t\rho_i\cos(\phi-\phi_i)+z_i^2]^{1/2},$$

and

$$M_1(t,\phi)=\frac{\mathcal{L}(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{a_1}\frac{y\,\mathrm{d}y}{\sqrt{y^2-t^2}}\,\mathcal{L}\left(\frac{1}{y^2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\rho\,\mathrm{d}\rho}{\sqrt{y^2-\rho^2}}\,\mathcal{L}(\rho)v_1(\rho,\phi). \tag{3.2.13}$$

Similar equations can be derived for the other disks thus forming a set of integral equations to be solved. One has to remember that each such equation is valid in the local set of coordinates related to the particular disk. It is also important to notice that during the derivation we only used the assumption that S_1 was a circular disk, equation (12) would remain unchanged if S_i (i>1) were arbitrary surfaces. It is possible to prove (see section 3.1) that the set of equations (12) can be solved by successive iterations but the most interesting feature of these equations is the ability to obtain the estimation for some integral characteristics without solving the equations. For example, the estimation of the total charge can be made in the following manner. Multiplying both sides of (11) by $t\,\mathrm{d}t\,\mathrm{d}\phi$ and integrating over the surface of the first disk, one gets

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n}\int\!\!\int_{S_i} \sin^{-1}\!\left(\frac{c_i(a_1)}{\rho_i}\right) q_i\,\mathrm{d}S_i = \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}. \qquad (3.2.14)$$

Introducing a new quantity $b_i(\rho)$ as

$$b_i(\rho) = \frac{1}{2}\{[(\rho+\rho_i)^2 + z_i^2]^{1/2} + [(\rho-\rho_i)^2 + z_i^2]^{1/2}\} \qquad (3.2.15)$$

with an obvious property $c_i(\rho)\,b_i(\rho) = \rho\rho_i$, expression (14) can be rewritten in the form

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n}\int\!\!\int_{S_i} \sin^{-1}\!\left(\frac{a_1}{b_i(a_1)}\right) q_i\,\mathrm{d}S_i = \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}. \qquad (3.2.16)$$

Evoking the mean value theorem which is valid when q_i does not change sign, expression (14) can be evaluated as follows

$$Q_1 + \frac{2}{\pi}\sum_{i=2}^{n} Q_i \sin^{-1}\!\left(\frac{a_1}{b_{i1}}\right) = B_1, \qquad (3.2.17)$$

where Q_i stands for the total charge on the i-th disk, and

$$B_1 = \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{a_1} \frac{v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_1^2-\rho^2}}, \qquad (3.2.18)$$

$$b_{i1} = \frac{1}{2}\{[(a_1+\rho_{i1})^2 + z_{i1}^2]^{1/2} + [(a_1-\rho_{i1})^2 + z_{i1}^2]^{1/2}\}. \qquad (3.2.19)$$

The physical meaning of b_{i1} is quite obvious: it represents a half of the sum of distances from a point inside S_i to the closest and the farthermost points of the first disk's edge.

Equation similar to (17) can be derived for the other disks, and the following set of linear algebraic equations with respect to the total charges Q_i can be written

$$Q_k + \frac{2}{\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} Q_i \sin^{-1}(a_k/b_{ik}) = B_k, \quad \text{for} \quad k = 1, 2, 3, ..., n; \tag{3.2.20}$$

where

$$B_k = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a_k} \frac{v_k(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a_k^2 - \rho^2}}, \tag{3.2.21}$$

$$b_{ik} = \frac{1}{2}\{[(a_k + \rho_{ik})^2 + z_{ik}^2]^{1/2} + [(a_k - \rho_{ik})^2 + z_{ik}^2]^{1/2}\}. \tag{3.2.22}$$

Of course, the exact values of ρ_{ik} and z_{ik} are not known but the fact that $(\rho_{ik}, z_{ik}) \subset S_i$ allows us to obtain the upper and the lower bounds for the total charges by solving the set (20) for the extreme points. It will be shown later that this estimation is sufficiently sharp and can be used for verification of the accuracy of various approximate solutions. Notice also that in the case $v_k(\rho,\phi) = v_k = \text{const.}$,

$$M_k v_k(\rho,\phi) = v_k/\sqrt{a_k^2 - \rho^2}, \quad B_k = \frac{2}{\pi} v_k a_k. \tag{3.2.23}$$

Since $b_{kk} = a_k$, the set of equations (20) can be rewritten in a uniform manner

$$\frac{2}{\pi}\sum_{i=1}^{n} Q_i \sin^{-1}(a_k/b_{ik}) = B_k, \quad \text{for} \quad k = 1, 2, 3, ..., n. \tag{3.2.24}$$

The possibility to assess the integral characteristics in such a simple manner is not limited to the quantity of total charge. One can multiply (11) by $t^m \mathrm{d}t\,\mathrm{d}\phi$ and integrate over the surface S_1. The result can always be expressed in elementary functions. For example, in the case $m = 2$, the result of integration is

$$\int_0^{2\pi}\int_0^{a_1} q_1(t,\phi)\, t^2\, \mathrm{d}t\, \mathrm{d}\phi + \sum_{i=2}^{n} \int\int_{S_i} \Big[\sqrt{\rho_i^2 + z_i^2} \;\; - \sqrt{b_i^2(a_1) - a_1^2}$$

$$+|z_i|\ln\frac{b_i(a_1)\,[\sqrt{b_i^2(a_1)-a_1^2}+|z_i|]}{\sqrt{b_i^2(a_1)-a_1^2}[|z_i|+\sqrt{\rho_i^2+z_i^2}]}\Bigg]q_i\,\mathrm{d}S_i$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{a_1}\left[\frac{a_1}{\sqrt{a_1^2-\rho^2}}-\cosh^{-1}\left(\frac{a_1}{\rho}\right)\right]v_1(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi.$$

Here, one can again evoke the mean value theorem and get the upper and the lower bounds for the quantities of interest.

Example 1. The simplest example to consider is the case of two disks of radii R_1 and R_2 lying in two planes intersecting at an angle α and whose centers are lying in one plane orthogonal to the line of intersection at the distances d_1 and d_2 from the line. Let the disks be conductors charged to the potentials V_1 and V_2 respectively. The total charges Q_1 and Q_2 are to be determined. The set of equations to be solved has the form

$$Q_1+\frac{2}{\pi}Q_2\sin^{-1}\left(\frac{a_1}{b_{21}}\right)=\frac{2}{\pi}V_1a_1$$

$$\frac{2}{\pi}Q_1\sin^{-1}\left(\frac{a_2}{b_{12}}\right)+Q_2=\frac{2}{\pi}V_2a_2. \tag{3.2.25}$$

Here

$$b_{12}=\frac{1}{2}\{[(d_2+x)^2+(d_1-R_1)^2-2(d_2+x)(d_1-R_1)\cos\alpha]^{1/2}$$

$$+[(d_2+x)^2+(d_1+R_1)^2-2(d_2+x)(d_1+R_1)\cos\alpha]^{1/2}\},$$

$$b_{21}=\frac{1}{2}\{[(d_1+y)^2+(d_2-R_2)^2-2(d_1+y)(d_2-R_2)\cos\alpha]^{1/2}$$

$$+[(d_1+y)^2+(d_2+R_2)^2-2(d_1+y)(d_2+R_2)\cos\alpha]^{1/2}\}, \tag{3.2.26}$$

where $-R_2\le x\le R_2$, and $-R_1\le y\le R_1$. The extreme points give the upper and the lower bounds for the total charges. It is logical to consider the central estimation corresponding to $x=y=0$. Calculations show that in some cases the central estimation is very close to the exact result.

The problem of two non-parallel disks was considered in (Rukhovets and Ufliand, 1971) using the Mehler-Fok transform. The following result was obtained for the total charge Q_1 in the assumption that

$$\mu_1/\sin(\alpha/2) \ll 1 \quad \text{and} \quad \mu_2/\sin(\alpha/2) \ll 1; \quad \mu_1 = R_1/d_1, \quad \mu_2 = R_2/d_2$$

$$Q_1 \approx \frac{2}{\pi} R_1 \left[V_1 - V_2 \frac{\mu_2}{\pi \sin(\alpha/2)} + V_1 \frac{\mu_1 \mu_2}{\pi^2 \sin^2(\alpha/2)} - V_2 \frac{\mu_1 \mu_2^2}{\pi^3 \sin^3(\alpha/2)} \right.$$

$$\left. + V_2 \frac{\mu_1^2 \mu_2 [2 + 3\sin^2(\alpha/2)]}{24\pi \sin^3(\alpha/2)} + V_2 \frac{\mu_2^3 [2 - 9\sin^2(\alpha/2)]}{24\pi \sin^3(\alpha/2)} \right]. \tag{3.2.27}$$

It is of interest to compare the results given by (25-26) with those by Rukhovets and Ufliand (27). Calculations were performed for the case $R_1 = R_2 = 1$, $V_1 = V_2 = 1$, $\alpha = \pi/4$, $d_1 = d_2 = l$. The value of $Q^* = \pi Q_1/2$ versus l is presented in Table 3.2.1.

Table 3.2.1. Comparison of our results with Rukhovets and Ufliand's

l	Upper bound for Q^*	Lower bound for Q^*	Central estimation	Rukhovets' result	Numerical solution
0.1	0.6521719	0.5000000	0.5114988	–	–
0.5	0.6858624	0.5000000	0.5612540	–	–
0.7	0.7013298	0.5000000	0.5871789	1.699512	–
1.0	0.7228720	0.5000000	0.6252061	1.023282	0.64925
1.5	0.7545003	0.6352847	0.6817026	0.8013288	0.70034
2.0	0.7811831	0.6964245	0.7271795	0.7774743	0.74027
3.0	0.8223547	0.7723489	0.7909440	0.8056570	0.79734
5.0	0.8731642	0.8500498	0.8595063	0.8626233	0.86149
7.0	0.9021068	0.8889649	0.8946465	0.8957692	0.89547
10.0	0.9273461	0.9203712	0.9235224	0.9239035	0.92383
15.0	0.9493124	0.9460188	0.9475592	0.9476710	0.94766

The numerical results were obtained by the method of iteration, with the accuracy 0.0001. As we can see, formula (27) gives good results for $l > 2.5$, the results sharply deviate from the admissible zone for $l < 2$. The central estimation gives reasonably good accuracy in the whole range $0 < l < \infty$.

The new approach allows a very simple treatment of complicated problems. The method is not limited to circular disks, it can be modified for other surfaces, for example, a system of arbitrarily located spherical caps can be treated in a similar manner.

Example 2. Consider the case of $n+1$ disks with their centers located at the plane $z = 0$. The plane of the first disk is horizontal, its center being placed at the coordinates system origin, and its radius being a_0. This disk will be called central. The centers of the remaining n equal disks are located at the apices of a regular polygon, their planes being orthogonal to the line connecting the coordinates origin with the apex, the length of this line being l. Let the central disk be charged to a potential V_0, and the rest being kept at a potential V_1, and their radius being a_1. We need to write the set of approximate linear algebraic equations for the total charges on the disks.

Due to the symmetry of the system, the problem of finding the total charge at each disk can be reduced to a set of just two linear algebraic equations

$$Q_0 + \frac{2}{\pi} n Q_1 \sin^{-1}\left(\frac{a_0}{b_0}\right) = \frac{2}{\pi} V_0 a_0,$$

$$\frac{2}{\pi} Q_0 \sin^{-1}\left(\frac{a_1}{b_{01}}\right) + Q_1\left[1 + \frac{2}{\pi}\sum_{i=2}^{n} \sin^{-1}\left(\frac{a_1}{b_{11}}\right)\right] = \frac{2}{\pi} V_1 a_1,$$

where, from elementary geometrical considerations,

$$b_0 = \frac{1}{2}\{[(l+a_0)^2 + x^2]^{1/2} + [(l-a_0)^2 + x^2]^{1/2}\},$$

$$b_{01} = [(l-y)^2 + a_1^2]^{1/2},$$

$$b_{i1} = \frac{1}{2}[\sqrt{x_{i1}^2 + y_{i1}^2} + \sqrt{X_{i1}^2 + Y_{i1}^2}],$$

$$x_{i1} = 2h\sum_{k=1}^{i-2} \sin\left(\frac{2\pi k}{n}\right) + (h-a_1)\sin\left(\frac{2\pi(i-1)}{n}\right), \quad h = l\tan(\pi/n),$$

$$y_{i1} = 2h\sum_{k=1}^{i-2} \cos\left(\frac{2\pi k}{n}\right) + (h-a_1)\cos\left(\frac{2\pi(i-1)}{n}\right) + h + x,$$

$$X_{i1} = 2h\sum_{k=1}^{i-2} \sin\left(\frac{2\pi k}{n}\right) + (h+a_1)\sin\left(\frac{2\pi(i-1)}{n}\right),$$

$$Y_{i1} = 2h\sum_{k=1}^{i-2} \cos\left(\frac{2\pi k}{n}\right) + (h+a_1)\cos\left(\frac{2\pi(i-1)}{n}\right) + h + x,$$

for $i=2,3,...,n$; where $-a_1 \le x \le a_1$ and $-a_0 \le y \le a_0$. The central estimation corresponds to the case $x=y=0$. Note that the accuracy of the central estimation improves, as the ratio l/a_0 and l/a_1 increases, tending to the *exact* results when this ratio tends to infinity.

3.3. Capacity of flat laminae

A new method is proposed for the evaluation of the capacity of flat laminae of arbitrary shape. Specific approximate formulae are derived for evaluating the capacity of a polygon, a triangle, a rectangle, a rhombus, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature, and a good accuracy is confirmed.

Capacity is one of the most important electrostatic characteristics. Of all two-dimensional shapes, the exact formulae are known at the moment for a circle and for an ellipse only. There seems to be only one approximate formula for the capacity of a rectangle Howe (1920) which is not very good for the aspect ratios close to unity but gives better results for the rectangles with high aspect ratio. A universal formula was suggested by Solomon (1964a), who considered a mathematically equivalent contact problem of a flat punch on an elastic half-space. It is exact for a circle and reasonably accurate for a regular polygon but it fails even in the case of an ellipse of high eccentricity. A new method is suggested here giving simple yet accurate formulae for the capacity of flat laminae of arbitrary shape.

Consider a flat conducting lamina S whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{3.3.1}$$

The lamina is charged to a potential v. The governing integral equation takes the form

$$\iint_S \frac{\sigma(M)\,\mathrm{d}S}{R(M,N)} = v(N). \tag{3.3.2}$$

Here σ is the charge density distribution, R is the distance between two points. By using (1.3.9), equation (2) can be written as

$$4\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_0^{2\pi} \mathrm{d}\phi_0 \int_x^{a(\phi_0)} \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \lambda\!\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) \sigma(\rho_0,\phi_0) = v(\rho,\phi). \tag{3.3.3}$$

It is also noteworthy that the change of the order of integration which led to (3) is valid inside the circle $\rho \le \min a(\phi)$ only. Nevertheless, one can obtain from (3) *exact* solution for an ellipse and sufficiently accurate formulae for the capacity of flat laminae of arbitrary shape.

In the case of a conducting disk, the potential v is constant, and the capacity is defined as the ratio of the total charge Q and v. Let the charge density distribution be

$$\sigma = \frac{c\, a(\phi)}{\sqrt{a^2(\phi) - \rho^2}}. \tag{3.3.4}$$

where c is a constant which can be easily defined from the condition that the integral of σ over S should give the total charge Q.

$$\int_0^{2\pi} d\phi \int_0^{a(\phi)} \frac{c\, a(\phi)}{\sqrt{a^2(\phi) - \rho^2}}\, \rho\, d\rho = c \int_0^{2\pi} a^2(\phi)\, d\phi = 2Ac = Q, \tag{3.3.5}$$

where A is the area of S. It is noteworthy that the total charge does not depend on the location of the coordinate system origin. This location can be defined from the condition that the dipole moment of the charge distribution (4) about the origin be zero which leads to two equations

$$\int_0^{2\pi} a^3(\phi) \cos\phi\, d\phi = 0\,, \qquad \int_0^{2\pi} a^3(\phi) \sin\phi\, d\phi = 0. \tag{3.3.6}$$

The left hand sides of both equations are proportional to the x and y coordinates of the center of gravity which means that the origin of the system of polar coordinates should be located at the center of gravity of the lamina S. One gets immediately from (5) that

$$\sigma = \frac{Q\, a(\phi)}{2A\sqrt{a^2(\phi) - \rho^2}}. \tag{3.3.7}$$

For the case of a perfectly conducting lamina $v = \delta = const$. Now substituting (7) in (3), we can verify how close to a constant will be the potential produced by the charge distribution (7). Integration with respect to ρ_0 gives

$$v(\rho,\phi)=\frac{Q}{2A}\sum_{n=-\infty}^{\infty}\int_0^{\rho}\left(\frac{x}{\rho}\right)^{|n|}\frac{x\,\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_0^{2\pi}\mathrm{e}^{in(\phi-\phi_0)}F\left(1-\frac{|n|}{2},\frac{1}{2};1;1-\frac{x^2}{a^2(\phi_0)}\right)\mathrm{d}\phi_0. \tag{3.3.8}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the potential can be done separately for each harmonic. The zero*th* harmonic has the form

$$v_0=\frac{Q}{2A}\frac{\pi}{2}\int_0^{2\pi}a(\phi)\,\mathrm{d}\phi. \tag{3.3.9}$$

It is important to note that the second harmonic is equal zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. Here is the expression for the fourth harmonic

$$v_4=\frac{Q}{2A}\frac{8}{35}\rho^3\int_0^{2\pi}\frac{\mathrm{e}^{4i(\phi-\phi_0)}\,\mathrm{d}\phi_0}{a^2(\phi_0)}. \tag{3.3.10}$$

The investigation of further harmonics shows that their amplitude decreases.

Now consider in more detail the case of a square with the side $2l$. The equation of the boundary in this case is $a(\phi)=l/\cos\phi$ for $-\pi/4<\phi<\pi/4$, and the pattern is repeated outside this range. We can evaluate several non-zero harmonics:

$$v_0=\frac{Q}{2A}4\pi l\ln(1+\sqrt{2}),\qquad v_4=\frac{Q}{2A}\frac{32\rho^3\cos4\phi}{105l^2},$$

$$v_8=-\frac{Q}{2A}\frac{64}{3465}\rho\cos8\phi\left[\left(\frac{\rho}{l}\right)^2+\frac{12}{13}\left(\frac{\rho}{l}\right)^4+\frac{20}{13}\left(\frac{\rho}{l}\right)^6\right], \tag{3.3.11}$$

If we assume that the potential $\delta\approx v_0$ then the remaining harmonics may be called the solution error. Direct computations show that the error is less than 3% inside the circle $\rho\leq l$. The error is reasonably small outside the circle reaching 20% at the apex, and decreasing very rapidly with the distance from the apex. Taking into consideration that the error sign fluctuation will result in even smaller error in the total charge value, we may assume (9) being the relationship between the potential value and the total charge which can be rewritten in the form

$$\delta = \frac{Q}{g\sqrt{A}}. \tag{3.3.12}$$

where A is the area of the lamina, and g is a dimensionless coefficient depending on the lamina geometry only

$$g = \frac{2\sqrt{A}}{\pi^2 r_a}, \tag{3.3.13}$$

where r_a can be called *average radius* with respect to the center of gravity

$$r_a = \frac{1}{2\pi}\int_0^{2\pi} a(\phi)\, d\phi. \tag{3.3.14}$$

One can easily deduce that our coefficient g is related to the capacity C of a flat lamina by

$$g = \frac{C}{\sqrt{A}}. \tag{3.3.15}$$

The problem now is to find the value of g for various shapes. One can easily compute the coefficient g for the square from the first equation (11) as

$$g = \frac{1}{\pi\, ln(1+\sqrt{2})} = 0.3611$$

which is very close to the value 0.3607 given by Maxwell for the capacitance of the square. Of course, closeness to the result by Maxwell does not mean that our result is so accurate. The value of g which seems to be accurate was obtained by Noble (1960), and is 0.367, so that our result is in error by 1.6% which is not bad. Now it seems logical to assume that formulae (12–14) are valid for an arbitrary lamina, and we shall verify how good they really are for each specific case in the next Section.

We have found in the literature only one general formula of the type (12) suggested by Solomon (1964a). His result expressed through the coefficient g reads

$$g = \frac{2^{9/8} I_0^{1/8}}{\pi^{11/8} A^{1/4}}, \tag{3.3.16}$$

where I_0 stands for the polar moment of inertia. One can easily verify that formula (16) is exact for a circle, so one should expect it to be sufficiently accurate for domains with the aspect ratio not far away from unity, but the error might be quite significant for oblong domains. For example, in the case of an ellipse with semi-axes a and b formula (16) gives

$$g = \frac{2^{7/8}}{\pi^{3/2}}\left(\frac{a}{b}+\frac{b}{a}\right)^{1/8}.$$

Our formulae (12–14) in the case of an ellipse are exact. Several specific applications are considered below.

Polygon. Consider a polygon with n sides, with the only limitation that the function a describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the k-th side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered k+1. It is clear that the value of index equal n+1 is understood as 1. Denote b_k the distance from the center of gravity to the k-th apex. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. Then formulae (13) and (14) yield the following expression for the coefficient g

$$g = \frac{2\sqrt{A}}{\pi\sum_{k=1}^{n}\frac{A_k}{a_k}\ln\frac{b_k+b_{k+1}+a_k}{b_k+b_{k+1}-a_k}}. \qquad (3.3.17)$$

In the case of a regular polygon formula (17) simplify to

$$g = \frac{4\sqrt{\tan(\pi/n)}}{\pi\sqrt{n}\ \ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}\ . \qquad (3.3.18)$$

Consider several particular values of n. For an equilateral triangle (n=3) formula (18) gives g = 0.3673. The value of g, which seems to be accurate, can be computed from (Solomon, 1964b), and is equal 0.3829, so that our result is in error by 4.1%. As we have seen earlier, the error of (18) for a square is 1.6%. Since formula (18) in the limiting case $n\to\infty$ gives the exact result for a circle $g = 2/\pi^{3/2}$= .35917, we should expect that the error of (18) will decrease with n. For a regular pentagon g = 0.3599. We did not find in the literature anything to compare with this result. It seems quite clear that the maximum possible error indeed decreases with n. It is noteworthy that the value of g does not change significantly in the whole range $3 \leq n < \infty$.

Triangle. In the case of a triangle with the sides a_1, a_2 and a_3, formula (17) simplifies as follows:

$$g = \frac{6}{\pi\sqrt{A}} \left[\frac{1}{a_1} \ln \frac{b_1 + b_2 + a_1}{b_1 + b_2 - a_1} + \frac{1}{a_2} \ln \frac{b_2 + b_3 + a_2}{b_2 + b_3 - a_2} + \frac{1}{a_3} \ln \frac{b_3 + b_1 + a_3}{b_3 + b_1 - a_3} \right]^{-1}. \tag{3.3.19}$$

The parameters in (19) can be defined by the well known formulae from geometry:

$$A = [\, p(p - a_1)(p - a_2)(p - a_3)\,]^{1/2}, \quad p = (a_1 + a_2 + a_3)/2,$$

$$b_1 = \frac{1}{3}[\, 2(a_1^2 + a_3^2) - a_2^2\,]^{1/2}, \qquad b_2 = \frac{1}{3}[\, 2(a_2^2 + a_1^2) - a_3^2\,]^{1/2},$$

$$b_3 = \frac{1}{3}[\, 2(a_3^2 + a_2^2) - a_1^2\,]^{1/2}.$$

We are unaware of any report treating a triangle of general type but certain particular cases have been considered, so we can compare the results. When $a_1 = a_2 = l$, and the angle between these two sides is equal α, formula for the coefficient g can be rewritten in the form

$$\mathrm{g} = \frac{6}{\pi}\sqrt{\tan(\alpha/2)} \left[2\sin(\frac{\alpha}{2}) \ln(\cot\frac{2\gamma - \alpha}{4} \cot\frac{\alpha}{4}) + \ln\tan(\frac{\pi}{4} + \frac{\gamma}{2}) \right]^{-1}, \tag{3.3.20}$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

In the case of $\alpha = \pi/2$, Okon and Harrington (1970) obtained $g = 0.3867$ as the most probable result. Our result is $g = 0.374$ which is within 3.3% from the numerical one.

Rectangle. Consider a rectangular lamina, a and b being its semiaxes. Introduce the aspect ratio $\varepsilon = a/b$. Formula (17) in this case reduces to

$$g = \frac{2}{\pi[\sqrt{\varepsilon}\,\sinh^{-1}(1/\varepsilon) + (1/\sqrt{\varepsilon})\,\sinh^{-1}\varepsilon]}. \tag{3.3.21}$$

Howe (1920) suggested an approximate formula for the capacitance of a rectangle which in terms of the coefficient g reads

$$g = \frac{1}{2\sqrt{\varepsilon}\left[\frac{1}{\varepsilon}\sinh^{-1}\varepsilon + \sinh^{-1}(\frac{1}{\varepsilon}) + \frac{\varepsilon}{3} + \frac{1}{3\varepsilon^2} - \frac{(\varepsilon^2+1)^{3/2}}{3\varepsilon^2}\right]}, \qquad (3.3.22)$$

The result due to Solomon (16) in this case takes the form

$$g = \frac{2^{(9/8)}}{\pi^{11/8}}\left[\frac{\varepsilon}{12} + \frac{1}{12\varepsilon}\right]^{1/8}. \qquad (3.3.23)$$

We have found in the literature some numerical results which seem to be more or less accurate. Noble (1960) investigated a problem of electric charge distribution on a rectangular lamina, and Borodachev and Galin (1974) have considered an equivalent problem of a narrow rectangular punch on an elastic half-space. Their data, expressed in terms of the coefficient g, is presented below and compared with our result (21) and those due to Howe (22) and Solomon (23). The following relationship was used between Borodachev-Galin's coefficient γ and our g: $\gamma=1/2\pi g\sqrt{\varepsilon}$.

ε =	0.020	0.050	0.100	0.125	0.150	0.200	0.250	0.500	1.000
Borodachev and Galin	0.7375	0.5661	0.4819	–	0.4458	0.4259	–	–	–
Noble	–	–	–	0.4543	–	–	0.4047	0.3762	0.3670
Formula (21)	0.8031	0.6072	0.5037	0.4771	0.4576	0.4306	0.4128	0.3742	0.3612
Howe (22)	0.6916	0.5317	0.4481	0.4268	0.4112	0.3899	0.3759	0.3462	0.3363
Solomon (23)	0.5402	0.4819	0.4423	0.4304	0.4211	0.4071	0.3969	0.3715	0.3613
Discrepancy %									
Formula (21)	-8.9	-7.3	-4.5	-5.0	-2.6	-1.1	-2.0	0.5	1.6
Howe (22)	6.2	6.1	7.0	6.1	7.8	8.5	7.1	8.0	8.4
Solomon (23)	26.7	14.9	8.2	5.3	5.5	4.4	1.9	1.3	1.6

Several useful conclusions can be drawn from the data presented. It seems logical to assume that the error of an approximate formula should change monotonously (or to have only one extremum) with respect to a certain parameter. The fact that the discrepancy due to each formula jumps when moving from the data due to Noble to those by Borodachev and Galin, indicates that the results of at least one author are not exact. Our own computations favor the results of Noble. Here are some of our numerical results: $\varepsilon=0.5$, $g=0.3763$; $\varepsilon=1/8$, $g=0.4543$; $\varepsilon=0.1$, $g=0.4752$; $\varepsilon=1/15$, $g=0.5200$; $\varepsilon=1/30$, $g=0.6207$; $\varepsilon=1/40$, $g=0.6729$; $\varepsilon=1/50$, $g=0.7192$. Our formula seems to perform better then the other two in a sufficiently wide range of the aspect ratio. As it was expected formula due to Solomon performs well when the aspect ratio is not far away from unity only. There seems to be little change in the error of Howe's formula (22). If this is really so, then its accuracy can be improved dramatically just by multiplication by a constant factor, say, 1.07.

Rhombus. Let α be the angle at one of the rhombus apexes. Formula (17) in this case yields

$$g = \frac{2}{\pi\sqrt{\sin\alpha}\,\ln\dfrac{\cos(\alpha/2)+\sin(\alpha/2)+1}{\cos(\alpha/2)+\sin(\alpha/2)-1}}. \tag{3.3.24}$$

The same formula in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = a/b$ has the form

$$g = \frac{\sqrt{2(\varepsilon+(1/\varepsilon))}}{\pi\ln\dfrac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}}}. \tag{3.3.25}$$

The capacity of a diamond was computed by Okon and Harrington (1970). Their result, expressed in terms of the coefficient g, for a diamond with the aspect ratio $a{:}b$ = 0.7:1.65 is g = 0.3855. Formula (25) gives g = 0.3744 which is within 3% from the result of Okon and Harrington. They also considered a rhombus with the aspect ratio 1:2. Their result g = 0.3705 practically coincides with ours g = 0.3698. Slightly different numerical results are given by De Smedt (1979). We compare his data expressed in terms of the coefficient g with those computed due to (25)

ε=	0.100	0.200	0.333	0.500	0.750	1.000
De Smedt (1979)	0.521	0.442	0.402	0.381	0.369	0.366
Formula (25)	0.462	0.409	0.383	0.370	0.363	0.361
Discrepancy %	11.4	7.6	4.8	2.8	1.6	1.3

Our formula (25) seems to perform quite satisfactory in a wide range of the aspect ratio ε.

Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}. \tag{3.3.26}$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r\,[-k\cos\phi + \sqrt{1-k^2\sin^2\phi}, \quad \text{for} \quad 0 \le \phi \le \pi-\gamma \quad \text{or} \quad \pi+\gamma \le \phi < 2\pi;$$

$$a(\phi) = r\,\frac{k-\cos\alpha}{\cos(\pi-\phi)}, \quad \text{for} \quad \pi-\gamma \le \phi \le \pi+\gamma. \tag{3.3.27}$$

Substitution of (27) into (13)–(14) yields

$$g = \frac{2\sqrt{\alpha - \sin\alpha\cos\alpha}}{\pi\{2E(k) - E(\gamma,k) - k\sin\gamma + (k - \cos\alpha\ln\tan[(\pi+2\gamma)/4]\}}. \tag{3.3.28}$$

where $\gamma = \tan^{-1}[\sin\alpha/(k - \cos\alpha)]$. We have found only one numerical example to verify the accuracy of (28): Okon and Harrington (1970) have computed the capacity of a semicircle. Their result expressed through the coefficient g is 0.3724. Formula (28) gives 0.3714 with the discrepancy of 0.3%.

Lune. Two equal circular segments of radius r and angle 2α joined along their chords give us a lamina of lune shape. The capacity of such lamina was considered by Lebedev *et. al.*(1986). They have obtained an asymptotic formula for narrow lune which reads

$$g = \frac{2\sin\alpha}{L\sqrt{2\alpha - \sin 2\alpha}}\left[1 - \frac{2}{L} + \frac{8}{L^2}\left(1 - \frac{\pi^2}{24}\right) + \frac{130}{3}\left(1 - \frac{3\pi^2}{65}\right)\frac{e^{-L}}{L^2}\right]. \tag{3.3.29}$$

Here

$$L = \frac{\pi K(\cos\alpha)}{K(\sin\alpha)},$$

and $K()$ stands for the complete elliptic integral of the first kind. Our method yields

$$g = \frac{\sqrt{2\alpha - \sin 2\alpha}}{\pi[E(\cos\alpha) - \cos\alpha]}. \tag{3.3.30}$$

Here $E()$ denotes a complete elliptic integral of the second kind. It is of interest to compare formulae (29) and (30) with an accurate numerical solution obtained by the method described in section 7.3. The results are given below

α (degrees)	5	7	10	15	20	30	45	60
g (29)	0.6277	0.5753	0.5300	0.4923	0.4763	0.4795	0.5624	0.8287
g (30)	0.5749	0.5265	0.4831	0.4426	0.4191	0.3927	0.3737	0.3647
g numerical	0.6283	0.5656	0.5135	0.4631	0.4340	0.4044	0.3784	0.3662
error of (30) (%)	8.5	6.9	5.9	4.4	3.4	2.9	1.2	0.4

Our formula (30) is sufficiently accurate for $\alpha > 15$ degrees. Lebedev's formula (29) is good only for very small $\alpha < 15$ degrees (although the authors claim it to be sufficiently accurate up to $\alpha = 30$ degrees). It deviates from reasonable behavior for $\alpha > 20$ degrees (g increases instead of decreasing). In view of these properties of (29), we can suggest a much more simple asymptotic formula, namely,

$$g = \frac{\sqrt{3}}{2\sqrt{\alpha}\,\ln(4/\alpha)}\left[1 - \frac{1}{\ln(4/\alpha)} + \frac{2}{\ln^2(4/\alpha)}\left(1 - \frac{\pi^2}{24}\right)\right]. \tag{3.3.31}$$

Besides being more simple than (29) formula (31) is also more accurate. Some numerical results for a 'shamrock'-shaped lamina can be found in section 7.3 where a mathematically equivalent contact problem is considered.

Circular sector. Repetition of the procedure, described earlier, leads to the following result for a circular sector with the angle 2α:

$$g = \frac{2\sqrt{\alpha}}{\pi\{E(\gamma, k) - k\sin\gamma + k\sin\alpha\,\ln[\cot(\alpha/2)\cot((\gamma-\alpha)/2)]\}}. \tag{3.3.32}$$

Here, $k=2\sin\alpha/(3\alpha)$, and $\gamma=\tan^{-1}(\sin\alpha/(\cos\alpha-k))$. Okon and Harrington (1970) in the case of a quadrant obtained $g=0.3668$. Formula (32) for $\alpha=\pi/4$ gives $g=0.3639$, with the discrepancy of 0.8%.

One can enquire whether there exists any contour, other than an ellipse, for which expression of the type (7) would be an exact solution to the integral equation (29). Expression (8) can provide the sufficient conditions:

$$\int_0^{2\pi} e^{in(\phi-\phi_0)} F\left(1 - \frac{|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) d\phi_0 \tag{3.3.33}$$

should be equal zero for $n\neq 0$. Integral (33) will vanish for all odd n if $a(\phi)$ contains even harmonics only. In the case of even n, the hypergeometric function in (33) represents a finite polynomial in $x/a(\phi)$ of degree not greater than $n-2$ which means that integral (33) vanishes if $[a(\phi)]^{-2}$ contains harmonics not higher than the second, which corresponds to an ellipse. The question whether these conditions will be necessary requires an additional investigation.

Solomon's formula (16) can be considered as a particular case of a more general one, namely

$$g = \frac{2}{\pi\sqrt{A}}\left[\frac{1}{2\pi}\int_0^{2\pi} (a(\phi))^m d\phi\right]^{1/2m}\left[\frac{1}{2\pi}\int_0^{2\pi} (a(\phi))^n d\phi\right]^{1/2n} \tag{3.3.34}$$

for $m=2$ and $n=4$. One may ask now whether this choice of the parameters m and n is in any sense optimal. Direct computations show that this is not the case. Different shapes require different values for m and n. Formulae of type (34) are of empirical type, and have very little physical background.

3.4. Magnetic polarizability of small apertures

Analysis of magnetic polarizability of small apertures of arbitrary shape is presented here. A general formula is derived for the coefficients of magnetic polarizability of small apertures. Specific formulae are obtained for the apertures of various shape. All the formulae are checked against the solutions known in the literature, and their accuracy is confirmed. The material follows (Fabrikant, 1987b).

Many years ago Bethe (1944) reduced the problem of diffraction by small apertures to an evaluation of the coefficient of electric polarizability and the tensor of magnetic polarizability. At the moment, closed-form expressions for these quantities are known for an elliptic aperture in a planar screen only. All non-elliptic shapes have been treated either experimentally (Cohn 1951) or numerically (Okon and Harrington 1981, de Smedt 1979, De Meulenaere and Van Bladel 1977), the variational approach was used by Fikhmanas and Fridberg (1973). Their results will be used for verification of the accuracy of the formulae to be derived here.

It is well known (Bethe, 1944) that the problem of diffraction by small apertures can be reduced to the solution of the following integral equation

$$w(N) = \int\int_S \frac{\sigma(M)}{R(M,N)}\,\mathrm{d}S, \tag{3.4.1}$$

where S is a two-dimensional domain of the aperture, $R(M,N)$ stands for the distance between the points M and N, w is a known function, and σ stands for the charge density (unknown function), Let the boundary of the aperture S in a planar screen be given in the polar coordinates as

$$\rho = a(\phi),$$

where the function $a(\phi)$ is bounded and single-valued. We use again the integral representation for the reciprocal distance established in Chapter 1

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0},\,\phi-\phi_0)\,\mathrm{d}x}{\sqrt{\rho^2-x^2}\,\sqrt{\rho_0^2-x^2}}, \tag{3.4.2}$$

where

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}. \tag{3.4.3}$$

Substitution of (2) into (1) gives, after changing the order of integration

$$w(\rho, \phi) = \frac{2}{\pi} \int_0^{\rho} \frac{dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} d\phi_0 \int_x^{a(\phi_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)}{(\rho_0{}^2 - x^2)^{1/2}} \sigma(\rho_0, \phi_0)\, \rho_0 d\rho_0. \qquad (3.4.4)$$

It is noteworthy that the change of the order of integration which led to (4) is valid inside the circle $\rho \le \min\{a(\phi)\}$ only. Nevertheless, one can obtain from (4) the *exact* solution for an ellipse and sufficiently accurate formulae for various specific apertures as it will be demonstrated further.

For the case of magnetic polarizability, it is sufficient to consider equation (1), with the function w taking the form

$$w = \alpha_x y - \alpha_y x, \qquad (3.4.5)$$

where α_x and α_y are constants. It is quite clear that in the case of a uniaxial excitation one of these constants can be put equal zero.

Let the charge distribution at the aperture be

$$\sigma(\rho, \phi) = \frac{a(\phi)\, \rho(p_1 \cos\phi + p_2 \sin\phi)}{[\, a^2(\phi) - \rho^2\,]^{1/2}}, \qquad (3.4.6)$$

where p_1 and p_2 are yet unknown constants. The main reason for this choice is the requirement that (6) be exact for an ellipse. Make use of the condition that the integral of σ over S should be equal zero. Since p_1 and p_2 are independent, this leads to two equations

$$\int_0^{2\pi} (a(\phi))^3 \cos\phi\, d\phi = 0\,, \qquad \int_0^{2\pi} (a(\phi))^3 \sin\phi\, d\phi = 0. \qquad (3.4.7)$$

One can note that the left-hand side of each equation (7) is proportional to the x or y coordinates of the center of gravity. This means that the origin of the system of coordinates should be located at the center of gravity of the aperture. The axis orientation will be discussed later.

The relationships between the dipole moments and the parameters p_1 and p_2 can be established from the conditions

$$M_x = \iint_S \sigma y\, dS\,, \qquad M_y = -\iint_S \sigma x\, dS,$$

which leads to

$$M_x = \frac{8}{3}(p_1 I_{xy} + p_2 I_x)\,, \qquad M_y = -\frac{8}{3}(p_1 I_y + p_2 I_{xy}), \tag{3.4.8}$$

where I_x, I_y and I_{xy} are the well known quantities of the moments of inertia and the product of inertia respectively.

$$I_x = \int\!\!\int_S y^2 \mathrm{d}S\,, \quad I_y = \int\!\!\int_S x^2 \mathrm{d}S\,, \quad I_{xy} = \int\!\!\int_S xy\,\mathrm{d}S.$$

Now it is necessary to relate p_1 and p_2 to the parameters α_x and α_y. This can be done by substitution of (6) into (4) which yields after integration with respect to ρ_0

$$w(\rho,\phi) = \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^2\,\mathrm{d}x}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \mathrm{e}^{in(\phi-\phi_0)}$$

$$\times F\left(\frac{3-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right)(p_1 \cos\phi_0 + p_2 \sin\phi_0)\,\mathrm{d}\phi_0. \tag{3.4.9}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the function w can be done separately for each harmonic. Note that the zero*th* and all the even harmonics of w will be zero if $a(\phi)$ contains only the even harmonics. The first harmonic will take the form

$$w_1(\rho,\phi) = \frac{\pi}{2}\rho \int_0^{2\pi} \cos(\phi - \phi_0)\,(p_1 \cos\phi_0 + p_2 \sin\phi_0\,)\,a(\phi_0)\,\mathrm{d}\phi_0,$$

which can be simplified as

$$w_1(\rho,\phi) = \frac{\pi}{2}\rho\,[(p_1 J_y + p_2 J_{xy})\cos\phi + (p_1 J_{xy} + p_2 J_x)\sin\phi]. \tag{3.4.10}$$

Here the following quantities were introduced

$$J_x = \int_0^{2\pi} a(\phi)\sin^2\phi\,\mathrm{d}\phi\,, \quad J_y = \int_0^{2\pi} a(\phi)\cos^2\phi\,\mathrm{d}\phi\,, \quad J_{xy} = \int_0^{2\pi} a(\phi)\sin\phi\cos\phi\,\mathrm{d}\phi. \tag{3.4.11}$$

These quantities do not seem to have been used before in engineering practice so they do not have an accepted name. Since their tensor properties are similar to those of the moments of inertia, we shall call J_x and J_y *the linear moments of a two-dimensional domain about the axes Ox and Oy* respectively, J_{xy} will be called *the linear product of a two-dimensional domain about the axes Ox and Oy*

It is important to note that the third harmonic is equal zero for an arbitrary contour. Here is the expression for the fifth harmonic

$$w_5(\rho, \phi) = \frac{128}{315}\rho^4 \int_0^{2\pi} \frac{\cos 5(\phi - \phi_0)}{a^2(\phi_0)} (p_1 \cos\phi_0 + p_2 \sin\phi_0)\, d\phi_0,$$

which can be modified as

$$w_5(\rho, \phi) = \frac{64}{315}\rho^4 \{[(A_{c6} + A_{c4})p_1 + (A_{s6} - A_{s4})p_2] \cos 5\phi$$

$$+ [(A_{s6} + A_{s4})p_1 + (A_{c4} - A_{c6})p_2] \sin 5\phi\}. \tag{3.4.12}$$

Here, the following geometrical characteristics of the aperture domain were introduced

$$A_{c4} = \int_0^{2\pi} \frac{\cos 4\phi\, d\phi}{(a(\phi))^2}, \qquad A_{c6} = \int_0^{2\pi} \frac{\cos 6\phi\, d\phi}{(a(\phi))^2},$$

$$A_{s4} = \int_0^{2\pi} \frac{\sin 4\phi\, d\phi}{(a(\phi))^2}, \qquad A_{s6} = \int_0^{2\pi} \frac{\sin 6\phi\, d\phi}{(a(\phi))^2}.$$

Investigation of further harmonics shows that their amplitude decreases. Since the amplitude of w_5 is significantly less than that of w_1, it seems natural to assume $w \approx w_1$, and the remaining harmonics may be called the solution error. Taking into consideration that the error sign fluctuation will result in even smaller error in the integral characteristics sought, a direct comparison of (5) and (10) leads to

$$\alpha_x = \frac{\pi}{2}(p_1 J_{xy} + p_2 J_x), \qquad \alpha_y = -\frac{\pi}{2}(p_1 J_y + p_2 J_{xy}). \tag{3.4.13}$$

The inversion of (13) gives

$$p_1 = -\frac{2}{\pi}\, \frac{J_{xy}\alpha_x + J_x\alpha_y}{J_x J_y - J_{xy}^2}, \qquad p_2 = \frac{2}{\pi}\, \frac{J_y\alpha_x + J_{xy}\alpha_y}{J_x J_y - J_{xy}^2}. \tag{3.4.14}$$

Substitution of (14) in (8) finally gives the required relationship

$$M_x = \frac{16}{3\pi}(m_{11}\alpha_x + m_{12}\alpha_y), \qquad M_y = \frac{16}{3\pi}(m_{21}\alpha_x + m_{22}\alpha_y), \tag{3.4.15}$$

where

$$m_{11} = \frac{J_y I_x - J_{xy} I_{xy}}{J_x J_y - J_{xy}^2}, \qquad m_{12} = \frac{J_{xy} I_x - J_x I_{xy}}{J_x J_y - J_{xy}^2},$$

$$m_{21} = \frac{J_{xy} I_y - J_y I_{xy}}{J_x J_y - J_{xy}^2}, \qquad m_{22} = \frac{J_x I_y - J_{xy} I_{xy}}{J_x J_y - J_{xy}^2}.$$

It is clear that all these results can be rewritten in a matrix or a tensor form. One can verify that formulae (15) are invariant with respect to an arbitrary rotation of the axes. The same property holds for $m_{11} + m_{22}$ and $m_{12} - m_{21}$. Strictly speaking, according to the reciprocal theorem, m_{12} should be equal m_{21}, so that formulae (15) generally do not satisfy this theorem. But we may state that this theorem is satisfied 'approximately'. We mean by this the following property which has been verified by several direct computations, namely, $|m_{12} - m_{21}|/m_{11} \ll 1$ and $|m_{12} - m_{21}|/m_{22} \ll 1$. This theorem will be satisfied exactly for any domain which has at least one axis of symmetry because in this case $m_{12} = m_{21} = 0$ provided that the coordinate axes coincide with the central principal axes of the domain of aperture. Since we have no numerical data for non-symmetrical domains which could be used to verify the accuracy of (15), we shall consider further only the case when the aperture has an axis of symmetry. In this case formulae (8), (13) and (15) simplify significantly

$$M_x = \frac{8}{3} I_x p_2, \qquad M_y = -\frac{8}{3} I_y p_1, \tag{3.4.16}$$

$$\alpha_x = \frac{\pi}{2} J_x p_2, \qquad \alpha_y = -\frac{\pi}{2} J_y p_1, \tag{3.4.17}$$

$$M_x = \frac{16}{3\pi}\frac{I_x}{J_x}\alpha_x, \qquad M_y = \frac{16}{3\pi}\frac{I_y}{J_y}\alpha_y. \tag{3.4.18}$$

Now, we can rewrite the expression for the charge distribution (6) in terms of the moments M_x and M_y in the form

$$\sigma = \frac{a(\phi)}{[a^2(\phi) - \rho^2]^{1/2}}\left[\frac{3}{4}\left(\frac{M_x y}{I_x} - \frac{M_y x}{I_y}\right)\right]. \tag{3.4.19}$$

An expression equivalent to (19) can be written in terms of the parameters α_x and α_y

$$\sigma = \frac{2a(\phi)}{\pi[\,a^2(\phi) - \rho^2\,]^{1/2}} \left[\frac{\alpha_x y}{J_x} - \frac{\alpha_y x}{J_y} \right]. \tag{3.4.20}$$

Expressions (19) and (20) are *exact* for an ellipse. We expect them to be reasonably accurate in the neighborhood of the coordinate origin for an arbitrary aperture with at least one axis of symmetry, while the error might become quite significant close to the boundary of the domain S.

Let us rewrite formulae (18) in the form

$$M_x = \frac{A^{3/2}}{2\pi} \nu_x \alpha_x\,, \qquad M_y = \frac{A^{3/2}}{2\pi} \nu_y \alpha_y, \tag{3.4.21}$$

where A is the aperture area, and

$$\nu_x = \frac{32 I_x}{3A^{3/2} J_x}\,, \qquad \nu_y = \frac{32 I_y}{3A^{3/2} J_y}. \tag{3.4.22}$$

We introduced the coefficients ν_x and ν_y for two reasons: since they are dimensionless they characterize the shape of S and do not depend on its size; both ν_x and ν_y are equal to the corresponding coefficients of magnetic polarizability which will simplify the comparison of our results with the numerical data available. The remaining part of the section will be devoted to the evaluation of the coefficients ν_x and ν_y for various aperture shapes.

Several specific aperture shapes are considered here. Each configuration is related to its central principal axes and assumed to have at least one axis of symmetry coinciding with the axis Ox.

Polygon. Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The origin of the coordinate system is located at the center of gravity, as before. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote b_k the distance from the center of gravity to the kth apex; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. The following expressions can be obtained for the moments of inertia

$$I_x = \sum_{k=1}^{n} -m_k \cos 2\psi_k + g_k \sin 2\psi_k + 2h_k \cos^2\psi_k ,$$

$$I_y = \sum_{k=1}^{n} m_k \cos 2\psi_k - g_k \sin 2\psi_k + 2h_k \sin^2\psi_k,$$

$$I_{xy} = \sum_{k=1}^{n} (m_k - h_k)\sin 2\psi_k + g_k \cos 2\psi_k, \tag{3.4.23}$$

where

$$m_k = \frac{2A_k^3}{a_k^2}, \quad g_k = A_k^2 \frac{b_{k+1}^2 - b_k^2}{2a_k^2}, \quad h_k = \frac{A_k[3(b_{k+1}^2 + b_k^2) - a_k^2]}{24}. \tag{3.4.24}$$

Formulae (23–24) are valid for an arbitrary polygon, not necessarily having an axis of symmetry.

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^{n} -q_k \cos 2\psi_k + s_k \sin 2\psi_k + 2t_k \cos^2\psi_k ,$$

$$J_y = \sum_{k=1}^{n} q_k \cos 2\psi_k - s_k \sin 2\psi_k + 2t_k \sin^2\psi_k , \tag{3.4.25}$$

$$J_{xy} = \sum_{k=1}^{n} (q_k - t_k) \sin 2\psi_k + s_k \cos 2\psi_k,$$

where

$$q_k = \frac{A_k}{a_k^2}\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right)[a_k^2 + (b_k - b_{k+1})^2] , \quad s_k = \frac{4A_k^2}{a_k^2}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right),$$

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}. \tag{3.4.26}$$

Substitution of (23–26) into (22) gives the coefficients ν_x and ν_y for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k-1)/n$, $A_k = [a^2\cot(\pi/n)]/4 = [b^2\sin(2\pi/n)]/2$, $A = nA_k$, and formulae (23–26) simplify to

$$I_x = I_y = \frac{na^4}{64}\cot\frac{\pi}{n}\left[\cot^2\frac{\pi}{n} + \frac{1}{3}\right] = \frac{nb^4}{24}\sin\frac{2\pi}{n}\left[2 + \cos\frac{2\pi}{n}\right], \tag{3.4.27}$$

$$J_x = J_y = \frac{1}{4}na\cot\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} = \frac{1}{2}nb\cos\frac{\pi}{n}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}. \tag{3.4.28}$$

Substituting (27) and (28) in (22) leads to

$$\nu_x = \nu_y = \frac{16(2+\cos\frac{2\pi}{n})}{9\left(n^3\sin\frac{\pi}{n}\cos^3(\frac{\pi}{n})\right)^{1/2}\ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}}. \tag{3.4.29}$$

Consider several particular values of n. For an equilateral triangle (n=3) formula (29) gives $\nu_x = \nu_y = 3^{1/4}16/[27\ln(2+\sqrt{3})] = 0.5922$. We did not find any numerical data to compare with this result. In the case of a square $n = 4$, and $\nu_x = \nu_y = 4/[9\ln(1+\sqrt{2})] = 0.5043$ which is inside the interval from 0.4973 to 0.5162 given by Okon and Harrington (1981) and within 3% from the result of de Smedt 0.5193. Since formula (29) in the limiting case $n \to \infty$ gives the exact result for a circle $\nu_x = \nu_y = 8/(3\pi^{3/2}) = 0.4789$, we should expect that the error of (29) will decrease with n. The value of the coefficients for a regular hexagon is $\nu_x = \nu_y = 40\sqrt{2}/(3^{1/4}81\ln 3) = 0.4830$ which differs by 1.4% from the result 0.49 due to Okon and Harrington (1981), and it is quite clear that the maximum possible error indeed decreases with n. It is noteworthy that the coefficients of magnetic polarizability do not change significantly in the whole range $3 \le n < \infty$.

Isosceles triangle. In the case of a triangle with the sides $a_1 = a_2 = l$ and the angle between them equal to α formulae (22–26) give

$$I_x = \frac{1}{12}l^4\sin\alpha\sin^2(\alpha/2), \qquad I_y = \frac{1}{36}l^4\sin\alpha\cos^2(\alpha/2),$$

$$J_x = \frac{2}{3}l\cos\frac{\alpha}{2}\left[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma\right.$$

$$+ 2\sin^3\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\Bigg]\Bigg],$$

$$J_y = \tfrac{2}{3}\,l\cos\frac{\alpha}{2}\left[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma + \sin\alpha\cos\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\right],$$

with the result for the coefficients

$$\nu_x = 8(\tan(\alpha/2))^{3/2}\Bigg\{3\Bigg[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma$$

$$+ 2\sin^3\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\Bigg]\Bigg\}^{-1},$$

$$\nu_y = 8\sqrt{\cot(\alpha/2)}\Bigg\{9\Bigg[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma$$

$$+ \sin\alpha\cos\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\Bigg]\Bigg\}^{-1}, \qquad (3.4.30)$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

The isosceles right triangle was considered by Okon and Harrington (1981) who gave the interval between 0.9829 and 1.021 for only one coefficient which in our notation is ν_x. Our result for ν_x is 0.9255 which differs less than 10% from theirs. The second formula (30) gives $\nu_y = 0.3995$, and there is nothing in the literature to compare with this result.

Rectangle. Consider a rectangular aperture, a_1 and a_2 being its semiaxes. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (23–26) in this case reduce to

$$I_x = \tfrac{4}{3}a_1a_2^3, \qquad I_y = \tfrac{4}{3}a_1^3a_2,$$

$$J_x = 4a_1\sinh^{-1}\varepsilon, \qquad J_y = 4a_2\sinh^{-1}(1/\varepsilon),$$

and formulae (22) yield

$$\nu_x = \frac{4\varepsilon^{3/2}}{9\sinh^{-1}\varepsilon}, \qquad \nu_y = \frac{4\varepsilon^{-3/2}}{9\sinh^{-1}(1/\varepsilon)}. \tag{3.4.31}$$

The coefficients of magnetic polarizability were computed by de Smedt (1979) for a rectangle with different aspect ratio ε. Here, we present his results along with those given by (31).

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x=	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (31) ν_x=	0.1408	0.2001	0.2612	0.3265	0.4165	0.4341	0.5043
de Smedt ν_y=	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (31) ν_y=	4.6876	2.1488	1.2701	0.8708	0.6228	0.5929	0.5043
Discrepancy in ν_x (%)	-9.4	-6.4	-3.2	-0.5	1.8	2.2	2.9
Discrepancy in ν_y (%)	-14.1	-6.1	-0.8	2.1	3.1	3.3	2.9

Our formula (31) seems to perform satisfactorily in a sufficiently wide range of aspect ratio. The approximate expression for the charge distribution at the aperture, according to (19), takes the form

$$\sigma = \frac{a(\phi)}{8a_1a_2[\,a^2(\phi) - \rho^2\,]^{1/2}}\left[\frac{9}{4}\left(\frac{M_x y}{a_2^2} - \frac{M_y x}{a_1^2}\right)\right]. \tag{3.4.32}$$

The results due to (32) can be compared with the numerical data received in personal communication from de Smedt. In order to make the comparison possible, one should put in (32) M_x=0, replace M_y by (21), with the result

$$\sigma = \frac{9\sqrt{\varepsilon}\,a(\phi)\,\nu_y\,x}{4a_1[\,a^2(\phi) - \rho^2\,]^{1/2}}. \tag{3.4.33}$$

Computations due to (33) were made for $\varepsilon = 0.5$ along the axis Ox, the value ν_y was taken 0.8708 (see the table above). Here are the results compared to those communicated by de Smedt

x/a_1=	0.0833	0.2500	0.3333	0.5000	0.6667	0.7500	0.9167
de Smedt σ=	0.1143	0.3501	0.4759	0.7523	1.1460	1.4304	2.8182
Formula (33) σ=	0.1159	0.3577	0.4898	0.7999	1.2392	1.5709	3.1777
Discrepancy (%)	-1.3	-2.2	-2.9	-6.3	-8.1	-9.8	-12.8

We can also compare the same values along the axis Oy. One can use a formula similar to (33) replacing all x by y and interchanging a_1 and a_2, the value of ν_x was taken to be 0.3265.

y/a_2=	0.1667	0.3333	0.5000	0.6667	0.8333
de Smedt σ=	0.1756	0.3663	0.6011	0.9014	1.6413
our result σ=	0.1756	0.3673	0.5998	0.9292	1.5662
Discrepancy (%)	0.0	-0.3	0.2	-3.1	4.6

The agreement is satisfactory.

Rhombus. Let α be the angle at one of the rhombus apexes, and l be its side. Formulae (23–26) in this case yield

$$I_x = \frac{1}{6} l^4 \sin\alpha \sin^2\frac{\alpha}{2}, \qquad I_y = \frac{1}{6} l^4 \sin\alpha \cos^2\frac{\alpha}{2}, \qquad A = l^2 \sin\alpha,$$

$$J_x = 2l \sin\alpha \left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} \ln\frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right],$$

$$J_y = 2l \sin\alpha \left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} \ln\frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right].$$

The coefficients will be defined as

$$\nu_x = \frac{8\sin^2\frac{\alpha}{2}}{9(\sin\alpha)^{3/2} \left[\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} \ln\frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right]},$$

$$\nu_y = \frac{8\cos^2\frac{\alpha}{2}}{9(\sin\alpha)^{3/2} \left[-\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} \ln\frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right]}. \qquad (3.4.34)$$

The same formulae in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = b/a$ has the form

$$\nu_x = \frac{2\sqrt{2\varepsilon}\,(1+\varepsilon^2)}{9\left[1 - \varepsilon + \frac{\varepsilon^2}{\sqrt{1+\varepsilon^2}} \ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}} \right]},$$

$$\nu_y = \frac{2\sqrt{2}\,(1+\varepsilon^2)}{9\varepsilon^{3/2}\left[\varepsilon - 1 + \frac{1}{\sqrt{1+\varepsilon^2}} \ln\frac{1+\varepsilon+\sqrt{1+\varepsilon^2}}{1+\varepsilon-\sqrt{1+\varepsilon^2}} \right]}. \qquad (3.4.35)$$

The coefficients of magnetic polarizability of a diamond were computed by de

Smedt (1979). Here, we present his results compared to those given by formula (35)

ε=	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x=	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (35) ν_x=	0.1078	0.1627	0.2258	0.2986	0.4026	0.4230	0.5043
de Smedt ν_y=	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (35) ν_y=	4.5987	2.1982	1.3254	0.9095	0.6388	0.6052	0.5043
Discrepancy of ν_x (%)	8.7	5.9	3.6	2.2	1.8	2.1	2.9
Discrepancy of ν_y (%)	25.6	18.8	13.0	8.6	4.7	4.3	2.9

The deterioration of the accuracy of (35) for small values of ε is the result of erroneous assumption of a square root singularity in (6) which is grossly incorrect for domains with sharp angles.

Circular segment. Let the radius r and the angle 2α be the segment parameters. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}. \tag{3.4.36}$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k\cos\phi + (1 - k^2\sin^2\phi)^{1/2}] \quad \text{for} \quad 0 \le \phi \le \pi - \gamma \ \text{ or } \ \pi + \gamma \le \phi < 2\pi,$$

and

$$a(\phi) = r\,\frac{k - \cos\alpha}{\cos(\pi - \phi)} \qquad \text{for} \quad \pi - \gamma \le \phi \le \pi + \gamma. \tag{3.4.37}$$

Computation of the moments yields

$$A = r^2(\alpha - \tfrac{1}{2}\sin 2\alpha), \quad I_x = \tfrac{1}{4}Ar^2(1 - k\cos\alpha), \quad I_y = \tfrac{1}{4}Ar^2(1 + 3k\cos\alpha - 4k^2),$$

$$J_x = \frac{2}{3}r\left\{-k\sin^3\gamma + (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\pi-\gamma, k)\right.$$

$$\left. + \frac{2k^2-1}{k^2}E(\pi-\gamma, k) + 3(k - \cos\alpha)\left[-\sin\gamma + \ln\tan\left(\frac{\pi}{4} + \frac{\gamma}{2}\right)\right]\right\},$$

$$J_y = \frac{2}{3} r \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1-k^2\sin^2\gamma)^{1/2}\cos\gamma \right] \right.$$

$$\left. - \frac{1-k^2}{k^2} F(\pi-\gamma, k) + \frac{1+k^2}{k^2} E(\pi-\gamma, k) \right\},$$

where $\gamma = \tan^{-1}(\sin\alpha/(k-\cos\alpha))$. Substituting in (22) leads to

$$\nu_x = \frac{4(1-k\cos\alpha)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ -k\sin^3\gamma + (1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\pi-\gamma, k) \right.$$

$$\left. + \frac{2k^2-1}{k^2}E(\pi-\gamma, k) + 3(k-\cos\alpha)\left[-\sin\gamma + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\right] \right\}^{-1},$$

$$\nu_y = \frac{4(1+3k\cos\alpha-4k^2)}{\left[\alpha - \frac{1}{2}\sin 2\alpha\right]^{1/2}} \left\{ \sin\gamma \left[k\sin^2\gamma - 3\cos\alpha - (1-k^2\sin^2\gamma)^{1/2}\cos\gamma \right] \right.$$

$$\left. - \frac{1-k^2}{k^2} F(\pi-\gamma, k) + \frac{1+k^2}{k^2} E(\pi-\gamma, k) \right\}^{-1}. \qquad (3.4.38)$$

A plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in Fig. 3.4.1. We are unaware of any data to verify the accuracy of (38).

Circular sector. Repetition of the procedure, described in the previous paragraph, leads to the following results for a circular sector with the angle 2α:

$$A = r^2\alpha, \quad I_x = \frac{1}{4} r^4(\alpha - \frac{1}{2}\sin 2\alpha), \quad I_y = r^4 \frac{9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha}{36\alpha},$$

$$J_x = \frac{2}{3} r \left\{ -k\sin^3\gamma - (1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\gamma, k) \right.$$

Fig. 3.4.1. Coefficients of magnetic polarizability for circular segment

$$+\frac{2k^2-1}{k^2}E(\gamma,k)+3k\sin\alpha\left[\cos\alpha+\cos(\alpha+\gamma)\right.$$

$$\left.\left.+\sin^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\right]\right\},$$

$$J_y=\frac{2}{3}r\left\{k\sin\gamma(\sin^2\gamma-3)+(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma-\frac{1-k^2}{k^2}F(\gamma,k)\right.$$

$$+\frac{1+k^2}{k^2}E(\gamma,k)+3k\sin\alpha\left[-\cos\alpha-\cos(\alpha+\gamma)\right.$$

$$\left.\left.+\cos^2\alpha\ln(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2})\right]\right\}.$$

(3.4.39)

Here, $k = 2\sin\alpha/(3\alpha)$, and $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$. The coefficients sought are expressed as follows

$$\nu_x = 2\alpha^{-3/2}(2\alpha - \sin 2\alpha)\Bigg\{ -k\sin^3\gamma - (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma$$

$$+ \frac{1-k^2}{k^2}F(\gamma, k) + \frac{2k^2-1}{k^2}E(\gamma, k) + 3k\sin\alpha\Bigg[\cos\alpha$$

$$+ \cos(\alpha+\gamma) + \sin^2\alpha \ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\Bigg]\Bigg\}^{-1},$$

$$\nu_y = \frac{4(9\alpha^2 + 9\alpha\sin\alpha\cos\alpha - 16\sin^2\alpha)}{9\alpha^{5/2}}\Bigg\{ k\sin\gamma(\sin^2\gamma - 3)$$

$$+ (1 - k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma - \frac{1-k^2}{k^2}F(\gamma, k) + \frac{1+k^2}{k^2}E(\gamma, k)$$

$$+ 3k\sin\alpha\Bigg[-\cos\alpha - \cos(\alpha+\gamma) + \cos^2\alpha \ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\Bigg]\Bigg\}^{-1}. \qquad (3.4.40)$$

Formulae (40) are exact for a complete circle ($\alpha = \pi$), and give the same results as (38) for a half-circle ($\alpha = \pi/2$). A plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in Fig. 3.4.2.

Cross. Consider an aperture obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon = b/a$. The area and the moments will take the form

$$A = 4a^2\varepsilon(2-\varepsilon), \qquad I_x = I_y = \frac{4}{3}a^4\varepsilon(1+\varepsilon^2-\varepsilon^3),$$

$$J_x = J_y = 4a\left[\ln(\varepsilon + \sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon}\right].$$

The coefficients will be defined as

$$\nu_x = \nu_y = \frac{4\varepsilon(1+\varepsilon^2-\varepsilon^3)}{9\varepsilon(2-\varepsilon)^{3/2}}\left[\ln(\varepsilon + \sqrt{1+\varepsilon^2}) + \varepsilon\ln\frac{1+\sqrt{1+\varepsilon^2}}{(1+\sqrt{2})\varepsilon}\right]^{-1}. \qquad (3.4.41)$$

Fig. 3.4.2. Coefficients of magnetic polarizability for circular sector

The comparison between the results due to (41) and those given by de Smedt (1979) are presented below

$\varepsilon=$	0.1000	0.2000	0.3333	0.4000	0.5000	0.6000	0.7500	0.8000	1.0000
de Smedt $\nu_x=\nu_y=$	1.5910	0.8720	0.6255	0.5725	0.5267	0.5069	0.4985	0.4997	0.5193
Formula (41) $\nu_x=\nu_y=$	1.7382	0.8758	0.6006	0.5465	0.5049	0.4890	0.4893	0.4926	0.5043
Discrepancy (%)	-9.3	-0.4	4.0	4.5	4.1	3.5	1.9	1.4	2.9

Taking into consideration the shape complexity, we should consider the results agreement as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$.

Variational approach. The accuracy of formulae (22) can be improved by taking into consideration the fifth harmonic (12) in combination with the variational approach (Noble 1960). The following functional assumes its maximum value at the exact solution of (1)

$$I(\sigma) = 2\iint_S \sigma(M)w(M)\mathrm{d}S_M - \iint_S \sigma(M)\left[\iint_S \frac{\sigma(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \qquad (3.4.42)$$

Taking

$$\iint_S \frac{\sigma(N)}{R(M,N)}\,\mathrm{d}S_N \approx w_1 + w_5, \qquad (3.4.43)$$

and substituting (6), (10), (12) and (43) in (42), one gets after integration with respect to ρ

$$I=\int_0^{2\pi}(a(\phi))^4\left\{(p_1\cos\phi+p_2\sin\phi)\left[\frac{4}{3}(\alpha_x\sin\phi-\alpha_y\cos\phi)-\frac{\pi}{3}(p_1J_y+p_2J_{xy})\cos\phi\right.\right.$$

$$-\frac{\pi}{3}(p_1J_{xy}+p_2J_x)\sin\phi-\frac{2\pi}{63}(a(\phi))^3([p_1(A_{c6}+A_{c4})$$

$$\left.\left.+p_2(A_{s6}-A_{s4})]\cos5\phi+[p_1(A_{s6}+A_{s4})+p_2(A_{c4}-A_{c6})]\sin5\phi)\right]\right\}\mathrm{d}\phi. \tag{3.4.44}$$

Considering now the functional I as a function of p_1 and p_2, the extremum conditions

$$\frac{\partial I}{\partial p_1}=0\,,\qquad \frac{\partial I}{\partial p_2}=0$$

give two linear algebraic equations with respect to the unknowns p_1 and p_2. The complete solution is pretty cumbersome. Here, we present the final result for the coefficients ν_x and ν_y which are valid only for domains having at least one axis of symmetry, and the central principal axes taken as the coordinate axes

$$\nu_x=\frac{32I_x}{3A^{3/2}J_x(1+\eta_x)}\,,\qquad \nu_y=\frac{32I_y}{3A^{3/2}J_y(1+\eta_y)} \tag{3.4.45}$$

where the correction terms

$$\eta_x=\frac{(B_{c4}-B_{c6})(A_{c4}-A_{c6})}{84I_xJ_x}\,,\qquad \eta_y=\frac{(B_{c4}+B_{c6})(A_{c4}+A_{c6})}{84I_yJ_y}\,, \tag{3.4.46}$$

and

$$B_{c6}=\int_0^{2\pi}(a(\phi))^7\cos6\phi\,\mathrm{d}\phi\,,\qquad B_{c4}=\int_0^{2\pi}(a(\phi))^7\cos4\phi\,\mathrm{d}\phi\,.$$

Since expression (43) is approximate, there is no guarantee that (45) will be more accurate than (22). We performed the necessary computations for a rectangle. Here are the results compared to those by de Smedt (1979)

$\varepsilon=$	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt $\nu_x=$	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (45) $\nu_x=$	0.1403	0.1980	0.2558	0.3175	0.4165	0.4396	0.5510

de Smedt ν_y=	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (45) ν_y=	4.5294	2.0709	1.2355	0.8717	0.6606	0.6350	0.5510
Discrepancy in ν_x (%)	-9.0	-5.3	-1.1	2.3	1.8	0.9	-6.1
Discrepancy in ν_y (%)	-10.3	-2.2	1.9	2.0	-2.8	-3.6	-6.1

Comparison with similar data computed on the basis of formula (31) shows that the correction terms η_x and η_y in this particular case resulted in decreasing of the maximum value of discrepancy. We caution again that there is no guarantee that this will be valid for an arbitrary domain. The following rule of thumb may be suggested for the user wishing to improve the accuracy: when the value of the correction coefficients η_x and η_y does not exceed small percentage of unity this generally means an improvement in accuracy, otherwise one should not use formulae (45).

It is worthwhile to give the solution due to (44) for the case when the aperture has no axis of symmetry, and only the first harmonic of w_1 is taken into consideration. The result is

$$p_1 = \frac{\alpha_x(c_{22}I_{xy} - c_{12}I_x) + \alpha_y(c_{12}I_{xy} - c_{22}I_y)}{c_{11}c_{22} - c_{12}^2},$$

$$p_2 = \frac{\alpha_x(c_{11}I_x - c_{12}I_{xy}) + \alpha_y(c_{12}I_y - c_{11}I_{xy})}{c_{11}c_{22} - c_{12}^2}, \tag{3.4.47}$$

where

$$c_{11} = \frac{\pi}{2}(J_y I_y + J_{xy}I_{xy}), \quad c_{22} = \frac{\pi}{2}(J_x I_x + J_{xy}I_{xy}),$$

$$c_{12} = \frac{\pi}{4}(J_{xy}(I_x + I_y) + I_{xy}(J_x + J_y)).$$

Formulae (47) look different from the equivalent set (14) derived earlier. In the absence of any numerical data related to a general domain, it is impossible to say whether formulae (47) are more accurate than (14), but they are definitely more complicated. It is noteworthy that in the case of a domain with an axis of symmetry both (47) and (14) simplify to the same equations (17).

3.5. Electrical polarizability of small apertures

The method of previous section is used here for analytical solution to the problem of electrical polarizability of arbitrarily shaped small apertures. A simple general formula is established for the computation of the coefficients of electrical polarizability. Specific formulae are derived for apertures of various shapes. A comparison is made with the results available in the literature.

At the present time closed form expression for electric polarizability is known for an elliptic aperture in a planar screen only. Some specific shapes were treated either experimentally (Cohn 1952) or numerically (Okon and Harrington 1981, De Meulenaere and Van Bladel 1977). A variational approach to the problem was proposed by Fikhmanas and Fridberg (1973) who also suggested an empirical formula for evaluating the coefficients of electrical polarizability which will be discussed further. No analytical approach for the case of non-elliptical apertures has been reported as yet. The first attempt to do so was made in (Fabrikant, 1987d).

Consider a flat screen conceived as the plane z=0, with an electrically small aperture S whose boundary is given in the polar coordinates as

$$\rho = a(\phi). \tag{3.5.1}$$

It is well known (*e.g.* De Meulenaere and Van Bladel 1977) that the governing integral equation for the electric polarizability density w can be written in the form

$$\sigma(N) = \Delta \int\int_S \frac{w(M)}{R(M,N)} \mathrm{d}S. \tag{3.5.2}$$

where Δ is the two-dimensional Laplace operator, S is the aperture domain, $R(M,N)$ stands for the distance between the points M and N, σ is a known function ($\sigma = -2\pi/\sqrt{A}$, where A is the aperture area). We use again the integral representation

$$\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right) \mathrm{d}x}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-x^2}}. \tag{3.5.3}$$

Substitution of (3) into (2) gives, after changing the order of integration

$$\sigma(\rho,\phi) = \frac{2}{\pi}\Delta\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_0^{2\pi} \mathrm{d}\phi_0 \int_x^{a(\phi_0)} \frac{\lambda\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right)}{\sqrt{\rho_0^2-x^2}} w(\rho_0,\phi_0)\,\rho_0 \mathrm{d}\rho_0. \tag{3.5.4}$$

Again we have to state that the change of the order of integration which led to (4) is valid inside the circle $\rho \le min\{a(\phi)\}$ only. Nevertheless, its usefulness will be demonstrated quite convincingly.

Let the electrical polarizability density w be

$$w = \frac{\delta}{a(\phi)} \sqrt{a^2(\phi) - \rho^2}, \tag{3.5.5}$$

where δ is a constant to be defined. Now substituting (5) in (4), we can verify how close to a constant will be the value of σ produced by (5). Integration with respect to ρ_0 gives

$$\sigma(\rho, \phi) = \frac{\delta}{2} \sum_{n=-\infty}^{\infty} \Delta \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x\,\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_0^{2\pi} \frac{a^2(\phi_0) - x^2}{a^2(\phi_0)}$$

$$\times F\left(2 - \frac{|n|}{2}, \frac{1}{2}; 2; 1 - \frac{x^2}{a^2(\phi_0)}\right) \mathrm{e}^{in(\phi - \phi_0)}\,\mathrm{d}\phi_0. \tag{3.5.6}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the value of σ can be done separately for each harmonic. The zero*th* harmonic has the form

$$\sigma_0 = \frac{\pi\delta G}{2}, \tag{3.5.7}$$

where the notation was introduced

$$G = \int_0^{2\pi} \frac{\mathrm{d}\phi}{a(\phi)}. \tag{3.5.8}$$

It is quite clear that the integral value in (8) will depend not only on the domain contour but also on the location of the system of coordinate origin. The following logic might be useful for establishing certain rule in this regard. According to (5), the coordinate origin location corresponds to the point where the electrical polarizability density attains its maximum. We shall call this point *the aperture center*. In the case of an aperture domain with one axis of symmetry, we may conclude from physical considerations that this point should be located at the axis. When this domain possesses two axes of symmetry the location of the aperture center is at their intersection, e.g. at the center of gravity of the domain. It is noteworthy that the integral (8) attains its minimum in this case. One can extend this rule to a general aperture, namely, the aperture center should be identified with the point inside S where the integral (8) reaches its minimum. Direct computations for various domains indicate that this minimum is, in general, sufficiently flat, so that in many cases one may locate the aperture center at the center of gravity, without significant loss in accuracy.

It is important to note that the second harmonic is equal to zero for an arbitrary contour, and that all the odd harmonics will be zero if the expression for $a(\phi)$ does not contain odd harmonics. Here is the expression for the fourth harmonic

$$\sigma_4 = \frac{16}{5}\delta\rho \int_0^{2\pi} \frac{\cos 4(\phi - \phi_0)\, d\phi_0}{a^2(\phi_0)}. \tag{3.5.9}$$

The investigation of further harmonics shows that their amplitude decreases for general domains, and they vanish in the case of an ellipse. If we assume that $\sigma_0 \approx -2\pi/\sqrt{A}$ then the remaining harmonics may be called the solution error. This means establishment of the following relationship

$$-\frac{2\pi}{\sqrt{A}} = -\frac{\pi\delta G}{2}. \tag{3.5.10}$$

where A is the aperture area. Immediate consequence of (10) is

$$\delta = \frac{4}{G\sqrt{A}}. \tag{3.5.11}$$

One can verify that in the case of an ellipse, the solution given by (5) and (11) is *exact*. We expect it to be reasonably accurate for an aperture of general shape. This assumption will be justified later on. We also expect (5) to be sufficiently accurate in the neighborhood of the aperture center, though the relative error might be quite significant close to the boundary.

Introduce the coefficient of electrical polarizability τ as the average

$$\tau = \frac{1}{A}\int\!\!\int_S w\, dS. \tag{3.5.12}$$

Substitution of (5) in the last expression yields

$$\tau = \frac{2}{3}\delta, \tag{3.5.13}$$

which gives, after comparison with (11)

$$\tau = \frac{8}{3\sqrt{A}G}. \tag{3.5.14}$$

One can deduce that the value of τ does not depend on the size of the domain S, and is defined by its shape only. It attains its maximum in the case of a circle, so that $0 \le \tau \le 4/(3\pi^{3/2}) = 0.2394$. Tabulation of the coefficient τ for various aperture shapes might prove very useful since its knowledge allows to find the maximum (or average) value of the electric polarizability density. An empirical formula for the coefficient of electrical polarizability was proposed by Fikhmanas and Fridberg (1973). This formula in our notation reads

$$\tau = \frac{8\sqrt{A}}{3\pi L} \tag{3.5.15}$$

where L stands for the perimeter of the domain S. Formula (15) is also exact for an ellipse. It would be interesting to compare its performance with our (14). Several aperture shapes are considered below.

Polygon. Consider a polygon with n sides, with the only limitation that the function $a(\phi)$ describing its boundary be continuous and single-valued. The origin of the coordinate system is located at the aperture center as it is defined above. Let us number the polygon sides in a counter-clockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote b_k the distance from the aperture center to the kth apex. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. Then formulae (8) and (14) yield the following expression for the coefficient τ

$$\tau = \frac{8}{3\sqrt{A}} \left\{ \sum_{k=1}^{n} \left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_k^2} \right]^{1/2} + \left[\frac{a_k^2}{4A_k^2} - \frac{1}{b_{k+1}^2} \right]^{1/2} \right\}^{-1}. \tag{3.5.16}$$

In the case of a regular polygon formula (16) simplifies to

$$\tau = \frac{4\sqrt{\cot(\pi/n)}}{3n^{3/2}\sin(\pi/n)}. \tag{3.5.17}$$

Formula (15) gives for a regular polygon

$$\tau = \frac{4}{3\pi} \left[\frac{\cot(\pi/n)}{n} \right]^{1/2} \tag{3.5.18}$$

It is of interest to compare the numerical results due to (17) and (18). Here the relevant computations are presented

$n=$	3	4	5	6	9	100
formula (17) $\tau=$	0.2251	0.2357	0.2380	0.2388	0.2393	0.2394
formula (18) $\tau=$	0.1862	0.2122	0.2227	0.2280	0.2345	0.2394
discrepancy %	17.3	10.0	6.5	4.5	2.0	0.0

While both formulae in the limiting case $n \to \infty$ give the same exact result for a circle, their discrepancy for small n is quite significant, so it is important to establish which one is more accurate. We did not find any reliable data for equilateral triangle. If one takes the experimental result by Cohn for a square $\tau = 0.2274$ as exact, then our formula (17) is in error by 3.6% while formula (18) due to Fikhmanas and Fridberg is in error by 6.7%. The numerical result due to Okon and Harrington for a square is 0.2258 which also favors our formula. In the case of a regular hexagon, the result by Okon and Harrington is 0.2375, so that our result is about 0.5% away while the error of (18) is 4%. It is noteworthy that the value of τ does not change significantly in the whole range $3 \le n < \infty$.

We can also compare the electric polarizability density distribution along a central line of a hexagon perpendicular to its side, given by Okon and Harrington (1981), with those due to (5). Here are the data

$\rho/a=$	0.	0.1667	0.3333	0.5000	0.6667	0.8333
Okon *et al* $w=$	0.351	0.346	0.331	0.305	0.263	0.210
formula (5) $w=$	0.357	0.352	.3366	0.3092	0.2660	0.1973
discrepancy %	-1.7	-1.7	-1.4	-1.4	-1.2	6.0

As we expected, the agreement is good, except for the points very close to the boundary.

Rectangle. Consider a rectangular aperture, a and b being its semiaxes along the axes Ox and Oy respectively. Introduce the aspect ratio $\varepsilon = b/a \le 1$. Formula (17) in this case reduces to

$$\tau = \frac{\sqrt{\varepsilon}}{3(1+\varepsilon^2)^{1/2}}. \tag{3.5.19}$$

Formula (15) in this case gives

$$\tau = \frac{4\sqrt{\varepsilon}}{3\pi(1+\varepsilon)} \tag{3.5.20}$$

We present below the results of computations due to (19) and (20) compared with the experimental results by Cohn (1952)

$\varepsilon=$	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
experiment $\tau=$	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (19) $\tau=$	0.1049	0.1277	0.1462	0.1749	0.2108	0.2309	0.2357
discrepancy %	12.7	9.5	6.6	2.3	-0.7	-2.6	-3.7

formula (20) τ=	0.1220	0.1429	0.1582	0.1788	0.2001	0.2100	0.2122
discrepancy %	-1.5	-1.3	-1.1	0.1	4.4	6.7	6.7

If one assumes the results by Cohn as exact then our formula performs better for $\varepsilon\geq 0.5$ while the formula by Fikhmanas and Fridberg is more accurate for $\varepsilon<0.5$. If we take the numerical results received in a personal communication from De Smedt as correct then the conclusion might be different. For example, his value of τ for $\varepsilon=0.1$ is 0.1142; now our result is in error by 8% while the result by Fikhmanas and Fridberg is in error by -7%. At this moment nobody seems to know which estimation is correct. We can also compare the distribution of w due to (5) with the numerical results received in a personal communication from De Smedt for a rectangle with the aspect ratio $\varepsilon=0.5$. Here are the data computed along the axis Ox for $y/b=0.025$.

x/a=	0.0250	0.2250	0.4250	0.6250	0.8250	0.9750
De Smedt w=	0.3161	0.3118	0.2989	0.2713	0.2107	0.0852
formula (5) w=	0.3158	0.3081	0.2862	0.2469	0.1787	0.0703
Discrepancy %	0.1	1.2	4.2	9.0	15.2	17.5

The agreement is not bad except for the zone close to the boundary. Here are the data computed along the axis Oy for $x/a=0.025$.

y/b=	0.0250	0.1250	0.2250	0.3250	0.4250	0.4750
De Smedt w=	0.3161	0.3067	0.2836	0.2424	0.1690	0.0976
formula (5) w=	0.3158	0.3062	0.2824	0.2403	0.1666	0.0987
Discrepancy %	0.1	0.2	0.4	0.8	1.4	-1.2

We observe here a good agreement even close to the boundary.

Rhombus. Consider the case when the domain S is a rhombus, a and b being its semiaxes along the axes Ox and Oy respectively. Introduce the aspect ratio $\varepsilon=b/a\leq 1$. Formula (16) in this case reduces to

$$\tau=\frac{\sqrt{2\varepsilon}}{3(1+\varepsilon)}. \tag{3.5.21}$$

The result due to Fikhmanas and Fridberg is

$$\tau=\frac{2\sqrt{2\varepsilon}}{3\pi(1+\varepsilon^2)^{1/2}}. \tag{3.5.22}$$

The coefficient of electrical polarizability for a diamond with the aspect ratio $\varepsilon=0.5$ was found numerically by Okon and Harrington as $\tau=0.2082$. Our result is 0.2222 (discrepancy 6.7%) while formula (22) gives 0.1898 (discrepancy 8.9%). We have received two sets of data in personal communications from De Smedt and Lee. Here are the data received as compared to formulae (21) and (22)

ε=	0.100	0.200	0.333	0.500	0.800	1.000

De Smedt τ=	0.111	0.151	0.182	0.204	0.219	0.221
formula (21) τ=	0.136	0.176	0.204	0.222	0.234	0.236
Discrepancy %	-21.9	-16.4	-12.0	-9.0	-6.8	-6.6
formula (22) τ=	0.094	0.132	0.164	0.190	0.210	0.212
Discrepancy %	15.1	12.8	9.8	6.9	4.4	4.1

The data received from Lee is given as a function of the angle $\alpha = \tan^{-1}\varepsilon$

α(deg.)=	10.0	15.0	20.0	25.0	30.0	40.0	45.0
Lee τ=	0.147	0.174	0.193	0.207	0.216	0.226	0.228
formula (21) τ=	0.168	0.192	0.209	0.220	0.227	0.235	0.236
Discrepancy %	-14.2	-10.6	-8.1	-6.3	-5.2	-3.8	-3.6
formula (22) τ=	0.124	0.150	0.170	0.186	0.197	0.211	0.212
Discrepancy %	15.8	13.7	11.8	10.1	8.5	6.9	6.7

We have presented both sets of data in order to underline the fact that there is no really reliable data as yet. The first set of data makes the formula by Fikhmanas and Fridberg more accurate, while the second set favors ours. It is noteworthy that formula (21) seems to give the upper bound, and formula (22) provides the lower bound, their average being very close to the numerical data.

We can also compare the value of electrical polarizability density due to our (5) with a similar result due to Okon and Harrington (1981). Here are the data computed along a central line parallel to its side

ρ/a=	.0	0.3333	0.6667
Okon *et al.* w=	0.335	0.304	0.257
formula (5) w=	0.3333	0.3142	0.2484
discrepancy %	0.5	-3.4	3.3

The agreement is worse if the comparison is made along the major axis. This is mainly due to the assumption of a square root singularity in (5) which does not hold for domains with sharp angles.

Circular segment. Let the radius r and the angle 2α be the segment parameters. Direct numerical computations show that the aperture center can be identified with the center of gravity, with the error comparable with the accuracy of the theory presented. The location of its center of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}.$$

The equation of the segment boundary with respect to its center of gravity takes the form

$$a(\phi) = r[-k\cos\phi + (1 - k^2\sin^2\phi)^{1/2}] \quad \text{for} \quad 0 \le \phi \le \pi - \gamma \ \text{ or } \ \pi + \gamma \le \phi < 2\pi,$$

and

$$a(\phi) = r\,\frac{k-\cos\alpha}{\cos(\pi-\phi)} \qquad \text{for} \quad \pi-\gamma \le \phi \le \pi+\gamma. \tag{3.5.23}$$

Feeding of (23) in (8) and (14) gives

$$\tau = \frac{4}{3(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}\left[\frac{k\sin\gamma + E(\pi-\gamma,k)}{1-k^2} + \frac{\sin\gamma}{k-\cos\alpha}\right]^{-1}, \tag{3.5.24}$$

where $\gamma = \tan^{-1}(\sin\alpha/(k - \cos\alpha))$. The formula due to Fikhmanas and Fridberg will take the form

$$\tau = \frac{4(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}{3\pi(\alpha+\sin\alpha)}. \tag{3.5.25}$$

The coefficient of electrical polarizability for a semi-circle was computed by Okon and Harrington as $\tau=0.2161$. Our result due to (24) is $\tau=0.2163$ which is practically identical to the previously mentioned one. The result due to (25) is $\tau=0.2069$ (discrepancy 4.3%). An additional confirmation of correctness of the new method can be obtained by observing the plot of the electrical polarizability density distribution for a semi-circle presented by Okon and Harrington (1981). Its maximum is located at the distance $\approx 0.47r$ from the circle's center. Our definition of the aperture center requiring the minimization of the integral (8) gives its coordinate at $0.48r$ which is very close. The center of gravity of the semi-circle is located at $0.42r$. Figure 3.5.1 plots the value of τ against α/π due to formulae (24) (solid line) and (25) (broken line).

Circular sector. Let r and 2α be its radius and the polar angle. The aperture center is assumed to be located on the axis of symmetry at a distance kr from the circle's center. Direct computations show that the aperture center may be located at the center of gravity for $0.1\pi<\alpha<0.6\pi$. In this case the value of k is defined by $k=2\sin\alpha/(3\alpha)$. In the range $\alpha<0.1\pi$ or $\alpha>0.6\pi$, the value of k should be found from the minimum condition for the integral (8). Repetition of the procedure, described in the previous paragraph, leads to the following result

$$\tau = \frac{4}{3\sqrt{\alpha}}\left[\frac{k\sin\gamma + E(\gamma,k)}{1-k^2} + \frac{\cos\alpha + \cos(\alpha-\gamma)}{k\sin\gamma}\right]^{-1}. \tag{3.5.26}$$

Here, $\gamma = \tan^{-1}(\sin\alpha/(\cos\alpha - k))$. The formula due to Fikhmanas and Fridberg reads

Fig. 3.5.1. Coefficient of electrical polarizability for circular segment

$$\tau = \frac{4\sqrt{\alpha}}{3\pi(1+\alpha)}. \tag{3.5.27}$$

Note that neither (26) nor (27) reduce to the exact value for a circle when $\alpha=\pi$. This is due to the fact that we do not really have the case of a complete circular aperture when α approaches π: we have a circular aperture which has its radius $\phi=\pi$ grounded. This case has not been considered by other authors so we can not say which formula is more accurate. Okon and Harrington in the case of a quadrant obtained $\tau=0.2269$, formula (26) gives $\tau=0.2308$ (discrepancy 1.7%), and formula (27) gives $\tau=0.2107$ (discrepancy 7%). Figure 3.5.2 plots the value of τ against α/π due to formulae (26) (solid line) and (27) (broken line).

Cross. Consider an aperture configuration obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon = b/a \le 1$. The area can be expressed as

$$A = 4a^2\varepsilon(2-\varepsilon),$$

The following expression can be obtained for τ, namely,

Fig. 3.5.2. Coefficient of electrical polarizability for circular sector

$$\tau = \frac{\sqrt{2\varepsilon}}{6\sqrt{2-\varepsilon}\,\{[2(1+\varepsilon^2)]^{1/2} - 1\}}\,. \tag{3.5.28}$$

The formula due to Fikhmanas and Fridberg is

$$\tau = \frac{2\sqrt{\varepsilon(2-\varepsilon)}}{3\pi}\,. \tag{3.5.29}$$

Here, we present the results given by formulae (28) and (29) compared to the experimental results by Cohn (1952) and the numerical results by De Meulenaere and Van Bladel (1977), and those received in personal communication from De Smedt

$\varepsilon=$	0.1000	0.2000	0.3000	0.4000	0.6000	0.8000	1.0000
experimental $\tau=$	0.0942	0.1333	0.1609	–	–	–	0.2274
De Meulenaere $\tau=$	–	–	–	0.19	0.22	0.23	0.238
De Smedt $\tau=$	0.0835	0.1183	–	0.1767	0.2084	0.2193	0.2212
formula (28) $\tau=$	0.1284	0.1777	0.2078	0.2252	0.2376	0.2372	0.2357
formula (29) $\tau=$	0.0925	0.1273	0.1515	0.1698	0.1944	0.2079	0.2122

We did not compute the discrepancy since the data disagreement is too big thus making all the data not very reliable. The general impression is that our (28) gives the upper bound for τ while the formula due to Fikhmanas and Fridberg provides the lower bound. This conclusion might be wrong if the numerical results received in the personal communication from De Smedt are correct. For example, his result for $\varepsilon=0.1$ is $\tau=0.08347$ which differs from the experimental result by 11%. All this proves one point: the existing numerical methods are too crude, and there is a need to develop some new and more reliable numerical methods. It should be noted that the function defined by (28) is not monotonic: a relatively flat maximum is observed for $\varepsilon\approx0.7$. The remaining data are monotonic. We have no rigorous proof to claim that the quantitative behavior of (28) is correct while the other data behavior is not, but we can indicate that the value of τ for a quadrant is also greater than that for a semi-circle, and this is mainly due to the fact that the shape of a quadrant is more close to a circle than the shape of a semi-circle. Similar statement can be made about a cross with the aspect ratio $\varepsilon\approx0.7$ as compared to a square.

Majority of the examples considered indicate that the exact result is sandwiched between those given by our (14) and by the formula due to Fikhmanas and Fridberg (15). In this sense both formulae act as an upper bound and a lower bound which leads to a conjecture: for a certain class of contours one of the inequalities holds, namely, either $\tau_{14}\le\tau_{exact}\le\tau_{15}$, or $\tau_{14}\ge\tau_{exact}\ge\tau_{15}$. A significant effort is required to find such class of contours for which one of the conjectures holds, and it is beyond the scope of this book.

The accuracy of formula (14) can be improved in some cases by taking into consideration the fourth harmonic (9) in combination with the variational approach (Noble 1960). The following functional assumes its stationary value at the exact solution of (2)

$$I(w) = 2\int\!\!\int_S \sigma(M)w(M)\mathrm{d}S_M - \int\!\!\int_S w(M)\left[\Delta\int\!\!\int_S \frac{w(N)}{R(M,N)}\mathrm{d}S_N\right]\mathrm{d}S_M. \tag{3.5.30}$$

Taking

$$\Delta\int\!\!\int_S \frac{w(N)}{R(M,N)}\mathrm{d}S_N \approx \sigma_0+\sigma_4$$

where σ_0 and σ_4 are defined by (7) and (9) respectively, and substituting them in (30), we obtain a functional which can be considered as a function of δ. From the extremum condition

$$\frac{\partial I}{\partial\delta}=0.$$

one finally gets

$$\tau = \frac{8}{3G\sqrt{A}(1-\eta)}, \tag{3.5.31}$$

where

$$\eta = \frac{3(F_c E_c + F_s E_s)}{5AG},$$

and the following geometrical characteristics were introduced

$$F_c = \int_0^{2\pi} \frac{\cos 4\phi \, d\phi}{a^2(\phi)}, \quad F_s = \int_0^{2\pi} \frac{\sin 4\phi \, d\phi}{a^2(\phi)},$$

$$E_c = \int_0^{2\pi} a^3(\phi) \cos 4\phi \, d\phi, \quad E_s = \int_0^{2\pi} a^3(\phi) \sin 4\phi \, d\phi,$$

The results of computations due to (31) for a rectangle are presented below against the experimental results by Cohn

ε=	0.1000	0.1500	0.2000	0.3000	0.5000	0.7500	1.0000
Cohn τ=	0.1202	0.1411	0.1565	0.1789	0.2093	0.2251	0.2274
formula (31) τ=	0.1054	0.1290	0.1484	0.1785	0.2125	0.2257	0.2278
discrepancy %	12.3	8.6	5.2	0.2	-1.5	-0.3	-0.2

Comparison of this table with a similar one above indicates that the variational approach does improve the accuracy, though the improvement is still not sufficient for small ε. This example can not be considered as a proof of better accuracy of the variational approach. We are quite sure that one can produce examples showing the opposite. It is up to the user to decide whether the more cumbersome computations are worth somewhat better accuracy.

3.6. Dirichlet problem for an annular disk

It is impossible even to mention all the publications related to the Dirichlet problem for a flat circular annulus. Their number is awesome. Tranter (1960) and Gubenko (1960) were among the first to consider the problem. One can find many references related to contact problem in (Borodachev, 1976), other references related to the equivalent electrostatic problem can be found in Love (1976). Majority of publications is devoted to the axisymmetric problems. Though some results related to consideration of specific harmonics have been published (Williams, 1963; Cooke, 1963), no general solution to the problem has been attempted as yet. This kind of solution is now possible. The problem is reduced to a set of two two-dimensional Fredholm integral equations with an elementary non-singular kernel which can be solved by iterations. This set can be easily uncoupled. The case of conducting circular annulus kept at constant potential and the problem of magnetic polarizability of such a disk are considered as examples. The governing integral equations are solved exactly in series involving the iterated kernels. Approximate formulae are derived for the case of a wide annulus.

Theory. It is convenient to reformulate the Dirichlet problem for a circular annulus as a mixed boundary value problem of potential theory for a half space $z \geq 0$. We need to find a harmonic function V vanishing at infinity and satisfying the following conditions at $z=0$:

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad b<\rho<a, \quad 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for} \quad \rho<b \quad \text{or} \quad \rho>a, \quad 0\leq\phi<2\pi. \tag{3.6.1}$$

Here v is a known function. The approach proposed here is inspired by the elegant solution for the capacity of an annulus (Love, 1976) which is based on the method described in (Clement and Love, 1974) for solving axisymmetric problems. It looked very challenging to generalize the approach for non-axisymmetric case. Such a generalization has been found after several trials and errors, and it is presented here. The general approach is based on the results presented in Chapter 1. Let us introduce two harmonic functions

$$V_1(\rho,\phi,z) = -\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{b} \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}}{R_0^2} f_1(\rho_0,\phi_0)\, d\rho_0 d\phi_0$$

$$= -\frac{2}{\pi}\int_0^{b} \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}}{l_2^2(\rho_0) - l_1^2(\rho_0)}\, \mathcal{L}\left(\frac{l_1(\rho_0)}{l_2(\rho_0)}\right) f_1(\rho_0,\phi)\, d\rho_0; \tag{3.6.2}$$

$$V_2(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}}{R_0^2} f_2(\rho_0,\phi_0)\, d\rho_0 d\phi_0$$

$$= \frac{2}{\pi}\int_a^{\infty} \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}}{l_2^2(\rho_0) - l_1^2(\rho_0)}\, \mathcal{L}\left(\frac{l_1(\rho_0)}{l_2(\rho_0)}\right) f_2(\rho_0,\phi)\, d\rho_0. \tag{3.6.3}$$

Here f_1 and f_2 are the as yet unknown functions, and the following notations were introduced:

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2} - \sqrt{(\rho-x)^2+z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho+x)^2+z^2} + \sqrt{(\rho-x)^2+z^2}\},$$

$$R_0 = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}. \tag{3.6.4}$$

We remind that the $\mathcal{L}$-operator for $k<1$ is understood as follows:

$$\mathcal{L}(k)f(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k,\phi-\phi_0)\, f(\rho,\phi_0)\, d\phi_0$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|}e^{in\phi}\int_0^{2\pi} e^{-in\phi_0} f(\rho,\phi_0)\, d\phi_0 = \sum_{n=-\infty}^{\infty} k^{|n|} f_n(\rho)e^{in\phi}. \tag{3.6.5}$$

Here f_n is the n-th Fourier coefficient of the function f, and

$$\lambda(k,\psi) = \frac{1-k^2}{1-2k\cos\psi+k^2}. \tag{3.6.6}$$

One can easily verify that the potential in (2) vanishes on the plane $z=0$ for $\rho>b$, while the potential in (3) vanishes on the boundary for $\rho<a$. These properties allow us to reformulate the problem as the Dirichlet problem for a half space, with the potential prescribed all over the plane $z=0$, namely,

$$V(\rho,\phi,0) = V_1(\rho,\phi,0), \quad \text{for} \quad 0\le\rho<b, \quad 0\le\phi<2\pi;$$

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad b\le\rho\le a, \quad 0\le\phi<2\pi;$$

$$V(\rho,\phi,0) = V_2(\rho,\phi,0), \quad \text{for} \quad a<\rho<\infty, \quad 0\le\phi<2\pi. \tag{3.6.7}$$

Thus, the first boundary condition in (1) is satisfied. The unknown functions f_1 and f_2 are to be chosen in such a way that the second boundary condition in (1) is satisfied too.

Formulae (2) and (3) on the plane $z=0$ take the form

$$V_1(\rho,\phi,0) = -\frac{2}{\pi}\int_\rho^b \mathcal{L}\left(\frac{\rho}{\rho_0}\right)\frac{f_1(\rho_0,\phi)\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}},$$

$$V_2(\rho,\phi,0) = \frac{2}{\pi}\int_a^\rho \mathcal{L}\left(\frac{\rho_0}{\rho}\right)\frac{f_2(\rho_0,\phi)\,\mathrm{d}\rho_0}{\sqrt{\rho^2-\rho_0^2}}. \tag{3.6.8}$$

On the other hand, from (1.3.47) by assuming $z=0$ the following expression can be obtained for the potential which is nonzero in the interval $[0,b]$ and zero outside this interval:

$$V_1(\rho,\phi,0) = 4\int_\rho^b \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_0^x \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi). \tag{3.6.9}$$

Here σ_1 denotes the charge density distribution. Comparison of (7) and (9) yields the following relationship between σ_1 and f_1:

$$f_1(\rho,\phi) = -2\pi\int_0^\rho \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma_1(\rho_0,\phi). \tag{3.6.10}$$

The inverse relationship is readily available, and is

$$\sigma_1(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_0^{\min(\rho,b)} \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}(\rho_0)f_1(\rho_0,\phi). \tag{3.6.11}$$

By comparing in the same manner the expression (1.4.33) for $z=0$

$$V_2(\rho,\phi,0) = 4\int_a^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^\infty \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma_2(\rho_0,\phi), \tag{3.6.12}$$

with (8), we obtain

$$f_2(\rho,\phi) = 2\pi\int_\rho^\infty \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi). \tag{3.6.13}$$

The inverse to (13) takes the form

$$\sigma_2(\rho,\phi) = -\frac{\mathcal{L}(\rho)}{\pi^2\rho}\frac{d}{d\rho}\int_{\max(\rho,a)}^{\infty}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi). \tag{3.6.14}$$

We have presented the resultant field as a superposition of three fields, namely, the field due to potential equal to $V_1(\rho,\phi,0)$ on the interval $0\le\rho\le b$, and zero outside the interval; the field due to potential equal to $v(\rho,\phi)$ in the interval $b\le\rho\le a$, and zero outside this interval; and the field due to potential equal to $V_2(\rho,\phi,0)$ on the interval $a\le\rho<\infty$, and zero outside the interval. The corresponding charge density distributions will be denoted as σ_1, σ_0, and σ_2 respectively. Now we can use the fact that the total charge $\sigma=\sigma_1+\sigma_0+\sigma_2=0$ in the intervals $\rho\le b$ and $\rho\ge a$. These conditions will give us two equations from which the as yet unknown functions f_1 and f_2 can be found. By using the result established in (Fabrikant, 1989a), we can write

$$\sigma_0(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{d}{dx}\int_b^{a}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi), \quad \text{for} \quad \rho\le b. \tag{3.6.15}$$

By using (11), (14), and (15) the following equation may be written for $\rho\le b$:

$$-\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{\rho_0\,d\rho_0}{\sqrt{\rho^2-\rho_0^2}}\mathcal{L}(\rho_0)f_1(\rho_0,\phi) - \frac{\mathcal{L}(\rho)}{\pi^2\rho}\frac{d}{d\rho}\int_a^{\infty}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi)$$

$$-\frac{1}{\pi^2\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_0^{\rho}\frac{x\,dx}{\sqrt{\rho^2-x^2}}\mathcal{L}(x^2)\frac{d}{dx}\int_b^{a}\frac{\rho_0\,d\rho_0}{\sqrt{\rho_0^2-x^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi) = 0. \tag{3.6.16}$$

Application of an operator

$$\int_0^{r}\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\mathcal{L}\left(\frac{\rho}{r}\right)$$

to both sides of (16) yields

$$\frac{1}{2\pi}f_1(r,\phi)+\frac{1}{\pi^2}\int_0^r\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}\int_a^\infty\frac{\rho_0\,d\rho_0}{(\rho_0^2-\rho^2)^{3/2}}\mathcal{L}\left(\frac{\rho^2}{\rho_0 r}\right)f_2(\rho_0,\phi)$$

$$+\frac{1}{2\pi}r\int_b^a\frac{\rho_0\,d\rho_0}{(\rho_0^2-r^2)^{3/2}}\mathcal{L}\left(\frac{r}{\rho_0}\right)v(\rho_0,\phi)=0. \tag{3.6.17}$$

We can interchange the order of integration in the second term of (17) and perform the integration with respect to ρ. Then equation (17) will take the form

$$f_1(\rho,\phi)+\frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty Q_1(\rho,\rho_0,\phi-\phi_0)\,f_2(\rho_0,\phi_0)\,d\rho_0\,d\phi_0=g_1(\rho,\phi), \tag{3.6.18}$$

where

$$g_1(\rho,\phi)=-\rho\int_b^a\frac{\rho_0\,d\rho_0}{(\rho_0^2-\rho^2)^{3/2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)v(\rho_0,\phi), \tag{3.6.19}$$

$$Q_1(\rho,\rho_0,\phi-\phi_0)=2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{(\rho_0 e^{i(\phi-\phi_0)}-\rho)R}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho/\rho_0)}{(\rho_0/\rho)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\}-\frac{\rho}{R^2}, \tag{3.6.20}$$

with

$$R=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{3.6.21}$$

Here the following integral was used

$$\int_0^r\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}\,(y^2-\rho^2)^{3/2}(1-m\rho^2)}=\frac{m}{(my^2-1)^{3/2}\sqrt{1-mr^2}}\tan^{-1}\left[\frac{r\sqrt{my^2-1}}{y\sqrt{1-mr^2}}\right]$$

$$-\frac{r}{y(y^2-r^2)(my^2-1)} \tag{3.6.22}$$

The second equation is obtained from the condition that $\sigma=0$ for $\rho>a$. We write from (Fabrikant, 1989a)

$$\sigma_0(\rho,\phi) = -\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{x\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{1}{x^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}(\rho_0)v(\rho_0,\phi),$$
$$\text{for} \quad \rho > a. \tag{3.6.23}$$

By using (11), (14), and (23) the following equation can be obtained:

$$-\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\mathcal{L}\left(\frac{1}{\rho_0}\right)f_2(\rho_0,\phi) + \frac{1}{\pi^2}\int_0^b \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)f_1(\rho_0,\phi)$$

$$-\frac{1}{\pi^2\rho}\mathcal{L}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{x\,\mathrm{d}x}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{1}{x^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}(\rho_0)v(\rho_0,\phi) = 0. \tag{3.6.24}$$

Let us apply the operator

$$\int_r^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\mathcal{L}\left(\frac{r}{\rho}\right)$$

to both sides of (24). The result is

$$\frac{1}{2\pi}f_2(r,\phi) + \frac{1}{\pi^2}\int_r^\infty \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\int_0^b \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\mathcal{L}\left(\frac{\rho_0 r}{\rho^2}\right)f_1(\rho_0,\phi)$$

$$+\frac{1}{2\pi}\mathcal{L}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\mathcal{L}(\rho_0)\,v(\rho_0,\phi) = 0. \tag{3.6.25}$$

Again we can interchange the order of integration in (25) and perform the integration with respect to ρ, with the result

$$f_2(\rho,\phi) + \frac{1}{\pi^2}\int_0^{2\pi}\int_0^b Q_2(\rho,\rho_0,\phi-\phi_0)f_1(\rho_0,\phi_0)\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = g_2(\rho,\phi). \tag{3.6.26}$$

Here

$$g_2(\rho,\phi) = \rho\int_b^a \frac{\rho_0\,\mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{3/2}}\,L\!\left(\frac{\rho_0}{\rho}\right)\!v(\rho_0,\phi), \tag{3.6.27}$$

$$Q_2(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}}}{(\rho e^{i(\phi-\phi_0)}-\rho_0)R}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\rho_0/\rho)}{(\rho/\rho_0)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\}-\frac{\rho_0}{R^2}, \tag{3.6.28}$$

and R is defined by (21). The following integral was used:

$$\int_r^\infty \frac{\rho^3\mathrm{d}\rho}{\sqrt{\rho^2-r^2}\,(\rho^2-y^2)^{3/2}(\rho^2-m)} = \frac{m}{(m-y^2)^{3/2}\sqrt{r^2-m}}\tan^{-1}\left[\frac{m-y^2}{r^2-m}\right]^{1/2}$$

$$-\frac{y^2}{(r^2-y^2)(m-y^2)}. \tag{3.6.29}$$

We note that $Q_1(\rho,\rho_0,\phi-\phi_0) = Q_2(\rho_0,\rho,\phi-\phi_0)$. This circumstance will allow us to decouple the equations by introduction of new variables. Indeed, substituting in (18) $t\sqrt{ab}$ instead of ρ and $\sqrt{ab}/x$ instead of ρ_0 yields

$$F_1(t,\phi) + \int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)\,F_2(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_1(t,\phi). \tag{3.6.30}$$

Here

$$F_1(t,\phi) = f_1(t\sqrt{ab},\phi),\quad F_2(t,\phi) = f_2(\sqrt{ab}/t,\phi)/t,\quad k=\sqrt{b/a}; \tag{3.6.31}$$

$$G_1(t,\phi) = g_1(t\sqrt{ab},\phi),\quad K(xt,\phi-\phi_0) = \frac{2}{\pi^2}\Re\left\{\frac{\gamma}{R_{xt}}\tan^{-1}\left(\frac{xt}{\gamma R_{xt}}\right)\right\}-\frac{xt}{\pi^2 R_{xt}^2}, \tag{3.6.32}$$

$$R_{xt} = \sqrt{1+x^2t^2-2xt\cos(\phi-\phi_0)},\quad \gamma = \frac{\sqrt{xte^{i(\phi-\phi_0)}}}{e^{i(\phi-\phi_0)}-xt}. \tag{3.6.33}$$

Equation (26) can be transformed by substitution $\rho=\sqrt{ab}\,/t$ and $\rho_0=x\sqrt{ab}$ to exactly the same form as (30), namely,

$$F_2(t,\phi) + \int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)\,F_1(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_2(t,\phi). \tag{3.6.34}$$

Here

$$G_2(t,\phi) = g_2(\sqrt{ab}/t)/t, \tag{3.6.35}$$

and all the remaining notations are given by (31)–(33). Equations (30) and (34) can be easily uncoupled by summation and subtraction

$$F_+(t,\phi) + \int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)\,F_+(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_+(t,\phi), \tag{3.6.36}$$

$$F_-(t,\phi) - \int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)\,F_-(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_-(t,\phi), \tag{3.6.37}$$

where

$$F_\pm = F_1 \pm F_2, \quad G_\pm = G_1 \pm G_2. \tag{3.6.38}$$

Thus the problem has been reduced to two independent integral equations (36) and (37) with elementary non-singular kernels which can be solved by iteration. Convergence of the iteration procedure is not guaranteed for k very close to unity which corresponds to the case of a very narrow annulus. Direct computation of the norm of the kernel in space L_2 gave the result of 0.41 for $k=0.9$, and it was less than 0.8 for $k=0.95$. It is then recommended to use an asymptotical solution for $k>0.95$. We note that the arguments of the kernel x and t do not enter it independently but only as a product xt. The following integral representation is useful for computation of various integrals of the kernel:

$$K(y,\psi) = \frac{y}{2\pi^2}\int_0^1 \frac{\lambda(yz,\psi)\,\mathrm{d}z}{\sqrt{1-z}\,(1-y^2z)^{3/2}}. \tag{3.6.39}$$

We recall that λ is defined by (6). Expression (39) shows that the kernel for each particular harmonic will also be an elementary function. For example, the kernels for the zero and first harmonic will be respectively

$$K_0(xt) = \frac{2}{\pi}\frac{xt}{1-x^2t^2}, \tag{3.6.40}$$

$$K_1(xt) = \frac{2}{\pi}\left[\frac{1}{1-x^2t^2} - \frac{1}{2xt}\ln\left(\frac{1+xt}{1-xt}\right)\right]. \tag{3.6.41}$$

Expression (40) is in agreement with the result of Clement and Love (1974). Expression (41) does not seem to have been reported in the literature. It is important to notice that various integral characteristics of interest can be expressed directly through the function f_2 (or F_2). For example, the total charge Q can be written as a limit

$$Q = \lim_{\rho\to\infty}\{\rho V_2(\rho,\phi,z)\}. \tag{3.6.42}$$

Substitution of (3) in (42) leads to

$$Q = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi = \frac{\sqrt{ab}}{\pi^2}\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\,\frac{\mathrm{d}x}{x}\,\mathrm{d}\phi. \tag{3.6.43}$$

The quantities proportional to magnetic polarizability can be found from

$$\nu_x = \lim_{\substack{\rho\to\infty\\ \phi=0}}\left[\rho^2\frac{\partial V_2}{\partial\phi}\right], \tag{3.6.44}$$

$$\nu_y = \lim_{\substack{\rho\to\infty\\ \phi=\pi/2}}\left[\rho^2\frac{\partial V_2}{\partial\phi}\right]. \tag{3.6.45}$$

Substitution of (3) in (44) and (45) yields respectively

$$\nu_x = \frac{2}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\sin\phi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = \frac{2}{\pi^2}ab\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\sin\phi\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi, \tag{3.6.46}$$

$$\nu_y = -\frac{2}{\pi^2}\int_0^{2\pi}\int_a^{\infty} f_2(\rho,\phi)\cos\phi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = -\frac{2}{\pi^2}ab\int_0^{2\pi}\int_0^{k} F_2(x,\phi)\cos\phi\,\frac{\mathrm{d}x}{x^2}\,\mathrm{d}\phi. \tag{3.6.47}$$

Formulae (30)–(35), (43), and (46)–(47) are the main new results of this section.

Conducting annular disk charged to a unit potential. The governing integral equations in this case will take the form

$$F_1(t) + \int_0^k K_0(xt)\, F_2(x)\, \mathrm{d}x = G_1(t), \tag{3.6.48}$$

$$F_2(t) + \int_0^k K_0(xt)\, F_1(x)\, \mathrm{d}x = G_2(t), \tag{3.6.49}$$

where K_0 is defined by (40),$F_{1,2}$ and $G_{1,2}$ are understood as zero harmonics of the relevant notations (31),(32), and (35). In this particular case

$$G_1 = -\frac{t}{\sqrt{k^2 - t^2}} + \frac{kt}{\sqrt{1 - k^2 t^2}}, \tag{3.6.50}$$

$$G_2 = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2 t^2}}. \tag{3.6.51}$$

Equations (48) and (49) were solved by Love (1976). We present here a slightly different version though based on the same idea, as well as a simple approximate treatment of the problem.

Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_0^{2n}(G_1 - K_0 G_2), \tag{3.6.52}$$

$$F_2 = \sum_{n=0}^{\infty} K_0^{2n}(G_2 - K_0 G_1). \tag{3.6.53}$$

Here K_0^m is understood as the m-th iteration of the kernel. The first iteration in (53) ($n = 0$) yields

$$F_2^{(1)}(t) = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2 t^2}} - \frac{2}{\pi}\, t \int_0^k \left[\frac{k}{\sqrt{1 - k^2 x^2}} - \frac{1}{\sqrt{k^2 - x^2}}\right] \frac{x^2\, \mathrm{d}x}{1 - x^2 t^2}$$

$$= \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t} + \theta(t), \tag{3.6.54}$$

where the notation was introduced

$$\theta(\mathrm{t}) = \frac{2}{\pi t}\left[\sin^{-1}(k^2) - \frac{k}{\sqrt{k^2 - t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2 - t^2}}{\sqrt{1 - k^2 t^2}}\right)\right]. \tag{3.6.55}$$

Now we need to consider the action of K_0^2 on the first iteration (54).

$$K_0^2 F_2^{(1)}(t) = \frac{4}{\pi^2}\int_0^k \frac{ts\,\mathrm{d}s}{1 - t^2 s^2}\int_0^k \left[\frac{k}{x\sqrt{k^2 - x^2}} - \frac{1}{x}\right]\frac{sx\,\mathrm{d}x}{1 - s^2 x^2} + K_0^2\theta(t)$$

$$= \frac{2}{\pi}kt\int_0^k \frac{s^2\,\mathrm{d}s}{(1 - t^2 s^2)\sqrt{1 - k^2 s^2}} - \frac{2}{\pi^2}\int_0^k \ln\left(\frac{1 + ks}{1 - ks}\right)\frac{ts\,\mathrm{d}s}{1 - t^2 s^2} + K_0^2\theta(t)$$

$$= -\theta(t) - \int_0^k K_0^2(xt)\frac{\mathrm{d}x}{x} + K_0^2\theta(t). \tag{3.6.56}$$

Substitution of (56) in (53) shows that all the expressions containing $\theta(t)$ will cancel out at each subsequent iteration. This allows us to write the exact solution in the form

$$F_2(t) = \frac{1}{t}\left[\frac{k}{\sqrt{k^2 - t^2}} - 1\right] - \sum_{n=1}^{\infty}\int_0^k K_0^{2n}(xt)\frac{\mathrm{d}x}{x}. \tag{3.6.57}$$

Since all iterated kernels are positive, the term in square brackets in (57) gives the upper bound for the solution. Substitution of (57) in (48) yields, after simplification,

$$F_1(t) = -\frac{t}{\sqrt{k^2 - t^2}} + \sum_{n=0}^{\infty}\int_0^k K_0^{2n+1}(xt)\frac{\mathrm{d}x}{x}. \tag{3.6.58}$$

Capacitance of the annulus can be found by substitution of (57) in (43), with the result

$$C = \frac{2}{\pi} a \left\{ 1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt)\, \frac{dx}{x} \frac{dt}{t} \right\}. \tag{3.6.59}$$

Taking into consideration that

$$C_0 = \frac{2}{\pi} a \tag{3.6.60}$$

is the capacity of a circular disk of radius a, we can write the expression for the dimensionless capacity C^* which is defined as the ratio

$$C^* = \frac{C}{C_0} = 1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt)\, \frac{dx}{x} \frac{dt}{t}. \tag{3.6.61}$$

The symmetry of x and t in (61) allows us to reduce the order of iterated kernel as follows:

$$C^* = 1 - k \sum_{n=1}^{\infty} \int_0^k \left[\int_0^k K_0^n(xt)\, \frac{dx}{x} \right]^2 dt. \tag{3.6.62}$$

Yet another approximate solution for the dimensionless capacity C^* can be found by a different method. Indeed, from (43) we have

$$C^* = k \int_0^k F_2(t)\, \frac{dt}{t}. \tag{3.6.63}$$

Multiplying both sides of (49) by k/t and integrating with respect to t from 0 to k, we obtain

$$k \int_0^k F_2(t)\, \frac{dt}{t} + \frac{k}{\pi} \int_0^k \ln\left(\frac{1 + kx}{1 - kx} \right) F_1(x)\, dx = \sqrt{1 - k^4} \tag{3.6.64}$$

We can express F_1 from (48) and substitute it into (64). The result is

$$k\int_0^k F_2(t)\,\frac{\mathrm{d}t}{t}-\frac{2k}{\pi^2}\int_0^k T(x)\,F_2(x)\,\frac{\mathrm{d}x}{x}=\frac{2}{\pi}\left[\cos^{-1}(k^2)+\frac{1}{2}\sqrt{1-k^4}\,\ln\left(\frac{1+k^2}{1-k^2}\right)\right], \tag{3.6.65}$$

where

$$T(x)=x^2\int_0^k \ln\left(\frac{1+kt}{1-kt}\right)\frac{t\,\mathrm{d}t}{1-x^2t^2}. \tag{3.6.66}$$

Now we can use the mean value theorem to obtain

$$C^*=\frac{2}{\pi}\left[\cos^{-1}(k^2)+\frac{1}{2}\sqrt{1-k^4}\,\ln\left(\frac{1+k^2}{1-k^2}\right)\right]\left(1-\frac{2}{\pi^2}\,T(X)\right)^{-1}. \tag{3.6.67}$$

According to the mean value theorem, we know about X only that it is located somewhere in the interval $[0,k]$. One needs to find an optimum value for X in order to make (67) useful. This exercise is beyond the scope of this book. When $X=0$, formula (67) coincides with the result of Smythe (1951) who obtained it from physical considerations. Since $T(X)$ is non-negative, the term in square brackets in (67) gives the lower bound. Note also that it is exact in two extreme cases, namely, $k=0$ and $k=1$.

Magnetic polarizability of a circular annulus. In this case we may assume, without loss of generality, that

$$v(\rho,\phi)=v_1\rho\cos\phi, \tag{3.6.68}$$

where v_1 is a constant. The governing integral equations will take the form

$$F_1(t)+\int_0^k K_1(xt)\,F_2(x)\,\mathrm{d}x=G_1(t), \tag{3.6.69}$$

$$F_2(t)+\int_0^k K_1(xt)\,F_1(x)\,\mathrm{d}x=G_2(t), \tag{3.6.70}$$

where K_1 is defined by (41),$F_{1,2}$ and $G_{1,2}$ are understood as first harmonics of the relevant notations (31),(32), and (35). In this particular case

$$G_1(t) = v_1\sqrt{ab}t^2\left[\frac{k}{\sqrt{1-k^2t^2}} - \frac{1}{\sqrt{k^2-t^2}}\right], \tag{3.6.71}$$

$$G_2(t) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - \frac{2-k^2t^2}{\sqrt{1-k^2t^2}}\right]. \tag{3.6.72}$$

Equations (69) and (70) have not been considered before. We employ the same method as above. Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_1^{2n}(G_1 - K_1G_2), \tag{3.6.73}$$

$$F_2 = \sum_{n=0}^{\infty} K_1^{2n}(G_2 - K_1G_1). \tag{3.6.74}$$

Here K_1^m is understood as the m-th iteration of the kernel. The first iteration in (74) yields

$$F_2^{(1)}(\mathrm{t}) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - 2\right] + \theta_1(t). \tag{3.6.75}$$

Here the notation was introduced

$$\theta_1(t) = \frac{2v_1\sqrt{ab}}{\pi t^2}\left[2\sin^{-1}(k^2) - \frac{t}{2k}\sqrt{1-k^4}\ \ \ln\left(\frac{1+kt}{1-kt}\right)\right.$$
$$\left.-\frac{2k^2-t^2}{k\sqrt{k^2-t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right)\right]. \tag{3.6.76}$$

Now we need to compute

$$K_1^2F_2^{(1)}(t) = \int_0^k K_1(ts)\,\mathrm{d}s\int_0^k K_1(sx)\frac{v_1\sqrt{ab}}{x^2}\left[\frac{2k^2-x^2}{k\sqrt{k^2-x^2}} - 2\right]\mathrm{d}x + K_1^2\theta_1(t). \tag{3.6.77}$$

Integration with respect to x in (77) yields

$$K_1^2F_2^{(1)}(t) = \frac{2}{\pi}v_1\sqrt{ab}\int_0^k\left[\frac{1}{k}+\frac{\pi}{2}\frac{ks^2}{\sqrt{1-k^2s^2}}\right.$$

$$\left.-\frac{1+k^2s^2}{2k^2s}\ln\left(\frac{1+ks}{1-ks}\right)\right]K_1(ts)\,\mathrm{d}s + K_1^2\theta_1(t). \tag{3.6.78}$$

One can easily verify that

$$\int_0^k \frac{ks^2}{\sqrt{1-k^2s^2}}K_1(st)\,\mathrm{d}s = -\frac{\theta_1(t)}{v_1\sqrt{ab}}. \tag{3.6.79}$$

Substitution of (79) in (78) gives

$$K_1^2F_2^{(1)}(t) = -\theta_1(t) + \frac{2}{\pi}v_1\sqrt{ab}\int_0^k\left[\frac{1}{k}-\frac{1+k^2s^2}{2k^2s}\ln\left(\frac{1+ks}{1-ks}\right)\right]K_1(st)\,\mathrm{d}s + K_1^2\theta_1(t). \tag{3.6.80}$$

By using the identity

$$\frac{1}{k}-\frac{1+k^2s^2}{2k^2s}\ln\left(\frac{1+ks}{1-ks}\right) = -\pi\int_0^k K_1(xs)\frac{\mathrm{d}x}{x^2}, \tag{3.6.81}$$

we can further simplify (80), namely,

$$K_1^2F_2^{(1)}(t) = -\theta_1(t) - 2v_1\sqrt{ab}\int_0^k K_1^2(st)\frac{\mathrm{d}s}{s^2} + K_1^2\theta_1(t). \tag{3.6.82}$$

Finally, substitution of (82) in (74) gives the solution

$$F_2(t) = v_1\sqrt{ab}\left\{\frac{1}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}}-2\right]-2\sum_{n=1}^{\infty}\int_0^k K_1^{2n}(st)\frac{\mathrm{d}s}{s^2}\right\}. \tag{3.6.83}$$

Since iterated kernels are all positive, the first term in (83) gives the upper bound for F_2. Substitution of (83) in (69) allows us to find

$$F_1(t) = v_1\sqrt{ab}\left\{-\frac{t^2}{\sqrt{k^2-t^2}} + 2\sum_{n=0}^{\infty}\int_0^k K_1^{2n+1}(st)\frac{\mathrm{d}s}{s^2}\right\}. \tag{3.6.84}$$

Formulae (83) and (84) give the complete solution to the problem. We introduce the dimensionless magnetic polarizability ν^* as the ratio of magnetic polarizability of the annulus ν_y to that of a circular disk of radius a $\nu_0 = 4v_1a^3/(3\pi)$. Substitution of (83) in (47) yields, after integration,

$$\nu^* = 1 - 3k^3\sum_{n=1}^{\infty}\int_0^k\int_0^k K_1^{2n}(st)\frac{\mathrm{d}s\,\mathrm{d}t}{s^2t^2}. \tag{3.6.85}$$

Again, the symmetry of (85) allows us to reduce the order of the kernel iteration by writing

$$\nu^* = 1 - 3k^3\sum_{n=1}^{\infty}\int_0^k\left[\int_0^k K_1^{n}(st)\frac{\mathrm{d}s}{s^2}\right]^2\mathrm{d}t. \tag{3.6.86}$$

Since all iterated kernels are positive, truncation in (86) gives the upper bound for ν^*. A simple approximate formula for small k can be derived by integration of (75), with the result

$$\nu^* = 1 - \frac{2}{\pi}\left\{\sin^{-1}(k^2) + \frac{\sqrt{1-k^4}}{4}\left[k^2 - \frac{5+k^4}{2}\ln\left(\frac{1+k^2}{1-k^2}\right)\right]\right\}. \tag{3.6.87}$$

Formula (87) is exact in two extreme cases, namely, for $k=0$ and for $k=1$. The series expansion of (87) gives

$$\nu^* = 1 - \frac{2}{\pi}\left(\frac{k^{10}}{5} + \frac{9k^{14}}{70} + \frac{233k^{18}}{2520}\right) + \mathrm{O}(k^{22}). \tag{3.6.88}$$

It is of interest to notice that all powers of k below 10 cancelled out as compared to the relevant expression for capacity where the series expansion starts with the sixth power of k. This means that ν^* will be not far away from unity even for not so small k. For example, ν^* is greater than 0.98 for $k=0.8$. Convergence of the iteration procedure was investigated by computing the norm of K_1. It was found to be much less than 1 up to the ratio $b/a = 0.9$. Even for $b/a = 0.999$ the norm is equal to 0.6 which still assures a good convergence.

3.7. Neumann problem for a circular annulus

The problem is reduced to a two-dimensional integral equation with an elementary non-singular kernel. Several specific examples are considered. Exact solution has been obtained in terms of the iterated kernel.

A number of papers have been published on the Neumann potential problem for a circular annulus. One of the first significant works on the subject was that of Collins (1963) who considered the axisymmetric problem by means of integral equations. A new and interesting approach was developed by Clements and Love (1974) where the reader can also find some additional references. Quite a few papers are devoted to the mathematically equivalent problem of an annular crack in an elastic space. One can find the relevant references in Clements and Ang (1988). All the works published are devoted to the axisymmetric problems only. No solution to the general problem has been attempted as yet. This kind of solution is now possible due to the results described in Chapter 1. The problem is reduced to a set of two two-dimensional Fredholm integral equations with an elementary non-singular kernel which can be solved by iterations. This set can be easily uncoupled. The case of a circular annulus, with a prescribed charge density distribution, and the rest of the plane being grounded, is considered in detail. The governing integral equations are solved exactly in series involving the iterated kernels.

Theory. We have a mixed boundary value problem of potential theory for a half space $z \geq 0$. We need to find a harmonic function V vanishing at infinity and satisfying the following conditions at $z = 0$:

$$-\frac{1}{2\pi}\frac{\partial V}{\partial z} \equiv \sigma(\rho,\phi) = p(\rho,\phi), \quad \text{for} \quad b < \rho < a, \quad 0 \leq \phi < 2\pi;$$

$$V = 0, \quad \text{for} \quad \rho < b \quad \text{or} \quad \rho > a, \quad 0 \leq \phi < 2\pi. \tag{3.7.1}$$

Here p is a known function. The approach proposed here is similar to the one used in previous section. Let us introduce two harmonic functions

$$V_1(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{b} \frac{\sqrt{l_2^2(\rho_0) - \rho_0^2}\; f_1(\rho_0,\phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2}, \tag{3.7.2}$$

$$V_2(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty} \frac{\sqrt{\rho_0^2 - l_1^2(\rho_0)}\; f_2(\rho_0,\phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2}. \tag{3.7.3}$$

Here the notation was introduced

$$l_1(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} - \sqrt{(\rho - x)^2 + z^2}\},$$

$$l_2(x) = \frac{1}{2}\{\sqrt{(\rho + x)^2 + z^2} + \sqrt{(\rho - x)^2 + z^2}\},$$

One can verify that the potential V_1 is due to some charge distribution σ_1 on the interval $\rho \le b$, and zero charge distribution outside this interval. The potential V_2 is due to σ_2 as the charge density distribution for $\rho \ge a$, and zero charge elsewhere. The functions f_1 and f_2 can be related to the relevant charge density distributions from the following considerations. On the plane $z = 0$ formula (2) gives

$$V_1(\rho, \phi, 0) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\min(\rho,b)} \frac{\sqrt{\rho^2 - \rho_0^2}\, f_1(\rho_0, \phi_0)\, d\rho_0\, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)} = 4\int_0^{\min(\rho,b)} \frac{dx}{\sqrt{\rho^2 - x^2}}\, \mathcal{L}\left(\frac{x}{\rho}\right) f_1(x, \phi). \tag{3.7.4}$$

The utility of the formal change of the notation ρ_0 to x will be clear from the comparison with (5). As before, the $\mathcal{L}$-operator is defined as

$$\mathcal{L}(k) f(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi - \phi_0)\, f(\phi_0)\, d\phi_0,$$

with

$$\lambda(k, \psi) = \frac{1 - k^2}{1 + k^2 - 2k\cos\psi}.$$

On the other hand, we have from (Fabrikant, 1989a) the following expression for the potential in the plane $z = 0$ due to an arbitrary charge distribution σ_1 inside a circle $\rho = b$:

$$V_1(\rho, \phi, 0) = 4\int_0^{\min(\rho,b)} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^b \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \sigma_1(\rho_0, \phi). \tag{3.7.5}$$

Comparison of (4) and (5) immediately gives

$$f_1(x,\phi) = \int_x^b \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma_1(\rho_0,\phi). \tag{3.7.6}$$

The inverse of (6) is readily available, and is

$$\sigma_1(\rho,\phi) = -\frac{2}{\pi}\frac{\mathcal{L}(\rho)}{\rho}\frac{d}{d\rho}\int_\rho^b \frac{x\,dx}{\sqrt{x^2-\rho^2}} \mathcal{L}\left(\frac{1}{x}\right) f(x,\phi). \tag{3.7.7}$$

On the plane $z=0$ formula (3) gives

$$V_2(\rho,\phi,0) = \frac{2}{\pi}\int_0^{2\pi}\int_{\max(a,\rho)}^{\infty} \frac{\sqrt{\rho_0^2-\rho^2}\; f_2(\rho_0\phi_0)\, d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)} = 4\int_{\max(a,\rho)}^{\infty} \frac{dx}{\sqrt{x^2-\rho^2}} \mathcal{L}\left(\frac{\rho}{x}\right) f_2(x,\phi). \tag{3.7.8}$$

In a similar manner, we can find from (Fabrikant, 1989a) that the potential on the plane $z=0$ due to the charge distribution σ_2 which is non-zero outside the circle $\rho=a$ and zero elsewhere, is given by

$$V_2(\rho,\phi,0) = 4\int_{\max(\rho,a)}^{\infty} \frac{dx}{\sqrt{x^2-\rho^2}}\int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi). \tag{3.7.9}$$

Again, comparison of (8) and (9) yields

$$f_2(x,\phi) = \int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma(\rho_0,\phi). \tag{3.7.10}$$

The inversion of (10) yields

$$\sigma_2(\rho,\phi) = \frac{2}{\pi\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_a^\rho \frac{\rho_0 d\rho_0}{\sqrt{\rho^2-\rho_0^2}} \mathcal{L}(\rho_0)\, f_2(\rho_0,\phi). \tag{3.7.11}$$

Functions f_1 and f_2 in (2) and (3) can be found from the second condition (1). We can assume now that the charge distribution is given all over the plane $z=0$, namely, it is assumed to be σ_1 on the interval $[0,b]$, it is equal to p on the annulus $b \le \rho \le a$, and it is equal to σ_2 for $\rho > a$. In this case the relevant potential can be related to the charge distribution in two different ways, namely, (Fabrikant, 1989a):

$$V(\rho,\phi) = 4\int_0^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^{\infty} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \tag{3.7.12}$$

$$V(\rho,\phi) = 4\int_{\rho}^{\infty} \frac{dx}{\sqrt{x^2 - \rho^2}} \int_0^{x} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{3.7.13}$$

Substitution of the second boundary condition (1) in (13) for $\rho > a$ yields

$$0 = 4\int_{\rho}^{\infty} \frac{dx}{\sqrt{x^2 - \rho^2}} \left\{ \int_0^{b} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_1(\rho_0,\phi) + \int_b^{a} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)p(\rho_0,\phi) \right.$$

$$\left. + \int_a^{x} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma_2(\rho_0,\phi) \right\},$$

which immediately leads to

$$\int_0^{b} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_1(\rho_0,\phi) + \int_b^{a} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)p(\rho_0,\phi)$$

$$+ \int_a^{x} \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma_2(\rho_0,\phi) = 0. \tag{3.7.14}$$

Substitution of (7) and (10) in (14) gives the first equation relating f_1 and f_2

$$f_2(x,\phi) - \frac{2}{\pi}\int_0^{b} \frac{d\rho}{\sqrt{x^2 - \rho^2}} \mathcal{L}\left(\frac{\rho^2}{x}\right)\frac{d}{d\rho}\int_{\rho}^{b} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - \rho^2}} \mathcal{L}\left(\frac{1}{\rho_0}\right)f_1(\rho_0,\phi)$$

$$= -\int_b^{a} \frac{\rho d\rho}{\sqrt{x^2 - \rho^2}} \mathcal{L}\left(\frac{\rho}{x}\right)p(\rho,\phi). \tag{3.7.15}$$

We can interchange the order of integration in the second term of (15) according to the rule

$$\int_0^b F(r)\,\mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{f(\rho)\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}} = -\int_0^b f(\rho)\,\mathrm{d}\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^b \frac{F(r)\,r\mathrm{d}r}{\sqrt{r^2-\rho^2}}, \tag{3.7.16}$$

and to integrate with respect to ρ. The result can be simplified as follows:

$$f_2(\rho,\phi) + \int_0^{2\pi}\int_0^b K_1(\rho,\rho_0,\phi-\phi_0)\,f_1(\rho_0,\phi_0)\,\mathrm{d}\rho_0\mathrm{d}\phi_0 = g_2(\rho,\phi), \tag{3.7.17}$$

where

$$K_1(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{\frac{\sqrt{\rho\rho_0\mathrm{e}^{i(\phi-\phi_0)}}}{\pi^2R\,(\rho-\rho_0\mathrm{e}^{i(\phi-\phi_0)})}\tan^{-1}\left[\frac{\mathrm{e}^{i(\phi-\phi_0)}-(\rho_0/\rho)}{(\rho/\rho_0)-\mathrm{e}^{i(\phi-\phi_0)}}\right]^{1/2}\right\} + \frac{\rho}{\pi^2R^2},$$

$$R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0), \quad g_2(\rho,\phi) = -\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho^2-\rho_0^2}}\,\mathcal{L}\!\left(\frac{\rho_0}{\rho}\right)p(\rho_0,\phi). \tag{3.7.18}$$

Here we used the following result:

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{\rho\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}\,\sqrt{x^2-\rho^2}\,(1-m\rho^2)}$$

$$= \frac{mr}{\sqrt{mx^2-1}(1-mr^2)^{3/2}}\tan^{-1}\frac{r\sqrt{mx^2-1}}{x\sqrt{1-mr^2}} + \frac{x}{(x^2-r^2)\,(1-mr^2)}. \tag{3.7.19}$$

In a similar manner, we get from (12) for $\rho<b$

$$\int_\rho^b \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)\sigma_1(\rho_0,\phi) + \int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)p(\rho_0,\phi)$$

$$+\int_a^\infty \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\,\mathcal{L}\!\left(\frac{\rho}{\rho_0}\right)\sigma_2(\rho_0,\phi) = 0. \tag{3.7.20}$$

Substitution of (6) and (11) in (20) yields, after interchanging the order of integration and subsequent simplification to

$$f_1(\rho,\phi) + \int_0^{2\pi}\int_a^{\infty} K_2(\rho,\rho_0,\phi-\phi_0)\, f_2(\rho_0,\phi_0)\, \mathrm{d}\rho_0 \mathrm{d}\phi_0 = g_1(\rho,\phi), \tag{3.7.21}$$

where

$$K_2(\rho,\rho_0,\phi-\phi_0) = 2\Re\left\{ \frac{\sqrt{\rho\rho_0 \mathrm{e}^{i(\phi-\phi_0)}}}{\pi^2 R\,(\rho_0 - \rho \mathrm{e}^{i(\phi-\phi_0)})} \tan^{-1}\left[\frac{\mathrm{e}^{i(\phi-\phi_0)} - (\rho/\rho_0)}{(\rho_0/\rho) - \mathrm{e}^{i(\phi-\phi_0)}}\right]^{1/2} \right\} + \frac{\rho_0}{\pi^2 R^2},$$

$$R^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0), \quad g_1(\rho,\phi) = -\int_b^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2-\rho^2}}\, \mathcal{L}\left(\frac{\rho}{\rho_0}\right) p(\rho_0,\phi). \tag{3.7.22}$$

Here we used the following result:

$$\int_a^{\infty} F(\rho)\, \mathrm{d}\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^{\infty} \frac{f(r)\, r\mathrm{d}r}{\sqrt{r^2-\rho^2}} = -\int_a^{\infty} f(r)\, \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\int_a^r \frac{F(\rho)\, \rho \mathrm{d}\rho}{\sqrt{r^2-\rho^2}} \tag{3.7.23}$$

in order to interchange the order of integration. The integration itself was performed by using

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_r^{\infty} \frac{\rho^3 \mathrm{d}\rho}{\sqrt{\rho^2-x^2}\sqrt{\rho^2-r^2}\,(\rho^2-m)}$$

$$= -\frac{mr}{\sqrt{m-x^2}\,(r^2-m)^{3/2}} \tan^{-1}\left[\frac{m-x^2}{r^2-m}\right]^{1/2} - \frac{mr}{(r^2-m)\,(r^2-x^2)}. \tag{3.7.24}$$

Though the kernels of integral equations (17) and (21) look somewhat different, a simple change of variables can make them identical. Indeed, introduce new variables x and t which are related to the old ones by the relationships $\rho = \sqrt{ab}/t$ and $\rho_0 = \sqrt{ab}\,x$. These substitutions are to be used in (17). The slightly different relationships, namely, $\rho = \sqrt{ab}\,t$ and $\rho_0 = \sqrt{ab}/x$ are to be substituted in (21). The final result is

$$F_1(t,\phi)+\int_0^{2\pi}\int_0^{k} K(xt,\phi-\phi_0)\,F_2(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_1(t,\phi),$$

$$F_2(t,\phi)+\int_0^{2\pi}\int_0^{k} K(xt,\phi-\phi_0)\,F_1(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_2(t,\phi), \qquad (3.7.25)$$

where

$$F_1(t,\phi)=f_1(\sqrt{ab}\,t,\phi), \quad F_2(t,\phi)=f_2(\sqrt{ab}/t,\phi)/t,$$

$$G_1(t,\phi)=g_1(\sqrt{ab}\,t,\phi), \quad G_2(t,\phi)=g_2(\sqrt{ab}/t,\phi)/t, \qquad (3.7.26)$$

$$K(xt,\phi-\phi_0)=\frac{1}{\pi^2}\left\{2\Re\left[\frac{\bar{\gamma}}{R_{xt}}\tan^{-1}\left(\frac{xt}{\gamma R_{xt}}\right)\right]+\frac{1}{R_{xt}^2}\right\}, \qquad (3.7.27)$$

$$k=\sqrt{b/a}, \quad \gamma=\frac{\sqrt{xt\mathrm{e}^{i(\phi-\phi_0)}}}{\mathrm{e}^{i(\phi-\phi_0)}-xt}, \quad R_{xt}=\sqrt{1+x^2t^2-2\,xt\cos(\phi-\phi_0)}. \qquad (3.7.28)$$

and the overbar denotes the complex conjugate value. Note also the relationship $xt/(\gamma R_{xt})=R_{xt}\bar{\gamma}$. Since integral equations (25) have the same kernel, they can be uncoupled by simple addition and subtraction. The result can be presented as

$$F_+(t,\phi)+\int_0^{2\pi}\int_0^{k} K(xt,\phi-\phi_0)\,F_+(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_+(t,\phi),$$

$$F_-(t,\phi)-\int_0^{2\pi}\int_0^{k} K(xt,\phi-\phi_0)\,F_-(x,\phi_0)\,\mathrm{d}x\,\mathrm{d}\phi_0 = G_-(t,\phi), \qquad (3.7.29)$$

with

$$F_\pm=F_1\pm F_2, \quad G_\pm=G_1\pm G_2. \qquad (3.7.30)$$

Formulae (25)–(30) are the main new results of this section. Some quantities which are of interest in applications can be expressed directly through functions F_1 and F_2. For example, in elastic crack problems the stress intensity factors are given by

$$K_a(\phi) = \frac{2kF_2(k,\phi)}{\pi\sqrt{2a}}, \quad K_b(\phi) = \frac{2F_1(k,\phi)}{\pi\sqrt{2b}}.$$

In electromagnetic problems, the total charge inside the circle $\rho = b$ is

$$Q_1 = \frac{2}{\pi}\sqrt{ab} \int_0^{2\pi}\int_0^k F_1(t,\phi)\,\mathrm{d}t\,\mathrm{d}\phi$$

The similar quantity outside the circle $\rho = a$ is

$$Q_2 = \sqrt{ab}\int_0^{2\pi} F_2(0,\phi)\,\mathrm{d}\phi.$$

In the annular crack problems quantities Q_1 and Q_2 correspond to the relevant resultant forces.

Uniform charge distribution. Let us consider in more detail the case of $p = p_0 = \text{const}$. The governing integral equations in this case will take the form

$$F_1(t) + \frac{2}{\pi}\int_0^k \frac{F_2(x)\,\mathrm{d}x}{1 - x^2t^2} = p_0\sqrt{ab}\left[\sqrt{k^2 - t^2} - \frac{\sqrt{1 - k^2t^2}}{k}\right],$$

$$F_2(t) + \frac{2}{\pi}\int_0^k \frac{F_1(x)\,\mathrm{d}x}{1 - x^2t^2} = p_0\frac{\sqrt{ab}}{t^2}\left[\frac{\sqrt{k^2 - t^2}}{k} - \sqrt{1 - k^2t^2}\right]. \tag{3.7.31}$$

Denote the kernel of (31) as

$$K_0(xt) = \frac{2}{\pi}\frac{1}{1 - x^2t^2}.$$

Equations (31) are in agreement with the results of Clements and Love (1974). Their numerical solution was given in Clements and Ang (1988). We present here an analytical solution. Let us use the method of iteration. Assuming the right hand sides in (31) as zero approximations, we can write for the first approximation

$$F_2^{(1)} = p_0\sqrt{ab}\left\{\frac{1}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k} - \sqrt{1-k^2t^2}\right] - \frac{2}{\pi}\int_0^k \frac{\sqrt{k^2-x^2} - \sqrt{(1/k^2)-x^2}}{1-x^2t^2}\,\mathrm{d}x\right\}. \tag{3.7.32}$$

Simplifications in (32) are elementary, and the final result is

$$F_2^{(1)}(t) = p_0\frac{\sqrt{ab}}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k} - 1\right] + \theta(t), \tag{3.7.33}$$

with

$$\theta(t) = p_0\frac{2\sqrt{ab}}{\pi t^2}\left[\sin^{-1}(k^2) - \frac{\sqrt{k^2-t^2}}{k}\sin^{-1}\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right]. \tag{3.7.34}$$

Now we need to investigate the action of the second iterated kernel K_0^2 on (33). The relevant expression is

$$K_0^2F_2^{(1)}(t) = \frac{4}{\pi^2}p_0\sqrt{ab}\int_0^k \frac{\mathrm{d}x}{1-x^2t^2}\int_0^k \frac{\mathrm{d}s}{1-s^2x^2}\frac{1}{s^2}\left[\frac{\sqrt{k^2-s^2}}{k} - 1\right] + K_0^2\theta(t). \tag{3.7.35}$$

Taking into consideration that

$$\theta(t) = \frac{1}{k}\int_0^k \sqrt{1-k^2x^2}\,K_0(xt)\,\mathrm{d}x,$$

expression (35) can be presented as

$$K_0^2F_2^{(1)}(t) = -\theta(t) + \frac{2}{\pi}p_0\sqrt{ab}\int_0^k \psi(x)\,K_0(xt)\,\mathrm{d}x + K_0^2\theta(t), \tag{3.7.36}$$

with

$$\psi(x) = \frac{1}{k} - \frac{x}{2}\ln\left(\frac{1+kx}{1-kx}\right). \tag{3.7.37}$$

Since the exact solution for F_2 can be represented as

$$F_2(t) = F_2^{(1)}(t) + \sum_{n=1}^{\infty} \int_0^k F_2^{(1)}(x)\, K_0^{2n}(x,t)\,\mathrm{d}x, \tag{3.7.38}$$

we can see that each next iteration will cancel expression containing θ from the previous iteration, so that θ would not enter the final solution which now takes the form

$$F_2(t) = p_0\sqrt{ab}\left\{\frac{1}{t^2}\left[\frac{\sqrt{k^2-t^2}}{k}-1\right]+\frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^k \psi(x)\, K_0^{2n+1}(x,t)\,\mathrm{d}x\right\}. \tag{3.7.39}$$

Substitution of (39) in the first equation of (31) yields

$$F_1(t) = p_0\sqrt{ab}\left\{\sqrt{k^2-t^2}-\frac{2}{\pi}\left[\psi(t)+\sum_{n=1}^{\infty}\int_0^k \psi(x)\, K_0^{2n}(x,t)\,\mathrm{d}x\right]\right\}. \tag{3.7.40}$$

Formulae (39) and (40) give an exact solution of (31) in terms of the iterated kernel. Actual computations were made for the set of values of $\{b/a\} := 0.04, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99$. The dimensionless quantities $F_1^* = F_1/(p_0\sqrt{ab}\,)$ and $F_2^* = F_2/(p_0\sqrt{ab}\,)$ are presented in Fig. 3.7.1 and Fig. 3.7.2 respectively versus $\xi = 1 + 300(t/k)$. This choice allowed us to plot all curves on the same base. In order to avoid overlapping, not all curves were actually plotted. A comparison was also made with the numerical data for the stress intensity factors presented by Clements and Ang. The agreement was found to be excellent.

The case of non-axisymmetric charge distribution prescribed over the annulus can be treated in a similar manner. For example, let

$$p(\rho,\phi) = p_1\rho\cos\phi. \tag{3.7.41}$$

The governing integral equations in this case will take the form

$$F_1(t) + \frac{2}{\pi}\int_0^k \left[\frac{xt}{1-x^2t^2}+\frac{1}{2}\ln\left(\frac{1+xt}{1-xt}\right)\right]F_2(x)\,\mathrm{d}x = p_1 abt\left[\sqrt{k^2-t^2}-\frac{\sqrt{1-k^2t^2}}{k}\right],$$

Fig. 3.7.1. Uniform charge distribution (solution for F_1)

Fig. 3.7.2. Uniform charge distribution (solution for F_2)

$$F_2(t)+\frac{2}{\pi}\int_0^k\left[\frac{xt}{1-x^2t^2}+\frac{1}{2}\ln\left(\frac{1+xt}{1-xt}\right)\right]F_1(x)\,\mathrm{d}x=\frac{p_1ab}{3t^2}\left[\frac{2k^2+t^2}{k^3}\sqrt{k^2-t^2}\right.$$

$$\left.-(2+k^2t^2)\sqrt{1-k^2t^2}\right]. \tag{3.7.42}$$

Equations (42) can be solved by any standard numerical procedure.

3.8. Alternative approach to the Dirichlet problem

The solution to the Dirichlet problem for a circular annulus, presented in section 3.6 is not the only one to use. Several alternatives may be suggested. The boundary conditions, as before, are

$$V(\rho,\phi,0)=v(\rho,\phi),\quad \text{for}\quad b<\rho<a,\quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for}\quad \rho<b\quad\text{or}\quad \rho>a,\quad 0\le\phi<2\pi. \tag{3.8.1}$$

The governing integral equation will take the form

$$\int_0^{2\pi}\int_b^a\frac{\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}=v(\rho,\phi). \tag{3.8.2}$$

By using the integral representation

$$\frac{1}{R}=\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}}=\frac{2}{\pi}\int_{max(\rho_0,\rho)}^{\infty}\frac{\lambda(\frac{\rho\rho_0}{x^2},\phi-\phi_0)\,\mathrm{d}x}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}, \tag{3.8.3}$$

The governing integral equation may be reduced to

$$4\int_\rho^a\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_b^x\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$+4\int_a^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)=v(\rho,\phi). \tag{3.8.4}$$

Application of the operator

$$\mathcal{L}(r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\mathcal{L}\left(\frac{1}{\rho}\right)$$

to both sides of (4) yields

$$-2\pi\int_b^r \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho_0}{r}\right)\sigma(\rho_0,\phi)-\frac{4r}{\sqrt{a^2-r^2}}\int_a^\infty \frac{\sqrt{x^2-a^2}\,\mathrm{d}x}{x^2-r^2}\int_b^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\mathcal{L}\left(\frac{r\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$=\mathcal{L}(r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\mathcal{L}\left(\frac{1}{\rho}\right)v(\rho,\phi). \tag{3.8.5}$$

We introduce a new unknown function

$$\psi(r,\phi)=\int_b^r \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{r^2-\rho_0^2}}\mathcal{L}\left(\frac{\rho_0}{r}\right)\sigma(\rho_0,\phi). \tag{3.8.6}$$

The inverse of (6) is readily available, and is

$$\sigma(\rho,\phi)=\frac{2}{\pi\rho}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_b^\rho \frac{r\,\mathrm{d}r}{\sqrt{\rho^2-r^2}}\mathcal{L}(r)\,\psi(r,\phi). \tag{3.8.7}$$

Substitution of (6) in (5) gives

$$-2\pi\psi(r,\phi)-\frac{8}{\pi}\frac{r}{\sqrt{a^2-r^2}}\int_a^\infty \frac{x^2-a^2}{x^2-r^2}\mathrm{d}x\int_b^a \frac{y\,\mathrm{d}y}{\sqrt{a^2-y^2}(x^2-y^2)}\mathcal{L}\left(\frac{yr}{x^2}\right)\psi(y,\phi)$$

$$= \mathcal{L}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)v(\rho,\phi). \tag{3.8.8}$$

One can interchange the order of integration in the second term of (8) and perform the integration with respect to x. The result is

$$-\psi(r,\phi)+\frac{1}{\pi^3}\int_0^{2\pi}\int_b^a \frac{K(y,r,\phi-\phi_0)-K(r,y,\phi-\phi_0)}{r^2-y^2}\,\psi(y,\phi_0)\,\mathrm{d}y\,\mathrm{d}\phi_0$$

$$= \mathcal{L}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)v(\rho,\phi). \tag{3.8.9}$$

The kernel of (9) can be expressed in terms of elementary functions as follows:

$$K(y,r,\phi-\phi_0) = ry\left(\frac{a^2-r^2}{a^2-y^2}\right)^{1/2}\left\{\lambda\left(\frac{r}{y},\phi-\phi_0\right)\frac{1}{r}\ln\left(\frac{a+r}{a-r}\right)\right.$$

$$\left. -\,2\Re\left[\frac{1}{\xi\left(1-\frac{r}{y}\mathrm{e}^{-i(\phi-\phi_0)}\right)}\ln\frac{a+\xi}{a-\xi}\right]\right\}, \tag{3.8.10}$$

where

$$\xi = \sqrt{yr\mathrm{e}^{i(\phi-\phi_0)}}. \tag{3.8.11}$$

Here $\Re$ denotes the real part of the expression to follow. Thus, the general problem of annular punch has been reduced to a Fredholm integral equation (9) with an elementary kernel which can be solved numerically. It is noteworthy that the governing equation for each specific harmonic will also have an elementary kernel. For example, the equation corresponding to the zero harmonic is

$$-\psi_0(r)+\frac{2}{\pi^2}\int_b^a \frac{K_0(y,r)-K_0(r,y)}{r^2-y^2}\,\psi_0(y)\,\mathrm{d}y = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{v_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}, \tag{3.8.12}$$

with

$$K_0(y,r) = y\left(\frac{a^2-r^2}{a^2-y^2}\right)^{1/2} \ln\frac{a+r}{a-r}. \tag{3.8.13}$$

There have been so many variations of the governing integral equation published for the case of axial symmetry, that there is no doubt that equation (12) coincides with some known result, though we have difficulty to pinpoint exactly which one. The governing integral equation for the first harmonic will take the form

$$-\psi_1(r) + \frac{2}{\pi^2}\int_b^a \frac{K_1(y,r)-K_1(r,y)}{r^2-y^2}\,\psi_1(y)\,\mathrm{d}y = \frac{1}{2\pi}\,r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{v_1(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}, \tag{3.8.14}$$

with

$$K_1(y,r) = \left(\frac{a^2-r^2}{a^2-y^2}\right)^{1/2}\left[\frac{y^2}{r}\ln\frac{a+r}{a-r} - 2a\right]. \tag{3.8.15}$$

There is no need to compute the charge distribution σ if one is interested in the integral characteristics only. Indeed, both the total charge Q and the moment M can be expressed through the new unknown function ψ as follows:

$$Q = \frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\psi(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a^2-\rho^2}} = 4\int_b^a \frac{\psi_0(\rho)\,\rho\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}}, \tag{3.8.16}$$

$$M = -\frac{2}{\pi}\int_0^{2\pi}\int_b^a \frac{\psi(\rho,\phi)\,\rho^2\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{a^2-\rho^2}} = -2\int_b^a \frac{\psi_1(\rho)\,\rho^2\,\mathrm{d}\rho}{\sqrt{a^2-\rho^2}}. \tag{3.8.17}$$

We note also that the kernels in (12) and (14) are finite at the point $y=r$. The following limits can be computed

$$\lim_{y\to r}\frac{K_0(y,r)-K_0(r,y)}{r^2-y^2} = \frac{1}{a^2-r^2}\left[a - \frac{r^2+a^2}{2r}\ln\frac{a+r}{a-r}\right],$$

$$\lim_{y\to r}\frac{K_1(y,r)-K_1(r,y)}{r^2-y^2} = \frac{1}{a^2-r^2}\left[3a - \frac{3a^2-r^2}{2r}\ln\frac{a+r}{a-r}\right]. \tag{3.8.18}$$

Equations (9), (10), (16), and (17) are the main new results of this section. The integral equations can be solved by any regular numerical method.

A similar approach can be extended to spherical coordinates. We consider a Dirichlet problem for a spherical annulus $\beta\le\theta\le\alpha$, $0\le\phi<2\pi$. The annulus is a part of a sphere of radius a. Let an arbitrary potential $v(\theta, \phi)$ be prescribed at the annulus, with no charges elsewhere. We need to find the charge density distribution σ at the spherical annulus. The governing integral equation may be written as

$$\int\!\!\int_S \frac{\sigma}{R}\,dS \equiv \frac{a}{2\cos(\theta/2)}$$

$$\times \int_\beta^\alpha \int_0^{2\pi} \frac{\sigma(\theta_0, \phi_0)\sin\theta_0\,d\theta_0\,d\phi_0}{\cos(\theta_0/2)\sqrt{\tan^2(\theta/2) + \tan^2(\theta_0/2) - 2\tan(\theta/2)\tan(\theta_0/2)\cos(\phi - \phi_0)}} = v(\theta, \phi).$$

(3.8.19)

By using the integral representation for the reciprocal of the distance (1.5.3) for $r = a$, equation (19) will take the form

$$\frac{a}{\pi}\Bigg[\int_\beta^\theta \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_\tau^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\tau - \cos\theta_0}}\, \mathcal{L}\!\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0, \phi_0)$$

$$+ \int_0^\beta \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_\beta^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\tau - \cos\theta_0}}\, \mathcal{L}\!\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0, \phi_0)\Bigg] = v(\theta, \phi).$$

(3.8.20)

Application of the operator

$$\mathcal{L}[\cot(\theta_1/2)]\,\frac{d}{d\theta_1}\int_\beta^{\theta_1} \frac{\sin\theta\,d\theta}{\sqrt{\cos\theta - \cos\theta_1}}\,\mathcal{L}\!\left(\tan\frac{\theta}{2}\right)$$

to both sides of (20) leads to

$$\frac{a}{\pi}\Bigg\{\pi\int_{\theta_1}^\alpha \frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_1 - \cos\theta_0}}\,\mathcal{L}\!\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)}\right)\sigma(\theta_0, \phi)$$

$$+ \int_0^\beta \frac{\sin\theta_1\sqrt{\cos\tau - \cos\beta}\,d\tau}{(\cos\tau - \cos\theta_1)\sqrt{\cos\beta - \cos\theta_1}}$$

$$\times \int_{\beta}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\tau - \cos\theta_0}}\, \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta_1/2)\tan(\theta_0/2)}\right)\sigma(\theta_0, \phi)\Bigg\}$$

$$= \mathcal{L}[\cot(\theta_1/2)]\, \frac{d}{d\theta_1} \int_{\beta}^{\theta_1} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta - \cos\theta_1}}\, \mathcal{L}\left(\tan\frac{\theta}{2}\right) v(\theta, \phi). \tag{3.8.21}$$

Introduction of a new unknown

$$\chi(\theta_1, \phi) = \int_{\theta_1}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_1 - \cos\theta_0}}\, \mathcal{L}\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)}\right)\sigma(\theta_0, \phi)$$

will lead to the new type of integral equations which would be the spherical equivalent to (9). This exercise is left to the reader.