# CHAPTER 2

## GENERALIZED POTENTIAL THEORY SOLUTIONS

We can generalize the Newton potential as $V = H/R^{1+\kappa}$, where $R$ is the distance between two points, $H$ is a constant depending on the physical constants of the space, and $-1 < \kappa < 1$. This potential has various applications in engineering, for example, in the theory of elasticity of inhomogeneous elastic body, with the modulus of elasticity $E$ being a power function of $z$: $E = E_0 z^{\kappa}$. Other applications can be found in Weinstein (1952) and Payne (1952) where only the axisymmetric case was considered. Closed form solution to various non-axisymmetric problems is given in this Chapter.

### 2.1. Interior problem for a half-space

The problem is called interior when the potential is prescribed inside a circle.

**Problem 1.** We consider generalization of the problem solved in section 1.3. The boundary conditions are

$$V = f(\rho,\phi), \quad \text{for} \quad \rho \le a, \quad 0 \le \phi < 2\pi;$$

$$\sigma = 0, \quad \text{for } \rho > a, \quad 0 \le \phi < 2\pi. \tag{2.1.1}$$

Here $V$ is the generalized potential, and $\sigma$ is the charge density distribution. The governing integral equation will take the form

$$H \int_0^{2\pi} \int_0^{a} \frac{\sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{(1+\kappa)/2}} = f(\rho,\phi), \tag{2.1.2}$$

Rostovtsev (1964) obtained an exact solution of (2) in Fourier series. Here we present a closed form solution.

By using the integral representation (1.2.1), integral equation (2) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2}\int_0^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)=f(\rho,\phi). \tag{2.1.3}$$

Integral equation (3) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of (1.3.9). The first operator to be applied to both sides of (3) is

$$\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho). \tag{2.1.4}$$

The result of application of (4) to both sides of (3) is

$$2\pi H t^\kappa\int_t^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-t^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{t}{\rho_0}\right)\sigma(\rho_0,\phi)=\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho)f(\rho,\phi). \tag{2.1.5}$$

The second operator to be applied to both sides of (5) is

$$\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{t}\right),$$

with the result

$$\sigma(y,\phi)=-\frac{\cos(\pi\kappa/2)}{\pi^2 H y}\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t^{1-\kappa}\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}\left(\frac{1}{t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}(\rho)f(\rho,\phi). \tag{2.1.6}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (6) in the form

$$\sigma(y,\phi)=\frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{(1-\kappa)/2}}-\int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{2.1.7}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t^{1+\kappa}} \int_0^t \frac{\rho^{1-\kappa}\mathrm{d}\rho}{(t^2-\rho^2)^{(1-\kappa)/2}} \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^{1+\kappa}\mathcal{L}\left(\frac{\rho y}{t^2}\right) f(\rho,\phi)\right]. \tag{2.1.8}$$

Yet another form of solution can be found in (Fabrikant, 1971e).

It becomes possible to compute various integral characteristics, like the total charge $Q$ and the polarizability moments $M_x$ and $M_y$ directly in terms of the prescribed potential. Since

$$Q = \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\, \rho \mathrm{d}\rho \mathrm{d}\phi, \tag{2.1.9}$$

substitution of (6) in (9) yields directly the total charge

$$Q = \frac{\cos(\pi\kappa/2)}{\pi^2 H} \int_0^{2\pi}\int_0^a \frac{f(\rho,\phi)\, \rho \mathrm{d}\rho \mathrm{d}\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}. \tag{2.1.10}$$

For computation of the moments $M_x$ and $M_y$, it is convenient to introduce the complex parameter

$$M = M_x + \mathrm{i}M_y = -\mathrm{i}\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\, \mathrm{e}^{i\phi}\, \rho^2 \mathrm{d}\rho \mathrm{d}\phi. \tag{2.1.11}$$

By using (6), the final expression for the moment is found to be

$$M = -\mathrm{i}\frac{2\cos(\pi\kappa/2)}{\pi^2 H(1+\kappa)} \int_0^{2\pi}\int_0^a \frac{f(\rho,\phi)\, \mathrm{e}^{i\phi}\rho^2 \mathrm{d}\rho \mathrm{d}\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}. \tag{2.1.12}$$

Expressions (11) and (12) are in agreement with similar results of Rostovtsev (1964).

By reviewing the derivation of expression (3), one may find that it is valid for evaluating the potential for $\rho > a$, if the upper limit of integration $\rho$ is replaced by $a$. Substitution of (6) into the modified form of (3) results in

$$V(\rho,\phi)=\frac{2\cos(\pi\kappa/2)}{\pi}\int_0^a\frac{\mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{\rho_0\mathrm{d}\rho_0}{(x^2-\rho_0^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\rho_0}{\rho}\right)f(\rho_0,\phi),$$
$$\text{for } \rho>a. \tag{2.1.13}$$

Performing differentiation of the integrand, and then integrating by parts, we obtain

$$V(\rho,\phi)=\frac{1}{\pi^2}\cos(\frac{\pi\kappa}{2})(\rho^2-a^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_0^a\frac{f(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(a^2-\rho_0^2)^{(1-\kappa)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]},$$
$$\text{for } \rho>a. \tag{2.1.14}$$

Here the following identities were employed (Bateman and Erdélyi, 1955)

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\zeta^{(1+\kappa)/2}F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};\zeta\right)\right]=\frac{1+\kappa}{2}\zeta^{-(1-\kappa)/2}(1-\zeta)^{-(1+\kappa)/2},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{f(t)t\,\mathrm{d}t}{(x^2-t^2)^{(1-\kappa)/2}}=f(0)x^{\kappa}+x\int_0^x\frac{\mathrm{d}f(t)}{(x^2-t^2)^{(1-\kappa)/2}}. \tag{2.1.15}$$

All the quantities of interest, namely, the charge distribution $\sigma$, the total charge $Q$, the moment $M$, and the potential outside the circle can be expressed directly through the prescribed potential $f$ by formulae (6), (10), (12), and (14) respectively.

**Problem 2.** In a similar manner, we can consider yet another mixed boundary value problem of a half-space, subject to the boundary conditions at $z$=0:

$$V=0,\quad \text{for}\quad 0\le\rho\le a,\ 0\le\phi<2\pi;$$

$$\sigma=\sigma(\rho,\phi),\quad \text{for } a<\rho<\infty,\ 0\le\phi<2\pi. \tag{2.1.16}$$

The governing integral equation takes the form (2), with the known function

$$f(\rho,\phi)=-H\int_0^{2\pi}\int_a^{\infty}\frac{\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \tag{2.1.17}$$

Its solution can be found in exactly the same manner as that of (1.4.27), and is

$$\sigma(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2(a^2-\rho^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_a^{\infty}\frac{(\rho_0^2-a^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \qquad (2.1.18)$$

Expression (18) gives the charge distribution outside the circle through its prescribed value inside. Note that (1.4.30) may be considered as a particular case of (18), when $\kappa=0$.

The potential outside the circle can be evaluated as a superposition of the potential due to the charge inside the circle and the prescribed charge distribution outside. By using a procedure analogous to the one described in section 1.4, we can obtain the expression

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\left\{\int_\rho^{\infty}\frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x\frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$\left.+\int_0^a\frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^a\frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\right\}, \quad \text{for } \rho>a. \qquad (2.1.19)$$

Substitution of (18) in (19) leads, after simplification, to

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\int_a^{\rho}\frac{x^\kappa dx}{(\rho^2-x^2)^{(1+\kappa)/2}}\int_x^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi), \qquad (2.1.20)$$

$$\text{for } \rho>a$$

The potential inside the circle is now defined in terms of the charge density distribution prescribed outside.

Introduce the *energy function*:

$$K_1(\rho,\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}}\int_\rho^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-\rho^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma(\rho_0,\phi). \qquad (2.1.21)$$

By using the property of the $\mathcal{L}$-operators (1.1.5) we may rewrite (20) as

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\int_a^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)\int_x^\infty \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= 2^{(3\text{-}\kappa)/2}\pi H\int_a^\rho \frac{x^{(1+\kappa)/2}\mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi). \qquad (2.1.22)$$

The energy $W$ may be defined by the integral

$$W = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)V(\rho,\phi)\rho\,\mathrm{d}\rho\mathrm{d}\phi. \qquad (2.1.23)$$

Substitution of (22) in (23) gives

$$W = 2^{(3\text{-}\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty \sigma(\rho,\phi)\rho\,\mathrm{d}\rho\int_a^\rho \frac{x^{(1+\kappa)/2}\,\mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi)$$

$$= 2^{(3\text{-}\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty x^{(1+\kappa)/2}\mathrm{d}x\int_x^\infty \frac{\sigma(\rho,\phi)\rho\,\mathrm{d}\rho}{(\rho^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)K_1(x,\phi)$$

$$= 2^{(3\text{-}\kappa)/2}\pi H\int_0^{2\pi}\mathrm{d}\phi\int_a^\infty K_1(x,\phi)x^{(1+\kappa)/2}\mathrm{d}x\int_x^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-x^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{x}{\rho}\right)\sigma(\rho,\phi). \qquad (2.1.24)$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (21) yields the final result

$$W = \frac{2^{1\text{-}\kappa}\pi^2 H}{\cos(\pi\kappa/2)}\int_0^{2\pi}\int_a^\infty [K_1(\rho,\phi)]^2\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \qquad (2.1.25)$$

Expression (25) gives a good explanation why we called $K_1$ energy function: its square is proportional to the energy per unit area. In the case of axial symmetry formula (25) simplifies as

$$W=\frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)}\int_a^\infty [K_1(\rho)]^2\rho\, d\rho,$$

with

$$K_1(\rho)=\frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}}\int_\rho^\infty \frac{\sigma(\rho_0,\phi_0)\rho_0\, d\rho_0}{(\rho_0^2-\rho^2)^{(1+\kappa)/2}}. \qquad (2.1.26)$$

## 2.2. Exterior problem for a half-space

We call a problem exterior when the potential is prescribed outside the circle $\rho = a$.

**Problem 1.** The problem is characterized by the following mixed boundary conditions on the plane $z$=0:

$$V=f(\rho,\phi), \quad \text{for} \quad \rho>a, \quad 0\le\phi<2\pi;$$

$$\sigma=0, \quad \text{for} \quad \rho<a, \quad 0\le\phi<2\pi. \qquad (2.2.1)$$

Here the same notation is used as in the previous section. The governing integral equation will take the form

$$H\int_0^{2\pi}\int_a^\infty \frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}=f(\rho,\phi), \qquad (2.2.2)$$

where $-1<\kappa<1$. The new method allows us to present a closed form solution.

By using the integral representation (1.2.17), for $z$=0, the integral equation (2) can be rewritten as

$$4H\cos\frac{\pi\kappa}{2}\int_\rho^\infty \frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)=f(\rho,\phi). \qquad (2.2.3)$$

Integral equation (3) represents a sequence of two Abel operators and one $\mathcal{L}$-operator. The solution procedure is similar to that of section 1.4. The first operator to be applied to both sides of (3) is

$$\mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\,\mathcal{L}\left(\frac{1}{\rho}\right). \tag{2.2.4}$$

The result of application of (4) to both sides of (3) is

$$-2\pi H t^{\kappa}\int_a^t \frac{\rho_0\,\mathrm{d}\rho_0}{(t^2-\rho_0^2)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi)$$

$$= \mathcal{L}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)f(\rho,\phi). \tag{2.2.5}$$

The second operator to be applied to both sides of (5) is

$$\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t^{1-\kappa}\,\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}\,\mathcal{L}(t),$$

with the result

$$\sigma(y,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2 H y}\,\mathcal{L}\left(\frac{1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_a^y \frac{t^{1-\kappa}\,\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}$$

$$\times\mathcal{L}(t^2)\,\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\,\mathcal{L}\left(\frac{1}{\rho}\right)f(\rho,\phi). \tag{2.2.6}$$

The rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (6) in the form

$$\sigma(y,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2 H}\left[\frac{\Phi(a,y,\phi)}{(y^2-a^2)^{(1-\kappa)/2}} - \int_a^y \frac{\mathrm{d}t}{(y^2-t^2)^{(1-\kappa)/2}}\,\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{2.2.7}$$

Here

$$\Phi(t,y,\phi) = t^{1-\kappa}\int_t^\infty \frac{d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\frac{d}{d\rho}\left[\mathcal{L}\left(\frac{t^2}{\rho y}\right)f(\rho,\phi)\right]. \tag{2.2.8}$$

Certain integral characteristics can be computed directly in terms of the given function $f$. Since the total charge

$$Q = \int_0^{2\pi}\int_a^\infty \sigma(\rho,\phi)\,\rho d\rho d\phi, \tag{2.2.9}$$

substitution of (6) in (9) yields directly the total charge

$$Q = \lim_{y\to\infty}\left\{-\frac{\cos(\pi\kappa/2)}{\pi^2 H}\int_a^y \frac{t^{2-\kappa}dt}{(y^2-t^2)^{(1-\kappa)/2}}\int_t^\infty \frac{d\rho}{(\rho^2-t^2)^{(1-\kappa)/2}}\frac{d}{d\rho}\int_0^{2\pi} f(\rho,\phi)\,d\phi\right\}. \tag{2.2.10}$$

The moment can be found in a similar manner. We can also express the potential inside the circle $\rho\le a$ directly in terms of the prescribed $f$ outside the circle. We substitute (6) in (3) keeping in mind that, for $\rho\le a$, the lower limit of integration of the first integral will be $a$ instead of $\rho$. By using the properties of Abel operators and the $\mathcal{L}$-operators, the following expression can be obtained

$$V(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\int_a^\infty \frac{dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\frac{d}{dx}\int_x^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-x^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi). \tag{2.2.11}$$

Carrying out the differentiation of the integrand, interchanging the order of integration, and then integrating with respect to $x$ yields

$$V(\rho,\phi) = -\frac{2\cos(\pi\kappa/2)}{\pi(1+\kappa)}\int_a^\infty \left(\frac{\rho_0^2-a^2}{\rho_0^2-\rho^2}\right)^{(1+\kappa)/2}$$

$$\times F\left(\frac{1+\kappa}{2},\frac{1+\kappa}{2};\frac{3+\kappa}{2};\frac{\rho_0^2-a^2}{\rho_0^2-\rho^2}\right)\frac{d}{d\rho_0}\left[\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi)\right]d\rho_0. \tag{2.2.12}$$

Integration by parts and the differential properties of the Gauss hypergeometric functions (2.1.15) allow us to simplify the last expression, namely,

$$V(\rho,\phi) = -\frac{2}{\pi}\cos\frac{\pi\kappa}{2}(a^2-\rho^2)^{(1-\kappa)/2}\int_a^\infty \frac{\rho_0 d\rho_0}{(\rho_0^2-a^2)^{(1-\kappa)/2}(\rho_0^2-\rho^2)}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)f(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)(a^2-\rho^2)^{(1-\kappa)/2}\int_0^{2\pi}\int_a^\infty \frac{f(\rho_0,\phi_0)\rho_0\, d\rho_0\, d\phi_0}{(\rho_0^2-a^2)^{(1-\kappa)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}. \quad (2.2.13)$$

Expression (13) gives the potential inside a circle $\rho\le a$ directly in terms of its prescribed value outside the circle. We note certain similarity between (2.1.14) and (13).

**Problem 2**. We consider yet another boundary value problem for a half-space, with the boundary conditions at $z$=0:

$$V = 0, \quad \text{for } a<\rho<\infty, \quad 0\le\phi<2\pi;$$

$$\sigma = \sigma(\rho,\phi), \quad \text{for } 0\le\rho\le a, \quad 0\le\phi<2\pi. \quad (2.2.14)$$

The governing integral equation takes the form (3), with the known function

$$f(\rho,\phi) = -H\int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\kappa)/2}}. \quad (2.2.15)$$

Its exact solution is

$$\sigma(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{\pi^2(\rho^2-a^2)^{(1-\kappa)/2}}\int_0^{2\pi}\int_0^a \frac{(a^2-\rho_0^2)^{(1-\kappa)/2}\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \quad (2.2.16)$$

Again, one should notice the similarity between (2.1.18) and (16). Expression (16) gives the charge density distribution in the plane $z$=0 outside the circle in terms of the charge density given inside.

The potential can be evaluated as a superposition of the potentials due to the charge inside and outside of the circle. By using a procedure analogous to the one described in section 1.3, we obtain the expression

$$V(\rho,\phi,0) = 4H\cos\frac{\pi\kappa}{2}\left\{\int_a^\infty \frac{x^\kappa dx}{(x^2-\rho^2)^{(1+\kappa)/2}}\int_a^x \frac{\rho_0 d\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)\right.$$

$$+\int_0^\rho \frac{x^\kappa \mathrm{d}x}{(\rho^2-x^2)^{(1+\kappa)/2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{(\rho_0^2-x^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)\Bigg\}, \text{ for } \rho<a. \tag{2.2.17}$$

Substitution of (16) in (17) leads, after simplification, to

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\int_\rho^a \frac{x^\kappa \mathrm{d}x}{(x^2-\rho^2)^{(1+\kappa)/2}} \int_0^x \frac{\rho_0 \mathrm{d}\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi), \tag{2.2.18}$$

The potential is now found in terms of the given charge distribution.

We can again introduce the energy function

$$K_1(\rho,\phi) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}}\int_0^\rho \frac{\rho_0 \mathrm{d}\rho_0}{(\rho^2-\rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi). \tag{2.2.19}$$

By using the property of the $\mathcal{L}$-operators (1.1.5) we may rewrite (18) as

$$V(\rho,\phi) = 4H\cos\frac{\pi\kappa}{2}\int_\rho^a \frac{x^\kappa \mathrm{d}x}{(x^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right)\int_0^x \frac{\rho_0 \mathrm{d}\rho_0}{(x^2-\rho_0^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho_0}{x}\right)\sigma(\rho_0,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H\int_\rho^a \frac{x^{(1+\kappa)/2}\mathrm{d}x}{(x^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right)K_1(x,\phi). \tag{2.2.20}$$

The energy $W$ may be defined by the integral

$$W = \int_0^{2\pi}\int_0^a \sigma(\rho,\phi)V(\rho,\phi)\rho\, \mathrm{d}\rho \mathrm{d}\phi. \tag{2.2.21}$$

Substitution of (20) in (21) gives

$$W = 2^{(3-\kappa)/2}\pi H\int_0^{2\pi} \mathrm{d}\phi \int_0^a \sigma(\rho,\phi)\rho\, \mathrm{d}\rho \int_\rho^a \frac{x^{(1+\kappa)/2}\,\mathrm{d}x}{(x^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right)K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} d\phi \int_0^a x^{(1+\kappa)/2}\, dx \int_0^x \frac{\sigma(\rho,\phi)\rho\, d\rho}{(x^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) K_1(x,\phi)$$

$$= 2^{(3-\kappa)/2}\pi H \int_0^{2\pi} d\phi \int_0^a K_1(x,\phi)\, x^{(1+\kappa)/2}\, dx \int_0^x \frac{\rho\, d\rho}{(x^2-\rho^2)^{(1+\kappa)/2}} \mathcal{L}\left(\frac{\rho}{x}\right) \sigma(\rho,\phi). \tag{2.2.22}$$

Here the interchange of the order of integration was used twice. Now comparison of the last expression with (19) yields the final result

$$W = \frac{2^{1-\kappa}\pi^2 H}{\cos(\pi\kappa/2)} \int_0^{2\pi} \int_0^a [K_1(\rho,\phi)]^2 \rho\, d\rho\, d\phi. \tag{2.2.23}$$

Expression (23) interprets the energy function squared as being proportional to the energy per unit area. In the case of axial symmetry, formula (23) simplifies to

$$W = \frac{2^{2-\kappa}\pi^3 H}{\cos(\pi\kappa/2)} \int_0^a [K_1(\rho)]^2 \rho\, d\rho,$$

with

$$K_1(\rho) = \frac{2\cos(\pi\kappa/2)}{\pi(2\rho)^{(1-\kappa)/2}} \int_0^\rho \frac{\sigma(\rho_0,\phi_0)\rho_0\, d\rho_0}{(\rho^2-\rho_0^2)^{(1+\kappa)/2}}. \tag{2.2.24}$$

## 2.3. Generalized problem for a spherical cap

We consider a spherical cap whose surface in spherical coordinates is given by $r=a$, $0\le\theta\le\alpha$, $0\le\phi<2\pi$. Arbitrary potential $v(\theta,\phi)$ is prescribed at each point of the cap. The charge density $\sigma(\theta,\phi)$ is to be defined. The potential can be described through a simple layer distribution as

$$V(r,\theta,\phi) = \int_0^\alpha \int_0^{2\pi} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, d\theta_0 d\phi_0}{\{r^2+a^2-2ar[\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)]\}^{(1+\kappa)/2}}. \tag{2.3.1}$$

Taken at the surface of the cap, expression (1) gives the governing integral equation

$$\frac{a^{1-\kappa}}{2^{(1+\kappa)/2}}\int_0^{\alpha}\int_0^{2\pi}\frac{\sigma(\theta_0,\phi_0)\sin\theta_0\,d\theta_0 d\phi_0}{\{1-[\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)]\}^{(1+\kappa)/2}}=v(\theta,\phi). \tag{2.3.2}$$

The denominator in (2) can be transformed as follows:

$$1-[\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)]=2\cos^2\frac{\theta}{2}\cos^2\frac{\theta_0}{2}\Big[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}$$
$$-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\Big]. \tag{2.3.3}$$

Substitution of (3) in (2) yields the governing equation in the form

$$\frac{a^{1-\kappa}}{2^{1+\kappa}}\int_0^{\alpha}\int_0^{2\pi}\frac{\sigma(\theta_0,\phi_0)\sin\theta_0\,d\theta_0 d\phi_0}{\left(\cos\frac{\theta}{2}\cos\frac{\theta_0}{2}\right)^{1+\kappa}\left[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\right]^{(1+\kappa)/2}}=v(\theta,\phi),$$
$$\text{for }\ 0\le\theta\le\alpha. \tag{2.3.4}$$

By using formally (1.2.1) we can write

$$\left[\tan^2\frac{\theta}{2}+\tan^2\frac{\theta_0}{2}-2\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\cos(\phi-\phi_0)\right]^{-(1+\kappa)/2}$$
$$=\frac{2}{\pi}\cos\frac{\pi\kappa}{2}\left(\cos\frac{\theta}{2}\cos\frac{\theta_0}{2}\right)^{1+\kappa}\int_0^{\min(\theta,\theta_0)}\frac{\lambda\left(\frac{\tan^2(\xi/2)}{\tan(\theta/2)\tan(\theta_0/2)},\phi-\phi_0\right)\sin^{\kappa}\xi\,d\xi}{[(\cos\xi-\cos\theta)(\cos\xi-\cos\theta_0)]^{(1+\kappa)/2}}. \tag{2.3.5}$$

Substitution of (5) in (4) yields

$$\frac{a^{1-\kappa}}{2^{\kappa-1}}\int_0^{\theta}\frac{\sin^{\kappa}\xi\,d\xi}{(\cos\xi-\cos\theta)^{(1+\kappa)/2}}\int_{\xi}^{\alpha}\frac{\sin\theta_0\,d\theta_0}{(\cos\xi-\cos\theta_0)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\tan^2(\xi/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)$$
$$=v(\theta,\phi). \tag{2.3.6}$$

Again we have an integral equation consisting of two Abel-type operators and the $\mathcal{L}$-operator. We apply to both sides of (6) the following operator:

$$\mathcal{L}\left(\cot\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right). \tag{2.3.7}$$

The result is

$$2^{-\kappa}a^{1-\kappa}\sin^{\kappa}\theta_1\int_{\theta_1}^{\alpha}\frac{\sin\theta_0\,d\theta_0}{(\cos\theta_1-\cos\theta_0)^{(1+\kappa)/2}}\,\mathcal{L}\left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)}\right)\sigma(\theta_0,\phi_0)$$

$$=\mathcal{L}\left(\cot\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right)v(\theta,\phi). \tag{2.3.8}$$

The following well-known integral was used here

$$\int_{\theta_1}^{\theta}\frac{\sin\xi\,d\xi}{(\cos\xi-\cos\theta)^{(1+\kappa)/2}(\cos\theta_1-\cos\xi)^{(1-\kappa)/2}}=\frac{\pi}{\cos(\pi\kappa/2)}. \tag{2.3.9}$$

The next operator to apply is

$$\mathcal{L}\left(\tan\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\theta_2}^{\alpha}\frac{\sin^{1-\kappa}\theta_1\,d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\cot\frac{\theta_1}{2}\right),$$

with the result

$$\sigma(\theta_2,\phi)=-\frac{\cos(\pi\kappa/2)}{(2a)^{1-\kappa}\pi^2\sin\theta_2}\,\mathcal{L}\left(\tan\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\theta_2}^{\alpha}\frac{\sin^{1-\kappa}\theta_1\,d\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\cot^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_0^{\theta_1}\frac{\sin\theta_1\,d\theta_1}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\,\mathcal{L}\left(\tan\frac{\theta}{2}\right)v(\theta,\phi). \tag{2.3.10}$$

Some further simplification of (10) is possible by using the following rules of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\theta_1}\int_0^{\theta_1}\frac{f(\theta)\sin\theta\,\mathrm{d}\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}=\sin\theta_1\left[\frac{f(0)}{(1-\cos\theta_1)^{(1-\kappa)/2}}+\int_0^{\theta_1}\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta_2}\int_{\theta_2}^{\alpha}\frac{f(\theta_1)\sin\theta_1\,\mathrm{d}\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}=\sin\theta_2\left[-\frac{f(\alpha)}{(\cos\theta_2-\cos\alpha)^{(1-\kappa)/2}}\right.$$

$$\left.+\int_{\theta_2}^{\alpha}\frac{\mathrm{d}f(\theta_1)}{\mathrm{d}\theta_1}\frac{\mathrm{d}\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\right]. \qquad (2.3.11)$$

The simplified result is

$$\sigma(\theta_2,\phi)=\frac{\cos(\pi\kappa/2)}{(2a)^{1-\kappa}\pi^2}\left\{\frac{\Phi(\alpha,\theta_2,\phi)}{(\cos\theta_2-\cos\alpha)^{(1-\kappa)/2}}-\int_{\theta_2}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}\theta_1}[\Phi(\theta_1,\theta_2,\phi)]\frac{\mathrm{d}\theta_1}{(\cos\theta_2-\cos\theta_1)^{(1-\kappa)/2}}\right\},$$

with

$$\Phi(\theta_1,\theta_2,\phi)=\sin^{1-\kappa}\theta_1\left\{\frac{1}{(1-\cos\theta_1)^{(1-\kappa)/2}}\int_0^{2\pi}v(0,\phi)\,\mathrm{d}\phi\right.$$

$$\left.+2\pi\int_0^{\theta_1}\frac{\mathrm{d}\theta}{(\cos\theta-\cos\theta_1)^{(1-\kappa)/2}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\mathcal{L}\left(\frac{\tan(\theta_2/2)\tan(\theta/2)}{\tan^2(\theta_1/2)}\right)v(\theta,\phi)\right]\right\}. \qquad (2.3.12)$$

In the case $\kappa\to 0$ a regular electrostatic solution is recovered.

It is of interest to establish a relationship between function $g$ introduced by Collins (1959) and the axisymmetric component of $\sigma$. His function $g$ was introduced by the expression

$$V(\theta)=\int_0^{\theta}\frac{g(\eta)\sec(\eta/2)\,\mathrm{d}\eta}{\sqrt{2(\cos\eta-\cos\theta)}}. \qquad (2.3.13)$$

Comparison of (13) with (6), taking into consideration $\sigma$ does not depend on $\phi$ and $\kappa=0$, leads to the sought relationship

$$g(\eta) = 2\sqrt{2}\, a \cos\frac{\eta}{2} \int_{\eta}^{\alpha} \frac{\sigma(\theta_0) \sin\theta_0 \, d\theta_0}{\sqrt{\cos\eta - \cos\theta_0}}. \tag{2.3.14}$$

## 2.4. Generalized potential problem for a surface of revolution

The following generalized potential problem is considered: given an arbitrary potential distribution on a surface of revolution, the charge density is to be determined. A closed form solution to the problem is obtained for a certain class of surfaces of revolution due to a special integral representation of the kernel of the governing integral equation. We consider an arbitrary surface of revolution, the axis $0z$ being the axis of revolution. Introduce a set of orthogonal curvilinear coordinates $u$ and $v$, with their relations to the cartesian set as

$$x = f(u)\cos(v), \quad y = f(u)\sin(v), \quad z = w(u). \tag{2.4.1}$$

Here the functions $f$ and $w$ are known and are defined by the shape of the surface of revolution. The distance $R$ between any two points $(u,v)$ and $(u_0,v_0)$ on the surface may be defined as

$$R^2 = [f(u)\cos(v) - f(u_0)\cos(v_0)]^2 + [f(u)\sin(v) - f(u_0)\sin(v_0)]^2$$
$$+ [w(u) - w(u_0)]^2 = h_1(u,u_0) - h(u,u_0)\cos(v - v_0);$$
$$h(u, u_0) = 2 f(u)f(u_0), \quad h_1(u, u_0) = f^2(u) + f^2(u_0) + [w(u) - w(u_0)]^2. \tag{2.4.2}$$

Let us represent

$$h_1(u,u_0) - h(u,u_0)\cos(v - v_0)$$
$$= h(u,u_0)\frac{[\xi_1(u,u_0) - 2\xi_1(u,u_0)\xi_2(u,u_0)\cos(v - v_0) + \xi_2(u,u_0)]}{2} \tag{2.4.3}$$

where

$$\xi_{1,2}(u,u_0) = \frac{h_1(u,u_0)}{h(u,u_0)} \pm \left[\frac{h_1^2(u,u_0)}{h^2(u,u_0)} - 1\right]^{1/2}. \tag{2.4.4}$$

Note that

$$\xi_1(u,u_0) = 1/\xi_2(u,u_0) \tag{2.4.5}$$

Using (2), (4) and (5), we establish also

$$\xi_1(u,u) = \xi_2(u,u) = 1 \tag{2.4.6}$$

Assume the existence of a function $\zeta$ such that

$$\xi_1(u,u_0) = \zeta(u)/\zeta(u_0), \quad \xi_2(u,u_0) = \zeta(u_0)/\zeta(u) \tag{2.4.7}$$

Later it will be shown how to find such a function for certain systems of coordinates. Substitution of (7) and (3) into (2) leads to the relationship

$$R^2 = \frac{f(u)\,f(u_0)}{\zeta(u)\,\zeta(u_0)}\,[\zeta^2(u) - 2\zeta(u)\zeta(u_0)\cos(v - v_0) + \zeta^2(u_0)]. \tag{2.4.8}$$

Now reformulating the problem as: given an arbitrary potential distribution $V(u,v)$ on the surface of revolution ($a \le u \le b$, $0 \le v < 2\pi$), the generalized charge distribution is to be determined. Using (8), one may write the governing integral equation in the form

$$\int_0^{2\pi}\int_a^b \frac{\sigma(u_0,v_0)g(u_0)du_0dv_0}{\}\,[f(u)f(u_0)/\zeta(u)\zeta(u_0)]\,[\zeta^2(u) - 2\zeta\,(u)\zeta(u_0)\cos(v - v_0) + \zeta^2(u_0)]\,\{^{(1+\kappa)/2}}$$

$$= V(u,v), \quad \text{for} \quad a \le u \le b, \quad 0 \le v < 2\pi \tag{2.4.9}$$

For the system of curvilinear coordinates (1), the Jacobian $g$ is given by

$$g(u) = f(u)\,[(df(u)/du)^2 + (dw(u)/du)^2\,]^{1/2}. \tag{2.4.10}$$

Making use of the integral representation established in (1.2.1)

$$\frac{1}{[r^2 - 2rr_0\cos(v - v_0) + r_0^2]^{(1+\kappa)/2}}$$

$$= \frac{2}{\pi}\cos(\frac{\pi\kappa}{2})\int_0^{min(r,r_0)} \frac{\lambda\left(\frac{x^2}{rr_0}, v - v_0\right)x^\kappa dx}{[(r^2 - x^2)(r_0^2 - x^2)]^{(1+\kappa)/2}}. \tag{2.4.11}$$

Assuming $r \equiv \zeta(u)$, $r_0 \equiv \zeta(u_0)$ and $x \equiv \zeta(t)$, one will have instead of (11),

$$\frac{1}{[\zeta^2(u)-2\zeta(u)\zeta(u_0)\cos(v-v_0)+\zeta^2(u_0)]^{(1+\kappa)/2}}$$

$$=\frac{2}{\pi}\cos(\frac{\pi\kappa}{2})\int_c^{min(u,u_0)}\frac{\chi(t,u,u_0,v,v_0)\,\zeta^\kappa(t)\,d\zeta(t)}{\}[\zeta^2(u)-\zeta^2(t)][\zeta^2(u_0)-\zeta^2(t)]\{^{(1+\kappa)/2}}$$

(2.4.12)

Here, $c=\zeta^{-1}(0)$,

$$\chi(t,u,u_0,v,v_0)=\lambda[\zeta^2(t)/\zeta(u)\zeta(u_0),v-v_0], \tag{2.4.13}$$

and $\zeta^{-1}$ denotes the function inverse of $\zeta$. Also, let

$$\zeta^{-1}(0)=a. \tag{2.4.14}$$

Later it will be shown that this requirement is not so limiting. Now substitution of (14) and (12) into (9) leads to the governing integral equation

$$\frac{2}{\pi}\cos(\frac{\pi\kappa}{2})\int_a^u\frac{\zeta^\kappa(t)d\zeta(t)}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\int_t^b\frac{G(u_0)du_0}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}$$

$$\times\int_0^{2\pi}\chi(t,u,u_0,v,v_0)\,\sigma(u_0,v_0)dv_0=\Omega(u,v). \tag{2.4.15}$$

Here

$$G(u_0)=[\zeta(u_0)/f(u_0)]^{(1+\kappa)/2}g(u_0),\quad \Omega(u,v)=[f(u)/\zeta(u)]^{(1+\kappa)/2)}V(u,v).$$

(2.4.16)

The following scheme of change of the order of integration was employed:

$$\int_a^b du_0\int_a^{min(u,u_0)}dt=\int_a^u du_0\int_a^{u_0}dt+\int_u^b du_0\int_a^u dt$$

$$=\int_a^u dt\int_t^u du_0+\int_a^u dt\int_u^b du_0=\int_a^u dt\int_t^b du_0.$$

Equation (15), despite looking more complicated than the original equation (9), admits exact solution in a closed form, using a technique developed in previous sections. Equation (15) may be rewritten by introducing the $\mathcal{L}$-operators

$$4\cos(\frac{\pi\kappa}{2})\int_a^u \frac{\zeta^\kappa(t)d\zeta(t)}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\int_t^b \frac{G(u_0)du_0}{[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}\times$$

$$\times\mathcal{L}\left(\frac{\zeta^2(t)}{\zeta(u)\zeta(u_0)}\right)\sigma(u_0,v)=\Omega(u,v) \quad \text{for} \quad a\le u\le b, \quad 0\le v\le 2\pi. \tag{2.4.17}$$

Now the governing equation (17) presents a sequence of two integral operators of Abel type and the $\mathcal{L}$-operator. The inverse operator to each one is known. An exact solution of (17) may now be constructed in several steps. Convergency at each step is not verified, but is assumed. Application of the operator

$$\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)\frac{d}{du_1}\int_a^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\mathcal{L}(\zeta(u))$$

to both sides of (17) gives

$$2\pi\zeta'(u_1)\,\zeta^\nu(u_1)\int_{u_1}^b\frac{G(u_0)du_0}{[\zeta^2(u_0)-\zeta^2(u_1)]^{(1-\kappa)/2}}\mathcal{L}\left(\frac{\zeta(u_1)}{\zeta(u_0)}\right)\sigma(u_0,v)$$

$$=\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)\frac{d}{du_1}\int_a^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\mathcal{L}[\zeta(u)]\,\Omega(u,v). \tag{2.4.18}$$

Hereafter, a prime indicates the derivative with respect to the variable given in brackets. The following integral is also used

$$\int_t^{u_1}\frac{\zeta(u)d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\nu)/2}\,[\zeta^2(u)-\zeta^2(t)]^{(1+\kappa)/2}}=\frac{\pi}{2\cos(\pi\kappa/2)}. \tag{2.4.19}$$

The next step is application of the operator

$$\mathcal{L}[\zeta(u_2)]\frac{d}{du_2}\int_{u_2}^b\frac{\zeta^{1-\kappa}(u_1)du_1}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{\zeta(u_1)}\right)$$

to both sides of equation (18). The result then is

$$\sigma(u_2,v) = -\frac{\cos(\pi\kappa/2)}{\pi^2 G(u_2)}\,\mathcal{L}(\zeta(u_2))\,\frac{d}{du_2}\int_{u_2}^{b}\frac{\zeta^{1-\kappa}(u_1)\,du_1}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}$$

$$\times\,\mathcal{L}\left(\frac{1}{\zeta^2(u_1)}\right)\frac{d}{du_1}\int_{a}^{u_1}\frac{\zeta(u)\,d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\,\mathcal{L}(\zeta(u))\,\Omega(u,v). \tag{2.4.20}$$

One may establish the following rules of differentiation under the integral sign:

$$\frac{d}{du_1}\int_{a}^{u_1}\frac{\phi(u)\zeta(u)\,d\zeta(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}} = \left\{\frac{\phi(a)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\right.$$

$$\left.+\int_{a}^{u_1}\frac{d\phi(u)}{[\zeta^2(u_1)-\zeta^2(u)]^{(1-\kappa)/2}}\right\}\zeta(u_1)\,\zeta'(u_1),$$

$$\frac{d}{du_2}\int_{u_2}^{b}\frac{\phi(u_1)\zeta(u_1)\,d\zeta(u_1)}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}} = \left\{-\frac{\phi(b)}{[\zeta^2(\mathrm{b})-\zeta^2(u_2)]^{(1-\kappa)/2}}\right.$$

$$\left.+\int_{u_2}^{b}\frac{d\phi(u_1)}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\right\}\zeta(u_2)\,\zeta'(u_2). \tag{2.4.21}$$

Further simplification of (20) is possible by using (21). The result then is

$$\sigma(u_2,v) = \frac{\cos(\pi\kappa/2)}{2\pi^3\,G\,(u_2)}\,\zeta(u_2)\,\zeta'\,(u_2)\left\{\frac{\Phi(b,u_2,v)}{[\zeta^2(b)-\zeta^2(u_2)]^{(1-\kappa)/2}}\right.$$

$$\left.-\int_{u_2}^{b}\frac{[d\Phi(u_1,u_2,v)\,/du_1]}{[\zeta^2(u_1)-\zeta^2(u_2)]^{(1-\kappa)/2}}\,du_1\right\},$$

$$\Phi(u_1, u_2, v) = \int_0^{2\pi} ]\Omega(a,v)\mathrm{d}v$$

$$+ 2\pi\zeta^{1-\kappa}(u_1)\int_a^{u_1} \frac{du}{[\zeta^2(u_1) - \zeta^2(u)]^{(1-\kappa)/2}} \frac{d}{du}\left[ \mathcal{L}\left(\frac{\zeta(u)\zeta(u_2)}{\zeta^2(u_1)}\right)\Omega(u,v)\right]. \tag{2.4.22}$$

The solution of the problem is now complete; expressions (20) and (22) give the two different forms of presentation of the solution.

The method of exact solution to the generalized potential problem, proposed in this section, may be applied only to those systems of curvilinear coordinates, which satisfy the requirements (7) and (14). This limitation is not that severe, as it might appear. We may consider a spherical cap as an example of a surface of revolution. If the cap radius is $r$, then

$$f(u) = r\sin u, \quad w(u) = r\cos u, \quad a = 0,$$

and formulae (2), (4) and (7) give

$$\xi_1(u,u_0) = \frac{\tan(u/2)}{\tan(u_0/2)} = \frac{1}{\xi_2(u,u_0)}, \quad \zeta(u) = \tan\frac{u}{2}.$$

Since $\zeta^{-1}(u) = 2\tan^{-1}u$, the requirement in (14) is satisfied and the solution (22) is valid.

This does not mean that the system of spherical coordinates is the only one to satisfy conditions (7) and (14). For example, the system of polar coordinates on a circular disk also satisfies (7) and (14); namely, for that case

$$f(u) = u, \quad w(u) = 0, \quad a = 0, \quad \xi_1(u,u_0) = 1/\xi_2(u,u_0) = u/u_0, \quad \zeta(u) = u$$

and formula (22) gives the solution of the generalized potential problem for a circular disk, corresponding to the one obtained in previous section.

When the surface of revolution is such that the system of curvilinear coordinates does not satisfy the requirements (7) and (14), the procedure of approximation of (4) in the form (7) may be recommended. If the procedure is successful, formula (22) will give an approximate solution to the problem. As an example of such an approximation, consider the case of a cylinder of radius $R$ and height $H$. In this case $f(u)=R$, $w(u)=u$ and formulae (2) and (4) give

$$\xi_{1,2}(u,u_0) = 1 + \frac{1}{2}\left(\frac{u-u_0}{R}\right)^2 \pm \left(\frac{u-u_0}{R}\right)\left[1 + \left(\frac{u-u_0}{2R}\right)^2\right]^{1/2}. \qquad (2.4.23)$$

If the condition $(H/R) \ll 1$ holds, the following approximation of (23) is valid:

$$\xi_{1,2}(u,u_0) = 1 \pm \left(\frac{u-u_0}{R}\right)$$

The last expression may be represented with the same degree of accuracy as

$$\xi_1(u,u_0) = \frac{R+u}{R+u_0}, \quad \xi_2(u,u_0) = \frac{R+u_0}{R+u}.$$

Now one may assume $\zeta(u)=R+u$, $a=-R$ and the requirements (7) and (14) are satisfied, then all the results of this section become applicable to the case of a short cylinder.

It may also be noticed that it is possible to solve the mathematically identical problems in hydrodynamics, nonhomogeneous elasticity, heat conduction, etc., using the same general approach.

## Exercises 2

1. The potential inside a circle is given by $V(\rho,\phi)=w_0+\theta\rho\cos\phi$, with $w_0$=const, and $\theta$=const. Find the corresponding charge distribution $\sigma$.

*Answer*:

$$\sigma(\rho,\phi) = \frac{\cos(\pi\kappa/2)}{\pi^2 H}\,\frac{w_0 + (2\theta\rho\cos\phi)/(1+\kappa)}{(a^2-\rho^2)^{(1-\kappa)/2}}.$$

2. In the problem above express the total charge $Q$ and the moment $M$, in terms of the parameters $w_0$ and $\theta$.

*Answer*:

$$Q = \frac{2w_0 a^{1+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)}, \quad M = \frac{4\theta a^{3+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}.$$

3. The potential inside a circle is given by $V(\rho,\phi)=w_0-c\rho^2$, with $w_0$=const, and $c$=const. Find the charge distribution $\sigma$ and the radius $a$, if it is known that $\sigma(a,\phi)=0$.

*Solution*: utilization of (2.1.6) yields

$$\sigma(\rho) = \frac{\cos(\pi\kappa/2)}{\pi^2 H(a^2-\rho^2)^{(1-\kappa)/2}}\left\{w_0 + \frac{2c[a^2(1-\kappa)-2\rho^2]}{(1+\kappa)^2}\right\}.$$

The radius $a$ is found from the condition $\sigma(a)=0$. The result is $a=[(1+\kappa)w_0/(2c)]^{1/2}$, and the final expression for the charge distribution is

$$\sigma(\rho) = \frac{2w_0\cos(\pi\kappa/2)}{\pi^2 a^2 H(1+\kappa)}(a^2-\rho^2)^{(1+\kappa)/2}.$$

4. In the problem above find the total charge $Q$.
*Answer*:

$$Q = \frac{4w_0\, a^{1+\kappa}\cos(\pi\kappa/2)}{\pi H(1+\kappa)(3+\kappa)}.$$

5. A circular disk of radius $a$ is grounded and kept at zero potential. A point charge $P$ is present at $(b,\psi)$, $b>a$. Find the induced charge distribution.
*Answer*:

$$\sigma(\rho,\phi) = -\frac{P}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)\left(\frac{b^2-a^2}{a^2-\rho^2}\right)^{(1-\kappa)/2}\frac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}.$$

6. A circular disk of radius $a$ is grounded and kept at zero potential. Let a uniform density charge $\sigma_0$ be prescribed at the annulus $b\le\rho\le c$, $(b>a)$, with no active charge elsewhere. Find the charge density $\sigma$ for $\rho<a$.
*Answer*:

$$\sigma(\rho) = -\frac{2\sigma_0\cos(\pi\kappa/2)}{\pi(a^2-\rho^2)^{(1-\kappa)/2}}\int_b^c \frac{(\rho_0^2-a^2)^{(1-\kappa)/2}\rho_0\, d\rho_0}{\rho_0^2-\rho^2}, \quad \text{for } \rho<a.$$

In general, the last integral can be computed in terms of hypergeometric functions. In the particular case of $\kappa=0$, the integral is computable in elementary functions:

$$\sigma(\rho) = -\frac{2}{\pi}\sigma_0\left[\frac{(c^2-a^2)^{1/2}-(b^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}} - \tan^{-1}\left(\frac{c^2-a^2}{a^2-\rho^2}\right)^{1/2}\right.$$
$$\left. + \tan^{-1}\left(\frac{b^2-a^2}{a^2-\rho^2}\right)^{1/2}\right].$$

7. In the preceding problem find the potential $V$.
*Answer*:

$$V(\rho) = \frac{4\cos(\pi\kappa/2)}{1-\kappa}\sigma_0\left\{\int_a^{\min(\rho,c)} \frac{(c^2-x^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1+\kappa)/2}}x^\kappa\,\mathrm{d}x\right.$$

$$\left.-\int_a^{\min(\rho,b)} \frac{(b^2-x^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1+\kappa)/2}}x^\kappa\,\mathrm{d}x\right\}, \text{ for } \rho>a.$$

8. By using formula (2.1.6), prove the identity

$$\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\rho^{1+|n|}\,\mathrm{e}^{in\phi}\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= \frac{(1+\kappa)\Gamma(1+|n|)}{2\pi^2\Gamma[|n|+(1+\kappa)/2]}\cos\frac{\pi\kappa}{2}\int_0^{2\pi}\int_0^a \frac{f(\rho,\phi)\rho^{1+|n|}\,\mathrm{e}^{in\phi}\,\mathrm{d}\rho\,\mathrm{d}\phi}{(a^2-\rho^2)^{(1-\kappa)/2}}.$$

*Note*: in the particular cases $n$=0 and $n$=–1, the last identity transforms into (2.1.10) and (2.1.12) respectively.

9. Define the energy function $K_1(\rho,\phi)$ in terms of the potential $V$ for interior problems.
*Answer*:

$$K_1(\rho,\phi) = \frac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2 H\rho^{(1+\kappa)/2}}\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{x\,\mathrm{d}x}{(\rho^2-x^2)^{(1-\kappa)/2}}\mathcal{L}(x)V(x,\phi).$$

*Hint*: perform the inversion of (2.1.22).

10. Prove the identity

$$\lim_{\rho\to a}\left\{\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{(x^2-a^2)^{(1-\kappa)/2}}{(\rho^2-x^2)^{(1-\kappa)/2}}f(x)\,\mathrm{d}x\right\} = \frac{\pi(1-\kappa)f(a)}{2\cos(\pi\kappa/2)}.$$

*Hint*: use the substitution $t=(\rho^2-x^2)/(x^2-a^2)$.

11. The following charge density distribution is prescribed over a circle $\rho\le a$: $\sigma(\rho,\phi)=\sigma_0+\sigma_1\rho\cos\phi$, with $\sigma_0$=const and $\sigma_1$=const. The diaphragm outside the circle is grounded. Find the charge induced on the diaphragm.
*Answer*:

$$\sigma(\rho,\phi) = -\frac{2\cos(\pi\kappa/2)}{\pi(3-\kappa)}\left(\frac{a}{\rho}\right)^{3-\kappa}\left\{\sigma_0 F\left(\frac{3-\kappa}{2}, \frac{3-\kappa}{2}; \frac{5-\kappa}{2}; (\frac{a}{\rho})^2\right)\right.$$

$$\left. + \frac{2}{5-\kappa}\left(\frac{a}{\rho}\right)^2 F\left(\frac{3-\kappa}{2}, \frac{5-\kappa}{2}; \frac{7-\kappa}{2}; (\frac{a}{\rho})^2\right)\sigma_1\rho\cos\phi\right\}, \text{ for } \rho>a.$$

The result can be expressed in elementary functions for $\kappa=0$

$$\sigma(\rho,\phi) = -\frac{2}{\pi}\left\{\left[\frac{a}{(\rho^2-a^2)^{1/2}} - \sin^{-1}(\frac{a}{\rho})\right]\sigma_0\right.$$

$$\left. + \left[\frac{3\rho^2-a^2}{3\rho(\rho^2-a^2)^{1/2}} - 3\frac{\rho}{a}\sin^{-1}(\frac{a}{\rho})\right]\sigma_1\rho\cos\phi\right\}.$$

12. In the problem above find the potential $V$.
*Answer*:

$$V = \frac{4\cos(\pi\kappa/2)}{(1-\kappa)^2}(a^2-\rho^2)^{(1-\kappa)/2}\left[\sigma_0 + \frac{2}{3-\kappa}\sigma_1\rho\cos\phi\right].$$

13. A point charge $P$ is located at the point $(b,\psi)$, $b<a$. The diaphragm outside the circle is grounded. Find the charge induced on the diaphragm.
*Answer*:

$$\sigma(\rho,\phi) = -\frac{P}{\pi^2}\cos\left(\frac{\pi\kappa}{2}\right)\left(\frac{a^2-b^2}{\rho^2-a^2}\right)^{(1-\kappa)/2}\frac{1}{\rho^2+b^2-2b\rho\cos(\phi-\psi)}.$$

14. Express the energy function $K_1(\rho,\phi)$ in terms of the potential $V$ in exterior problems.
*Answer*:

$$K_1(\rho,\phi) = -\frac{\cos(\pi\kappa/2)}{2^{(1-\kappa)/2}\pi^2 H\rho^{(1+\kappa)/2}}\mathcal{L}(\rho)\frac{d}{d\rho}\int_\rho^a \frac{x\,dx}{(x^2-\rho^2)^{(1-\kappa)/2}}\mathcal{L}\left(\frac{1}{x}\right)V(x,\phi).$$

15. In the generalized potential theory, consider a spherical cap of radius $r$ with uniform potential prescribed over the surface, namely $V(u,v)=V_0=\text{const.}$ Find the charge density distribution.
*Answer:*

$$\sigma(u_2, v) = \frac{V_0 \cos(\pi\kappa/2)}{(2r)^{1-\kappa}\pi^2}\left\{\frac{1-\kappa}{1+\kappa} z^{(1+\kappa)/2} F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}, \frac{3+\kappa}{2}; z\right)\right.$$

$$\left. + \frac{\cos^{1-\kappa}(b/2)}{[\cos^2(u_2/2) - \cos^2(b/2)]^{(1-\kappa)/2}}\right\},$$

$$z = 1 - \frac{\cos^2(b/2)}{\cos^2(u_2/2)}.$$

Here $F(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function, and the following integral was employed.

$$\int_{u_2}^{b} \frac{\cos^{\beta}\left(\frac{u_1}{2}\right) d(\cos(u_1/2))}{\left[\cos^2\left(\frac{u_2}{2}\right) - \cos^2\left(\frac{u_1}{2}\right)\right]^{(1-\kappa)/2}} = -\frac{\cos^{\beta-1}(u_2/2)}{1+\kappa}\left[\cos^2\left(\frac{u_2}{2}\right) - \cos^2\left(\frac{b}{2}\right)\right]^{(1+\kappa)/2}$$

$$\times F\left(\frac{1-\beta}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right).$$

16. In the generalized potential theory consider the case of a grounded spherical cap or radius $r$ in a uniform electric field of intensity $E$, acting along the $0z$ axis. For this case, $V(u,v) = E r \cos u$. Find the charge density distribution.

*Answer:*

$$\sigma(u_2, v) = \frac{E r^{\kappa} \cos(\pi\kappa/2)}{2^{1-\kappa}(1+\kappa)\pi^2}\left\{\left(\cos^{1-\kappa}\frac{b}{2}\right)\frac{(3-\kappa)\cos u_2 - (1-\kappa)(1+\cos b)}{(\cos^2(u_2/2) - \cos^2(b/2))^{(1-\kappa)/2}}\right.$$

$$\left. + \frac{(3-\kappa)(1-\kappa)}{1+\kappa} z^{(1+\kappa)/2} \cos u_2\, F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)\right\},$$

$$z = 1 - \frac{\cos^2(b/2)}{\cos^2(u_2/2)}$$

Here integral from Example 15 was employed, as well as the following properties of the hypergeometric functions

$$F\left(-\frac{1-\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$= \frac{1+\kappa}{2}(1-z)^{(1-\kappa)/2} + \frac{1-\kappa}{2} F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right),$$

$$F\left(-\frac{3-\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$= \frac{(1-\kappa)(3-\kappa)}{8} F\left(\frac{1+\kappa}{2}, \frac{1+\kappa}{2}; \frac{3+\kappa}{2}; z\right)$$

$$-\frac{1+\kappa}{8}(2z-5+\kappa)(1-z)^{(1-\kappa)/2},$$

$$F(d, \beta; \beta; z) = (1-z)^{-d}.$$

17. In the generalized potential theory, consider the problem of a grounded spherical cap in a uniform electrical field, acting in the $0x$ direction. For this case $V(u,v) = E r \sin u \cos v$. Find the charge density distribution

*Answer:*

$$\sigma(u_2, v) = \frac{2E(2r)^{\kappa}\cos(\pi\kappa/2)}{\pi^2(1+\kappa)} \cos v \tan\left(\frac{u_2}{2}\right)$$

$$\times\left\{\frac{3-\kappa}{1+\kappa}\cos^2\left(\frac{u_2}{2}\right) z^{(1+\kappa)/2} F\left(-\frac{1-\kappa}{2}, \frac{1+\kappa}{2}, \frac{3+\kappa}{2}; z\right)\right.$$

$$\left. + \frac{\cos^{3-\kappa}(b/2)}{[\cos^2(u_2/2) - \cos^2(b/2)]^{(1-\kappa)/2}}\right\},$$

with

$$z = 1 - \frac{\cos^2(b/2)}{\cos^2(u_2/2)}$$