INTRODUCTION

This book may be considered as logical continuation of the previously published (V.I. Fabrikant, *Applications of Potential Theory in Mechanics*, Kluwer Academic, 1989), where a new and elementary method was described for solving mixed boundary value problems. The method can solve *non-axisymmetric problems* as easily as axisymmetric ones, *exactly and in closed form*. It enables us to treat *analytically* non-classical domains. The method also provides, as a bonus, a tool for exact evaluation of various two-dimensional integrals involving distances between two or more points.

The main emphasis of the first book was on solid mechanics problems. Here, we describe various applications of the new method to electromagnetics, acoustics and diffusion. Also included in this book are some results in fracture mechanics and elastic contact problems which were obtained just recently and could not be included in the first book.

The book is addressed to a wide audience ranging from engineers to mathematical physicists. While an engineer can find in the book some elementary, ready to use formulae for solving various practical problems, a mathematical physicist might become interested in new applications of the mathematical apparatus presented. The book should be of interest to specialists in electromagnetics, acoustics, diffusion, solid and fluid mechanics, etc.

The book is *accessible to anyone* with a background in university undergraduate calculus, but should be of interest to established scientists as well. Though the method is elementary, the transformations involved are sometimes very non-trivial and cumbersome, while the final result is usually very simple. The reader who is interested only in application of the general results to his/her particular problems may skip the long derivations and use the final formulae which requires little effort. The reader who wants to master the method in order to solve new problems has to repeat the derivations which are given in sufficient detail. The exercises are important in this regard.

The book is based entirely on the author's results, and this is why the work of other scientists is mentioned only when such a quotation is inevitable for some reason, like numerical data needed to verify the accuracy of approximate results, comparison with existing results, or pointing out some errors in publications. There are several books and review articles presenting an adequate account of the state-of-the-art in the field. Appropriate references are given for the reader's convenience. The purpose of this book was neither to repeat nor to compete with them.

For the reader's convenience, it was attempted to make each chapter (and section, wherever possible) self-contained. The reader can skip several sections and continue reading, without loosing the ability to understand material. On the other hand, this resulted in repetition of some definitions and descriptions. The author thinks that the additional convenience is worth several extra pages in the book.

A new and more general definition of the $\mathcal{L}$-operator is given in Chapter 1. This definition gives rigorous justification to the mathematical formalism involved. Various forms of integral representation for the reciprocal of the distance between two points follows. A general solution is presented to basic mixed boundary value problems for a half-space in cylindrical coordinates. These results are generalized for the case of spherical and toroidal coordinates.

We can generalize the Newton potential as $V = H/R^{1+\kappa}$, where R is the distance between two points, H is a constant depending on the physical properties of the space, and $-1 < \kappa < 1$. This potential has various applications in engineering, for example, in the theory of elasticity of inhomogeneous elastic body, with the modulus of elasticity E being a power function of z: $E = E_0 z^{\kappa}$. Other applications include fluid mechanics and heat transfer. Closed form solution to various non-axisymmetric problems is given in Chapter 2.

The general results of Chapter 1 are applied in Chapter 3 to investigation of interaction of several charged coaxial and arbitrarily located disks. New type of governing integral equation is derived for the Dirichlet and Neumann problems for a circular annulus domain. Simple yet accurate formulae are derived for the capacity of flat laminae. Similar results were obtained for the electrical and magnetic polarizability of small apertures of general shape.

Advances in bioengineering have generated wide interest in the diffusion mechanism of biological membranes. The diffusion process through a thin membrane, perforated by several holes of arbitrary shape, is considered in Chapter 4. A general theorem is established which relates the total flux through each hole, with the concentration distribution of some chemical species prescribed in the hole, to a system of linear algebraic equations. The theorem is applied to the case of arbitrarily located circular and elliptical holes. The influence of the pore length is investigated by a new method. Application of the main results to the problems of sound transmission through an arbitrary aperture in a soft and rigid screen is also presented.

Chapter 5 contains complete solutions to several contact problems which were obtained recently, and could not be included in (Fabrikant, 1989). Those comprise complete elastic fields around axisymmetric and inclined bonded punch. These fundamental solutions allow us to solve various problems of interaction between punches and anchor loads. Two of such solutions are included. A new approach is presented to a general annular punch problem, with analytical, numerical and asymptotic solutions derived and compared.

Some new results in fracture mechanics are presented in Chapter 6. A complete solution is given for the first time to the case of general antisymmetric loading of internal and external circular cracks. All the relevant Green's functions are given explicitly and in closed form. A superposition of antisymmetric solution with a symmetric allows us to solve the problem of one-sided loading of a crack as well as various interactions between a crack and a general external force. A new approach is presented to the problem of a flat crack in the shape of a circular annulus, subjected to a general normal load. The problem is reduced to two two-dimensional integral equations with elementary kernels. The equations are non-singular, and can be easily uncoupled. An accurate numerical solution can be obtained by any standard method.

Chapter 7 is devoted to the numerical methods of solution of one-dimensional integral and integro-differential equations. The solution is represented in the form of power series with undetermined coefficients multiplied by a function in which the essential features of the singularity of the solution are preserved. The method of collocations is used to determine the unknown coefficients. The examples show that the method suggested is more general and gives good results even in the case when the form of solution does not exactly preserve the essential features of singularity. The method is simpler than others which use the properties of orthogonal polynomials. A standard FORTRAN subroutine is presented for solving general one-dimensional integro-differential equations. An algorithm and a standard subroutine are developed for computer evaluation of two-dimensional singular integrals. The software is used in numerical analysis of various non-classical two- and three-dimensional contact problems.

The book contains so much new material that some misprints and errors are inevitable, though every effort was made to eliminate them. The author would be grateful for every communication in this regard. All the readers' comments are welcome.

CHAPTER 1

NEW RESULTS IN POTENTIAL THEORY

1.1. The $\mathcal{L}$-operator and its properties

Let $f(r,\phi)$ be an arbitrary function which belongs to $L^1[0,2\pi]$ as a function of ϕ for any fixed $r \geq 0$. Let us associate with $f(r,\phi)$ the sequence

$$\{f_n(r)\} \qquad -\infty < n < \infty \tag{1.1.1}$$

of its Fourier coefficients. Consider $\mathcal{L}(k)$ as an operator on the linear space of sequences $\{f_n(r)\}$. We do not define any topology on this space. The algebraic operations are defined naturally as follows:

$$c_1\{f_n^{(1)}\} + c_2\{f_n^{(2)}\} := \{c_1 f_n^{(1)} + c_2 f_n^{(2)}\}, \tag{1.1.2}$$

$$\{f_n^{(1)}\} = \{f_n^{(2)}\} \iff f_n^{(1)} = f_n^{(2)} \quad \forall\, n \tag{1.1.3}$$

We define

$$\mathcal{L}(k)\{f_n\} = \{k^{|n|} f_n\}. \tag{1.1.4}$$

This definition makes sense for any $k \in \mathbb{C}$, and it implies that

$$\mathcal{L}(k_1)\,\mathcal{L}(k_2)\{f\} = \mathcal{L}(k_1 k_2)\,\{f\} \quad \forall \;\; k_1, k_2 \in \mathbb{C}, \tag{1.1.5}$$

$$\mathcal{L}(1)\,\{f\} = \{f\}. \tag{1.1.6}$$

Equation (6) is a particular case of (5) corresponding to $k_1 = k \neq 0$ and $k_2 = k^{-1}$.

Consider now the operator $\prod_{j=1}^{m} \mathcal{L}(k_j)$. It is well defined for any k_j, in particular, in the case when some of the k_j are greater than 1. An obvious corollary is: if $|\prod_{j=1}^{m} k_j| < 1$ then $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ is a sequence of the Fourier coefficients of some function belonging to $L^1[0,2\pi]$ if $\{f_n\}$ is a sequence of the Fourier coefficients of a function $f(\phi) \in L^1[0,2\pi]$, and, moreover, the Fourier series corresponding to the sequence $\prod_{j=1}^{m} \mathcal{L}(k_j)\{f_n\}$ converges absolutely and uniformly in $\phi \in [0,2\pi]$.

In the case when $k<1$, formula (4) can be rewritten as

$$\mathcal{L}(k)f(\phi) = \sum_{n=-\infty}^{\infty} k^{|n|} f_n e^{in\phi} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} k^{|n|} e^{in\phi} \int_0^{2\pi} e^{-in\phi_0} f(\phi_0)\, d\phi_0. \tag{1.1.7}$$

Summation in (7) yields

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \lambda(k, \phi - \phi_0)\, f(\phi_0)\, d\phi_0, \tag{1.1.8}$$

where the notation was introduced

$$\lambda(k,\psi) = \frac{1 - k^2}{1 - 2k\cos\psi + k^2}\,. \tag{1.1.9}$$

Note that the $\mathcal{L}$-operator, as it is presented in (8), coincides with the one which was introduced by Poisson for solving the two-dimensional boundary value problem of potential theory for a circle. We are going to use it for solving the relevant three-dimensional problems. Whenever the operator $\mathcal{L}(k)$ is applied, with no limitation $k < 1$ assumed, its general definition (4) is valid, allowing the properties (5) and (6) to be used. As soon as it becomes clear that $k \le 1$, the representation (8) becomes valid thus making it possible to write the final result in a closed and simplified form.

1.2. Integral representation for the reciprocal of the distance between two points

It was proven in (Fabrikant, 1971a) that

$$\frac{1}{R^{1+u}} = \frac{1}{(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0))^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, x^u \mathrm{d}x}{\left[(\rho^2-x^2)(\rho_0^2-x^2)\right]^{(1+u)/2}}. \tag{1.2.1}$$

Here λ is defined by (1.1.9). The identity (1) can be verified by the introduction of a new variable

$$\eta(x) = [(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}/x, \tag{1.2.2}$$

Substitution of the identities

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = -\frac{\rho^2\rho_0^2-x^4}{x^3\eta}, \qquad \lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0) = -\frac{x\eta}{R^2+\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

in (1) transforms it into

$$\frac{1}{R^{1+u}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{\infty}\frac{\eta^{-u}\mathrm{d}\eta}{R^2+\eta^2}. \tag{1.2.3}$$

The integral in (3) can be evaluated by using formula (3.241.4) from (Gradshtein and Ryzhik, 1963), thus proving the identity. All the results above are related to the distance between two points in the plane z=0. We need to generalize them to represent

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}}. \tag{1.2.4}$$

One can observe that representation (1) remains valid if we formally substitute ρ and ρ_0 by arbitrary quantities l_1 and l_2. We need to choose them so that

$$\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2 = l_1^2+l_2^2-2l_1l_2\cos(\phi-\phi_0). \tag{1.2.5}$$

This leads to two equations

$$l_1^2 + l_2^2 = \rho^2 + \rho_0^2 + z^2, \qquad l_1 l_2 = \rho\rho_0. \tag{1.2.6}$$

Their solution will take the form

$$l_1(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho+\rho_0)^2 + z^2]^{1/2} - [(\rho-\rho_0)^2 + z^2]^{1/2}\}, \tag{1.2.7}$$

$$l_2(\rho_0,\rho,z) = \frac{1}{2}\{[(\rho+\rho_0)^2 + z^2]^{1/2} + [(\rho-\rho_0)^2 + z^2]^{1/2}\}. \tag{1.2.8}$$

Hereafter the following abbreviations will be used:

$$l_1(x) \equiv l_1(x,\rho,z), \qquad l_2(x) \equiv l_2(x,\rho,z), \tag{1.2.9}$$

$$l_1 \equiv l_1(a,\rho,z), \qquad l_2 \equiv l_2(a,\rho,z). \tag{1.2.10}$$

Note the limiting properties

$$\lim_{z\to 0} l_1(x) = \min(x,\rho), \qquad \lim_{z\to 0} l_2(x) = \max(x,\rho). \tag{1.2.11}$$

In view of the properties above, the representation (1) can be generalized as follows:

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, x^u \mathrm{d}x}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{(1+u)/2}}. \tag{1.2.12}$$

Formula (12) simplifies when u=0

$$\frac{1}{R_0} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, \mathrm{d}x}{\{[l_1^2(\rho_0) - x^2][l_2^2(\rho_0) - x^2]\}^{1/2}}. \tag{1.2.13}$$

Again, one can notice that the integral in (13) may be evaluated as indefinite

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{\{[l_1^2(\rho_0)-x^2][l_2^2(\rho_0)-x^2]\}^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{\{[l_1^2(\rho_0)-x^2][l_2^2(\rho_0)-x^2]\}^{1/2}}{xR_0}. \tag{1.2.14}$$

The last representation is very important and will be widely used throughout the book.

Another series of useful formulae can be obtained from those above by a simple change of variables, namely,

$$\int \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}} = \frac{1}{R_0}\tan^{-1}\frac{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}}{xR_0}, \tag{1.2.15}$$

$$\frac{1}{R_0^{1+u}} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_{l_2(\rho_0)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, x^u dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{(1+u)/2}}, \tag{1.2.16}$$

$$\frac{1}{R_0} = \frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}} = \frac{2}{\pi}\int_{l_2(\rho_0)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\{[x^2-l_1^2(\rho_0)][x^2-l_2^2(\rho_0)]\}^{1/2}}, \tag{1.2.17}$$

$$\int \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi-\phi_0)\, dx}{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}} = \frac{1}{R}\tan^{-1}\left[\frac{\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}}{xR}\right]. \tag{1.2.18}$$

The representations above are useful for solving external mixed boundary value problems.

Several modifications of (14) are available. For example, we can write

$$\int \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi-\phi_0)\, dx}{(\rho^2-x^2)^{1/2}[\rho_0^2-g^2(x)]^{1/2}} = -\frac{1}{R_0}\tan^{-1}\frac{(\rho^2-x^2)^{1/2}[\rho_0^2-g^2(x)]^{1/2}}{xR_0}. \tag{1.2.19}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}. \tag{1.2.20}$$

It is important to notice that the function $g(x)$ is inverse to l_1 for $x^2 < \rho^2$, and is inverse to l_2 for $x^2 > \rho^2 + z^2$. Introduction of a new variable $x = l_1(y)$, $y = g(x)$ transforms (19) into

$$\int \frac{[l_2^2(y) - y^2]^{1/2}}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - l_1^2(y)]} \lambda\left(\frac{l_1^2(y)}{\rho\rho_0}, \phi - \phi_0\right) dy$$

$$= -\frac{1}{R_0} \tan^{-1} \frac{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - y^2]^{1/2}}{yR_0} \tag{1.2.21}$$

A particular case of (13), when $z = 0$, reads

$$\frac{1}{R} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]^{1/2}} = \frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\lambda(\frac{x^2}{\rho\rho_0}, \phi - \phi_0)\, dx}{\sqrt{\rho^2 - x^2}\sqrt{\rho_0^2 - x^2}}. \tag{1.2.22}$$

The same result takes another form due to (17)

$$\frac{1}{R} = \frac{2}{\pi} \int_{max(\rho_0,\rho)}^{\infty} \frac{\lambda(\frac{\rho\rho_0}{x^2}, \phi - \phi_0)\, dx}{\sqrt{x^2 - \rho^2}\sqrt{x^2 - \rho_0^2}}. \tag{1.2.23}$$

1.3. Internal mixed boundary value problem

The problem is called internal when the non-zero boundary conditions are prescribed inside a circle.

Problem 1. Let us consider a typical problem solved by our method. We need to find a function V such that

$$\Delta V = 0 \quad \text{in} \quad \mathbb{R}^3_+ := \{x: x_3 > 0\}, \quad x = (x_1, x_2, x_3) \tag{1.3.1}$$

subject to the following boundary conditions at $z = 0$

$$V = v(\rho, \phi), \quad \text{if} \quad \rho < a, \quad \partial V/\partial z = 0 \quad \text{if} \quad \rho > a, \quad \rho := x_1, \;\; \phi := x_2, \;\; z := x_3. \tag{1.3.2}$$

$$V(\infty) = 0. \tag{1.3.3}$$

Here (ρ, ϕ) are polar coordinates in the plane $P = \{x\colon\ x_3 = 0\}$; and v is a given function.

The problem can be interpreted as an electrostatic one of a charged disc, with a certain potential prescribed on its surface, or it can be interpreted as an elastic contact problem of a circular punch pressed against an elastic half-space; other interpretations are also possible. We call the problem internal because the non-zero conditions are prescribed inside the disc. The potential function V can be represented through a simple layer as follows:

$$V(\rho,\phi,z) = \int_0^{2\pi}\int_0^a \frac{\sigma(\rho_0,\phi_0)}{R_0}\rho_0 \mathrm{d}\rho_0 \mathrm{d}\phi_0 \tag{1.3.4}$$

Here

$$R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z^2]^{1/2}, \quad \text{and} \quad \sigma = -\frac{1}{2\pi}\frac{\partial V}{\partial z}\Big|_{z=0}. \tag{1.3.5}$$

Substitution of (13) in (4) yields, after changing the order of integration

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - g^2(x)}}\, \mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) \tag{1.3.6}$$

Here

$$g(x) = x[1 + z^2/(\rho^2 - x^2)]^{1/2}, \tag{1.3.7}$$

the $\mathcal{L}$-operator is defined by (1.1.4), the abbreviations l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a \mathrm{d}\rho_0 \int_0^{l_1(\rho_0)} \mathrm{d}x = \int_0^{l_1} \mathrm{d}x \int_{g(x)}^a \mathrm{d}\rho_0. \tag{1.3.8}$$

Substitution of the boundary condition (2) in (6) leads to the governing integral equation

$$4\int_0^{\rho} \frac{\mathrm{d}x}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 \mathrm{d}\rho_0}{\sqrt{\rho_0^2 - x^2}}\, \mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.3.9}$$

Expression (9) is now presented as a sequence of two Abel-type operators and one $\mathcal{L}$-operator. We recall that the general Abel integral equation

$$\int_x^a \frac{F(y)\,\mathrm{d}y}{(y^2-x^2)^{(1+u)/2}} = f(x) \tag{1.3.10}$$

has the solution

$$F(r) = -\frac{2\cos(\pi u/2)}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{f(x)\,x\mathrm{d}x}{(x^2-r^2)^{(1-u)/2}}. \tag{1.3.11}$$

Since the variables in the Abel operators of (9) are interwoven with those of the $\mathcal{L}$-operator, we need to apply their combination, in order to invert (9). In view of the new definition of the $\mathcal{L}$-operator (1.1.4), equation (9) can be rewritten as

$$4\int_0^\rho \frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\left\{\left(\frac{x^2}{\rho\rho_0}\right)^{|n|}\sigma_n(\rho_0)\right\} = \{v_n(\rho)\}. \tag{1.3.12}$$

We have here a sequence of one-dimensional integral equations. The first operator to be applied to both sides of (12) is

$$\mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\,\mathcal{L}(\rho), \tag{1.3.13}$$

with the result

$$2\pi\int_t^a \frac{\rho_0\mathrm{d}\rho_0}{(\rho_0^2-t^2)^{1/2}}\,\mathcal{L}\left(\frac{t}{\rho_0}\right)\{\sigma_n(\rho_0)\} = \mathcal{L}\left(\frac{1}{t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\,\mathcal{L}(\rho)\,\{v_n(\rho)\}. \tag{1.3.14}$$

The second operator to be applied to both sides of (14) is

$$\mathcal{L}(y)\,\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t\mathrm{d}t}{(t^2-y^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{t}\right),$$

with the result

$$\{\sigma_n(y)\} = -\frac{1}{\pi^2 y}\mathcal{L}(y)\frac{\mathrm{d}}{\mathrm{d}y}\int_y^a \frac{t\mathrm{d}t}{(t^2-y^2)^{1/2}}\mathcal{L}\left(\frac{1}{t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}(\rho)\{v_n(\rho)\}. \tag{1.3.15}$$

Taking into consideration that $(\rho y/t^2) < 1$, the rules of differentiation of integrands and the properties of the $\mathcal{L}$-operators allow us to rewrite (15) as follows:

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}} - \int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi)\right]. \tag{1.3.16}$$

Here

$$\Phi(t,y,\phi) = \frac{1}{t}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\mathcal{L}\left(\frac{\rho y}{t^2}\right)v(\rho,\phi)\right]. \tag{1.3.17}$$

Using integration by parts and the fact that $\lambda(k,\psi)$ satisfies the two-dimensional Laplace equation in polar coordinates, the following identity can be established

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,y,\phi) = \int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi), \tag{1.3.18}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Substitution of (18) in (16) leads to another form of solution, namely,

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left[\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}} - \int_y^a \frac{\mathrm{d}t}{(t^2-y^2)^{1/2}}\int_0^t \frac{\rho\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\mathcal{L}\left(\frac{\rho y}{t^2}\right)\Delta v(\rho,\phi)\right], \tag{1.3.19}$$

Interchange of the order of integration in (19) and integration with respect to t yields

$$\sigma(y,\phi) = \frac{1}{\pi^2}\left\{\frac{\Phi(a,y,\phi)}{(a^2-y^2)^{1/2}}\right.$$

$$\left. -\frac{1}{2\pi}\int_0^{2\pi}\int_0^a \tan^{-1}\left[\frac{\sqrt{a^2-\rho^2}(a^2-y^2)^{1/2}}{a[\rho^2+y^2-2\rho y\cos(\phi-\psi)]^{1/2}}\right]\frac{\Delta v(\rho,\psi)\,\rho\mathrm{d}\rho\mathrm{d}\psi}{[\rho^2+y^2-2\rho y\cos(\phi-\psi)]^{1/2}}\right\}.$$

(1.3.20)

The solution obtained here consists of two parts: the first part is singular at the boundary while the second one vanishes at the boundary. In various applications it is required that the solution be nonsingular at the boundary. The necessary and sufficient condition then is $\Phi(a,a,\phi)=0$. In elastic contact problems this condition defines the radius of the contact domain. Notice also that in the case when v is a two-dimensional harmonic function, the non-trivial solution is singular.

Now it is of interest to express the potential V in the half-space directly through its value v prescribed inside the disc $\rho=a$. Substitution of (15) in (6) yields, after subsequent integration

$$V(\rho,\phi,z)=\frac{2}{\pi}\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\mathcal{L}\left(\frac{x^2}{\rho g^2(x)}\right)\frac{\mathrm{d}}{\mathrm{d}g(x)}\int_0^{g(x)}\frac{r\,\mathrm{d}r}{[g^2(x)-r^2]^{1/2}}\mathcal{L}(r)v(r,\phi). \tag{1.3.21}$$

Here the following property of the Abel operators was used

$$\int_y^a\frac{\mathrm{d}r}{(r^2-y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a\frac{tf(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}}=-\frac{\pi}{2}f(y). \tag{1.3.22}$$

Introduction of a new variable $t=g(x)$, $x=l_1(t)$, transforms (21) into

$$V(\rho,\phi,z)=\frac{2}{\pi}\int_0^a\frac{\mathrm{d}l_1(t)}{[\rho^2-l_1^2(t)]^{1/2}}\mathcal{L}\left(\frac{l_1^2(t)}{\rho t^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{\rho_0\mathrm{d}\rho_0}{(t^2-\rho_0^2)^{1/2}}\mathcal{L}(\rho_0)v(\rho_0,\phi). \tag{1.3.23}$$

By changing the order of integration in (23), according to the rule

$$\int_0^a F(r)\mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r\frac{\rho f(\rho)\,\mathrm{d}\rho}{(r^2-\rho^2)^{1/2}}=-\int_0^a f(\rho)\mathrm{d}\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^a\frac{F(r)r\mathrm{d}r}{(r^2-\rho^2)^{1/2}}, \tag{1.3.24}$$

the following expression can be obtained

$$V(\rho,\phi,z)=-\frac{2}{\pi}\int_0^a\left\{\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^a\frac{t\mathrm{d}l_1(t)}{(t^2-\rho_0^2)^{1/2}[\rho^2-l_1^2(t)]^{1/2}}\mathcal{L}\left(\frac{\rho}{l_2^2(t)}\right)\right\}v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.3.25}$$

The integral in curly brackets can be evaluated in closed form. Consider the following equivalent integral

$$I_1 = -\frac{1}{\rho_0}\,\mathcal{L}(\rho_0)\,\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{x\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\,\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho}{l_2^2(x)},\,\phi-\phi_0\right). \tag{1.3.26}$$

Make use of the rule of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^a\frac{F(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-x^2}} = -\frac{F(a)x}{a(a^2-x^2)^{1/2}} + x\int_x^a\frac{\mathrm{d}\rho}{\sqrt{\rho^2-x^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= -\frac{F(a)a}{x(a^2-x^2)^{1/2}} + \frac{1}{x}\int_x^a\frac{\rho\mathrm{d}\rho}{\sqrt{\rho^2-x^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}F(\rho). \tag{1.3.27}$$

Expression (26) will take the form

$$I_1 = \frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}(l_2^2-l_1^2)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2},\,\phi-\phi_0\right) - \int_{\rho_0}^{a}\frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2(x)},\,\phi-\phi_0\right)\right]. \tag{1.3.28}$$

Introduce a new variable

$$h(x) = \frac{\sqrt{x^2-\rho_0^2}\,[x^2-l_1^2(x)]^{1/2}}{x},$$

$$h'(x) = \frac{\mathrm{d}h(x)}{\mathrm{d}x} = \frac{[x^2-l_1^2(x)]^{1/2}[x^2l_2^2(x)-\rho_0^2l_1^2(x)]}{x^2\sqrt{x^2-\rho_0^2}\,[l_2^2(x)-l_1^2(x)]}. \tag{1.3.29}$$

The expression for λ can be presented as

$$\lambda\left(\frac{\rho\rho_0}{l_2^2(x)},\,\phi-\phi_0\right) = \frac{[l_2^2(x)-l_1^2(x)]\sqrt{x^2-\rho_0^2}}{[x^2-l_1^2(x)]^{1/2}}\,\frac{h'(x)}{R_0^2+h^2(x)}. \tag{1.3.30}$$

Substitution of (29) and (30) in (28) yields

$$I_1 = \frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}(l_2^2-l_1^2)}\,\lambda\left(\frac{\rho\rho_0}{l_2^2},\,\phi-\phi_0\right)$$

$$-\int_{\rho_0}^{a}\frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xz\sqrt{x^2-\rho_0^2}\,h'(x)}{[x^2-l_1^2(x)][R_0^2+h^2(x)]}\right]$$

$$=\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho_0^2}(l_2^2-l_1^2)}\lambda\left(\frac{\rho\rho_0}{l_2^2},\phi-\phi_0\right)-z\int_{\rho_0}^{a}\frac{\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(x^2-\rho_0^2)^{3/2}h'(x)}{xh^2(x)[R_0^2+h^2(x)]}\right].$$

(1.3.31)

Integration by parts in (31) yields

$$-\frac{1}{\rho_0}\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{a}\frac{x\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{\rho}{l_2^2(x)},\phi-\phi_0\right)=\frac{z}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right].$$

(1.3.32)

Here h stands for $h(a)$, as defined by the first expression of (29). In the limiting case, when $a\to\infty$, expression (32) gives yet another representation for z/R_0^3, namely,

$$\frac{z}{R_0^3}=-\frac{2}{\pi\rho_0}\mathcal{L}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_{\rho_0}^{\infty}\frac{x\mathrm{d}x}{\sqrt{x^2-\rho_0^2}}\frac{\sqrt{l_2^2(x)-x^2}}{l_2^2(x)-l_1^2(x)}\lambda\left(\frac{\rho}{l_2^2(x)},\phi-\phi_0\right). \tag{1.3.33}$$

Now substitution of (32) in (25) yields

$$V(\rho,\phi,z)=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]\frac{z}{R_0^3}v(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.3.34}$$

Here R_0 is defined by (5) and

$$h=(a^2-l_1^2)^{1/2}(a^2-\rho_0^2)^{1/2}/a. \tag{1.3.35}$$

Formulae (23) and (34) define the potential function V in the half-space $z\geq 0$, expressed directly through its value v prescribed inside the disc $\rho=a$, $z=0$. Expression (23) is useful when an explicit evaluation of the integrals is possible, while expression (34) is more convenient for numerical integration.

Note that in the limiting case, when $z=0$, equation (34) transforms into a known result, namely,

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for} \quad \rho \le a; \quad \text{and}$$

$$V(\rho,\phi,0) = \frac{(\rho^2 - a^2)^{1/2}}{\pi^2} \int_0^{2\pi} \int_0^a \frac{v(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{(a^2-\rho_0^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0)]}, \quad \text{for} \quad \rho > a.$$

The solution of the first mixed boundary value problem is completed.

Problem 2. Consider now another internal problem, characterized by the following mixed conditions on the boundary $z=0$:

$$\frac{\partial V}{\partial z} = -2\pi\sigma(\rho,\phi), \quad \text{for } \rho \le a, \quad \text{and} \quad 0 \le \phi < 2\pi;$$

$$V = 0, \qquad \text{for } \rho > a, \quad \text{and} \quad 0 \le \phi < 2\pi. \tag{1.3.36}$$

The problem (36) can be interpreted as an electrostatic one of a charged disc $\rho \le a$ inside an infinite grounded diaphragm $\rho > a$. Mathematically similar problem arises in the consideration of a penny-shaped crack subjected to an arbitrary pressure σ.

The potential function V can be represented through the simple layer as follows:

$$V(\rho,\phi,z) = \int_0^{2\pi} \int_0^a \frac{\sigma(\rho_0,\phi_0)}{R_0} \rho_0 d\rho_0 d\phi_0 + \int_0^{2\pi} \int_a^{\infty} \frac{\sigma(\rho_0,\phi_0)}{R_0} \rho_0 d\rho_0 d\phi_0. \tag{1.3.37}$$

Substitution of (1.2.13) and (1.2.17) in (37) yields, after interchanging the order of integration

$$V(\rho,\phi,z) = 4\int_0^{l_1} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^{a} \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - g^2(x)}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+ 4\int_{l_2}^{\infty} \frac{dx}{\sqrt{x^2 - \rho^2}} \int_a^{g(x)} \frac{\rho_0 d\rho_0}{\sqrt{g^2(x) - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.3.38}$$

Here the $\mathcal{L}$-operator is defined by (1.1.4), g is given by (1.2.20), the abbreviations l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively; and the following rule is used for changing the order of integration:

$$\int_0^a d\rho_0 \int_0^{l_1(\rho_0)} dx = \int_0^{l_1} dx \int_{g(x)}^a d\rho_0, \quad \int_a^\infty d\rho_0 \int_{l_2(\rho_0)}^\infty dx = \int_{l_2}^\infty dx \int_a^{g(x)} d\rho_0. \tag{1.3.39}$$

Substitution of (36) in (38) leads to the integral equation, for $\rho > a$,

$$\int_0^a \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^a \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+ \int_\rho^\infty \frac{dx}{\sqrt{x^2 - \rho^2}} \int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi) = 0. \tag{1.3.40}$$

Notice that σ in the first term of (40) is known from (36), while σ in the second term is yet to be determined. By using the integral representations (1.2.23) and (1.2.22), equation (40) can be rewritten as

$$\int_\rho^\infty \frac{dx}{\sqrt{x^2 - \rho^2}} \int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$= -\int_\rho^\infty \frac{dx}{\sqrt{x^2 - \rho^2}} \int_0^a \frac{\rho_0 d\rho_0}{\sqrt{x^2 - \rho_0^2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.3.41}$$

Operation on both sides of (41) by

$$\mathcal{L}(t)\frac{d}{dt}\int_t^\infty \frac{\rho d\rho}{(\rho^2 - t^2)^{1/2}} \mathcal{L}\left(\frac{1}{\rho}\right)$$

leads to

$$\int_a^t \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi) = -\int_0^a \frac{\rho_0 d\rho_0}{(t^2 - \rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho_0}{t}\right)\sigma(\rho_0,\phi). \tag{1.3.42}$$

The next operator to apply is

$$\mathcal{L}\left(\frac{1}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\int_a^\rho \frac{t\mathrm{d}t}{(\rho^2-t^2)^{1/2}}\,\mathcal{L}(t),$$

and the final result takes the form

$$\sigma(\rho,\phi) = -\frac{2}{\pi(\rho^2-a^2)^{1/2}}\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,\rho_0\mathrm{d}\rho_0}{\rho^2-\rho_0^2}\,\mathcal{L}\left(\frac{\rho_0}{\rho}\right)\sigma(\rho_0,\phi)$$

$$= -\frac{1}{\pi^2(\rho^2-a^2)^{1/2}}\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2-\rho_0^2}\,\sigma(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \quad (1.3.43)$$

Formula (43) defines the value of σ outside the circle $\rho = a$ directly through its value inside. Now σ is known all over the plane $z = 0$, and substitution of (43) in the second term of (38) allows us to express the potential function V directly through the prescribed value of σ. The first integration yields

$$V(\rho,\phi,z) = 4\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_{g(x)}^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$-4\int_{l_2}^{\infty}\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_0^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \quad (1.3.44)$$

Here the following integral was employed

$$\int_a^\rho \frac{y\mathrm{d}y}{(\rho^2-y^2)^{1/2}(y^2-a^2)^{1/2}(y^2-r^2)} = \frac{\pi}{2(\rho^2-r^2)^{1/2}(a^2-r^2)^{1/2}}, \quad \text{for } r < a.$$

(1.3.45)

The first term in (44) can be transformed by using (1.2.17), in the following manner:

$$\int_0^{l_1}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_{g(x)}^a \frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$= \int_0^a \rho_0 d\rho_0 \int_{l_2(\rho_0)}^{\infty} \frac{dx}{\sqrt{x^2-\rho^2}[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi)$$

$$= \int_{l_2(0)}^{l_2} \frac{dx}{\sqrt{x^2-\rho^2}} \int_0^{g(x)} \frac{\rho_0 d\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi)$$

$$+ \int_{l_2}^{\infty} \frac{dx}{\sqrt{x^2-\rho^2}} \int_0^{a} \frac{\rho_0 d\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi). \tag{1.3.46}$$

Substitution of (46) in (44) yields

$$V(\rho,\phi,z) = 4\int_{l_2(0)}^{l_2} \frac{dx}{\sqrt{x^2-\rho^2}} \int_0^{g(x)} \frac{\rho_0 d\rho_0}{[g^2(x)-\rho_0^2]^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0\phi). \tag{1.3.47}$$

Introduction of a new variable $t = g(x)$, $x = l_2(t)$, transforms (47) into

$$V(\rho,\phi,z) = 4\int_0^a \frac{dl_2(t)}{[l_2^2(t)-\rho^2]^{1/2}} \int_0^t \frac{\rho_0 d\rho_0}{(t^2-\rho_0^2)^{1/2}} \mathcal{L}\left(\frac{\rho\rho_0}{l_2^2(t)}\right)\sigma(\rho_0,\phi). \tag{1.3.48}$$

An interchange of the order of integration in (48), and integration with respect to t (see 1.2.15), yields

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_0^a \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0. \tag{1.3.49}$$

Formulae (47–49) give three equivalent representations of the potential function V, the first two being more convenient for explicit evaluation of the integrals involved, while the third one has some advantages for numerical integration. Two examples are considered below.

Example 1. Let the potential prescribed inside the disc be $v(\rho,\phi) = v_n\rho^n\cos n\phi$, $v_n = \text{const}$. The solution due to (1.3.21) is

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\,\rho^n}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\cos n\phi\int_0^{l_1}\frac{x^{2n}\mathrm{d}x}{\sqrt{\rho^2-x^2}}$$

$$= v_n\rho^n\cos n\phi\left[1-\frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\frac{\sqrt{l_2^2-a^2}}{l_2}F\left(\frac{1}{2}-n,\frac{1}{2};\frac{3}{2};\frac{l_2^2-a^2}{l_2^2}\right)\right]. \tag{1.3.50}$$

The hypergeometric function in (50) can be expressed in elementary functions (Bateman and Erdelyi, 1955)

$$F(\frac{1}{2}-n,\frac{1}{2};\frac{3}{2};\zeta) = \frac{(1-\zeta)^{n+1/2}}{\Gamma(n+1)}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[\frac{\zeta^{n-1/2}}{\sqrt{1-\zeta}}\sin^{-1}\sqrt{\zeta}\right]. \tag{1.3.51}$$

Example 2. Let the charge distribution be prescribed in the form $\sigma(\rho,\phi) = \sigma_n\rho^n\cos n\phi$, $\sigma_n = \text{const.}$ The solution is given by (48)

$$V(\rho,\phi,z) = 2\sqrt{\pi}\;\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\sigma_n\rho^n\cos n\phi\int_0^b\frac{x^{2n+2}\mathrm{d}x}{(x^2+z^2)^{n+1}}$$

$$= \sqrt{\pi}\;\frac{\Gamma(n+1)}{\Gamma(n+\frac{5}{2})}\sigma_n\rho^n\cos n\phi\,\frac{b^{2n+3}}{z^{2n+2}}F(n+1,n+\frac{3}{2};n+\frac{5}{2};-\frac{b^2}{z^2}), \tag{1.3.52}$$

where $b = \sqrt{a^2-l_1^2}$, and the hypergeometric function can be expressed in elementary (Bateman and Erdelyi, 1955)

$$F(n+1,n+\frac{3}{2};n+\frac{5}{2};\zeta) = \frac{2n+3}{\Gamma(n+1)}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left\{\frac{1}{\zeta}\left[\frac{1}{2\sqrt{\zeta}}\ln\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}-1\right]\right\}. \tag{1.3.53}$$

1.4. External mixed boundary value problem

The problem is called external when the non-zero boundary conditions are prescribed outside the disc. As in the previous section, we consider two types of problem.

Problem 1. It is necessary to find a function, harmonic in the half-space $z \geq 0$, vanishing at infinity, and subject to the mixed boundary conditions on the plane $z = 0$, namely,

$$\frac{\partial V}{\partial z}\Big|_{z=0} = 0, \quad \text{for } \rho < a,\ 0 \le \phi < 2\pi;$$

$$V = v(\rho,\phi), \quad \text{for } \rho \ge a,\ 0 \le \phi < 2\pi. \tag{1.4.1}$$

The problem (1) can be interpreted as an electrostatic one of a charged diaphragm, or as an external elastic contact problem. The potential V is presented through a simple layer distribution (1.3.38). Substitution of the boundary conditions (1) in (1.3.38) leads to the governing integral equation

$$4\int_\rho^\infty \frac{dx}{\sqrt{x^2-\rho^2}}\int_a^x \frac{\rho_0 d\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi) = v(\rho,\phi). \tag{1.4.2}$$

Its solution is obtained in exactly the same manner as that of (1.3.12), and is

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\rho}\,\mathcal{L}\left(\frac{1}{\rho}\right)\frac{d}{d\rho}\int_a^\rho \frac{x\,dx}{\sqrt{\rho^2-x^2}}\,\mathcal{L}(x^2)\,\frac{d}{dx}\int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \tag{1.4.3}$$

The rules of differentiation allow us to rewrite (3) as follows

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2-a^2)^{1/2}} + \int_a^\rho \frac{dx}{\sqrt{\rho^2-x^2}}\,\frac{\partial}{\partial x}\chi(x,\rho,\phi)\right\}, \tag{1.4.4}$$

where

$$\chi(x,\rho,\phi) = x\int_x^\infty \frac{d\rho_0}{\sqrt{\rho_0^2-x^2}}\,\frac{\partial}{\partial\rho_0}\left[\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)v(\rho_0\phi)\right]. \tag{1.4.5}$$

The following transformation can now be performed:

$$\frac{\partial}{\partial x}\chi(x,\rho,\phi) = \frac{\partial}{\partial x}\left[x\int_x^\infty \frac{d\rho_0}{\sqrt{\rho_0^2-x^2}}\,(\mathcal{L}v)'\right]$$

$$= \int_x^\infty \frac{d\rho_0}{\sqrt{\rho_0^2-x^2}}\,[(\mathcal{L}v)' + \rho_0(\mathcal{L}v)'' - 2(\mathcal{L}'\rho_0 v)']$$

$$= \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \left[\mathcal{L}\left(v'' + \frac{1}{\rho_0} v' \right) - \left(\mathcal{L}'' + \frac{1}{\rho_0} \mathcal{L}' \right) v \right]. \tag{1.4.6}$$

Here, for the sake of brevity, the primes (′) indicate the partial derivatives with respect to ρ_0, $\mathcal{L}$ stands for $\mathcal{L}(x^2/\rho\rho_0)$, $v \equiv v(\rho_0,\phi)$, and the following identity was used

$$\frac{\partial}{\partial x} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) = -2\left(\frac{\rho_0}{x}\right) \frac{\partial}{\partial \rho_0} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right).$$

Since

$$\mathcal{L} \frac{1}{\rho_0^2} \frac{\partial^2 v}{\partial \phi^2} = \frac{1}{\rho_0^2} \frac{\partial^2 \mathcal{L}}{\partial \phi^2} v,$$

its addition to and subtraction from (6) yields

$$\frac{\partial}{\partial x} \chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \left[\mathcal{L}\Delta v - (\Delta \mathcal{L}) v \right], \tag{1.4.7}$$

where Δ is the two-dimensional Laplace operator in polar coordinates. Since λ is a harmonic function, $\Delta\mathcal{L} = 0$, and (7) simplifies to

$$\frac{\partial}{\partial x} \chi(x,\rho,\phi) = \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\Delta v. \tag{1.4.8}$$

Substitution of (8) in (4) yields

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2} \left\{ \frac{\chi(a,\rho,\phi)}{(\rho^2 - a^2)^{1/2}} + \int_a^\rho \frac{dx}{\sqrt{\rho^2 - x^2}} \int_x^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - x^2}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right) \Delta v(\rho_0,\phi) \right\}. \tag{1.4.9}$$

It should be noticed that the first term in (9) becomes singular when $\rho \to a$, while the second term vanishes at the edge of the disc. In the case of v being a harmonic function, the second term in (105) vanishes, and the solution is represented by the first term only. Further integration with respect to x becomes possible in (9), after interchanging the order of integration, with the result

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2}\left\{\frac{\chi(a,\rho,\phi)}{(\rho^2-a^2)^{1/2}}\right.$$

$$\left.+\frac{1}{2\pi}\int_0^{2\pi}\int_a^{\infty}\frac{\Delta v(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}\tan^{-1}\frac{(\rho^2-a^2)^{1/2}(\rho_0^2-a^2)^{1/2}}{a[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}\right\}. \tag{1.4.10}$$

Solutions, like (3) and (9), are appropriate for use when an exact evaluation of the integrals is possible, while the solution in the form (10) has some advantages when numerical integration is to be employed.

Now we can express the potential function V directly through its boundary value v. Since $\sigma = 0$ inside the circle $\rho = a$, the potential function (1.3.38) takes the form

$$V(\rho,\phi,z) = 4\int_{l_2}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\int_a^{g(x)}\frac{\rho_0 d\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.11}$$

Substitution of (3) in (11) yields, after the first integration,

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_{l_2}^{\infty}\frac{dx}{\sqrt{x^2-\rho^2}}\mathcal{L}\left(\frac{\rho g^2(x)}{x^2}\right)\frac{\partial}{\partial g(x)}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \tag{1.4.12}$$

Here the properties of the $\mathcal{L}$-operators (1.1.5) were used, along with the following identity, valid for the Abel-type operators

$$\int_a^{\rho}\frac{dx}{\sqrt{\rho^2-x^2}}\frac{d}{dx}\int_a^{x}\frac{f(t)\,t dt}{(x^2-t^2)^{1/2}} = \frac{\pi}{2}f(\rho). \tag{1.4.13}$$

Introduction of a new variable $y = g(x)$, $x = l_2(y)$, in (12), allows us to rewrite (12)

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^{\infty}\frac{dl_2(y)}{[l_2^2(y)-\rho^2]^{1/2}}\,\mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\frac{d}{dy}\int_y^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-y^2)^{1/2}}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)v(\rho_0,\phi). \tag{1.4.14}$$

Interchange of the order of integration in (14) yields

$$V(\rho,\phi,z) = -\frac{2}{\pi}\int_a^\infty \left\{ \mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{y\,\mathrm{d}l_2(y)}{(\rho_0^2 - y^2)^{1/2}[l_2^2(y) - \rho^2]^{1/2}}\,\mathcal{L}\left(\frac{l_1^2(y)}{\rho}\right)\right\} v(\rho_0,\phi)\mathrm{d}\rho_0. \tag{1.4.15}$$

Here the general formula was used

$$\int_a^\infty F(\rho)\mathrm{d}\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\infty \frac{xf(x)\mathrm{d}x}{\sqrt{x^2-\rho^2}} = -\int_a^\infty f(x)\,\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{\rho F(\rho)\,\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\,. \tag{1.4.16}$$

The integral in curly brackets of (15) can be evaluated in a closed form. We consider the following equivalent integral

$$I_2 = \frac{1}{\rho_0}\,\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{x\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\,\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\left(\frac{l_1(x)x}{l_2(x)}\,,\phi-\phi_0\right). \tag{1.4.17}$$

Make use of the rule of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{F(\rho)\,\mathrm{d}\rho}{\sqrt{x^2-\rho^2}} = \frac{F(a)x}{a(x^2-a^2)^{1/2}} + x\int_a^x \frac{\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{F(\rho)}{\rho}\right]$$

$$= \frac{F(a)a}{x(x^2-a^2)^{1/2}} + \frac{1}{x}\int_a^x \frac{\rho\mathrm{d}\rho}{\sqrt{x^2-\rho^2}}\,\frac{\mathrm{d}}{\mathrm{d}\rho}F(\rho). \tag{1.4.18}$$

Expression (17) will take the form

$$I_2 = \frac{\sqrt{a^2 - l_1^2}}{\sqrt{\rho_0^2 - a^2}[l_2^2 - l_1^2]}\,\lambda\left(\frac{l_1^2}{\rho\rho_0}\,,\phi-\phi_0\right)$$

$$+\int_a^{\rho_0} \frac{\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\,\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\,\lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0}\,,\phi-\phi_0\right)\right]. \tag{1.4.19}$$

By introducing the notation

$$F(y) = \frac{z}{R_0^3}\left[\frac{R_0}{j(y)} + \tan^{-1}\left(\frac{j(y)}{R_0}\right)\right], \tag{1.4.20}$$

where $j(x)$ is defined by

$$j(x) = \frac{(\rho_0^2 - x^2)^{1/2}\sqrt{l_2^2(x) - x^2}}{x},$$

$$j'(x) = \frac{\mathrm{d}j(x)}{\mathrm{d}x} = -\frac{\sqrt{l_2^2(x) - x^2}\,[\rho_0^2 l_2^2(x) - x^2 l_1^2(x)]}{x^2[l_2^2(x) - l_1^2(x)]\,(\rho_0^2 - x^2)^{1/2}}. \tag{1.4.21}$$

expression (19) can be rewritten as

$$I_2 = \frac{\rho_0^2 - a^2}{a}\frac{\mathrm{d}F(a)}{\mathrm{d}a} + \int_a^{\rho_0} \frac{\mathrm{d}y}{(\rho_0^2 - y^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{(\rho_0^2 - y^2)^{3/2}}{y}\frac{\mathrm{d}F(y)}{\mathrm{d}y}\right]. \tag{1.4.22}$$

Integration in (22) can be performed by parts, with a simple result $F(a)$, which means establishment of another integral representation

$$\frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right] = \frac{1}{\rho_0}\mathcal{L}\left(\frac{1}{\rho_0}\right)\frac{\mathrm{d}}{\mathrm{d}\rho_0}\int_a^{\rho_0} \frac{x\mathrm{d}x}{(\rho_0^2 - x^2)^{1/2}}\frac{[x^2 - l_1^2(x)]^{1/2}}{l_2^2(x) - l_1^2(x)}\lambda\left(\frac{l_1(x)x}{l_2(x)}, \phi - \phi_0\right). \tag{1.4.23}$$

As before, we use the convention $j \equiv j(a)$. Utilization of (23) allows us to simplify (15) as

$$V(\rho,\phi,z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} \frac{z}{R_0^3}\left[\frac{R_0}{j} + \tan^{-1}\left(\frac{j}{R_0}\right)\right] v(\rho_0,\phi_0)\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0. \tag{1.4.24}$$

In the particular case, when $z = 0$, expression (24) simplifies to

$$V(\rho,\phi,0) = \frac{1}{\pi^2}\sqrt{a^2 - \rho^2}\int_0^{2\pi}\int_a^{\infty} \frac{v(\rho_0,\phi_0)\,\rho_0\mathrm{d}\rho_0\mathrm{d}\phi_0}{(\rho_0^2 - a^2)^{1/2}[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]}, \quad \text{for } \rho < a;$$

$$V(\rho,\phi,0) = v(\rho,\phi), \quad \text{for } \rho \geq a. \tag{1.4.25}$$

The general solution is completed. The charge density σ is given by the two equivalent expressions (3) and (10), while the potential is in the two forms (14) and (24), the first one being more convenient for exact evaluation of the integrals involved, while the second is better suited for numerical integration.

Problem 2. Consider the problem of finding a harmonic function, vanishing at infinity, and subject to the mixed conditions on the plane $z=0$

$$V=0,\quad \text{for } \rho\le a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=-2\pi\sigma(\rho,\phi),\ \text{for } \rho>a,\ 0\le\phi<2\pi. \tag{1.4.26}$$

The problem may be interpreted as an electrostatic one of a charged infinite diaphragm, with a grounded disc inside, or as an external crack problem in elasticity. Substitution of the boundary conditions (26) in (1.3.38) leads to the governing integral equation

$$\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^{a}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$=-\int_a^{\infty}\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\int_a^{x}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{x^2-\rho_0^2}}\,\mathcal{L}\!\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.27}$$

One should notice that σ in the second term of (27) is known from the boundary condition (26), while the value of σ in the first term is yet to be determined. The right hand side of (27) can be transformed, by using (1.2.22) and (1.2.23),

$$\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_x^{a}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$=-\int_0^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}}\int_a^{\infty}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\!\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi),$$

with an immediate result

$$\int_x^{a}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\!\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi)=-\int_a^{\infty}\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{\rho_0^2-x^2}}\,\mathcal{L}\!\left(\frac{x}{\rho_0}\right)\sigma(\rho_0,\phi). \tag{1.4.28}$$

Application of the operator

$$\mathcal{L}(\rho)\frac{d}{d\rho}\int_\rho^a \frac{x\,dx}{\sqrt{x^2-\rho^2}}\,\mathcal{L}\left(\frac{1}{x}\right)$$

to both sides of (28) gives, after necessary transformations

$$\sigma(\rho,\phi) = -\frac{2}{\pi\sqrt{a^2-\rho^2}}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}}{\rho_0^2-\rho^2}\,\mathcal{L}\left(\frac{\rho}{\rho_0}\right)\sigma(\rho_0,\phi)\,\rho_0 d\rho_0, \quad \text{for } \rho<a, \tag{1.4.29}$$

or, interpreting the $\mathcal{L}$-operator, we obtain

$$\sigma(\rho,\phi) = -\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\int_0^{2\pi}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}\sigma(\rho_0,\phi_0)\,\rho_0 d\rho_0 d\phi_0}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}. \tag{1.4.30}$$

Now the value of σ is known all over the plane $z=0$, and (1.3.38) can be used in order to express the potential V directly through the prescribed σ. Substitution of (29) in (1.3.38) yields, after the first integration

$$V(\rho,\phi,z) = -4\int_0^{l_1}\frac{dx}{\sqrt{\rho^2-x^2}}\int_a^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+\,4\int_{l_2}^\infty\frac{dx}{\sqrt{x^2-\rho^2}}\int_a^{g(x)} \frac{\rho_0 d\rho_0}{\sqrt{g^2(x)-\rho_0^2}}\,\mathcal{L}\left(\frac{\rho\rho_0}{x^2}\right)\sigma(\rho_0,\phi). \tag{1.4.31}$$

The second term in (31) is equivalent to the second term in (1.3.38), which, in turn, can be represented by using (1.2.13), as

$$4\int_a^\infty\left\{\int_0^{l_1(\rho_0)}\frac{dx}{\sqrt{\rho^2-x^2}\sqrt{\rho_0^2-g^2(x)}}\,\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\right\}\sigma(\rho_0,\phi)\,\rho_0 d\rho_0.$$

The following scheme of changing the order of integration is enacted

$$\int_a^\infty d\rho_0\int_0^{l_1(\rho_0)} dx = \int_0^{l_1} dx\int_a^\infty d\rho_0 + \int_{l_1}^{l_1(\infty)} dx\int_{g(x)}^\infty d\rho_0,$$

and the second term in (31) can be rewritten as

$$4\int_0^{l_1} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_a^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - g^2(x)}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi)$$

$$+ 4\int_{l_1}^{l_1(\infty)} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - g^2(x)}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \qquad (1.4.32)$$

Substitution of (32) in (31) gives, by virtue of $l_1(\infty) = \rho$,

$$V(\rho,\phi,z) = 4\int_{l_1}^{\rho} \frac{dx}{\sqrt{\rho^2 - x^2}} \int_{g(x)}^\infty \frac{\rho_0 d\rho_0}{\sqrt{\rho_0^2 - g^2(x)}} \mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)\sigma(\rho_0,\phi). \qquad (1.4.33)$$

Interchange of the order of integration in (33), and integration with respect to x, according to (1.2.19), results in

$$V(\rho,\phi,z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^\infty \frac{1}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right) \sigma(\rho_0,\phi_0)\, \rho_0 d\rho_0 d\phi_0, \qquad (1.4.34)$$

where R_0 is defined by (1.3.5), and j stands for $j(a)$, as defined by (21).

The second problem is now solved. Expression (30) defines the charge density σ inside a circle directly in terms of its values outside. The potential V is given by two equivalent expressions (33) and (34), the first one to be used for exact evaluation of the integrals, while the second has some advantages in the case of numerical integration. Some specific examples are considered below.

Example 1. Consider an external mixed problem with the following boundary conditions at $z = 0$

$$V = v_0/\rho^n, \quad \text{for } \rho \geq a, \quad 0 \leq \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, \quad 0 \leq \phi < 2\pi. \qquad (1.4.35)$$

The conditions (35) correspond to those of Problem 1. The solution is given by (14) and (3). Substitution of (35) in (14) yields, after the integration

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}}\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)}\int_{l_2}^{\infty}\frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}\,g^n(x)}, \tag{1.4.36}$$

where $g(x)$ is defined by (1.2.20), and the following integral was employed (Gradshtein and Ryzhik, 1963)

$$\int_{x}^{\infty}\frac{\mathrm{d}\rho}{\rho^n\sqrt{\rho^2-x^2}} = \frac{\sqrt{\pi}\,\Gamma(n/2)}{2\Gamma[(n+1)/2]\,x^n}. \tag{1.4.37}$$

The integral in (36) can be evaluated in terms of elementary functions for any integer n, but the procedure is slightly different for even and odd values of n. For example, for even $n=2k$, the problem reduces to the evaluation of the integral

$$\int_{l_2}^{\infty}\frac{(x^2-\rho^2)^{k-1/2}\mathrm{d}x}{x^{2k}(x^2-\rho^2-z^2)^k},$$

which can be evaluated by introduction of a new variable $t = x/\sqrt{x^2 - \rho^2}$. The final result is

$$V(\rho,\phi,z) = \frac{2v_0}{\sqrt{\pi}}\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)z^n}\left\{\sum_{m=1}^{k}\frac{A_m}{2m-1}[1-Q_0^{2m-1}]\right.$$

$$\left.+\,2B_1\ln Q + \sum_{m=2}^{k}\frac{B_m}{1-m}\left[(Q_1^{m-1}-Q_2^{m-1})-(Q_3^{m-1}-Q_4^{m-1})\right]\right\}, \tag{1.4.38}$$

where

$$A_{k-m+1} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\eta^{m-1}}\left[\frac{(\eta-1)^{k-1}}{(r^2-\eta)^k}\right],\ \text{for } \eta = 0, \text{and } r^2=1+\rho^2/z^2;$$

$$B_{k-m+1} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left[\frac{(t^2-1)^{k-1}}{t^{2k}(r^2+t)^k}\right],\ \text{for } t=\sqrt{1+\rho^2/z^2};$$

$$Q_0 = \frac{\sqrt{l_2^2-\rho^2}}{l_2},\quad Q = \frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]},$$

$$Q_1 = \frac{z[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{l_1^2}, \quad Q_2 = \frac{z[(\rho^2+z^2)^{1/2}-\sqrt{l_2^2-a^2}]}{l_1^2},$$

$$Q_3 = \frac{z[(\rho^2+z^2)^{1/2}+z]}{\rho^2}, \quad Q_4 = \frac{z[(\rho^2+z^2)^{1/2}-z]}{\rho^2}. \tag{1.4.39}$$

For the case of an odd $n = 2k+1$, the integration can be performed by using the substitution $t = (x^2 - \rho^2 - z^2)^{1/2}$, and the final result is

$$V(\rho,\phi,z) = \frac{v_0}{\sqrt{\pi}}\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)}\left\{\sum_{m=1}^{k}\frac{C_m}{(2m-1)(a^2-l_1^2)^{m-1/2}} + \sum_{m=1}^{k+1} D_m E_m\right\}, \tag{1.4.40}$$

where

$$C_m = \frac{1}{(k-m)!}\frac{\mathrm{d}^{k-m}}{\mathrm{d}t^{k-m}}\left[\frac{(t+z^2)^k}{(t+\rho^2+z^2)^{k+1}}\right], \text{ for } t = 0;$$

$$D_m = \frac{1}{(k+1-m)!}\frac{\mathrm{d}^{k+1-m}}{\mathrm{d}t^{k+1-m}}\left[\frac{(t+z^2)^k}{t^k}\right], \text{ for } t = -(\rho^2+z^2);$$

$$E_m = \frac{(-1)^m}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left[\frac{1}{\sqrt{t}}\tan^{-1}\frac{\sqrt{t}}{\sqrt{a^2-l_1^2}}\right], \text{ for } t = \rho^2+z^2. \tag{1.4.41}$$

Substitution of (35) in (3) yields, after integration,

$$\sigma(\rho,\phi) = \frac{v_0\Gamma[(n+1)/2]}{\pi^{3/2}\Gamma(n/2)}\left\{\frac{1}{a^n(\rho^2-a^2)^{1/2}} - \frac{n(\rho^2-a^2)^{1/2}}{\rho^{n+2}}F\left(\tfrac{1}{2}n+1, \tfrac{1}{2}; \tfrac{3}{2}; 1-\frac{a^2}{\rho^2}\right)\right\}, \tag{1.4.42}$$

and the Gauss hypergeometric function can be expressed in elementary functions (Bateman and Erdelyi, 1955), namely, for even $n = 2k$, $k = 1,2,3\ \ldots$,

$$F\left(k+1, \frac{1}{2}; \frac{3}{2}; t\right) = \frac{1}{2k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[t^{k-1/2}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}\right],$$

and for odd $n = 2k+1$, $k = 0,1,2, \ldots$,

$$F\left(k+\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; t\right) = \frac{\sqrt{\pi}}{2\sqrt{t}\,\Gamma(k+\frac{3}{2})}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{t^{k+1/2}}{\sqrt{1-t}}\right]. \tag{1.4.43}$$

Fig. 1.4.1. Charge density for $n=1,2,3,4.$

The dimensionless charge density distribution $\sigma^*=\sigma a^{n+1}/v_0$, evaluated due to (42) for $n=1,2,3,4$, is presented on Fig. 1.4.1 versus $\rho^*=\rho/a$. It is non-negative for $n=1$, and changes sign when $n\geq 2$, its negative maximum increases with n, while the total charge stays at zero. Some specific formulae may be found in Exercises 1 (Examples 23-26). The equipotential lines for $n=2$ (formula 40) are presented in Fig. 1.4.2.

Fig. 1.4.2. Equipotential lines for $n=2$

Example 2. Consider the boundary conditions at $z=0$

$$V=(v_n/\rho^n)\,\mathrm{e}^{in\phi},\quad \text{for}\ \rho\geq a,\ \ 0\leq\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for}\ \rho<a,\ \ 0\leq\phi<2\pi, \tag{1.4.44}$$

where v_n is constant. The solution is given by (3) and (14). Substitution of (44) in (14) yields, after integration,

$$V(\rho,\phi,z)=\frac{2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n)}\rho^n\mathrm{e}^{in\phi}\int_{l_2}^{\infty}\frac{\mathrm{d}x}{x^{2n}\sqrt{x^2-\rho^2}}\,. \tag{1.4.45}$$

The final integration gives

$$V(\rho,\phi,z) = \frac{2v_n}{\sqrt{\pi}\rho^n} e^{in\phi} \sum_{k=1}^{n} \frac{(-1)^{k-1}\Gamma(n+1/2)}{(2k-1)\Gamma(k)\Gamma(n+1-k)} (1 - Q_0^{2k-1}), \tag{1.4.46}$$

where Q_0 is defined by (39). Some particular cases of (46) can be found in Exercises 1 (Examples 27-29). Substituting (44) in (3), we get, after integration,

$$\sigma(\rho,\phi) = \frac{\Gamma(n+1/2)}{\pi^{3/2}\Gamma(n)} \frac{v_n e^{in\phi}}{\rho^n\sqrt{\rho^2 - a^2}}. \tag{1.4.47}$$

Evidently, expression (47) can also be obtained by differentiation of (46) with respect to z for $z = 0$. The equipotential lines at the plane $\phi = 0$ for $n = 2$ are presented in Fig. 1.4.3.

Fig. 1.4.3. Equipotential lines for $n = 2$

Example 3. Consider a case related to Problem 2, with the boundary conditions

$$V = 0, \quad \text{for } \rho \le a, \ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\frac{\sigma_0}{\rho^n} \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi; \tag{1.4.48}$$

The solution is given by (29) and (33). Substitution of (48) in (33) yields, after integration using (37),

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0\frac{\Gamma[(n-1)/2]}{\Gamma(n/2)}\int_{l_1}^{\rho}\frac{\mathrm{d}x}{\sqrt{\rho^2-x^2}\,g^{n-1}(x)}, \tag{1.4.49}$$

where $g(x)$ is defined by (1.2.20). The technique used in the previous example can be employed here for further integration. The final result depends on the value of n being even or odd. For even $n = 2k$, $k = 1,2,3,\ \ldots$, the potential is

$$V(\rho,\phi,z) = 2\sqrt{\pi}\sigma_0\frac{\Gamma[(n-1)/2]}{\Gamma(n/2)}\Bigg\{2B_1\ln Q$$

$$+\sum_{m=1}^{k-1}\frac{A_m}{(2m-1)\,z^{2m-1}}\left[1-\left(\frac{\sqrt{a^2-l_1^2}}{a}\right)^{2m-1}\right]$$

$$+\sum_{m=1}^{k-1}\frac{B_{m+1}}{mz^m}[Q_1^m-Q_2^m-(Q_3^m-Q_4^m)]\Bigg\}. \tag{1.4.50}$$

Here Q, Q_1, Q_2, Q_3, and Q_4 are defined by (39), and

$$A_{k-m} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left[\frac{(t-z^2)^{k-1}}{(\rho^2+z^2-t)^k}\right] \text{ for } t=0;$$

$$B_{k-m+1} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left[\frac{(t^2-z^2)^{k-1}}{t^{2k-2}[(\rho^2+z^2)^{1/2}-t]^k}\right],\ \text{for } t = -(\rho^2+z^2)^{1/2}. \tag{1.4.51}$$

For odd $n = 2k+1$, the result is

$$V(\rho,\phi,z) = 2\sqrt{\pi}\ \frac{\sigma_0\Gamma[(n-1)/2]}{\Gamma(n/2)\,z^{n-1}}\sum_{m=1}^{k}\left\{\frac{G_m}{2m-1}\left(\frac{\sqrt{l_2^2-a^2}}{a}\right)^{2m-1} - H_m L_m\right\}. \tag{1.4.52}$$

Here

$$G_{k\text{-}m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{(1+t)^{k\text{-}1}}{(\xi + t)^k} \right], \quad \text{for } t = 0, \xi = (\rho^2 + z^2)/z^2;$$

$$H_{k\text{-}m+1} = \frac{1}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{(1+t)^{k\text{-}1}}{t^k} \right], \quad \text{for } t = -(\rho^2 + z^2)/z^2;$$

$$L_m = \frac{(-1)^m}{\Gamma(m)} \frac{\mathrm{d}^{m\text{-}1}}{\mathrm{d}t^{m\text{-}1}} \left[\frac{1}{\sqrt{t}} \tan^{-1}\left(\frac{\sqrt{t}}{a} \sqrt{l_2^2 - a^2} \right) \right], \quad \text{for } t = (\rho^2 + z^2)/z^2. \tag{1.4.53}$$

Fig. 1.4.4. Charge distribution for $n = 2, 3, 4$

Some particular cases of (50) and (52) can be found in Exercises 1 (Examples 30-32). The dimensionless charge density distribution $\sigma^* = \sigma a^n/\sigma_0$ is given in Fig. 1.4.4 versus $\rho^* = \rho/a$ for $n = 2, 3, 4$. The equipotential lines for $n = 3$ are presented in Fig. 1.4.5. The dimensionless potential $v^* = Va^2/\sigma_0$ was varied from 0.5 to 1.3. We note that the equipotential lines for $v^* < 0.92$ have two branches.

Example 4. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a, \ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_n/\rho^n)\mathrm{e}^{in\phi}, \quad \text{for } \rho > a, \ 0 \le \phi < 2\pi; \tag{1.4.54}$$

Fig. 1.4.5. Equipotential lines for $n = 3$

Substitution of (54) in (33) yields, after integration,

$$V(\rho,\phi,z) = 2\sqrt{\pi}\ \frac{\sigma_n \Gamma(n-1/2)}{\Gamma(n)\rho^n}\, \mathrm{e}^{in\phi} \Bigg\{ \sqrt{l_2^2 - a^2} - z$$

$$- z \sum_{k=2}^{n} \frac{(-1)^k \Gamma(n)}{\Gamma(k)\, \Gamma(n-k+1)(2k-3)} \left[1 - (1 - l_1^2/a^2)^{k-3/2}\right] \Bigg\}, \qquad (1.4.55)$$

and on the plane $z = 0$

$$V(\rho,\phi,0) = 2\sqrt{\pi}\ \frac{\sigma_n \Gamma(n-1/2)}{\Gamma(n)\rho^n}\, \mathrm{e}^{in\phi} \Re \sqrt{\rho^2 - a^2}.$$

The symbol $\Re$ indicates the real part sign. The charge density is defined, according to (29),

$$\sigma(\rho,\phi) = \frac{\sigma_n \Gamma(n-1/2)}{\sqrt{\pi}\, \Gamma(n)\rho^n}\, \mathrm{e}^{in\phi} \Re \left\{ 1 - \frac{a}{\sqrt{a^2 - \rho^2}} \right.$$

$$+\sum_{k=2}^{n}\frac{(-1)^k\Gamma(n)}{\Gamma(k)\,\Gamma(n-k+1)(2k-3)}\,[1-(1-\rho^2/a^2)^{k-3/2}]\Bigg\}. \tag{1.4.56}$$

A more general case of boundary conditions, namely,

$$V=0,\quad \text{for}\ \rho\le a,\quad 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=-2\pi(\sigma_{jn}/\rho^j)e^{in\phi},\quad \text{for}\ \rho>a,\quad 0\le\phi<2\pi; \tag{1.4.57}$$

can also be considered, by using the same technique as in the previous examples, and the final result can always be expressed in elementary functions. The form of the result will be different for $(j+n)$ even, and for $(j+n)$ odd. As an example, the following expression can be obtained by substituting (57) in (29), for the case when $j+n=2k$

$$\sigma(\rho,\phi)=\frac{\sigma_{jn}}{\rho^j}\,e^{in\phi}\Re\left\{1-\frac{a}{\sqrt{a^2-\rho^2}}\left[1-\sum_{m=2}^{k}\frac{\Gamma(m-3/2)}{2\sqrt{\pi}\Gamma(m)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.4.58}$$

and for odd $j+n=2k+1$

$$\sigma(\rho,\phi)=\frac{2\sigma_{jn}}{\pi\rho^j}\,e^{in\phi}\,\Re\left\{\sin^{-1}\left(\frac{\rho}{a}\right)-\frac{a}{\sqrt{a^2-\rho^2}}\left[1-\sum_{m=2}^{k}\frac{\sqrt{\pi}\Gamma(m-1)}{4\Gamma(m+1/2)}\left(\frac{\rho}{a}\right)^{2m-2}\right]\right\}, \tag{1.4.59}$$

Expressions (58) and (59) represent general formulae which cover all the particular cases considered in Examples 3 and 4.

The examples above have demonstrated the simplicity of the method. The generation of the solution is reduced to a straightforward and elementary procedure.

1.5. Mixed Problems in Spherical Coordinates

Exact solution in closed form is obtained to the following mixed problem for a charged sphere: an arbitrary charge density distribution is prescribed at the surface of a spherical cap while an arbitrary potential is given at the rest of the sphere. The new method makes the solution straightforward and elementary, with no special functions or integral transforms involved. A new type of solution is obtained for the Dirichlet problem with discontinuous boundary conditions.

Integral representation for the reciprocal of the distance between two points in spherical coordinates. Consider two points in spherical coordinates $M(r,\theta,\phi)$ and $N(a,\theta_0,\phi_0)$. The parameters l_1 and l_2 introduced in section 1.2 have the geometrical interpretation as the difference and the sum of the shortest and the longest distance from a point to the edge of a circular disk. In spherical coordinates the same quantities with respect to a spherical cap can be expressed as

$$m_1(\theta_0,\theta,a,r) = \frac{1}{2}\left[\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} - \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right]$$

$$m_2(\theta_0,\theta,a,r) = \frac{1}{2}\left[\sqrt{a^2+r^2-2ar\cos(\theta+\theta_0)} + \sqrt{a^2+r^2-2ar\cos(\theta-\theta_0)}\right] \tag{1.5.1}$$

The following properties can be easily established:

$$m_1 m_2 = ra\sin\theta\sin\theta_0, \quad m_1^2 + m_2^2 = r^2 + a^2 - 2ar\cos\theta\cos\theta_0, \tag{1.5.2}$$

so that the distance between two points M and N can be expressed as $R_0^2 = m_1^2 + m_2^2 - 2m_1m_2\cos(\phi-\phi_0)$. This property allows us to use formulae from section 1.2. For example, we can derive the following integral representations

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_0^{t_1(\theta_0)} \frac{\lambda\left(\dfrac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\tau-\cos\theta}\sqrt{\cos\gamma(\tau)-\cos\theta_0}}, \tag{1.5.3}$$

$$\frac{1}{R_0} = \frac{1}{\pi\sqrt{ar}}\int_{t_2(\theta_0)}^{\pi} \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta-\cos\tau}\sqrt{\cos\theta_0-\cos\gamma(\tau)}}, \tag{1.5.4}$$

where

$$t_1 \equiv t_1(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_1(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_1(\theta_0)}{m_2(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2},$$

$$t_2 \equiv t_2(\theta_0,\theta,a,r) = 2\tan^{-1}\left(\frac{m_2(\theta_0)}{2\sqrt{ar}\cos(\theta/2)\cos(\theta_0/2)}\right)$$

$$= 2\tan^{-1}\left[\frac{m_2(\theta_0)}{m_1(\theta_0)}\tan\frac{\theta}{2}\tan\frac{\theta_0}{2}\right]^{1/2}, \tag{1.5.5}$$

$$\cos\gamma(\tau) = \cos\tau - \frac{(r-a)^2}{4ar}\frac{\sin^2\tau}{\cos\tau - \cos\theta}, \tag{1.5.6}$$

and hereafter $m_1(x)$ and $m_2(x)$ are understood as abbreviations for $m_1(x,\theta,a,r)$ and $m_2(x,\theta,a,r)$ respectively. It is possible to show that both t_1 and t_2 are inverse to γ, i.e. $\gamma[t_{1,2}(\theta_0]=\theta_0$. Note also that $t_1 \le \min(\theta,\theta_0)$ and $t_2 \ge \max(\theta,\theta_0)$. We can see certain analogy between the notations and their properties used in cylindrical coordinates and those in spherical coordinates: l corresponds to t, g corresponds to γ, etc.

By using analogy with section 1.2, we can derive the following indefinite integrals:

$$\int \frac{\lambda\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\tau - \cos\theta}\sqrt{\cos\gamma(\tau) - \cos\theta_0}} = -\frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_1(\tau)}{R_0}\right), \tag{1.5.7}$$

$$\int \frac{\lambda\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi-\phi_0\right)d\tau}{\sqrt{\cos\theta - \cos\tau}\sqrt{\cos\theta_0 - \cos\gamma(\tau)}} = \frac{2\sqrt{ar}}{R_0}\tan^{-1}\left(\frac{y_2(\tau)}{R_0}\right), \tag{1.5.8}$$

where

$$y_1(\tau) = 2\sqrt{ar}\ \sqrt{\cos\tau - \cos\theta}\ \sqrt{\cos\gamma(\tau) - \cos\theta_0}/\sin\tau,$$

$$y_2(\tau) = 2\sqrt{ar}\ \sqrt{\cos\theta - \cos\tau}\ \sqrt{\cos\theta_0 - \cos\gamma(\tau)}/\sin\tau, \tag{1.5.9}$$

$$R_0^2 = r^2 + a^2 - 2ar[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)]. \tag{1.5.10}$$

The integrals in (7) and (8) can be verified by using the identity

$$\lambda\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}, \phi-\phi_0\right) = \frac{\sin\tau\ y_1(\tau)}{R_0^2 + y_1^2(\tau)}\frac{dy_1(\tau)}{d\tau}. \tag{1.5.11}$$

A similar relationship can be established to verify (8).

Formulation of the problem. We consider the following general problem: it is necessary to find the electrostatic field of a charged sphere of radius a when an arbitrary potential v is prescribed over a spherical cap $0 \le \theta \le \alpha$, while an arbitrary charge distribution σ is given at the rest of the sphere. As before, we represent the potential through a simple layer

$$V(r,\theta,\phi) = \int_0^{2\pi} d\phi_0 \int_0^{\alpha} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, d\theta_0}{R_0} + \int_0^{2\pi} d\phi_0 \int_{\alpha}^{\pi} \frac{\sigma(\theta_0,\phi_0)\, a^2 \sin\theta_0\, d\theta_0}{R_0}, \tag{1.5.12}$$

where R_0 is defined by (10).

Substitution of (3) and (4) in (12) yields, after interchanging the order of integration

$$V(r,\theta,\phi) = \frac{2a^2}{\sqrt{ar}} \left\{ \int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}}\, \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right) \sigma(\theta_0,\phi) \right.$$

$$\left. + \int_{t_2(\alpha)}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_{\alpha}^{\gamma(\tau)} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\gamma(\tau)}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right) \sigma(\theta_0,\phi) \right\}. \tag{1.5.13}$$

It is convenient at this stage to split our problem in two: *i*) to find the electrostatic potential of a charged sphere when an arbitrary charge density is given at a spherical cap, and the zero potential is prescribed elsewhere; *ii*) to find the potential when the zero charge density is prescribed at a spherical cap, and an arbitrary potential is given elsewhere. Both problems are treated separately.

Problem 1. Consider the boundary value problem, with the following mixed conditions at $r = a$:

$$\sigma = \sigma(\theta,\phi), \quad \text{for } 0 \le \phi < 2\pi, \; 0 \le \theta < \alpha;$$

$$V(a,\theta,\phi) = 0, \quad \text{for } 0 \le \phi < 2\pi, \; \alpha < \theta \le \pi. \tag{1.5.14}$$

Substitution of the boundary conditions (14) in (13) yields

$$0 = \int_0^{\alpha} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\tau}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\tau - \cos\theta_0}}\, \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)}\right) \sigma(\theta_0,\phi)$$

$$+ \int_{\theta}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_{\alpha}^{\tau} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right) \sigma(\theta_0,\phi). \tag{1.5.15}$$

Notice that the value of σ in the first term of (15) is known from (14) while σ in the second term of (15) is as yet unknown. It is then necessary to express one through the other. By using (4), we can rewrite (15) in the following manner:

$$\int_{\theta}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_{\alpha}^{\tau} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0, \phi)$$

$$= -\int_{\theta}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_{0}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0, \phi),$$

which immediately simplifies as

$$\int_{\alpha}^{\tau} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\tau/2)}\right)\sigma(\theta_0, \phi) = -\int_{0}^{\alpha} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\tau}} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\tau/2)}\right)\sigma(\theta_0, \phi). \tag{1.5.16}$$

Application of the operator

$$-\frac{1}{\sin\theta} \mathcal{L}\left(\frac{1}{\tan(\theta/2)}\right)\frac{d}{d\theta}\int_{\alpha}^{\theta} \frac{\sin\tau\, d\tau}{\sqrt{\cos\tau - \cos\theta}} \mathcal{L}[\tan(\tau/2)]$$

to both sides of (16) gives

$$\sigma(\theta, \phi) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_{0}^{\alpha} \frac{\sqrt{\cos\theta_0 - \cos\alpha}\, \sin\theta_0\, d\theta_0}{\cos\theta_0 - \cos\theta} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta/2)}\right)\sigma(\theta_0, \phi). \tag{1.5.17}$$

Expression (17) can also be rewritten in the form

$$\sigma(\theta, \phi) = -\frac{1}{2\pi^2\sqrt{\cos\alpha - \cos\theta}} \int_{0}^{2\pi}\int_{0}^{\alpha} \frac{\sqrt{\cos\theta_0 - \cos\alpha}\; \sigma(\theta_0, \phi_0)\sin\theta_0\, d\theta_0\, d\phi_0}{1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\phi - \phi_0)}, \tag{1.5.18}$$

which corresponds to the Green's function found in a geometric form by Lord Kelvin who used his method of images. Certain simplification occurs in the case of axial symmetry, namely,

$$\sigma(\theta) = -\frac{1}{\pi\sqrt{\cos\alpha - \cos\theta}} \int_0^\alpha \frac{\sqrt{\cos\theta_0 - \cos\alpha}}{\cos\theta_0 - \cos\theta} \sigma(\theta_0) \sin\theta_0 \, d\theta_0. \tag{1.5.19}$$

The charge density is now known all over the sphere from (14) and (17). Substitution of (17) in the second term (13) yields, after simplification,

$$V(r,\theta,\phi) = \frac{2a^2}{\sqrt{ar}} \left\{ \int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \right.$$

$$\times \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi)$$

$$\left. + \int_{t_2(\alpha)}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}} \int_0^{\alpha} \frac{\sin\theta_1 \, d\theta_1}{\sqrt{\cos\theta_1 - \cos\gamma(\tau)}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_1/2)}{\tan^2(\tau/2)} \right) \sigma(\theta_1, \phi) \right\}. \tag{1.5.20}$$

The first term in (20) can be transformed in the following manner:

$$\int_0^{t_1(\alpha)} \frac{d\tau}{\sqrt{\cos\tau - \cos\theta}} \int_{\gamma(\tau)}^{\alpha} \frac{\sin\theta_0 \, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}} \mathcal{L}\left(\frac{\tan^2(\tau/2)}{\tan(\theta/2)\tan(\theta_0/2)} \right) \sigma(\theta_0, \phi)$$

$$= \int_0^{\alpha} \sin\theta_0 \, d\theta_0 \int_{t_2(\theta_0)}^{\pi} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}\sqrt{\cos\theta_0 - \cos\gamma(\tau)}} \mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)} \right) \sigma(\theta_0, \phi). \tag{1.5.21}$$

The interchange of the order of integration in (21) can be performed according to the scheme

$$\int_0^{\alpha} d\theta_0 \int_{t_2(\theta_0)}^{\pi} d\tau = \int_{t_2(0)}^{t_2(\alpha)} d\tau \int_0^{\gamma(\tau)} d\theta_0 + \int_{t_2(\alpha)}^{\pi} d\tau \int_0^{\alpha} d\theta_0, \tag{1.5.22}$$

and the back substitution in (20) yields

$$V(r,\theta,\phi)=\frac{2a^2}{\sqrt{ar}}\int_{t_2(0)}^{t_2(\alpha)}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_0^{\gamma(\tau)}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\gamma(\tau)}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi). \tag{1.5.23}$$

Interchange in the order of integration in (23) and subsequent integration with respect to τ (see (8)) results in

$$V(r,\theta,\phi)=\frac{2}{\pi}\int_0^{2\pi}d\phi_0\int_0^{\alpha}\frac{\sigma(\theta_0,\phi_0)}{R_0}\tan^{-1}\left(\frac{\xi}{R_0}\right)a^2\sin\theta_0\,d\theta_0, \tag{1.5.24}$$

where R_0 is defined by (10) and

$$\xi=\frac{\sqrt{2}\sqrt{\cos\theta_0-\cos\alpha}\sqrt{m_2^2(\alpha)-\cos^2(\alpha/2)\,m_2^2(0)}}{\sin\alpha}. \tag{1.5.25}$$

Notice certain similarity between (25) and (1.3.49). Expressions (23) and (24) give two equivalent solutions to the problem 1, the first one being more convenient for the exact evaluation of the integrals involved while the second one has certain advantages when a numerical integration is required.

Problem 2. Consider a charged sphere with the following boundary conditions at its surface $r=a$

$$\sigma(\theta,\phi)=0,\ \text{for}\ \ 0\le\phi<2\pi,\ \ 0\le\theta<\alpha;$$

$$V(a,\theta,\phi)=v(\theta,\phi),\ \text{for}\ 0\le\phi<2\pi,\ \ \alpha\le\theta\le\pi. \tag{1.5.26}$$

The following integral equation results after substituting (26) in (13).

$$2a\int_{\theta}^{\pi}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{\alpha}^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.5.27}$$

Application of the operator

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)$$

to both sides of (27) yields

$$-2\pi a \int_{\alpha}^{\theta_1} \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_0 - \cos\theta_1}} \mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta_1/2)}\right)\sigma(\theta_0,\phi) = \mathcal{L}\left(\tan\frac{\theta_1}{2}\right)$$

$$\times \frac{d}{d\theta_1}\int_{\theta_1}^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta_1 - \cos\theta}} \mathcal{L}\left(\cot\frac{\theta}{2}\right) v(\theta,\phi). \qquad (1.5.28)$$

The next operator to apply is

$$\frac{1}{\sin\theta_2}\mathcal{L}\left(\cot\frac{\theta_2}{2}\right)\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2} \frac{\sin\theta_1\, d\theta_1}{\sqrt{\cos\theta_1 - \cos\theta_2}} \mathcal{L}\left(\tan\frac{\theta_1}{2}\right),$$

with the final result

$$\sigma(\theta_2,\phi) = -\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a \sin\theta_2}\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2} \frac{\sin\theta_1\, d\theta_1}{\sqrt{\cos\theta_1 - \cos\theta_2}} \mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times \frac{d}{d\theta_1}\int_{\theta_1}^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta_1 - \cos\theta}} \mathcal{L}\left(\cot\frac{\theta}{2}\right) v(\theta,\phi). \qquad (1.5.29)$$

Now the following rules of differentiation can be used

$$\frac{d}{d\theta_2}\int_{\alpha}^{\theta_2} \frac{f(\theta_1)\sin\theta_1\, d\theta_1}{\sqrt{\cos\theta_1 - \cos\theta_2}} = \sin\theta_2\left[\frac{f(\alpha)}{\sqrt{\cos\alpha - \cos\theta_2}}\right.$$

$$\left. + \int_{\alpha}^{\theta_2} \frac{df(\theta_1)}{d\theta_1}\frac{d\theta_1}{\sqrt{\cos\theta_1 - \cos\theta_2}}\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta_1}\int_{\theta_1}^{\pi}\frac{f(\theta)\sin\theta\,\mathrm{d}\theta}{\sqrt{\cos\theta_1-\cos\theta}}=2\tan\frac{\theta_1}{2}\int_{\theta_1}^{\pi}\frac{\cos(\theta/2)\,\mathrm{d}\theta}{\sqrt{\cos\theta_1-\cos\theta}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\cos\left(\frac{\theta}{2}\right)f(\theta)\right].\tag{1.5.30}$$

By using (7) and (30), expression (29) can be simplified as follows:

$$\sigma(\theta_2,\phi)=-\frac{1}{2\pi^2 a}\left\{\frac{\Phi(\alpha,\theta_2,\phi)}{\sqrt{\cos\alpha-\cos\theta_2}}+\int_{\alpha}^{\theta_2}\frac{\mathrm{d}\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\frac{\partial}{\partial\theta_1}\Phi(\theta_1,\theta_2,\phi)\right\}.\tag{1.5.31}$$

Here

$$\Phi(\theta_1,\theta_2,\phi)=2\tan\frac{\theta_1}{2}\int_{\theta_1}^{\pi}\frac{\cos(\theta/2)\,\mathrm{d}\theta}{\sqrt{\cos\theta_1-\cos\theta}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\cos\left(\frac{\theta}{2}\right)\mathcal{L}\left(\frac{\tan^2(\theta_1/2)}{\tan(\theta/2)\tan(\theta_2/2)}\right)v(\theta,\phi)\right].\tag{1.5.32}$$

Formulae (29) and (31)–(32) give two equivalent forms of solution of the integral equation (27). We note two different terms in (31): the first one is singular at $\theta_2\to\alpha$, while the second one tends to zero at $\theta_2\to\alpha$.

The potential in space due to a charged sphere can be obtained by substitution of (29) into (13). The result is

$$V(r,\theta,\phi)=-\frac{\sqrt{a}}{\pi\sqrt{r}}\int_{t_2(\alpha)}^{\pi}\frac{\mathrm{d}\tau}{\sqrt{\cos\theta-\cos\tau}}\mathcal{L}\left(\frac{\tan^2[\gamma(\tau)/2]\tan(\theta/2)}{\tan^2(\tau/2)}\right)$$

$$\times\frac{\partial}{\partial\gamma(\tau)}\int_{\gamma(\tau)}^{\pi}\frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos\gamma(\tau)-\cos\theta_0}}\mathcal{L}\left(\cot\frac{\theta_0}{2}\right)v(\theta_0,\phi).\tag{1.5.33}$$

Interchanging the order of integration in (33), we obtain, after subsequent integration with respect to τ,

$$V(r,\theta,\phi)=\frac{a|r^2-a^2|}{2\pi^2}\int_0^{2\pi}\mathrm{d}\phi_0\int_{\alpha}^{\pi}\left[\frac{R_0}{\chi}+\tan^{-1}\left(\frac{\chi}{R_0}\right)\right]\frac{v(\theta_0,\phi_0)}{R_0^3}\sin\theta_0\,\mathrm{d}\theta_0,\tag{1.5.34}$$

where

$$\chi=y_1[t_1(\alpha)]=\frac{2\sqrt{ar}\sqrt{\cos t_1(\alpha)-\cos\theta}\sqrt{\cos\alpha-\cos\theta_0}}{\sin t_1(\alpha)}.\tag{1.5.35}$$

It can be proven that (35) can be obtained from (25) by a formal substitution of θ_0, θ, and α by $\pi-\theta_0$, $\pi-\theta$, and $\pi-\alpha$ respectively.

The derivation of (34) can be outlined as follows. By introducing a new variable $\tau=t_2(x)$, $x=\gamma(\tau)$, expression (33) can be reduced to

$$V(r,\theta,\phi)=-\frac{\sqrt{a}}{\pi\sqrt{r}}\int_\alpha^\pi \frac{t_2'(x)\,\mathrm{d}x}{\sqrt{\cos\theta-\cos t_2(x)}}\,\mathcal{L}\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\pi \frac{\sin\theta_0\,\mathrm{d}\theta_0}{\sqrt{\cos x-\cos\theta_0}}\,\mathcal{L}\left(\cot\frac{\theta_0}{2}\right)v(\theta_0,\phi), \tag{1.5.36}$$

where

$$t_2'(x)=\frac{\partial t_2(x)}{\partial x}. \tag{1.5.37}$$

Integration by parts in (36) yields

$$V(r,\theta,\phi)=-\frac{\sqrt{a}}{\pi\sqrt{r}}\int_\alpha^\pi \left\{\frac{t_2'(\alpha)\,\mathcal{L}\left(\frac{\tan^2[t_1(\alpha)/2]}{\tan(\theta/2)\tan(\theta_0/2)}\right)}{\sqrt{\cos\theta-\cos t_2(\alpha)}\sqrt{\cos\alpha-\cos\theta_0}}\right.$$

$$\left.+\int_\alpha^{\theta_0}\frac{\mathrm{d}x}{\sqrt{\cos x-\cos\theta_0}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{t_2'(x)\,\mathcal{L}\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)\tan(\theta_0/2)}\right)}{\sqrt{\cos\theta-\cos t_2(x)}}\right]\right\}v(\theta_0,\phi)\sin\theta_0\,\mathrm{d}\theta_0. \tag{1.5.38}$$

Introducing the function

$$F(x)=\frac{\sqrt{ar}\,|r^2-a^2|}{R_0^3}\left[\frac{R_0}{y_1[t_1(x)]}+\tan^{-1}\left(\frac{y_1[t_1(x)]}{R_0}\right)\right], \tag{1.5.39}$$

where y_1 and R_0 are defined by (9) and (10) respectively. We can prove the following identity:

$$\frac{t_2'(x)\,\lambda\left(\frac{\tan^2[t_1(x)/2]}{\tan(\theta/2)\tan(\theta_0/2)},\phi-\phi_0\right)}{\sqrt{\cos\theta-\cos t_2(x)}}=\frac{2(\cos x-\cos\theta_0)^{3/2}}{\sin x}\frac{\mathrm{d}F(x)}{\mathrm{d}x}. \tag{1.5.40}$$

In order to prove (40) one should use (5), (6), (11), and the following identities:

$$[\cos\theta - \cos t_2(x)][\cos t_1(x) - \cos\theta] = \left(\frac{a-r}{a+r}\right)^2 \sin^2\theta,$$

$$[\cos x - \cos t_2(x)][\cos t_1(x) - \cos x] = \left(\frac{a-r}{a+r}\right)^2 \sin^2 x,$$

$$\frac{\partial t_2(x)}{\partial x} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos t_1(x) - \cos\theta]}\frac{\partial t_1(x)}{\partial x} = \frac{\sin x\,[\cos\theta - \cos t_2(x)]}{\sin\theta\,[\cos x - \cos t_2(x)]}\frac{\partial t_1(x)}{\partial x},$$

$$\sin t_1(x)\sin t_2(x) = \frac{4ar}{(a+r)^2}\sin x\sin\theta,$$

$$\frac{\sin t_1(x)}{\sin t_2(x)} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos\theta - \cos t_2(x)]} = \frac{\sin x\,[\cos t_1(x) - \cos\theta]}{\sin\theta\,[\cos x - \cos t_2(x)]}$$

$$\cos t_1(x)\cos t_2(x) = \frac{4ar\cos\theta\cos x - (r-a)^2}{(r+a)^2},$$

$$\cos t_1(x) + \cos t_2(x) = \frac{4ar}{(r+a)^2}(\cos x + \cos\theta),$$

$$\frac{[\cos t_1(x) - \cos x]\,[\cos t_1(x) - \cos\theta]}{\sin^2 t_1(x)} = \frac{[\cos x - \cos t_2(x)]\,[\cos\theta - \cos t_2(x)]}{\sin^2 t_2(x)} = \frac{(a-r)^2}{4ar}.$$

Substitution of (40) in (38) yields

$$V(r,\theta,\phi) = \frac{\sqrt{a}}{2\pi^2\sqrt{r}}\int_0^{2\pi} d\phi_0 \int_\alpha^\pi \left\{ \frac{2(\cos\alpha - \cos\theta_0)}{\sin\alpha}\frac{dF(\alpha)}{d\alpha} \right.$$

$$\left. + \int_\alpha^{\theta_0} \frac{dx}{\sqrt{\cos x - \cos\theta_0}}\frac{d}{dx}\left[\frac{2(\cos x - \cos\theta_0)^{3/2}}{\sin x}\frac{dF(x)}{dx}\right]\right\} v(\theta_0,\phi_0)\sin\theta_0\, d\theta_0. \tag{1.5.41}$$

Integration by parts in (41) gives

$$V(r,\theta,\phi) = \frac{\sqrt{a}}{2\pi^2\sqrt{r}} \int_0^{2\pi} d\phi_0 \int_\alpha^\pi F(\alpha)\, v(\theta_0,\phi_0) \sin\theta_0\, d\theta_0,$$

which is equivalent to (34). We note the analogy between (34) and (1.3.34). In the case $\alpha \to 0$ formula (34) transforms into the well-known Poisson's solution to the Dirichlet problem for a sphere.

Certain integral characteristics can be evaluated without solving any integral equation. For example, the total charge in Problem 1 can be found by integration of both sides of (17), with the result

$$Q_1 = \frac{2}{\pi} a^2 \int_0^{2\pi} d\phi \int_0^\alpha \sigma(\theta,\phi) \cos^{-1}\left(\frac{\cos(\alpha/2)}{\cos(\theta/2)}\right) \sin\theta\, d\theta. \tag{1.5.42}$$

The total charge in Problem 2 can be obtained from (29) as

$$Q_2 = \frac{a}{2\pi^2} \int_0^{2\pi} d\phi \int_\alpha^\pi \left[\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha - \cos\theta}} + \tan^{-1}\frac{\sqrt{\cos\alpha - \cos\theta}}{\sqrt{1-\cos\alpha}}\right] v(\theta,\phi) \sin\theta\, d\theta. \tag{1.5.43}$$

Dirichlet problem with discontinuous boundary conditions. In many practical cases, the boundary conditions for Dirichlet problem are changing so rapidly that they can be modelled as discontinuous. The spherical harmonic expansion solution converges very badly in those cases, and it is usually divergent on the surface of the sphere, thus making the solution unfit for practical purposes. On the other hand, the closed form solution, given by Poisson, is very inconvenient for practical evaluation of the integrals. The new method allows us to obtain an alternative solution, which is equivalent to the one obtained by Poisson, but is easy amenable for the exact evaluations of the integrals involved.

We consider a charged sphere with the following conditions at its surface $r = a$

$$V(a,\theta,\phi) = v(\theta,\phi), \text{ for } 0 \le \theta \le \alpha, \ 0 \le \phi < 2\pi;$$

$$V(a,\theta,\phi) = 0, \text{ for } \alpha \le \theta \le \pi, \ 0 \le \phi < 2\pi. \tag{1.5.44}$$

The problem, in a sense, is inverse to problem 1, therefore, substitution of (44) in (23) leads to the governing integral equation

$$2a\int_{\theta}^{\alpha}\frac{d\tau}{\sqrt{\cos\theta-\cos\tau}}\int_{0}^{\tau}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\tau}}\,\mathcal{L}\left(\frac{\tan(\theta/2)\,\tan(\theta_0/2)}{\tan^2(\tau/2)}\right)\sigma(\theta_0,\phi)=v(\theta,\phi). \tag{1.5.45}$$

The exact solution of (45) can be obtained in a manner similar to that of (27). We apply the operator

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)$$

to both sides of (45). The result is

$$-2\pi a\int_{0}^{\theta_1}\frac{\sin\theta_0\,d\theta_0}{\sqrt{\cos\theta_0-\cos\theta_1}}\mathcal{L}\left(\frac{\tan(\theta_0/2)}{\tan(\theta_1/2)}\right)\sigma(\theta_0,\phi)=$$

$$\mathcal{L}\left(\tan\frac{\theta_1}{2}\right)\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.5.46}$$

the next operator to apply is

$$\frac{\mathcal{L}[\cot(\theta_2/2)]}{\sin\theta_2}\frac{d}{d\theta_2}\int_{0}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan\frac{\theta_1}{2}\right),$$

thus giving the solution

$$\sigma(\theta_2,\phi)=\frac{\mathcal{L}[\cot(\theta_2/2)]}{2\pi^2 a\sin\theta_2}\frac{d}{d\theta_2}\int_{0}^{\theta_2}\frac{\sin\theta_1\,d\theta_1}{\sqrt{\cos\theta_1-\cos\theta_2}}\,\mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times\frac{d}{d\theta_1}\int_{\theta_1}^{\alpha}\frac{\sin\theta\,d\theta}{\sqrt{\cos\theta_1-\cos\theta}}\,\mathcal{L}\left(\cot\frac{\theta}{2}\right)v(\theta,\phi). \tag{1.5.47}$$

Expression (47) is valid in the interval $0 \le \theta_2 \le \alpha$. The charge density distribution at the rest of the sphere can be obtained by substitution of (47) in (17), with the result for $\alpha < \theta < \pi$

$$\sigma(\theta,\phi) = \frac{\mathcal{L}[\cot(\theta/2)]}{4\pi^2 a} \int_0^\alpha \frac{\sin\theta_1\, d\theta_1}{(\cos\theta_1 - \cos\theta)^{3/2}}\, \mathcal{L}\left(\tan^2\frac{\theta_1}{2}\right)$$

$$\times \frac{d}{d\theta_1} \int_{\theta_1}^\alpha \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\theta_1 - \cos\theta_0}}\, \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \tag{1.5.48}$$

The elementary analysis of (47) and (48) shows that both charge density distributions have non-integrable singularities of opposite sign at $\theta \to \alpha$, when $v(\alpha - 0, \phi) \ne 0$, otherwise expression (47) has no singularities, and formula (48) can give an integrable singularity. The total charge can be obtained by integration (47) and (48), with the result

$$Q = \frac{a}{4\pi} \int_0^{2\pi} d\phi \int_0^\alpha v(\theta,\phi)\sin\theta\, d\theta. \tag{1.5.49}$$

The potential in space due to a charged sphere can be obtained by substitution of (47) in (23) which yields, after simplification,

$$V(r,\theta,\phi) = -\frac{\sqrt{a}}{\pi\sqrt{r}} \int_{t_2(0)}^{t_2(\alpha)} \frac{d\tau}{\sqrt{\cos\theta - \cos\tau}}\, \mathcal{L}\left(\frac{\tan(\theta/2)\tan^2[\gamma(\tau)/2]}{\tan^2(\tau/2)}\right)$$

$$\times \frac{\partial}{\partial\gamma(\tau)} \int_{\gamma(\tau)}^\alpha \frac{\sin\theta_0\, d\theta_0}{\sqrt{\cos\gamma(\tau) - \cos\theta_0}}\, \mathcal{L}\left(\cot\frac{\theta_0}{2}\right) v(\theta_0,\phi). \tag{1.5.50}$$

Interchange of the order of integration in (50) and subsequent integration with respect to τ result in the well-known Poisson formula, namely,

$$V(r,\theta,\phi) = -\frac{a|r^2 - a^2|}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^\alpha \frac{v(\theta_0,\phi_0)}{R_0^3} \sin\theta_0\, d\theta_0. \tag{1.5.51}$$

Expression (50) is equivalent to (51) and has definite advantages when an exact evaluation of the integrals is possible.

Influence of a point charge. We consider the interaction between a point charge q located at the point with spherical coordinates (r_0, θ_0, ϕ_0) and a grounded spherical cap $\alpha \leq \theta \leq \pi$ of radius a. The potential in space can be represented as a sum

$$V = V_q + V_c, \tag{1.5.52}$$

where V_q is the potential due to the point charge q, and V_c is the potential of the charge induced on the spherical cap. At the surface of the cap holds the condition $V = 0$, which implies that

$$V_c = -V_q = -q/R, \tag{1.5.53}$$

with

$$R^2 = r_0^2 + a^2 - 2ar_0\,[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)].$$

Now we have a mixed problem with the following conditions at the surface of the sphere $r = a$:

$$\sigma(\theta, \phi) = 0, \quad \text{for} \quad 0 \leq \theta < \alpha, \ \ 0 \leq \phi < 2\pi;$$

$$V(a, \theta, \phi) = -q/R, \quad \text{for} \quad \alpha \leq \theta \leq \pi, \ \ 0 \leq \phi < 2\pi. \tag{1.5.54}$$

Conditions (54) correspond to Problem 2, which has already been solved. According to (4), we can write the following integral representation

$$\frac{1}{R} = \frac{1}{\pi\sqrt{ar_0}} \int_{t_{20}(\theta)}^{\pi} \frac{\lambda\left(\dfrac{\tan(\theta/2)\tan(\theta_0/2)}{\tan^2(\tau/2)}, \phi - \phi_0\right) d\tau}{\sqrt{\cos\theta_0 - \cos\tau}\,\sqrt{\cos\theta - \cos\gamma_0(\tau)}}, \tag{1.5.55}$$

where, according to (5) and (6)

$$t_{20}(\theta) \equiv t_2(\theta, \theta_0, a, r_0), \quad \cos\gamma_0(\tau) = \cos\tau - \frac{(r_0 - a)^2}{4ar_0}\,\frac{\sin^2\tau}{\cos\tau - \cos\theta_0}, \tag{1.5.56}$$

The induced charge density distribution is given by (29) which after substitution of (55) simplifies for $\alpha < \theta < \pi$ to

$$\sigma(\theta,\phi) = -q\,\frac{\mathcal{L}[\cot(\theta/2)]}{2\pi^2 a\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_\alpha^\theta \frac{\sin\theta_1\,\mathrm{d}\theta_1}{\sqrt{\cos\theta_1-\cos\theta}}$$

$$\times\lambda\left(\frac{\tan^2(\theta_1/2)\tan(\theta_0/2)}{\tan^2[t_{20}(\theta_1)/2]},\,\phi-\phi_0\right)\frac{\partial t_{20}(\theta_1)/\partial\theta_1}{\sqrt{ar_0}\,\sqrt{\cos\theta_0-\cos t_{20}(\theta_1)}}. \qquad (1.5.57)$$

The integral in (57) can be evaluated exactly in the same manner as before, with the result

$$\sigma(\theta,\phi) = -\frac{q|r^2-a^2|}{2\pi^2 aR^3}\left[\frac{R}{\chi_0}+\tan^{-1}\frac{\chi_0}{R}\right], \qquad (1.5.58)$$

where

$$\chi_0 = \frac{2\sqrt{ar_0}\,\sqrt{\cos t_{10}(\alpha)-\cos\theta_0}\,\sqrt{\cos\alpha-\cos\theta}}{\sin t_{10}(\alpha)}, \qquad (1.5.59)$$

$$t_{10}(x)\equiv t_1(x,\theta_0,a,r_0),\quad t_{20}(x)\equiv t_2(x,\theta_0,a,r_0),$$

Note certain similarity between (58)–(59) and (34)–(35). In the particular case $r_0\to a$ and $\theta_0<\alpha$, formula (58) simplifies as

$$\sigma(\theta,\phi) = -q\,\frac{\sqrt{\cos\theta_0-\cos\alpha}}{2\pi^2\sqrt{\cos\alpha-\cos\theta}}\frac{1}{R^2}, \qquad (1.5.60)$$

which is in agreement with (18). Expression (58) is convenient for a direct evaluation of the induced charge density distribution but, if some further mathematical transformations are needed, then the equivalent formula (57) has definite advantages. For example, to evaluate the total charge Q using (58) would be quite difficult, while (57) immediately gives

$$Q = -\frac{q\sqrt{a}}{\pi\sqrt{r_0}}\int_\alpha^\pi \frac{\sin\theta_1}{\sqrt{1+\cos\theta_1}\,\sqrt{\cos\theta_0-\cos t_{20}(\theta_1)}}\frac{\partial t_{20}(\theta_1)}{\partial\theta_1}\,\mathrm{d}\theta_1. \qquad (1.5.61)$$

Introducing a new variable $\tau = t_{20}(\theta_1)$, $\theta_1=\gamma_0(\tau)$ in (61) the following simplification occurs:

$$Q = -\frac{q\sqrt{a}}{\pi\sqrt{r_0}} \int_{t_{20}(\alpha)}^{\pi} \frac{\sqrt{1-\cos\gamma_0(\tau)}}{\sqrt{\cos\theta_0 - \cos\tau}}\, d\tau, \tag{1.5.62}$$

the evaluation of which is elementary, and the final result is

$$Q = -\frac{q}{\pi r_0}[(a+r_0)\sin^{-1}A_{10} - |a-r_0|\sin^{-1}A_{20}] \tag{1.5.63}$$

where

$$A_{10} = \frac{(r_0+a)\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha) + 4ar_0\cos^2(\theta/2)\cos^2(\alpha/2)}},$$

$$A_{20} = \frac{|r_0-a|\cos(\alpha/2)}{\sqrt{m_{20}^2(\alpha) - 4ar_0\sin^2(\theta/2)\cos^2(\alpha/2)}}, \quad m_{20}(\alpha) = m_2(\alpha, \theta, a, r_0). \tag{1.5.64}$$

When the point charge is located at the axis ($\theta_0 = \pi$), formula (63) simplifies as

$$Q = \frac{q}{\pi r_0}\left[|r_0-a|\left(\frac{\pi-\alpha}{2}\right) - (r_0+a)\tan^{-1}\left(\frac{r_0+a}{|r_0-a|}\cot\frac{\alpha}{2}\right)\right]. \tag{1.5.65}$$

In the case of a complete sphere we have $\alpha = 0$, and formula (65) simplifies further

$$Q = \frac{q}{2r_0}[|r_0-a| - (r_0+a)].$$

Similar formulae can be obtained for $\theta_0 = 0$.

The potential V_c due to the induced charge can be obtained by substitution of (55) in (33) which gives, after the first integration

$$V_c(r,\theta,\phi) = -\frac{q}{\pi\sqrt{rr_0}} \int_{\alpha}^{\pi} \frac{\lambda\left(\dfrac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi-\phi_0\right) t_2' t_{20}'\, dx}{\sqrt{\cos\theta - \cos t_2}\sqrt{\cos\theta_0 - \cos t_{20}}}. \tag{1.5.66}$$

Here the abbreviations t_1, t_2, t_{10}, and t_{20} are understood as $t_1(x)$, $t_2(x)$, $t_{10}(x)$, $t_{20}(x)$ respectively, the prime signs indicate the partial derivatives with respect to x. The integral in (66) can be evaluated exactly, and the final result is

$$V(r,\theta,\phi) = V_q + V_c = q\left\{\frac{1}{2R_1}\left[1+\frac{2}{\pi}\tan^{-1}\frac{\eta_1(\alpha)}{R_1}\right] - \frac{1}{2R_2}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\eta_2(\alpha)}{R_2}\right]\right\}, \tag{1.5.67}$$

which is in agreement with a similar result of Hobson (1900) in toroidal coordinates. The notations are

$$R_1^2 = r^2 + r_0^2 - 2rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)],$$

$$R_2^2 = \frac{r^2 r_0^2}{a^2} + a^2 - 2rr_0[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)],$$

$$\eta_{1,2} = \frac{(r+a)(r_0+a)}{2a}S(x) \pm \frac{(r-a)(r_0-a)}{2aS(x)},$$

$$S(x) = \sqrt{\cos t_1 - \cos x}\ \sqrt{\cos t_{10} - \cos x}/\sin x. \tag{1.5.68}$$

The following identities were used to perform integration in (66)

$$\lambda\left(\frac{\tan(t_1/2)\tan(t_{10}/2)}{\tan(t_2/2)\tan(t_{20}/2)}, \phi-\phi_0\right) = \frac{(a+r)^2(a+r_0)^2}{16a^2\sin^2 x}[(1-\cos t_1\cos t_2)(\cos t_{10}-\cos t_{20})$$

$$+ (1-\cos t_{10}\cos t_{20})(\cos t_1 - \cos t_2)]\left[\frac{1}{R_1^2+\eta_1^2(x)} + \frac{1}{R_2^2+\eta_2^2(x)}\right], \tag{1.5.69}$$

$$R_1^2 + \eta_1^2(x) = R_2^2 + \eta_2^2(x), \tag{1.5.70}$$

$$\frac{\partial}{\partial x}S(x) = \frac{S(x)}{2\sin x}\left[\frac{1-\cos t_1\cos t_2}{\cos t_1 - \cos t_2} + \frac{1-\cos t_{10}\cos t_{20}}{\cos t_{10}-\cos t_{20}}\right], \tag{1.5.71}$$

$$\frac{\partial t_2}{\partial x} = \frac{2\sqrt{ar}}{a+r}\frac{\sqrt{\cos t_1 - \cos x}\sqrt{\cos\theta - \cos t_2}}{\cos t_1 - \cos t_2}. \tag{1.5.72}$$

An expression similar to (72) can be written for the derivative of t_{20}. Substitution of (69)–(72) in (66) makes the procedure of integration very simple.

1.6. Mixed problems in toroidal coordinates

Further extension of previously obtained results to the case of toroidal coordinates is presented here. It is based on a new integral representation for the reciprocal of the distance between two points. Its substitution in the governing integral equation reduces the problem to sequence of two consecutive Abel type operators combined with the $\mathcal{L}$-operator. Each can be inverted exactly and in closed form, thus giving the solution. Some integrals of fundamental value, involving distances between several points, are established. The complete set of systems of coordinates, where the new method can be applied, is not known at this time and can constitute a subject for a separate investigation.

Mathematical preliminaries. The following relationships exist between the cartesian (x,y,z) and toroidal (v,u,ϕ) coordinates

$$x=\frac{c\sinh v\cos\phi}{\cosh v-\cos u},\quad y=\frac{c\sinh v\sin\phi}{\cosh v-\cos u},\quad z=\frac{c\sin u}{\cosh v-\cos u}. \tag{1.6.1}$$

Here c is a dimensional parameter. The surfaces $u=$constant are spherical caps

$$x^2+y^2+(z-c\cot u)^2=\left(\frac{c}{\sin u}\right)^2, \tag{1.6.2}$$

with the common line of intersection along the circle $\rho=c$, $z=0$. The surfaces $v=$constant are tori

$$(\sqrt{x^2+y^2}-c\coth v)^2+z^2=\left(\frac{c}{\sinh v}\right)^2. \tag{1.6.3}$$

The properties of toroidal coordinates allow us to use this system of coordinates for solving mixed boundary value problems for various geometries including the case of several spherical caps.

Consider two points M and N in a three-dimensional space. Let their cylindrical coordinates be respectively (ρ,ϕ,z) and (r,ψ,z_0). From the results of section 1.2, the following integral is valid

$$\int\frac{\lambda\left(\dfrac{y^2}{l_1(r)l_2(r)},\phi-\psi\right)\mathrm{d}y}{\sqrt{l_1^2(r)-y^2}\sqrt{l_2^2(r)-y^2}}=-\frac{1}{R_0}\tan^{-1}\left[\frac{\sqrt{l_1^2(r)-y^2}\sqrt{l_2^2(r)-y^2}}{yR_0}\right]. \tag{1.6.4}$$

Here, as before,

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}, \quad \text{for } k<1, \tag{1.6.5}$$

$$l_1(r) = \frac{1}{2}[\sqrt{(\rho+r)^2+(z-z_0)^2} - \sqrt{(\rho-r)^2+(z-z_0)^2}],$$

$$l_2(r) = \frac{1}{2}[\sqrt{(\rho+r)^2+(z-z_0)^2} + \sqrt{(\rho-r)^2+(z-z_0)^2}], \tag{1.6.6}$$

$$R_0 \equiv R(M,N) = [\rho^2 + r^2 - 2\rho r\cos(\phi-\psi) + (z-z_0)^2]. \tag{1.6.7}$$

The integral representation for reciprocal of the distance between two points can be obtained from (4) by applying the limits of integration from 0 to $l_1(r)$. This representation is fundamental for the new method in cylindrical coordinates. One can verify that the distance between two points $R(M,N)$ can be presented as

$$R_0 = \sqrt{l_1^2(r) + l_2^2(r) - 2l_1(r)l_2(r)\cos(\phi-\psi)}. \tag{1.6.8}$$

In order to be able to apply (4) in toroidal coordinates, we need to present the distance between two points in a similar form, namely, as a sum of two squares minus double product of those quantities and cosine of the difference between the appropriate angles. Let the toroidal coordinates of M and N be (v,u,ϕ) and (x,β,ψ) respectively. The distance between two points in toroidal coordinates is

$$R_0 \equiv R(M,N) = \frac{\sqrt{2}c\sqrt{\cosh v\cosh x - \sinh v\sinh x\cos(\phi-\psi) - \cos(u-\beta)}}{\sqrt{\cosh v - \cos u}\,\sqrt{\cosh x - \cos\beta}}. \tag{1.6.9}$$

Clearly, (9) does not look like (8), but we can transform it into

$$R_0 = \frac{2c\cosh(v/2)\cosh(x/2)}{\sqrt{\cosh v - \cos u}\,\sqrt{\cosh x - \cos\beta}}\left[\tanh^2\left(\frac{v}{2}\right) + \tanh^2\left(\frac{x}{2}\right)\right.$$

$$\left. - 2\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\cos(\phi-\psi) + \frac{\sin^2[(u-\beta)/2]}{\cosh^2(v/2)\cosh^2(x/2)}\right]^{1/2}. \tag{1.6.10}$$

Expression (10) gives us a hint that we can introduce some quantities t_1 and t_2 in such a way that the distance between two points will be proportional to the expression

$$\left[\tanh^2\left(\frac{t_1}{2}\right) + \tanh^2\left(\frac{t_2}{2}\right) - 2\tanh\left(\frac{t_1}{2}\right)\tanh\left(\frac{t_2}{2}\right)\cos(\phi-\psi)\right]^{1/2},$$

so that these quantities could play in toroidal coordinates the same role as the parameters l_1 and l_2 play in cylindrical coordinates. Indeed, this can be achieved by defining

$$t_1 = 2\tanh^{-1}\left[\frac{l_1}{l_2}\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\right]^{1/2},$$

$$t_2 = 2\tanh^{-1}\left[\frac{l_2}{l_1}\tanh\left(\frac{v}{2}\right)\tanh\left(\frac{x}{2}\right)\right]^{1/2}. \tag{1.6.11}$$

We can now introduce a new variable τ according to the expression

$$y = \left[\, l_1 l_2 \coth\left(\frac{v}{2}\right)\coth\left(\frac{x}{2}\right)\right]^{1/2}\tanh\left(\frac{\tau}{2}\right). \tag{1.6.12}$$

Substitution of (12) in (4) yields

$$\int \frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}, \phi-\psi\right)d\tau}{\sqrt{\cosh v - \cosh\tau}\,\sqrt{\cosh x - \cosh\gamma(\tau)}} = -\frac{2c}{R_0\sqrt{\cosh v - \cos u}\,\sqrt{\cosh x - \cos\beta}}$$

$$\times\tan^{-1}\left[\frac{2c\sqrt{\cosh v - \cosh\tau}\sqrt{\cosh x - \cosh\gamma(\tau)}}{R_0\sinh\tau\,\sqrt{\cosh v - \cos u}\sqrt{\cosh x - \cos\beta}}\right]. \tag{1.6.13}$$

Here

$$\cosh\gamma \equiv \cosh\gamma(\tau) \equiv \cosh\gamma(\tau,\ \beta,\ v,\ u) = \cosh\tau + \sin^2\left(\frac{u-\beta}{2}\right)\frac{\sinh^2\tau}{\cosh v - \cosh\tau}, \tag{1.6.14}$$

We intentionally use in this section the same notation t_1, t_2, and γ in order to demonstrate certain analogy between the toroidal and spherical coordinates. We hope the reader will not be confused. Introduce the following notation

$$t_1 \equiv t_1(x,\beta,v,u)$$

$$= 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)} - \sqrt{\cosh(x-v)-\cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\}, \tag{1.6.15}$$

$$t_2 \equiv t_2(x,\beta,v,u)$$

$$= 2\tanh^{-1}\left\{\frac{\sqrt{\cosh(x+v)-\cos(u-\beta)}+\sqrt{\cosh(x-v)-\cos(u-\beta)}}{2\sqrt{2}\cosh(x/2)\cosh(v/2)}\right\},$$

(1.6.16)

For the brevity sake, we use the following conventions: the parameters of γ, t_1 and t_2, given respectively in (14), (15), and (16), are considered as the default parameters. This would allow us to write, for example, $\gamma(y,\delta)$ instead of $\gamma(y,\delta,v,u)$. The rule is rather simple: the parameters which are not given explicitly assumed to be the default ones.

One can verify that (15) and (16) are in agreement with (11). Notice that both t_1 and t_2 are inverse to γ. This means that $\gamma(t_1)=x$ and $\gamma(t_2)=x$. The following property is valid $t_1\le\min\{v,x\}$, $t_2\ge\max\{v,x\}$, the equality sign holds for $u=\beta$. By using previous results we can obtain the following integral representation for the reciprocal of the distance between two points:

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c}\int_0^{t_1}\frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)d\tau}{\sqrt{\cosh v-\cosh\tau}\sqrt{\cosh x-\cosh\gamma}}.$$

(1.6.17)

We can derive several variations of (17). For example, introducing a new variable $\tau=t_1(y)$, expression (17) will take the form

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c}\int_0^{x}\frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)t_1'(y)\,dy}{\sqrt{\cosh v-\cosh t_1(y)}\sqrt{\cosh x-\cosh y}}.$$

(1.6.18)

Here the symbol ($'$) stands for the partial derivative with respect to the parameter in brackets. By using (A12), one can rewrite (18) as

$$\frac{1}{R_0}=\frac{\sqrt{\cosh v-\cos u}\sqrt{\cosh x-\cos\beta}}{\pi c\,|\cos[(u-\beta)/2]|}$$

$$\times\int_0^{x}\frac{\lambda\left(\dfrac{\tanh^2(t_1(y)/2)}{\tanh(v/2)\tanh(x/2)},\phi-\psi\right)\sqrt{\cosh t_2(y)-\cosh y}\,dy}{[\cosh t_2(y)-\cosh t_1(y)]\sqrt{\cosh x-\cosh y}}. \qquad (1.6.19)$$

We can also compute a more general indefinite integral, namely,

$$I_1 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$

$$\times \int \frac{\sqrt{\cosh t_2 - \cosh x}\sqrt{\cosh t_{20} - \cosh x}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})} \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) \mathrm{d}x \,. \tag{1.6.20}$$

Here $t_{10} = t_1(x, \beta, v_0, u_0)$ and $t_{20} = t_2(x, \beta, v_0, u_0)$ respectively. Introduce new variables

$$\eta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)S(x) \pm \frac{1}{S(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.6.21}$$

where

$$S(x) = \frac{\sqrt{\cosh t_2 - \cosh x}\sqrt{\cosh t_{20} - \cosh x}}{\sinh x} \tag{1.6.22}$$

The following identities may be established by using formulae from Appendix:

$$\frac{\mathrm{d}S(x)}{\mathrm{d}x} = -\frac{S(x)}{2\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{S(x)}\right) = \frac{1}{2\,S(x)\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.24}$$

$$\lambda\left(\frac{\tanh(t_1/2\tanh(t_{10}/2))}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right) = \frac{\cos^2[(u-\beta)/2]\cos^2[(u_0-\beta)/2]}{2\sinh^2 x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1}\right.$$

$$\left. + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right](\cosh t_2 - \cosh t_1)$$

$$\times (\cosh t_{20} - \cosh t_{10})\left[\frac{1}{\cosh w - \cos(u-u_0) + 2\eta_1^2}\right.$$

$$\left. + \frac{1}{\cosh w - \cos(u+u_0-2\beta) + 2\eta_2^2}\right]. \tag{1.6.25}$$

Here

$$\cosh w = \cosh v \cosh v_0 - \sinh v \sinh v_0 \cos(\phi - \phi_0). \tag{1.6.26}$$

The transformations leading to (23)–(25) are very non-trivial. One has to use the appropriate formulae from Appendix in an ingenious way in order to repeat the results. Taking into consideration that

$$\frac{d\eta_1(x)}{dx} = \frac{\eta_2(x)}{S(x)}\frac{dS(x)}{dx}, \qquad \frac{d\eta_2(x)}{dx} = \frac{\eta_1(x)}{S(x)}\frac{dS(x)}{dx},$$

$$\cosh w - \cos(u-u_0) + 2\eta_1^2 = \cosh w - \cos(u+u_0-2\beta) + 2\eta_2^2, \tag{1.6.27}$$

the substitution of (23)–(25) and (27) in (20) leads to

$$I_1 = -\int \left[\frac{d\eta_1}{\cosh w - \cos(u-u_0) + 2\eta_1^2} + \frac{d\eta_2}{\cosh w - \cos(u+u_0-2\beta) + 2\eta_2^2}\right]. \tag{1.6.28}$$

The last integral can be computed in an elementary way, and the final result is

$$I_1 = -\frac{1}{\sqrt{2[\cosh w - \cos(u-u_0)]}}\tan^{-1}\left[\frac{\sqrt{2}\eta_1(x)}{\sqrt{\cosh w - \cos(u-u_0)}}\right]$$
$$-\frac{1}{\sqrt{2[\cosh w - \cos(u+u_0-2\beta)]}}\tan^{-1}\left[\frac{\sqrt{2}\eta_2(x)}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\right]. \tag{1.6.29}$$

Here the reader may ask us two questions. First, why have we decided that the integral (20) is computable, and second, how did we come up with expressions (21) and the properties (23)–(25)? The integral (20) was encountered in solving the problem of influence of a point charge on a spherical bowl which, as we know, has an elementary solution. This meant that the integral (20) has to be computable. The hints on how to compute it can be taken from similar integral in section 1.5. One has just to replace the appropriate trigonometric functions by the hyperbolic ones.

Yet another integral can be computed in a similar manner, namely,

$$I_2 = \frac{1}{\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]}$$
$$\times\int \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh x - \cosh t_{10}}}{(\cosh t_2 - \cosh t_1)(\cosh t_{20} - \cosh t_{10})}\lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right)dx. \tag{1.6.30}$$

The same integral (30) can be rewritten as

$$I_2 = \int \lambda\left(\frac{\tanh(t_1/2)\tanh(t_{10}/2)}{\tanh(t_2/2)\tanh(t_{20}/2)}, \phi - \phi_0\right)\frac{t_2'(x)t_{20}'(x)\,\mathrm{d}x}{\sqrt{\cosh t_2 - \cosh v}\,\sqrt{\cosh t_{20} - \cosh v_0}}. \tag{1.6.31}$$

Introduction of new variables

$$\Theta_{1,2} = \cos\left(\frac{u-\beta}{2}\right)\cos\left(\frac{u_0-\beta}{2}\right)T(x) \pm \frac{1}{T(x)}\sin\left(\frac{u-\beta}{2}\right)\sin\left(\frac{u_0-\beta}{2}\right), \tag{1.6.32}$$

with

$$T(x) = \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh x - \cosh t_{10}}}{\sinh x},$$

and use of the identities (25) and

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = \frac{T(x)}{2\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{T(x)}\right) = -\frac{1}{2\,T(x)\sinh x}\left[\frac{\cosh t_1 \cosh t_2 - 1}{\cosh t_2 - \cosh t_1} + \frac{\cosh t_{10}\cosh t_{20} - 1}{\cosh t_{20} - \cosh t_{10}}\right], \tag{1.6.34}$$

allow us to compute the integral

$$I_2 = \frac{1}{\sqrt{2[\cosh w - \cos(u-u_0)]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_1(x)}{\sqrt{\cosh w - \cos(u-u_0)}}\right]$$
$$+ \frac{1}{\sqrt{2[\cosh w - \cos(u+u_0-2\beta)]}}\tan^{-1}\left[\frac{\sqrt{2}\Theta_2(x)}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\right]. \tag{1.6.35}$$

One may deduce from (A2) that

$$T(x) = |\tan\left(\frac{u-\beta}{2}\right)\tan\left(\frac{u_0-\beta}{2}\right)\frac{1}{S(x)}.$$

This property gives us various relationships between η and Θ depending on the signs of the trigonometric functions. For example, when $\cos[(u-\beta)/2]\cos[(u_0-\beta)/2]>0$ and $\sin[(u-\beta)/2]\sin[(u_0-\beta)/2]>0$, we have $\eta_1=\Theta_1$ and $\eta_2=-\Theta_2$. The derived integrals will be used in solving various mixed boundary value problems.

Problem description. Consider two spherical caps defined in the toroidal coordinates (v, u, ϕ) as follows:

$$0 \le v \le b_0, \qquad u = u_0, \qquad 0 \le \phi \le 2\pi;$$

$$0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi. \tag{1.6.36}$$

In the limiting case $b \to \infty$ and $b_0 \to \infty$ the spherical caps intersect along a circle of radius c which is the basic circle of the system of coordinates. Consider an electrostatic problem when an arbitrary charge distribution σ is prescribed on the first spherical cap ($u = u_0$), and an arbitrary potential distribution V is given on the surface of the second cap. It is then necessary to find the electrostatic field in the whole space. It is convenient to split the problem in two: the first problem assumes that $\sigma = 0$ and $V \neq 0$, while in the second problem we take $V = 0$ and $\sigma \neq 0$. The linear superposition of the two solutions would give us the general solution to the problem.

Problem 1. Since the first cap is not charged, we have to solve the Dirichlet problem for a spherical cap with the following condition on its surface:

$$V = V(v, \phi), \text{ for } \; 0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi. \tag{1.6.37}$$

The as yet unknown potential in space can be represented through a simple layer distribution

$$V(v, u, \phi) = c^2 \int_0^{2\pi} \int_0^{b} \frac{\sigma(x, \psi) \sinh x \, \mathrm{d}x \, \mathrm{d}\psi}{(\cosh x - \cos\beta)^2 R_0}. \tag{1.6.38}$$

Here σ is the charge distribution and R_0 is defined by (9). Substitution of (17) in (38) yields, after interchanging the order of integration

$$V(v, u, \phi) = 2c\sqrt{\cosh v - \cos u} \int_0^{t_1(b)} \frac{\mathrm{d}\tau}{\sqrt{\cosh v - \cosh \tau}}$$

$$\times \int_\gamma^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right) \sigma(x, \phi) \sinh x \, \mathrm{d}x}{(\cosh x - \cos\beta)^{3/2} \sqrt{\cosh x - \cosh \gamma}}. \tag{1.6.39}$$

The following scheme of interchanging the order of integration was used in (39):

$$\int_0^b \mathrm{d}x \int_0^{t_1} \mathrm{d}\tau = \int_0^{t_1(b)} \mathrm{d}\tau \int_\gamma^b \mathrm{d}x. \tag{1.6.40}$$

Substituting the boundary condition (37) in (39) results in the governing integral equation

$$V(v,\phi) = 2c\sqrt{\cosh v - \cos\beta} \int_0^v \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\times \int_\tau^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}. \tag{1.6.41}$$

The integral equation (41) represents a sequence of two Abel type operators and the $\mathcal{L}$-operator. Each can be inverted in a manner similar to the one employed in previous sections. Let us apply the following operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta}\sqrt{\cosh y - \cosh v}}$$

to both sides of (41) in the following manner:

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta}\sqrt{\cosh y - \cosh v}} = \mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}$$

$$\times \int_0^y \frac{\sinh v\, dv}{\sqrt{\cosh y - \cosh v}} \int_0^v \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}} \int_\tau^b \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}. \tag{1.6.42}$$

By using the general property

$$\int_0^y \frac{\sinh x\, dx}{\sqrt{\cosh y - \cosh x}} \int_0^{t_1} \frac{f(\tau)\, d\tau}{\sqrt{\cosh x - \cosh\gamma}} = \pi \int_0^{t_1(y)} f(\tau)\ d\tau, \tag{1.6.43}$$

expression (42) can be simplified, namely,

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

$$=\pi\int_y^b \frac{\mathcal{L}\left(\frac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \qquad (1.6.44)$$

The next operator to apply to both sides of (44) is

$$\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\,\mathcal{L}\left(\coth\frac{y}{2}\right). \qquad (1.6.45)$$

The final result is

$$\sigma(s,\phi)=-\frac{(\cosh s-\cos\beta)^{3/2}}{2\pi^2 c\sinh s}\,\mathcal{L}\left(\tanh\frac{s}{2}\right)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\,\mathcal{L}\left(\coth^2\frac{y}{2}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cos\beta}\,\sqrt{\cosh y-\cosh v}}. \qquad (1.6.46)$$

Here the following property was used

$$\int_s^b \frac{\sinh y\,\mathrm{d}y}{\sqrt{\cosh y-\cosh s}}\int_y^b \frac{f(x)\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}}=\pi\int_s^b f(x)\,\mathrm{d}x. \qquad (1.6.47)$$

Formula (46) gives the expression for the charge density in terms of the prescribed potential V.

We can now substitute (46) in (39) in order to obtain the potential in space through its value on the spherical cap. By using the property

$$\int_\gamma^b \frac{\mathrm{d}x}{\sqrt{\cosh x-\cosh\gamma}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{f(v)\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cosh x}}=-\pi f(\gamma), \qquad (1.6.48)$$

the following result can be obtained

$$V(v,u,\phi)=\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^{t_1(b)}\frac{\mathrm{d}\tau}{\sqrt{\cosh v-\cosh\tau}}\,\mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh^2(\gamma/2)\tanh(v/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}\gamma}\int_0^{\gamma}\frac{\sinh y\;\mathcal{L}[\tanh(y/2)]\;V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}\sqrt{\cosh\gamma-\cosh y}}. \qquad (1.6.49)$$

Introduction in (49) of a new variable $x=\gamma$, (which is equivalent to $\tau=t_1$), allows us to rewrite (49) as

$$V(v,u,\phi)=\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^{b}\frac{\mathrm{d}t_1}{\sqrt{\cosh v-\cosh t_1}}\,\mathcal{L}\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)}\right)$$

$$\times\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x}\frac{\sinh y\;\mathcal{L}[\tanh(y/2)]\;V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}\sqrt{\cosh x-\cosh y}}. \qquad (1.6.50)$$

We interchange the order of integration in (50) according to the scheme

$$\int_0^b F(x)\,\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{\sinh y\,f(y)\,\mathrm{d}y}{\sqrt{\cosh x-\cosh y}}=-\int_0^b f(y)\,\mathrm{d}y\,\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b\frac{\sinh x\,F(x)\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}}. \qquad (1.6.51)$$

The result is

$$V(v,u,\phi)=-\frac{1}{\pi}\sqrt{\cosh v-\cos u}\int_0^b\mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}y}\right.$$

$$\left.\times\int_y^b\frac{t_1'(x)\sinh x\,\mathcal{L}\left(\dfrac{\tanh(v/2)}{\tanh^2(t_2/2)}\right)\mathrm{d}x}{\sqrt{\cosh x-\cosh y}\sqrt{\cosh v-\cosh t_1}}\right]\frac{V(y,\phi)\,\mathrm{d}y}{\sqrt{\cosh y-\cos\beta}}. \qquad (1.6.52)$$

The interior integral in (52) can be computed in closed form. Indeed, consider the expression

$$I_3 = \mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{t_1'(x)\sinh x\,\lambda\left(\frac{\tanh(v/2)}{\tanh^2(t_2/2)},\phi-\psi\right)\mathrm{d}x}{\sqrt{\cosh x-\cosh y}\,\sqrt{\cosh v-\cosh t_1}}\right] \tag{1.6.53}$$

The differentiation can be performed according to the rule

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{f(x)\sinh x\,\mathrm{d}x}{\sqrt{\cosh x-\cosh y}} = -\frac{f(b)\sinh y}{\sqrt{\cosh b-\cosh y}} + \sinh y\int_y^b \frac{\mathrm{d}f(x)}{\sqrt{\cosh x-\cosh y}}, \tag{1.6.54}$$

with the result

$$I_3 = -\frac{t_1'(b)\,\lambda\left(\frac{\tanh(y/2)\tanh(v/2)}{\tanh^2[t_2(b)/2]},\phi-\psi\right)\sinh y}{\sqrt{\cosh v-\cosh t_1(b)}}$$

$$+\sinh y\int_y^b \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{t_1'(x)\,\lambda\left(\frac{\tanh(y/2)\tanh(v/2)}{\tanh^2(t_2/2)},\phi-\psi\right)}{\sqrt{\cosh v-\cosh t_1}}\right\}\frac{\mathrm{d}x}{\sqrt{\cosh x-\cosh y}} \tag{1.6.55}$$

Introduce the following notation:

$$F(x) = \frac{4c^3\,|\sin(u-\beta)|}{(\cosh v-\cos u)^{3/2}\,(\cosh y-\cos\beta)^{3/2}}\,\frac{1}{R_y^3}\left[\frac{R_y}{\chi(x)} + \tan^{-1}\left(\frac{\chi(x)}{R_y}\right)\right], \tag{1.6.56}$$

where

$$R_y = \frac{\sqrt{2}c\sqrt{\cosh v\cosh y-\sinh v\sinh y\cos(\phi-\psi)-\cos(u-\beta)}}{\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}, \tag{1.6.57}$$

and

$$\chi(x) = \frac{2c\,|\cos[(u-\beta)/2]|\,\sqrt{\cosh x-\cosh y}\,\sqrt{\cosh x-\cosh t_1}}{\sinh x\,\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}$$

$$= \frac{2c\sqrt{\cosh t_2-\cosh v}\,\sqrt{\cosh x-\cosh y}}{\sinh t_2\,\sqrt{\cosh v-\cos u}\,\sqrt{\cosh y-\cos\beta}}$$

$$= \frac{2c\,|\sin[(u-\beta)/2]|\,\sqrt{\cosh x - \cosh y}}{\sqrt{\cosh t_2 - \cosh x}\,\sqrt{\cosh v - \cos u}\,\sqrt{\cosh y - \cos\beta}}. \tag{1.6.58}$$

Equivalence of the expressions (58) can be proven by using formulae from Appendix. The following identity is valid:

$$\frac{t_1'(x)\,\lambda\!\left(\dfrac{\tanh(y/2)\tanh(v/2)}{\tanh^2(t_2/2)}, \phi-\psi\right)}{\sqrt{\cosh v - \cosh t_1}} = -\frac{(\cosh x - \cosh y)^{3/2}}{\sinh x}\frac{dF(x)}{dx}. \tag{1.6.59}$$

Substitution of (59) in (55) and integration by parts yield

$$\mathcal{L}\left(\tanh\frac{y}{2}\right)\left[\frac{d}{dy}\int_y^b \frac{t_1'(x)\sinh x\,\lambda\!\left(\dfrac{\tanh(v/2)}{\tanh^2(t_2/2)}, \phi-\psi\right)dx}{\sqrt{\cosh x - \cosh y}\,\sqrt{\cosh v - \cosh t_1}}\right] = -\frac{1}{2}F(b)\sinh y$$

$$= -\frac{2c^3\,|\sin(u-\beta)|\sinh y}{(\cosh v - \cos u)^{3/2}(\cosh y - \cos\beta)^{3/2}}\frac{1}{R_y^3}\left[\frac{R_y}{\chi(b)} + \tan^{-1}\left(\frac{\chi(b)}{R_y}\right)\right], \tag{1.6.60}$$

While integrating by parts in (60), one should notice that substitution of the lower limit of integration y leads to the uncertainty of the type $\infty-\infty$ which has to be dealt with properly.

Now substitution of (60) in (52) allows us to rewrite it in the form

$$V(v,u,\phi) = \frac{c^3\,|\sin(u-\beta)|}{\pi^2(\cosh v - \cos u)}\int_0^{2\pi}\int_0^b \frac{1}{R_y^3}\left[\frac{R_y}{\chi(b)} + \tan^{-1}\left(\frac{\chi(b)}{R_y}\right)\right]\frac{V(y,\psi)\sinh y\,dy\,d\psi}{(\cosh y - \cos\beta)^2}. \tag{1.6.61}$$

The last formula is in agreement with the classical result of Hobson (1900).

Problem 2. The boundary conditions in this case take the form

$$\sigma = \sigma_0(v,\phi), \qquad \text{for } 0 \le v \le b_0, \qquad u = u_0, \qquad 0 \le \phi \le 2\pi. \tag{1.6.62}$$

$$V = 0, \qquad \text{for } 0 \le v \le b, \qquad u = \beta, \qquad 0 \le \phi \le 2\pi, \tag{1.6.63}$$

Denote the surface of the first cap as S_0, and the surface of the second cap as S. Introduce the following points, with their toroidal coordinates: $M(v,u,\phi)$, $N(x,\beta,\psi)$, $N_0(v_0,u_0,\phi_0)$, and $K(v,\beta,\phi)$. The potential in the space can be presented again through the simple layer distributions

$$V(M)=\iint\limits_{S}\frac{\sigma(N)\,\mathrm{d}S}{R(M,N)}+\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(M,N_0)}. \tag{1.6.64}$$

We note that σ_0 in (64) is known from (62) while σ is not yet known. It can be found from the integral equation which results from substitution of the second boundary condition (63) in (64), namely,

$$0=\iint\limits_{S}\frac{\sigma(N)\,\mathrm{d}S}{R(K,N)}+\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \tag{1.6.65}$$

By using the procedure similar to (38)–(41), we can rewrite (65) as

$$2c\sqrt{\cosh v-\cos\beta}\int_0^v\frac{\mathrm{d}\tau}{\sqrt{\cosh v-\cosh\tau}}\int_\tau^b\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh\tau}}$$

$$=-\iint\limits_{S_0}\frac{\sigma_0(N_0)\,\mathrm{d}S_0}{R(K,N_0)}. \tag{1.6.66}$$

The general solution to (66) is given in (46), we just need to substitute the right-hand side. Assuming that the order of integration is interchangeable we need to compute first

$$J=\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y\frac{\sinh v\,\mathrm{d}v}{\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}\,\mathcal{L}\left(\tanh\frac{v}{2}\right)\left[\frac{1}{R(K,N_0)}\right]. \tag{1.6.67}$$

Make use of the integral representation

$$\frac{1}{R(K,N_0)}=\frac{\sqrt{\cosh v_0-\cos u_0}\sqrt{\cosh v-\cos\beta}}{\pi c}$$

$$\times\int_0^{t_{10}(v)}\frac{\lambda\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v_0/2)\tanh(v/2)},\phi-\phi_0\right)\mathrm{d}\tau}{\sqrt{\cosh v_0-\cosh\tau}\sqrt{\cosh v-\cosh\gamma_0}}. \tag{1.6.68}$$

Here $\gamma_0=\gamma(\tau,\beta,v_0,u_0)$ as it is defined by (14). Substitution of (68) in (67) yields

$$J = \frac{\sqrt{\cosh v_0 - \cos u_0}}{\pi c} \frac{d}{dy} \int_0^y \frac{\sinh v \, dv}{\sqrt{\cosh y - \cosh v}}$$

$$\times \int_0^{t_{10}(v)} \frac{\lambda\left(\frac{\tanh^2(\tau/2)}{\tanh(v_0/2)}, \phi - \phi_0\right) d\tau}{\sqrt{\cosh v_0 - \cosh \tau} \sqrt{\cosh v - \cosh \gamma_0}}$$

$$= \frac{\sqrt{\cosh v_0 - \cos u_0}\, t_{10}{}'(y)}{c\sqrt{\cosh v_0 - \cosh t_{10}(y)}} \lambda\left(\frac{\tanh^2[t_{10}(y)/2]}{\tanh(v_0/2)}, \phi - \phi_0\right). \quad (1.6.69)$$

Here we used the identity (43). The next step is to compute

$$J_1 = \mathcal{L}\left(\tanh\frac{s}{2}\right) \frac{d}{ds} \int_s^b \frac{\sinh y \, dy}{\sqrt{\cosh y - \cosh s}} \mathcal{L}\left(\coth^2\frac{y}{2}\right)\{J\}, \quad (1.6.70)$$

where J is defined by (69). The elementary simplification results in

$$J_1 = \frac{\sqrt{\cosh v_0 - \cos u_0}}{c} \mathcal{L}\left(\tanh\frac{s}{2}\right) \frac{d}{ds} \int_s^b \lambda\left(\frac{\tanh(v_0/2)}{\tanh^2[t_{20}(y)/2]}, \phi - \phi_0\right)$$

$$\times \frac{t_{10}{}'(y) \sinh y \, dy}{\sqrt{\cosh y - \cosh s} \sqrt{\cosh v_0 - \cosh t_{10}(y)}}. \quad (1.6.71)$$

The integral (71) has already been computed in (60), so that we can write the solution of (66) in the form

$$\sigma(N) = -\int\int_{S_0} G(N, N_0)\, \sigma(N_0)\, dS_0, \quad (1.6.72)$$

where the Green's function G is defined by

$$G(N, N_0) = \frac{c\, |\sin(u_0 - \beta)|}{\pi^2(\cosh v_0 - \cos u_0)\, R^3(N, N_0)} \left[\frac{R(N, N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N, N_0)}\right)\right], \quad (1.6.73)$$

with

$$\chi_0(y) = \frac{2c\sqrt{\cosh t_{20}(y) - \cosh v_0}\,\sqrt{\cosh y - \cosh x}}{\sinh t_{20}(y)\sqrt{\cosh v_0 - \cos u_0}\,\sqrt{\cosh x - \cos\beta}}. \tag{1.6.74}$$

The back substitution of (72) in (64) allows us to express the potential in space directly in terms of the prescribed charge distribution σ_0. The integrals involved, though looking quite formidable, can be computed in terms of elementary functions. The main integral to be computed is

$$J_2 = \iint_S \frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]\frac{\mathrm{d}S_N}{R(M,N)}$$

$$= \frac{c}{\pi}\sqrt{\cosh v - \cos u}\int_0^{2\pi}\mathrm{d}\psi\int_0^b \frac{\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}}$$

$$\times\int_0^x \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)\tanh(x/2)}, \phi - \psi\right)\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}\,\sqrt{\cosh x - \cosh y}}$$

$$\times\frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]. \tag{1.6.75}$$

Interchanging the order of integration and using the integral representation (60), we obtain

$$J_2 = -\frac{c}{\pi}\sqrt{\cosh v - \cos u}\int_0^{2\pi}\mathrm{d}\psi\int_0^b \frac{\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}}$$

$$\times\int_y^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)\tanh(x/2)}, \phi - \psi\right)\sinh x\,\mathrm{d}x}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh y}}\left\{\frac{(\cosh x - \cos\beta)^{3/2}(\cosh v_0 - \cos u_0)^{3/2}}{2c^3\,|\sin(\beta - u_0)|\sinh x}\right.$$

$$\left.\times\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{t_{10}'(s)\,\lambda\left(\dfrac{\tanh(v_0/2)}{\tanh^2[t_{20}(s)/2]}, \psi - \phi_0\right)\sinh s\,\mathrm{d}s}{\sqrt{\cosh s - \cosh x}\,\sqrt{\cosh v_0 - \cosh t_{10}(s)}}\right\}. \tag{1.6.76}$$

Some obvious simplifications can be made, with the result

$$J_2 = -\frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2\pi c^2\,|\sin(\beta - u_0)|}\int_0^{2\pi} \mathrm{d}\psi \int_0^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]}{\tanh(v/2)}, \phi - \psi\right)\mathrm{d}t_1(y)}{\sqrt{\cosh v - \cosh t_1(y)}}$$

$$\times \int_y^b \frac{\mathrm{d}x}{\sqrt{\cosh x - \cosh y}}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{t_{10}'(s)\,\lambda\left(\dfrac{\tanh(v_0/2)}{\tanh^2[t_{20}(s)/2]}, \psi - \phi_0\right)\sinh s\,\mathrm{d}s}{\sqrt{\cosh s - \cosh x}\,\sqrt{\cosh v_0 - \cosh t_{10}(s)}}. \tag{1.6.77}$$

Further simplification is due to (48):

$$J_2 = \frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2c^2\,|\sin(\beta - u_0)|}$$

$$\times \int_0^b \frac{\lambda\left(\dfrac{\tanh^2[t_1(y)/2]\,\tanh(v_0/2)}{\tanh^2[t_{20}(y)/2]\,\tanh(v/2)}, \phi - \phi_0\right)t_1'(y)\,t_{10}'(y)\,\mathrm{d}y}{\sqrt{\cosh v - \cosh t_1(y)}\,\sqrt{\cosh v_0 - \cosh t_{10}(y)}}. \tag{1.6.78}$$

The integral in (78) is similar to the one computed in (20). The final result is

$$\int\!\!\int_S \frac{1}{R^3(N,N_0)}\left[\frac{R(N,N_0)}{\chi_0(b)} + \tan^{-1}\left(\frac{\chi_0(b)}{R(N,N_0)}\right)\right]\frac{\mathrm{d}S_N}{R(M,N)}$$

$$= \frac{\sqrt{\cosh v - \cos u}\,(\cosh v_0 - \cos u_0)^{3/2}}{2c^2\,|\sin(\beta - u_0)|}$$

$$\times \left\{\frac{1}{\sqrt{2[\cosh w - \cos(u - u_0)]}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{2}\eta_1(b)}{\sqrt{\cosh w - \cos(u - u_0)}}\right)\right]\right.$$

$$\left. + \frac{1}{\sqrt{2[\cosh w - \cos(u + u_0 - 2\beta)]}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{2}\eta_2(b)}{\sqrt{\cosh w - \cos(u + u_0 - 2\beta)}}\right)\right]\right\}, \tag{1.6.79}$$

with $\eta_{1,2}$ defined by (21) and $\cosh w$ given by (26). Substitution of (79) in (64) allows us to express the potential in the form

$$V(M) = \int\int_{S_0} \left\{1 + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_1(b)}{\sqrt{\cosh w - \cos(u-u_0)}}\right) - \frac{\sqrt{\cosh w - \cos(u-u_0)}}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\left[1\right.\right.$$

$$\left.\left. - \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\eta_2(b)}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\right)\right]\right\} \frac{\sigma_0(N_0)\,dS_0}{2R(M,N_0)} \qquad (1.6.80)$$

In the particular case of $b \to \infty$ formula (80) simplifies as follows:

$$V(M) = \int\int_{S_0} \left\{1 + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u-u_0)/2]}{\sqrt{\cosh w - \cos(u-u_0)}}\right) - \frac{\sqrt{\cosh w - \cos(u-u_0)}}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\left[1\right.\right.$$

$$\left.\left. + \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{2}\cos[(u+u_0-2\beta)/2]}{\sqrt{\cosh w - \cos(u+u_0-2\beta)}}\right)\right]\right\} \frac{\sigma_0(N_0)\,dS_0}{2R(M,N_0)} \qquad (1.6.81)$$

The last formula is in agreement with the long standing result of Hobson (1900). Several examples are considered below.

Spherical cap charged to a uniform potential. Consider a spherical cap defined by $0 \le v \le b$, $u = \beta$, with a uniform potential V_0 prescribed at its surface. The charge distribution can be found from (46), and is

$$\sigma = \frac{V_0\sqrt{1-\cos\beta}}{2\pi^2 c}\left\{\left[\frac{(\cosh b + 1)(\cosh v - \cos\beta)}{\cosh b - \cosh v}\right]^{1/2}\right.$$

$$\left. + \sqrt{1+\cos\beta}\,\tan^{-1}\left[\frac{(1+\cos\beta)(\cosh b - \cosh v)}{(1+\cosh b)(\cosh v - \cos\beta)}\right]^{1/2}\right\} \qquad (1.6.82)$$

The potential in space is conveniently defined by (50). The final result is

$$V = \frac{\sqrt{2}\,V_0|\sin(\beta/2)|\,\sqrt{\cosh v - \cos u}}{\pi\,|\cos[(u-\beta)/2]|\,(x_1^2 + x_2^2)}\left[\left(1\right.\right.$$

$$\left. + \frac{x_1^2}{\sin^2(\beta/2)}\right)^{1/2} \sin^{-1}\left(\frac{\sinh[t_1(b)/2]\sqrt{x_1^2 + \sin^2(\beta/2)}}{x_1\sqrt{x_2^2 + \sinh^2[t_1(b)/2)]}}\right)$$

$$-\left(1-\frac{x_2^2}{\sin^2(\beta/2)}\right)^{1/2}\sin^{-1}\left(\frac{\sinh^2[t_1(b)/2)]\sqrt{\sin^2(\beta/2)-x_2^2}}{x_2\sqrt{x_1^2-\sinh^2[t_1(b)/2)]}}\right)\Bigg]. \tag{1.6.83}$$

Here

$$x_{1,2}^2=(m^2+n^2)^{1/2}\pm m,\qquad n=\frac{\sinh(v/2)\sin(\beta/2)}{\cos[(u-\beta)/2]},$$

$$m=\frac{\sinh^2(v/2)-\sin^2(\beta/2)+\sin^2[(u-\beta)/2]}{2\cos^2[(u-\beta)/2]}. \tag{1.6.84}$$

Application of the reciprocal theorem to (82) allows us to express the total charge on a spherical cap with an *arbitrary* potential $V(v,\phi)$ prescribed at its surface as follows:

$$Q=\frac{c\sqrt{1-\cos\beta}}{2\pi^2}\int_0^{2\pi}\int_0^b\left\{\left[\frac{(\cosh b+1)(\cosh v-\cos\beta)}{\cosh b-\cosh v}\right]^{1/2}\right.$$

$$\left.+\sqrt{1+\cos\beta}\tan^{-1}\left[\frac{(1+\cos\beta)(\cosh b-\cosh v)}{(1+\cosh b)(\cosh v-\cos\beta)}\right]^{1/2}\right\}\frac{V(v,\phi)\sinh v\,\mathrm{d}v\,\mathrm{d}\phi}{(\cosh v-\cos\beta)^2}. \tag{1.6.85}$$

The same result can be obtained by a direct integration of both sides of (46). Formulae (82) and (83) in the case of $b\to\infty$ take the form

$$\sigma=\frac{V_0\sqrt{1-\cos\beta}}{2\pi^2 c}\left\{\sqrt{\cosh v-\cos\beta}+\sqrt{1+\cos\beta}\tan^{-1}\left[\frac{1+\cos\beta}{\cosh v-\cos\beta}\right]^{1/2}\right\} \tag{1.6.86}$$

$$V=\frac{\sqrt{2}\,V_0|\sin(\beta/2)|\sqrt{\cosh v-\cos u}}{\pi|\cos[(u-\beta)/2]|(x_1^2+x_2^2)}\left[\left(1+\frac{x_1^2}{\sin^2(\beta/2)}\right)^{1/2}\cos^{-1}\left(\frac{x_2\sin[(u-\beta)/2]}{\sqrt{x_2^2+\sinh^2(v/2)}}\right)\right.$$

$$\left.-\left(1-\frac{x_2^2}{\sin^2(\beta/2)}\right)^{1/2}\cos^{-1}\left(\frac{x_1\sin[(u-\beta)/2]}{\sqrt{x_1^2-\sinh^2(v/2)}}\right)\right]. \tag{1.6.87}$$

Electrified spherical ring. Consider the Dirichlet problem for a spherical ring $b \le v < \infty$, with the following boundary condition prescribed at its surface:

$$V = V(v,\phi), \quad \text{for} \quad u = \beta, \quad b \le v < \infty, \quad 0 < \phi \le 2\pi. \tag{1.6.88}$$

By using the procedure similar to the one employed in Problem 1 above, we come to the following expression for the potential in space:

$$V(v,u,\phi) = 2c\sqrt{\cosh v - \cos u}\left\{\int_0^{t_1(b)} \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}\right.$$

$$\times \int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\gamma}} + \int_{t_1(b)}^{v} \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\left.\times \int_\gamma^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\gamma}}\right\}. \tag{1.6.89}$$

Substitution of the boundary condition (88) in (89) leads to the integral equation

$$V(v,\phi) = 2c\sqrt{\cosh v - \cos u}\left\{\int_0^{b} \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}\right.$$

$$\times \int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}} + \int_{b}^{v} \frac{d\tau}{\sqrt{\cosh v - \cosh\tau}}$$

$$\left.\times \int_\tau^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta)^{3/2}\sqrt{\cosh x - \cosh\tau}}\right\}. \tag{1.6.90}$$

Let us apply the operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y \frac{\mathcal{L}[\tanh(v/2)]\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

to both sides of (90). The result is

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y \frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}$$

$$=\int_0^b \frac{\sqrt{\cosh b-\cosh\tau}\,\sinh y\,\mathrm{d}\tau}{\sqrt{\cosh y-\cosh b}\,(\cosh y-\cosh\tau)}$$

$$\times\int_b^\infty \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh\tau}}$$

$$+\pi\int_y^\infty \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.91}$$

We introduce a new unknown

$$p(y,\phi)=\int_y^\infty \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\,\mathrm{d}x}{(\cosh x-\cos\beta)^{3/2}\sqrt{\cosh x-\cosh y}}. \tag{1.6.92}$$

Expression (92) can be easily inverted, namely,

$$\sigma(x,\phi)=-\frac{(\cosh x-\cos\beta)^{3/2}}{\pi\sinh x}\,\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\mathcal{L}\left(\coth\dfrac{z}{2}\right)p(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh x}}. \tag{1.6.93}$$

Substitution of (93) in (91) leads to the governing integral equation

$$p(y,\phi)+\frac{1}{\pi^2}\int_b^\infty\left\{\int_0^b \mathcal{L}\left(\frac{\tanh^2(\tau/2)}{\tanh(y/2)\tanh(z/2)}\right)\frac{(\cosh b-\cosh\tau)\,\mathrm{d}\tau}{(\cosh y-\cosh\tau)(\cosh z-\cosh\tau)}\right\}$$

$$\times\frac{\sinh y\, p(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh y-\cosh b}\sqrt{\cosh z-\cosh b}}$$

$$=\frac{1}{\pi}\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y\frac{\mathcal{L}[\tanh(v/2)]\,V(v,\phi)\sinh v\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}. \qquad (1.6.94)$$

The following rule of interchange of the order of integration was used:

$$\int_b^\infty f(x)\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty\frac{F(z)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh x}}=-\int_b^\infty F(z)\,\mathrm{d}z\frac{\mathrm{d}}{\mathrm{d}z}\int_b^z\frac{f(x)\sinh x\,\mathrm{d}x}{\sqrt{\cosh z-\cosh x}}.$$

It is important to notice that the integral in curly brackets of (94) can be computed exactly in terms of elementary functions for any specific harmonic. For example, in the case of axial symmetry, equation (94) takes the form

$$p(y)+\frac{\sinh y}{\pi^2\sqrt{\cosh y-\cosh b}}\int_b^\infty\frac{K(y)-K(z)}{\cosh y-\cosh z}\frac{p(z)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z-\cosh b}}$$

$$=\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_b^y\frac{V(v)\sinh v\mathrm{d}v}{2c\sqrt{\cosh v-\cos\beta}\sqrt{\cosh y-\cosh v}}. \qquad (1.6.95)$$

Here

$$K(x)=\frac{\cosh x-\cosh b}{\sinh x}\ln\left(\frac{\sinh[(x+b)/2]}{\sinh[(x-b)/2]}\right). \qquad (1.6.96)$$

If one is interested in the quantity of total charge Q only, it can be expressed directly through function p as follows:

$$Q=\frac{c^2}{\pi}\sqrt{\cosh b-\cos\beta}\int_0^{2\pi}\int_b^\infty\frac{p(y,\phi)\sinh y\,\mathrm{d}y\,\mathrm{d}\phi}{\sqrt{\cosh y-\cosh b}\,(\cosh y-\cos\beta)}. \qquad (1.6.97)$$

A complete solution to the problem is beyond the scope of this book.

Interaction of several charged spherical caps. Consider n spherical caps $u=\beta_k$, $0<v\le b_k$, $k=1,2,\ldots,n$, with arbitrary potentials prescribed on their surfaces. The boundary conditions can be formulated in the form

$$V=V_k(v,\phi),\quad \text{for}\quad u=\beta_k,\quad 0\le v\le b_k,\quad 0\le\phi<2\pi,\quad k=1,2,\ldots,n. \tag{1.6.98}$$

Again, the procedure outlined in the solution of Problem 1 leads to the expression for the potential

$$V(v,u,\phi)=2c\sqrt{\cosh v-\cos u}\sum_{k=1}^{n}\int_0^{t_{1k}(b_k)}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times\int_{\gamma_k}^{b_k}\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_k)^{3/2}\sqrt{\cosh x-\cosh\gamma_k}}\,. \tag{1.6.99}$$

Here σ_k is the yet unknown charge distribution over the surface of the k-th cap, and the following notation was introduced:

$$t_{1k}\equiv t_1(x,\beta_k,v,u),\quad t_{2k}\equiv t_2(x,\beta_k,v,u),\quad \gamma_k\equiv\gamma(\tau,\beta_k,v,u). \tag{1.6.100}$$

We recall that the notations $t_{1,2}$ and γ were first introduced by (15) and (14) respectively.

Substitution of the boundary conditions (98) in (99) leads to a system of n integral equations. We can single out, without loss of generality, the cap number one. The corresponding integral equation takes the form

$$V_1(v,\phi)=2c\sqrt{\cosh v-\cos\beta_1}\int_0^{v}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times\int_{\tau}^{b_1}\frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_1(x,\phi)\sinh x\,dx}{(\cosh x-\cos\beta_1)^{3/2}\sqrt{\cosh x-\cosh\tau}}+\sum_{k=2}^{n}\int_0^{t_{1k1}(b_k)}\frac{d\tau}{\sqrt{\cosh v-\cosh\tau}}$$

$$\times \int_{\gamma_{k1}}^{b_k} \frac{\mathcal{L}\left(\dfrac{\tanh^2(\tau/2)}{\tanh(v/2)\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta_k)^{3/2}\sqrt{\cosh x - \cosh\gamma_{k1}}}. \tag{1.6.101}$$

Here

$$\gamma_{kl} \equiv \gamma(\tau, \beta_k, v, \beta_l), \quad t_{1kl} \equiv t_1(x, \beta_k, u, \beta_l). \tag{1.6.102}$$

Application of the integral operator

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta_1}\sqrt{\cosh y - \cosh v}}$$

to both sides of (101) results in

$$\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{d}{dy}\int_0^y \frac{\mathcal{L}[\tanh(v/2)]\, V_1(v,\phi)\sinh v\, dv}{2c\sqrt{\cosh v - \cos\beta_1}\sqrt{\cosh y - \cosh v}}$$

$$= \pi\int_y^{b_1} \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta_1)^{3/2}\sqrt{\cosh x - \cosh y}}$$

$$+ \pi\sum_{k=2}^{n}\int_0^{b_k} \mathcal{L}\left(\frac{\tanh^2[t_1(x,\beta_k,y,\beta_1)/2]}{\tanh(y/2)\tanh(x/2)}\right)$$

$$\times \frac{[\partial t_1(x,\beta_k,y,\beta_1)/\partial y]\,\sigma_k(x,\phi)\sinh x\, dx}{\sqrt{\cosh x - \cosh t_1(x,\beta_k,y,\beta_1)}(\cosh x - \cos\beta_k)^{3/2}}. \tag{1.6.103}$$

Introduce a new variable q_k as follows

$$q_k(y,\phi) = \int_y^{b_k} \frac{\mathcal{L}\left(\dfrac{\tanh(y/2)}{\tanh(x/2)}\right)\sigma_k(x,\phi)\sinh x\, dx}{(\cosh x - \cos\beta_k)^{3/2}\sqrt{\cosh x - \cosh y}}. \tag{1.6.104}$$

Expression (104) can be easily inverted, namely,

$$\sigma_k(x,\phi) = -\frac{(\cosh x - \cos\beta_k)^{3/2}}{\pi \sinh x}\,\mathcal{L}\left(\tanh\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{b_k} \frac{\mathcal{L}\left(\coth\frac{z}{2}\right)q_k(z,\phi)\sinh z\,\mathrm{d}z}{\sqrt{\cosh z - \cosh x}} \quad (1.6.105)$$

Substitution of (104) in (103) leads to the governing integral equation

$$q_1(y,\phi) + \frac{1}{\pi}\sum_{k=2}^{n}\int_0^{b_k}\left\{\mathcal{L}\left(\coth\frac{z}{2}\right)\frac{\mathrm{d}}{\mathrm{d}z}\int_0^{z}\frac{[\partial t_1(x,\beta_k,y,\beta_1)/\partial y]\sinh x\,\mathrm{d}x}{\sqrt{\cosh x - \cosh t_1(x,\beta_k,y,\beta_1)}\sqrt{\cosh z - \cosh x}}\right.$$

$$\left.\times\,\mathcal{L}\left(\frac{\tanh^2[t_1(x,\beta_k,y,\beta_1)/2]}{\tanh(y/2)}\right)\right\}q_k(z,\phi)\,\mathrm{d}z$$

$$= \frac{1}{\pi}\mathcal{L}\left(\frac{1}{\tanh(y/2)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{\mathcal{L}[\tanh(v/2)]\,V_1(v,\phi)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v - \cos\beta_1}\sqrt{\cosh y - \cosh v}} \quad (1.6.106)$$

We can note again that the integral in curly brackets in (106) can be computed in terms of elementary functions for any specific harmonic. For example, in the case of axial symmetry, equation (106) takes the form

$$q_1(y) + \sum_{k=2}^{n}\frac{2\,|\sin[(\beta_1-\beta_k)/2]|\cosh(y/2)}{\pi\cos^4[(\beta_1-\beta_k)/2]}$$

$$\times\int_0^{b_k}\frac{\{\cosh z + \cosh y - 2\cos^2[(\beta_1-\beta_k)/2]\}\cosh(z/2)\,q_k(z)\,\mathrm{d}z}{[\cosh t_2(z,\beta_k,y,\beta_1) - \cosh t_1(z,\beta_k,y,\beta_1)]^2}$$

$$= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y}\frac{V_1(v)\sinh v\,\mathrm{d}v}{2c\sqrt{\cosh v - \cos\beta_1}\,\sqrt{\cosh y - \cosh v}}. \quad (1.6.107)$$

Expression (107) can also be rewritten as

$$q_1(y) + \sum_{k=2}^{n} \frac{|\sin[(\beta_1-\beta_k)/2]|}{\pi} \int_0^{b_k} \left[\frac{\cosh[(y+z)/2]}{\cosh(y+z)-\cos(\beta_1-\beta_k)} \right.$$

$$\left. + \frac{\cosh[(y-z)/2]}{\cosh(y-z)-\cos(\beta_1-\beta_k)} \right] q_k(z)\, \mathrm{d}z$$

$$= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y} \int_0^{y} \frac{V_1(v)\sinh v\, \mathrm{d}v}{2c\sqrt{\cosh v - \cos\beta_1}\,\sqrt{\cosh y - \cosh v}}. \tag{1.6.108}$$

The remaining $n-1$ integral equations can be derived in a similar manner. Equation (108) is in agreement with the result of Ufliand (1977) who obtained it by using the Mehler-Fok integral transform. The last formula simplifies when V_1 is a constant, namely,

$$q_1(y) + \sum_{k=2}^{n} \frac{|\sin[(\beta_1-\beta_k)/2]|}{\pi} \int_0^{b_k} \left[\frac{\cosh[(y+z)/2]}{\cosh(y+z)-\cos(\beta_1-\beta_k)} \right.$$

$$\left. + \frac{\cosh[(y-z)/2]}{\cosh(y-z)-\cos(\beta_1-\beta_k)} \right] q_k(z)\, \mathrm{d}z = \frac{V_1 \cosh(y/2)\,|\sin(\beta_1/2)|}{\pi c(\cosh y - \cos\beta_1)} \tag{1.6.109}$$

The total charge Q_k can be expressed directly through function q as follows:

$$Q_k = \frac{2}{\pi} c^2 |\sin(\beta_k/2)| \int_0^{2\pi} \int_0^{b_k} \frac{q_k(v,\phi)\cosh(v/2)\, \mathrm{d}v\, \mathrm{d}\phi}{\cosh v - \cos\beta_k}. \tag{1.6.110}$$

The reader is advised to try to obtain a complete solution to the problem.

Appendix. Some essential formulae used in the main body of this section are presented here.

$$\tanh\left(\frac{t_1}{2}\right)\tanh\left(\frac{t_2}{2}\right) = \tanh\left(\frac{x}{2}\right)\tanh\left(\frac{v}{2}\right), \tag{1.6.A1}$$

$$(\cosh t_2 - \cosh v)(\cosh v - \cosh t_1) = \sinh^2 v \tan^2\left(\frac{u-\beta}{2}\right), \tag{1.6.A2}$$

$$\cos^2\left(\frac{u-\beta}{2}\right)(\cosh t_1 - \cosh\tau)(\cosh t_2 - \cosh\tau) = (\cosh v - \cosh\tau)[\cosh x - \cosh\gamma(\tau)], \quad (1.6.\text{A}3)$$

$$\cosh t_1 \cosh t_2 = \frac{\sin^2[(u-\beta)/2] + \cosh v\,\cosh x}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}4)$$

$$\cosh t_1 + \cosh t_2 = \frac{\cosh v + \cosh x}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}5)$$

$$(\cosh x - \cosh t_1)(\cosh v - \cosh t_1) = \sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_1, \quad (1.6.\text{A}6)$$

$$(\cosh t_2 - \cosh x)(\cosh t_2 - \cosh v) = \sin^2\left(\frac{u-\beta}{2}\right)\sinh^2 t_2, \quad (1.6.\text{A}7)$$

$$\sinh t_1 \sinh t_2 = \frac{\sinh x \sinh v}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}8)$$

$$\frac{\sinh t_1}{\sinh t_2} = \frac{\sinh v(\cosh x - \cosh t_1)}{\sinh x(\cosh t_2 - \cosh v)} = \frac{\sinh x(\cosh v - \cosh t_1)}{\sinh v(\cosh t_2 - \cosh x)}, \quad (1.6.\text{A}9)$$

$$(\cosh t_1 - 1)(\cosh t_2 - 1) = \frac{(\cosh v - 1)(\cosh x - 1)}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}10)$$

$$\cosh t_2 - \cosh t_1 = \frac{\sqrt{\cosh(x+v) - \cos(u-\beta)}\sqrt{\cosh(x-v) - \cos(u-\beta)}}{\cos^2[(u-\beta)/2]}, \quad (1.6.\text{A}11)$$

$$\frac{\partial t_1}{\partial x} = \frac{\sqrt{\cosh v - \cosh t_1}\sqrt{\cosh t_2 - \cosh x}}{|\cos[(u-\beta)/2]|\,(\cosh t_2 - \cosh t_1)} = \frac{\sinh x(\cosh v - \cosh t_1)}{\sinh t_1(\cosh t_2 - \cosh t_1)\cos^2[(u-\beta)/2]}$$

$$= \frac{|\sin[(u-\beta)/2]|\,\sinh x\,\sqrt{\cosh v - \cosh t_1}}{\cos^2[(u-\beta)/2]\,(\cosh t_2 - \cosh t_1)\sqrt{\cosh x - \cosh t_1}}, \quad (1.6.\text{A}12)$$

$$\frac{\partial t_2}{\partial x} = \frac{\sinh x(\cosh t_2 - \cosh v)}{\cos^2[(u-\beta)/2]\,\sinh t_2(\cosh t_2 - \cosh t_1)} = \frac{\sqrt{\cosh x - \cosh t_1}\sqrt{\cosh t_2 - \cosh v}}{|\cos[(u-\beta)/2]|\,(\cosh t_2 - \cosh t_1)}$$

$$= \frac{|\sin[(u-\beta)/2]|\,\sinh x\,\sqrt{\cosh t_2 - \cosh v}}{\cos^2[(u-\beta)/2]\,(\cosh t_2 - \cosh t_1)\sqrt{\cosh t_2 - \cosh x}}, \quad (1.6.\text{A}13)$$

$$\frac{t_2'}{t_1'}=\frac{\sqrt{\cosh x-\cosh t_1}\sqrt{\cosh t_2-\cosh v}}{\sqrt{\cosh v-\cosh t_1}\sqrt{\cosh t_2-\cosh x}}=\frac{\sinh v(\cosh x-\cosh t_1)}{\sinh x(\cosh v-\cosh t_1)}$$

$$=\frac{\sinh x(\cosh t_2-\cosh v)}{\sinh v(\cosh t_2-\cosh x)}=\frac{\sinh t_1}{\sinh t_2}\sinh^2 v\tan^2[(u-\beta)/2], \qquad (1.6.\text{A}14)$$

Exercises 1

1. Prove the identity $\mathcal{L}(p)\lambda(q,\phi)=\lambda(pq,\phi)$, for $p<1$ and $q<1$.
Hint: use (1.1.5)

2. Evaluate the integral

$$\int_0^{2\pi}\frac{d\phi}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)][r^2+r_0^2-2rr_0\cos(\phi-\psi)]}, \quad \text{for } \rho>\rho_0 \text{ and } r>r_0.$$

Answer: $\dfrac{2\pi(\rho^2r^2-\rho_0^2r_0^2)}{(\rho^2-\rho_0^2)(r^2-r_0^2)[\rho^2r^2+\rho_0^2r_0^2-2\rho\rho_0 rr_0\cos(\phi_0-\psi)]}$.

3. Evaluate the integral

$$\int_0^{2\pi}\frac{d\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]}, \quad \text{for } k<1 \text{ and } k_1<1.$$

Answer:
$$\frac{2\pi}{1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)}\left\{\frac{2k^2}{(1-k^2)^3}\right.$$
$$\left.+\frac{1}{1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)}\left[\frac{k_1^2}{1-k_1^2}+\frac{1-k^4k_1^2}{(1-k^2)^2}\right]\right\}.$$

4. Evaluate the integral

$$\int_0^{2\pi}\frac{e^{i\phi}\,d\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]}, \quad \text{for } k<1 \text{ and } k_1<1.$$

Answer:
$$\frac{2\pi}{[1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)]^2}\left\{\frac{k_1^3e^{i\psi}}{1-k_1^2}\right.$$

$$+\frac{ke^{i\phi_0}[2(1+k^4k_1^2)-kk_1(1+k^2)e^{-i(\psi-\phi_0)}]+k_1e^{i\psi}(1-3k^2)}{(1-k^2)^3}\Bigg\}.$$

5. Evaluate the integral

$$\int_0^{2\pi}\frac{e^{2i\phi}\,d\phi}{[1+k^2-2k\cos(\phi-\phi_0)]^2[1+k_1^2-2k_1\cos(\phi-\psi)]},\quad \text{for } k<1 \text{ and } k_1<1.$$

Answer: $\dfrac{2\pi}{[1+k^2k_1^2-2kk_1\cos(\phi_0-\psi)]^2}\left\{\dfrac{k_1^2e^{2i\psi}}{1-k_1^2}\right.$

$$\left.+\frac{2kk_1e^{-i(\psi-\phi_0)}[e^{2i\psi}(k^4-3k^2+1)-k^2e^{2i\phi_0}]-k^2e^{2i\phi_0}(k^2+k_1^2-3-3k^2k_1^2)}{(1-k^2)^3}\right\}.$$

6. Prove the identity (η is defined by 1.2.2)

$$\frac{d\eta}{dx}=-\frac{\rho^2\rho_0^2-x^4}{x^3\eta}.$$

7. Prove the identity

$$\lambda\left(\frac{x^2}{\rho\rho_0},\phi-\phi_0\right)=-\frac{x\eta}{R^2+\eta^2}\frac{d\eta}{dx},\quad \text{where } R=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0))}.$$

Hint: use the identity: $x^2+\rho^2\rho_0^2/x^2-2\rho\rho_0\cos(\phi-\phi_0)=R^2+\eta^2$, and the result above.

8. Let $\eta=\sqrt{x^2-\rho^2}\sqrt{x^2-\rho_0^2}/x$. Prove the identity

$$\lambda\left(\frac{\rho\rho_0}{x^2},\phi-\phi_0\right)=\frac{x\eta}{R^2+\eta^2}\frac{d\eta}{dx}.$$

9. Prove the identities

$$\sqrt{l_2^2-\rho^2}\sqrt{l_2^2-a^2}=zl_2,\quad \sqrt{a^2-l_1^2}(\rho^2-l_1^2)^{1/2}=zl_1,$$

$$\sqrt{a^2-l_1^2}\sqrt{l_2^2-a^2}=za,\quad \sqrt{l_2^2-\rho^2}(\rho^2-l_1^2)^{1/2}=z\rho.$$

Reminder: l_1 and l_2 are understood as $l_1(a,\rho,z)$ and $l_2(a,\rho,z)$ respectively.
Hint: use (1.2.6)

10. Prove that $g(x)$ is inverse to both l_1 and l_2, namely, prove that $g(l_1)=a$, and $g(l_2)=a$.

11. Prove the identities

$$\frac{\partial l_1}{\partial z} = -\frac{z l_1}{l_2^2 - l_1^2}, \quad \frac{\partial l_2}{\partial z} = \frac{z l_2}{l_2^2 - l_1^2},$$

$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2 - l_1^2} = \frac{\rho(a^2 - l_1^2)}{l_1(l_2^2 - l_1^2)}, \quad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2 - l_1^2} = \frac{\rho(l_2^2 - a^2)}{l_2(l_2^2 - l_1^2)}.$$

Hint: use the properties above.

12. Evaluate the integral

$$\int \frac{dx}{\sqrt{\rho_0^2 - x^2}} \frac{\sqrt{l_2^2(x) - x^2}}{l_2^2(x) - l_1^2(x)} \lambda\left(\frac{l_1(x)x}{l_2(x)\rho_0}, \phi - \phi_0\right)$$

Answer: $-\frac{1}{R_0}\tan^{-1}\frac{\sqrt{\rho_0^2 - x^2}\sqrt{l_2^2(x) - x^2}}{xR_0}$.

Hint: use (1.2.21)

13. Evaluate the integral

$$\int \frac{dx}{\sqrt{x^2 - \rho_0^2}} \frac{(x^2 - l_1^2(x))^{1/2}}{[l_2^2(x) - l_1^2(x)]} \lambda\left(\frac{\rho\rho_0}{l_2^2(x)}, \phi - \phi_0\right)$$

Answer: $\frac{1}{R_0}\tan^{-1}\frac{(x^2 - l_1^2(x))^{1/2}\sqrt{x^2 - \rho_0^2}}{xR_0}$.

Hint: use (1.2.15)

14. Establish the integral representation

$$\frac{2}{\pi}\int_0^{min(\rho_0,\rho)} \frac{\cos[\kappa\sqrt{\rho^2 - x^2}\sqrt{\rho_0^2 - x^2}/x + (\pi\nu/2)]}{[(\rho^2 - x^2)(\rho_0^2 - x^2)]^{(1+\nu)/2}} \lambda\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) x^\nu dx = \frac{e^{-\kappa R}}{R^{1+\nu}}.$$

15. Prove that h in (1.3.35) can be defined by any of the expressions

$$h \equiv h(a) = \frac{\sqrt{a^2 - \rho_0^2}\sqrt{a^2 - l_1^2}}{a} = \frac{z\sqrt{a^2 - \rho_0^2}}{\sqrt{l_2^2 - a^2}}$$

$$= \frac{[l_2^2 - l_1^2(\rho_0)]^{1/2}[l_2^2 - l_2^2(\rho_0)]^{1/2}}{l_2} = \frac{\sqrt{a^2 - \rho_0^2}\sqrt{l_2^2 - \rho^2}}{l_2}.$$

16. Prove that j in (1.4.21) can be defined in several equivalent ways:

$$j \equiv j(a) = \frac{\sqrt{\rho_0^2 - a^2}\sqrt{l_2^2 - a^2}}{a} = \frac{\sqrt{\rho_0^2 - a^2}\sqrt{\rho^2 - l_1^2}}{l_1}$$

$$= \frac{z\sqrt{\rho_0^2 - a^2}}{\sqrt{a^2 - l_1^2}} = \frac{[l_1^2(\rho_0) - l_1^2]^{1/2}[l_2^2(\rho_0) - l_1^2]^{1/2}}{l_1}.$$

17. A circular conducting disc is kept at the potential v_0. Find the potential function V.

Answer: $V(\rho,z) = \frac{2}{\pi} v_0 \sin^{-1}(l_1/\rho) = \frac{2}{\pi} v_0 \sin^{-1}(\frac{a}{l_2})$.

Hint: use formula (1.3.23)

18. Subject to the conditions of the previous problem, find the charge distribution σ by using formulae (1.3.15) and (1.3.5). Prove that in both cases the result is the same.

Answer: $\sigma = \frac{v_0}{\pi^2(a^2 - \rho^2)^{1/2}}$.

19. Solve problems 17 and 18 for $v = v_1\rho\cos\phi$, $v_1 = \text{const}$.

Answer: $V(\rho,\phi,z) = \frac{2}{\pi}\rho v_1 \cos\phi \left[\sin^{-1}(\frac{a}{l_2}) - \frac{a}{l_2}\sqrt{1 - (a/l_2)^2}\right]$,

$$\sigma(\rho,\phi) = \frac{2v_1\rho\cos\phi}{\pi^2(a^2 - \rho^2)^{1/2}}.$$

20. A uniform charge density $\sigma = \sigma_0 = \text{const}$ is prescribed over a circular disc of radius a, and potential $v = 0$ for $\rho > a$. Find the potential function.

Answer: $V(\rho,z) = 4\sigma_0\left[\sqrt{a^2 - l_1^2} - z\sin^{-1}(\frac{a}{l_2})\right]$.

21. Solve the previous problem for the case where $\sigma = \sigma_1\rho\cos\phi$, $\sigma_1 = \text{const}$.

Answer: $V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[\sqrt{a^2 - l_1^2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}(\frac{a}{l_2})\right].$

22. The potential function is given by the expression

$$V(\rho,\phi,z) = \frac{8}{3}\sigma_1\rho\cos\phi\left[\sqrt{a^2 - l_1^2}\left(\frac{3}{2} - \frac{a^2}{2l_2^2}\right) - \frac{3}{2}z\sin^{-1}(\frac{a}{l_2})\right].$$

Find the charge distribution on the plane $z = 0$.
Answer: $\sigma = \sigma_1\rho\cos\phi$, for $\rho \le a$;

$$\sigma = -\frac{4}{3\pi}\sigma_1\rho\cos\phi\left[\frac{a}{\sqrt{\rho^2 - a^2}}\left(\frac{3}{2} - \frac{a^2}{2\rho^2}\right) - \frac{3}{2}\sin^{-1}\left(\frac{a}{\rho}\right)\right]. \quad , \text{ for } \rho > a.$$

Hint: use the second formula of (1.3.5).

23. The following boundary conditions are prescribed at $z = 0$

$$V = v_0/\rho, \text{ for } \rho \ge a, 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{2v_0}{\pi(\rho^2 + z^2)^{1/2}}\sin^{-1}\left(\frac{(\rho^2 + z^2)^{1/2}}{l_2}\right),$

$$\sigma(\rho,\phi) = \frac{v_0 a}{\pi^2\rho^2\sqrt{\rho^2 - a^2}}.$$

The total charge is equal v_0.

24. The following boundary conditions are prescribed at $z = 0$

$$V = v_0/\rho^2, \text{ for } \rho \ge a, 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a, 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{v_0}{\rho^2 + z^2}\left[1 - \frac{\sqrt{l_2^2 - \rho^2}}{l_2}\right.$

$$+\frac{z}{(\rho^2+z^2)^{1/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]}\Bigg],$$

$$\sigma(\rho,\phi)=\frac{v_0}{2\pi\rho^2}\left[\frac{1}{\sqrt{\rho^2-a^2}}-\frac{1}{\rho}\ln\frac{\rho+\sqrt{\rho^2-a^2}}{a}\right].$$

25. The following boundary conditions are prescribed at $z=0$

$$V=v_0/\rho^3,\ \text{ for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z)=\dfrac{4v_0}{(\rho^2+z^2)^2}\left[\dfrac{z^2}{\sqrt{a^2-l_1^2}}-\dfrac{l_1^2\sqrt{a^2-l_1^2}}{2a^2}\right.$

$$\left.+\frac{\rho^2-2z^2}{2(\rho^2+z^2)^{1/2}}\sin^{-1}\left(\frac{(\rho^2+z^2)^{1/2}}{l_2}\right)\right].$$

Note that the potential at the coordinate origin is finite, namely, $V(0,0,0)=4v_0/(3\pi a^3)$.

$$\sigma(\rho,\phi)=\frac{2v_0(2a^2-\rho^2)}{\pi^2 a\rho^4\sqrt{\rho^2-a^2}}.$$

26. The following boundary conditions are prescribed at $z=0$

$$V=v_0/\rho^4,\ \text{ for } \rho\ge a,\ 0\le\phi<2\pi;$$

$$\frac{\partial V}{\partial z}=0,\quad \text{for } \rho<a,\ 0\le\phi<2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z)=\dfrac{3v_0}{2(\rho^2+z^2)^2}\left\{\dfrac{\rho^2-z^2}{\rho^2+z^2}\left[1-\dfrac{\sqrt{l_2^2-\rho^2}}{l_2}\right]\right.$

$$-\frac{1}{3}\left[1-\left(\frac{\sqrt{l_2^2-\rho^2}}{l_2}\right)^3\right]+\frac{z}{2(\rho^2+z^2)}\left[\frac{l_2^2}{a^2}\sqrt{l_2^2-a^2}-z\right]$$

$$-\frac{z(2z^2-3\rho^2)}{2(\rho^2+z^2)^{3/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]},$$

The last expression simplifies at $z = 0$:

$$V(\rho,\phi,0) = \frac{v_0}{\rho^4}\left[1-\frac{\sqrt{a^2-\rho^2}}{a}-\frac{\rho^2\sqrt{a^2-\rho^2}}{2a^3}\right],\ \text{for } \rho \le a;$$

and $V(\rho,\phi,0) = v_0/\rho^4$, for $\rho > a$. Note that the potential at the coordinate origin is finite, namely, $V(0,0,0) = 3v_0/(8a^4)$.

$$\sigma(\rho,\phi) = \frac{3v_0}{8\pi\rho^4}\left\{\frac{3a^2-\rho^2}{a^2\sqrt{\rho^2-a^2}}-\frac{3}{\rho}\ln\left[\frac{\rho+\sqrt{\rho^2-a^2}}{a}\right]\right\}.$$

27. The following boundary conditions are prescribed at $z = 0$

$$V = (v_1/\rho)\, e^{i\phi},\ \text{for } \rho \ge a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0,\quad \text{for } \rho < a,\ 0 \le \phi < 2\pi.$$

Find the potential function.

Answer: $V(\rho,\phi,z) = \dfrac{v_1}{\rho}\, e^{i\phi}\left[1-\dfrac{\sqrt{a^2-l_1^2}}{a}\right]$

28. The following boundary conditions are prescribed at $z = 0$

$$V = (v_2/\rho^2)\, e^{2i\phi},\ \text{for } \rho \ge a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0,\quad \text{for } \rho < a,\ 0 \le \phi < 2\pi.$$

Find the potential function.

Answer: $V(\rho,\phi,z) = \dfrac{3v_2}{2\rho^2}\, e^{2i\phi}\left\{\left[1-\dfrac{\sqrt{a^2-l_1^2}}{a}\right]-\dfrac{1}{3}\left[1-\left(\dfrac{\sqrt{a^2-l_1^2}}{a}\right)^3\right]\right\}.$

29. The following boundary conditions are prescribed at $z = 0$

$$V = (v_3/\rho^3)\, e^{3i\phi},\ \text{for } \rho \ge a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = 0, \quad \text{for } \rho < a,\, 0 \le \phi < 2\pi.$$

Find the potential function.

Answer: $V(\rho,\phi,z) = \frac{15 v_3}{4\rho^3} e^{3i\phi} \left\{ \frac{1}{2}\left[1 - \frac{\sqrt{a^2 - l_1^2}}{a}\right] \right.$

$$\left. - \frac{1}{3}\left[1 - \left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^3\right] + \frac{1}{10}\left[1 - \left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^5\right]\right\}.$$

30. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \ \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^2,\ \sigma_0 = \text{const}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{2\pi\sigma_0}{(\rho^2+z^2)^{1/2}} \ln \frac{l_2[(\rho^2+z^2)^{1/2} + \sqrt{l_2^2 - a^2}]}{a[(\rho^2+z^2)^{1/2} + z]}$,

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^2}\Re\left[1 - \frac{a}{\sqrt{a^2-\rho^2}}\right], \quad \sigma(0,0) = -\frac{\sigma_0}{2a^2}.$$

31. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \ \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^3,\ \sigma_0 = \text{const}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{4\sigma_0}{\rho^2+z^2}\left[\frac{\sqrt{l_2^2 - a^2}}{a} - \frac{z}{(\rho^2+z^2)^{1/2}} \sin^{-1}\left(\frac{(\rho^2+z^2)^{1/2}}{l_2}\right)\right]$,

$$\sigma(\rho,\phi) = \frac{2\sigma_0}{\pi\rho^2}\left[\frac{1}{\rho}\sin^{-1}\left(\frac{\rho}{a}\right) - \frac{1}{\sqrt{a^2-\rho^2}}\right], \text{ for } \rho < a;\ \sigma(0,0) = -2\sigma_0/(3\pi a^3).$$

32. Let the following boundary conditions be prescribed at $z = 0$:

$$V = 0, \ \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi\sigma_0/\rho^4,\, \sigma_0 = \text{const}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \dfrac{\pi\sigma_0}{2(\rho^2+z^2)^2}\left\{\dfrac{2z}{a}\sqrt{a^2-l_1^2} \;\; -3z + \dfrac{l_2^2}{a^2}\sqrt{l_2^2-a^2}\right.$

$$\left.\frac{\rho^2-2z^2}{(\rho^2+z^2)^{1/2}}\ln\frac{l_2[(\rho^2+z^2)^{1/2}+\sqrt{l_2^2-a^2}]}{a[(\rho^2+z^2)^{1/2}+z]}\right\},$$

$$\sigma(\rho,\phi) = \frac{\sigma_0}{\rho^4}\Re\left[1 - \frac{2a^2-\rho^2}{2a\sqrt{a^2-\rho^2}}\right], \quad \sigma(0,0) = -\frac{\sigma_0}{8a^4}.$$

33. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_1/\rho)e^{i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = 2\pi(\sigma_1/\rho)\, e^{i\phi}[\sqrt{l_2^2-a^2} - z]$,

$$\sigma(\rho,\phi) = (\sigma_1/\rho)e^{i\phi}\Re[1 - a/\sqrt{a^2-\rho^2}].$$

34. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_2/\rho^2)e^{2i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \pi(\sigma_2/\rho^2)\, e^{2i\phi}[\sqrt{l_2^2-a^2} - 2z + z\sqrt{a^2-l_1^2}/a]$,

$$\sigma(\rho,\phi) = (\sigma_2/\rho^2)e^{i\phi}\Re\left[1 - \frac{2a^2-\rho^2}{2a\sqrt{a^2-\rho^2}}\right].$$

35. Consider the boundary conditions on the plane $z = 0$:

$$V = 0, \quad \text{for } \rho \le a,\ 0 \le \phi < 2\pi;$$

$$\frac{\partial V}{\partial z} = -2\pi(\sigma_3/\rho^3)e^{3i\phi}, \quad \text{for } \rho > a,\ 0 \le \phi < 2\pi.$$

Find the potential function and the charge distribution.

Answer: $V(\rho,\phi,z) = \frac{3\pi\sigma_3}{4\rho^3}e^{3i\phi}\Big[\sqrt{l_2^2 - a^2} - \frac{8}{3}z$

$$+ 2\frac{z}{a}\sqrt{a^2 - l_1^2} - \frac{1}{3}z\left(\frac{\sqrt{a^2 - l_1^2}}{a}\right)^3\Big],$$

$$\sigma(\rho,\phi) = \frac{\sigma_3}{\rho^3}e^{3i\phi}\Re\left[1 - \frac{3a}{8\sqrt{a^2-\rho^2}} - \frac{3\sqrt{a^2-\rho^2}}{4a} + \frac{(a^2-\rho^2)^{3/2}}{8a^3}\right].$$

36. Prove that the total charge Q_T in Problem 2 (1.4.26) can be expressed directly in terms of the given charge density σ as

$$Q_T = \frac{2}{\pi}\int_0^{2\pi}\int_a^{\infty} \sigma(\rho,\phi)\cos^{-1}\left(\frac{a}{\rho}\right)\rho d\rho d\phi.$$

Hint: integrate (1.4.29).

37. Solve the problem above in the case when $\sigma = \sigma_0/\rho^n$.

Answer: $Q_T = \frac{2\sigma_0\sqrt{\pi}\,\Gamma[(n-1)/2]}{(n-2)\,\Gamma(n/2)a^{n-2}}.$

38. Prove that parameter χ given in (1.5.35) can be defined by an alternative expression

$$\chi = \frac{\sqrt{2}\sqrt{m_2^2(\alpha) - \sin^2(\alpha/2)\,m_2^2(\pi)}\sqrt{\cos\alpha - \cos\theta_0}}{\sin\alpha},$$

with m_1 and m_2 defined by (1.5.1).

39. Prove that in the limiting case $r \to a$ formula (1.5.34) takes the form

$$V(a,\theta,\phi) = \frac{\sqrt{\cos\theta - \cos\alpha}}{2\pi^2}\int_0^{2\pi} d\phi_0$$

$$\times \int_{\alpha}^{\pi} \frac{v(\theta_0, \phi_0) \sin\theta_0 \, d\theta_0}{\sqrt{\cos\alpha - \cos\theta_0}\,[1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\phi - \phi_0)]}.$$

40. Prove that in the case of axial symmetry formula in Example 39 above simplifies as follows:

$$V(a,\theta) = \frac{\sqrt{\cos\theta - \cos\alpha}}{\pi} \int_{\alpha}^{\pi} \frac{v(\theta_0) \sin\theta_0 \, d\theta_0}{\sqrt{\cos\alpha - \cos\theta_0}\,(\cos\theta - \cos\theta_0)}.$$

41. Find the charge density distribution on a spherical cap $\alpha \le \theta \le \pi$ kept at a constant potential v_0.

Answer: $\sigma(\theta) = \frac{v_0}{2\pi^2 a}\left[\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha - \cos\theta}} + \tan^{-1}\frac{\sqrt{\cos\alpha - \cos\theta}}{\sqrt{1-\cos\alpha}}\right].$

42. Consider a mixed boundary value problem for a sphere, subject to the boundary conditions at $r = a$

$$\sigma(\theta,\phi) = q_0 \cos\theta, \text{ for } 0 \le \theta < \alpha, \ 0 \le \phi < 2\pi;$$

$$V(a,\theta,\phi) = v_0 \sin\theta\cos\phi, \text{ for } \alpha \le \theta \le \pi, \ 0 \le \phi < 2\pi,$$

with q_0 and v_0 being constant. Find: $a)$ charge density distribution for $\alpha \le \theta \le \pi$; $b)$ the total charge on the sphere; $c)$ the potential in space and on the sphere for $\theta < \alpha$.

Answers:

a) $$\sigma(\theta,\phi) = \frac{2}{\pi} q_0 \cos\theta\left[-\left(1 + \frac{1-\cos\alpha}{3\cos\theta}\right)\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha - \cos\theta}} + \tan^{-1}\frac{\sqrt{1-\cos\alpha}}{\sqrt{\cos\alpha - \cos\theta}}\right]$$

$$+ \frac{v_0 \sin\theta\cos\phi}{2\pi^2 a}\left[\mu_1 \frac{3-\mu_1^2}{\sqrt{1-\mu_1^2}} + 3\cos^{-1}\mu_1\right],$$

with $\mu_1 = \sin(\alpha/2)/\sin(\theta/2)$, for $\theta > \alpha$;

b) $$Q = \frac{4}{3} q_0 a^2 \sin\alpha(1 - \cos\alpha);$$

c) $$V(r,\theta,\phi)=\frac{4q_0}{3r^2}\Big\{[(r^3+a^3)\cos^{-1}A_1-|r^3-a^3|\cos^{-1}A_2]\cos\theta$$

$$+\,2arB\left(A_1^2\cos^2\frac{\theta}{2}-A_2^2\sin^2\frac{\theta}{2}\right)\Big\}$$

$$+\frac{v_0\sin\theta\cos\phi}{\pi a r^2}[(r^3+a^3)\sin^{-1}A_1-|r^3-a^3|\sin^{-1}A_2-arB(A_1^2+A_2^2)],$$

where

$$A_1=\frac{(r+a)\cos(\alpha/2)}{\sqrt{m_2^2(\alpha)+4ar\cos^2(\theta/2)\cos^2(\alpha/2)}},$$

$$A_2=\frac{|r-a|\cos(\alpha/2)}{\sqrt{m_2^2(\alpha)-4ar\sin^2(\theta/2)\cos^2(\alpha/2)}},$$

$$B=\sqrt{m_2^2(\alpha)/\cos^2(\alpha/2)-m_2^2(0)}.$$

On the surface of the sphere

$$V(a,\theta,\phi)=\frac{8}{3}aq_0\left[\cos\theta\cos^{-1}\mu_2+2\cos\frac{\alpha}{2}\cos\frac{\theta}{2}\sqrt{1-\mu_2^2}\right]$$

$$+\frac{2}{\pi}v_0\sin\theta\cos\phi[\sin^{-1}\mu_2-\mu_2\sqrt{1-\mu_2^2}],$$

with $\mu_2=\cos(\alpha/2)/\cos(\theta/2)$, for $\theta<\alpha$

43. Prove that χ_0 in (1.5.59) can also be presented as

$$\chi_0=\frac{\sqrt{2}\sqrt{m_{20}^2(\alpha)-\sin^2(\alpha/2)\,m_{20}^2(\pi)}\sqrt{\cos\alpha-\cos\theta}}{\sin\alpha}.$$

44. Prove that $\eta_{1,2}$ in (1.5.68) can also be presented as

$$\eta_{1,2}(x)=\frac{T(x)}{2a}\pm\frac{(r^2-a^2)(r_0^2-a^2)}{2aT(x)},$$

with

$$T(x) = 2\sqrt{m_2^2(x) - \cos^2(x/2)\, m_2^2(0)}\,\sqrt{m_{20}^2(x) - \cos^2(x/2)\, m_{20}^2(0)}/\sin x.$$

45. Consider a boundary value problem for a charged sphere of radius a, with discontinuous boundary conditions at $r=a$

$$V(a,\theta,\phi) = v_0 = \text{const.}, \quad \text{for} \quad 0 \le \theta < \alpha,\ 0 \le \phi < 2\pi;$$

$$V(a,\theta,\phi) = 0, \quad \text{for} \quad \alpha \le \theta < \pi,\ 0 \le \phi < 2\pi;$$

Find a) charge density distribution, b) the total charge, c) potential due to the charged sphere.
Answers:
a)

$$q(\theta) = \frac{v_0}{2\pi^2 a}\left\{\frac{\Pi(-\kappa_1^2,\kappa_2) - K(\kappa_2)}{\sin(\alpha/2)\cos(\theta/2)} + \frac{2\sin(\alpha/2)\cos(\theta/2)}{\cos\theta - \cos\alpha}E(\kappa_2)\right\}, \quad \text{for} \quad \theta < \alpha;$$

$$q(\theta) = \frac{v_0}{2\pi^2 a}\left\{\frac{\Pi(-\kappa_0^2,\kappa_2^{-1})}{\cos(\alpha/2)\sin(\theta/2)} + \frac{2\cos(\alpha/2)\sin(\theta/2)}{\cos\theta - \cos\alpha}E(\kappa_2^{-1})\right\},$$

$$\text{for} \quad \pi > \theta > \alpha,$$

with $\kappa_0 = \tan(\alpha/2)$, $\kappa_1 = \tan(\theta/2)$, $\kappa_2 = \kappa_1/\kappa_0$; K, E, Π are the complete elliptic integrals of the first, second and third kind respectively.

b) $\quad Q = v_0 a \sin^2(\alpha/2);$

c) $$V(r,\theta,\phi) = \frac{2v_0}{\pi r}\frac{|r-a|}{\sqrt{1-c_1}\sqrt{1+c_3}}[\Pi(\kappa_3^2,\kappa_4) - \Pi(-\kappa_5^2,\kappa_4)],$$

where

$$c_1 = \cos t_2(\alpha), \quad c_3 = \cos t_1(\alpha), \quad \kappa_3 = \tan(\alpha/2)/\tan[t_2(\alpha)/2],$$

$$\kappa_4 = \tan[t_1(\alpha)/2]/\tan[t_2(\alpha)/2], \quad \kappa_5 = \tan[t_1(\alpha)/2].$$

Hint: for details see (Fabrikant, 1987a,e)

46. Prove the identities

$$[\cos\theta - \cos t_2(x)][\cos t_1(x) - \cos\theta] = \left(\frac{a-r}{a+r}\right)^2 \sin^2\theta,$$

$$[\cos x - \cos t_2(x)][\cos t_1(x) - \cos x] = \left(\frac{a-r}{a+r}\right)^2 \sin^2 x,$$

$$\frac{\partial t_2(x)}{\partial x} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos t_1(x) - \cos\theta]}\frac{\partial t_1(x)}{\partial x} = \frac{\sin x\,[\cos\theta - \cos t_2(x)]}{\sin\theta\,[\cos x - \cos t_2(x)]}\frac{\partial t_1(x)}{\partial x},$$

$$\sin t_1(x)\sin t_2(x) = \frac{4ar}{(a+r)^2}\sin x\sin\theta,$$

$$\frac{\sin t_1(x)}{\sin t_2(x)} = \frac{\sin\theta\,[\cos t_1(x) - \cos x]}{\sin x\,[\cos\theta - \cos t_2(x)]} = \frac{\sin x\,[\cos t_1(x) - \cos\theta]}{\sin\theta\,[\cos x - \cos t_2(x)]}$$

$$\cos t_1(x)\cos t_2(x) = \frac{4ar\cos\theta\cos x - (r-a)^2}{(r+a)^2},$$

$$\cos t_1(x) + \cos t_2(x) = \frac{4ar}{(r+a)^2}(\cos x + \cos\theta),$$

where $t_1(x)$ and $t_2(x)$ are defined by (1.5.5)