

Tests of time-invariance based on quantile indicators

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Abstract

Quantiles provide a comprehensive description of the properties of a variable and tracking changes in quantiles over time using signal extraction methods can be informative. It is shown here how stationarity tests can be generalized to test the null hypothesis that a particular quantile is constant over time by using weighted indicators. Corresponding tests based on expectiles are also proposed; these might be expected to be more powerful for distributions that are not heavy-tailed. Tests for changing dispersion and asymmetry may be based on contrasts between particular quantiles or expectiles. We report Monte Carlo experiments investigating the effectiveness of the proposed tests and then move on to consider how to test for relative time invariance, based on residuals from fitting a time-varying level or trend. Empirical examples, using stock returns and U.S. inflation, provide an indication of the practical importance of the tests.

KEYWORDS: Dispersion; expectiles; quantiles; skewness; stationarity tests; stochastic volatility, value at risk.

JEL classification: C12, C22

1 Introduction

Tests for serial correlation and stationarity are mostly applied to the level of a time series. For financial time series, movements in the variance are

often of interest so serial correlation and stationarity tests are applied to squares or absolute values (possibly after first removing the mean). However, the distribution of a variable may display changes over time that are not captured by the level or variance. For example, there may be movements in skewness or the tails. Quantiles provide a more comprehensive description of the properties of a variable and tracking changes in quantiles over time can be informative. Indeed, lower tail quantiles are used to define Value at Risk (VaR), a measure that is of considerable importance in financial regulation; see, for example, Engle and Manganelli (2004). In a recent paper, De Rossi and Harvey (2006) explain how time-varying quantiles may be estimated using state space signal extraction techniques. Figure 1, taken from that paper, shows quantiles, modeled as random walks, fitted to General Motors daily stock returns.¹ The present paper generalizes serial correlation and stationarity tests so as to test the null hypothesis that a particular quantile is constant over time. These tests are based on statistics constructed from quantile residuals coded as weighted indicators. The main concern is with stationarity tests, where the test statistics can be shown to have the usual (asymptotic) Cramér von Mises distribution under the null hypothesis. The proof is a generalization of that given by De Jong, Amsler and Schmidt (2007) for the indicator test based on the median.

De Jong, Amsler and Schmidt (2007) show that for most distributions with finite variance the indicator-based stationarity test is less powerful than the standard test based on residuals from the mean. We therefore seek to generalize the standard stationarity test to provide alternatives to the indicator tests for the constancy of quantiles. We do this by first noting that expectiles have been proposed as an alternative - or complement - to quantiles; see, for example, Newey and Powell (1987), Efron (1991) and, in a time series context, De Rossi and Harvey (2007). We then define residuals based on expectiles and show that they can be used to construct stationarity tests with the usual asymptotic distribution. By matching up an expectile with

¹The stock returns data used as illustrations are taken from Engle and Manganelli (2004). Their sample runs from April 7th, 1986, to April 7th, 1999, but the figure uses only the first 2000. The large (absolute) values near the beginning of figure 1 are associated with the great crash of 1987.

The histogram of the series from observation 501 to 2000 (avoiding the 1987 crash) shows heavy tails but no clear evidence of skewness. The excess kurtosis is 1.547 and the associated test statistic, distributed as χ_1^2 under normality, is 149.5. On the other hand, skewness is 0.039 with an associated test statistic of only 0.37.

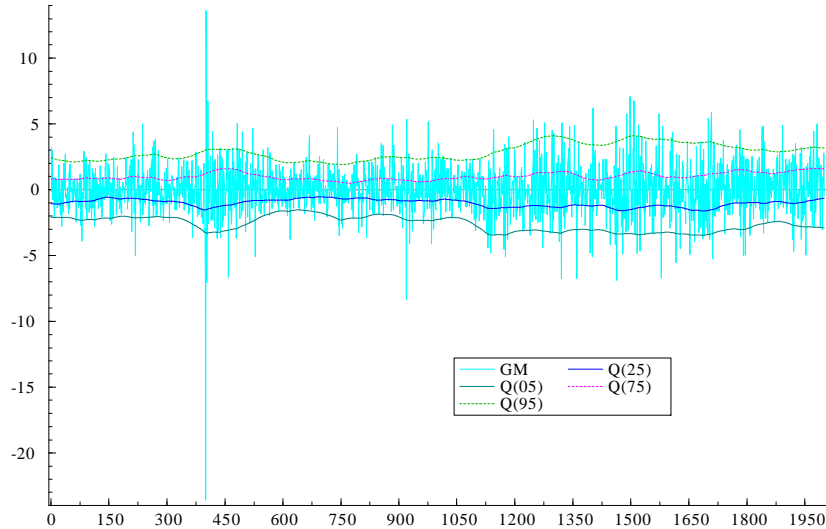


Figure 1: Quantiles fitted to GM returns

a quantile, an expectile based test can be used to test the constancy of a specific quantile. We investigate the asymptotic properties of such tests and carry out Monte Carlo experiments to compare their performance with that of quantile indicator tests. We also propose ways in which they may be made robust to heavy tails, a property that is a feature of quantile indicator tests.

If the level, as represented by the median, is constant, we can focus on other quantiles and contrasts between them. Stationarity tests built around contrasts allow inferences to be made about possible time variation in dispersion and asymmetry. Multivariate tests can capture movements in different parts of the distribution. We report Monte Carlo experiments investigating the effectiveness of various new tests and comparing them with standard procedures and with the Box-Ljung tests based on quantile indicators that are proposed in Linton and Whang (2007). We then move on to consider how to test for relative time invariance, in other words whether the distribution around a possibly nonstationary level is constant over time. The tests are based on residuals from fitting a time-varying mean or median. We discuss the implications of working with such residuals and investigate test performance by simulations. Monthly figures on the US rate of inflation are used

as an illustration.

2 Quantile stability tests

Let $\xi(\tau)$ denote the τ -th quantile. The probability that an observation is less than $\xi(\tau)$ is τ , where $0 < \tau < 1$. Given a set of T observations, $y_t, t = 1, \dots, T$, the sample quantile, $\tilde{\xi}(\tau)$, can be obtained by sorting the observations in ascending order. However, it is also given as the solution to minimizing

$$S_\tau = \sum_t (\tau - I(y_t - \xi(\tau) < 0))(y_t - \xi(\tau))$$

with respect to $\xi(\tau)$, where $I(\cdot)$ is the indicator function. Differentiating (minus) S_τ at all points where this is possible gives

$$\sum_{t=1}^T IQ(y_t - \xi(\tau)),$$

where

$$IQ(y_t - \xi(\tau)) = \begin{cases} \tau - 1, & \text{if } y_t < \xi(\tau) \\ \tau, & \text{if } y_t > \xi(\tau) \end{cases} \quad (1)$$

defines the *quantile indicator function*. Clearly, the quantile indicator function is not continuous at 0 and thus the minimization problem cannot be solved by the first order conditions. A sample τ -quantile is a point such that the number of observations smaller, that is $y_t < \tilde{\xi}(\tau)$, is no more than $[T\tau]$ while the number greater is no more than $[T(1 - \tau)]$, where $[x]$ denotes the integer less than or equal to x .

When evaluated at the sample quantile, we call (1) the τ -*quantic*, i.e. $IQ(y_t - \tilde{\xi}(\tau))$, $t = 1, \dots, T$. The quantics sum to zero if, when an observation is equal to $\xi(\tau)$, the corresponding quantic is assigned a value between $\tau - 1$ and τ to ensure that this is the case. If $T\tau$ is an integer, the sample quantile will normally lie at some point between observations, in which case the mean of the quantics is automatically zero, while the sample variance is equal to $\tau(1 - \tau)$.

The properties of the quantile indicator (1) are summarised by the following proposition, that can be proved by standard arguments.

Proposition 1 (A) If $\xi(\tau)$ is the unique population τ -quantile and the probability density function is continuous positive in the neighbourhood of $\xi(\tau)$, then the τ -th quantile indicator has a mean of zero and a variance of $\tau(1-\tau)$. (B) If y_t are strictly stationary random variables then the quantile indicator is both strictly stationary and covariance stationary since its moments always exist.

Covariance stationarity of y_t also induces covariance stationarity of the quantile indicator. However, transforming to indicators tends to weaken the correlation structure. Any autocorrelation for the τ -th quantile indicator is related to the corresponding autocorrelation of y_t , denoted ρ_y , by the formula

$$\rho_\tau(\rho_y) = \frac{\tau L(\xi(\tau); \rho_y) + (1-\tau)L(-\xi(\tau); \rho_y)}{\tau(1-\tau)} - 1 \quad (2)$$

where $L(\xi(\tau); \rho) = \Pr(y_t > \xi(\tau), y_{t-k} > \xi(\tau); \rho)$ with k any integer; see appendix A. Note that $\rho_\tau(0) = 0$, as it should be. The left-hand panel of Table 1 shows how the correlations of the quantile indicators vary with ρ_y and τ for a Gaussian process²; the figures are obtained by simulation. The correlation is significantly weaker as we move towards the tails and there is an interesting asymmetry in that negative correlations weaken much more than positive ones.

De Rossi and Harvey (2006) have recently proposed a signal extraction procedure based on the model

$$y_t = \xi_t(\tau) + \varepsilon_t(\tau), \quad t = 1, \dots, T, \quad (3)$$

where the disturbances $\varepsilon_t(\tau)$ are independently and identically distributed with $\Pr(\varepsilon_t(\tau) < 0) = \tau$ for $0 < \tau < 1$. The assumption that $\xi_t(\tau)$ follows a stochastic process, such as a random walk,

$$\xi_t(\tau) = \xi_{t-1}(\tau) + \eta_t(\tau), \quad \eta_t(\tau) \sim NID(0, \sigma_{\eta(\tau)}^2), \quad (4)$$

where $\eta_t(\tau)$ is normally and independently distributed with mean zero and variance $\sigma_{\eta(\tau)}^2$, and is independent of $\varepsilon_t(\tau)$, is essentially a device for inducing local weighting of the observations.

²For a Gaussian process, the formula for the median simplifies to $\rho_{0.5} = 2\pi^{-1} \arcsin(\rho_y)$ since $L(0; \rho) = 0.25 + (1/2\pi) \arcsin(\rho)$; cf. Abramowitz and Stegun (1964, p 936-9),

A test of the null hypothesis that a quantile is constant may be based on the quantics, $IQ(y_t - \tilde{\xi}(\tau))$, $t = \dots, T$. If the alternative hypothesis is that the quantile follows a random walk, as in (4), a modified version of the basic stationarity test of Nyblom and Mäkeläinen (1983) is appropriate. The test statistic of Nyblom and Mäkeläinen uses residuals from a sample mean and its asymptotic distribution is a Cramér-von Mises (CvM) distribution; the 1%, 5% and 10% critical values are 0.743, 0.461 and 0.347 respectively. Nyblom and Harvey (2001) show that the test has high power against an integrated random walk while Harvey and Streibel (1998) note that it also has a locally best invariant interpretation as a test of constancy against a highly persistent stationary AR(1) process. This makes a modified version of the test entirely appropriate for the kind of situation we have in mind for time-varying quantiles; see our figure 1 and also figure 9 in Linton and Whang (2007).

Allowance for a weak dependence structure in the data can be made by using a non-parametric estimator of the long run variance of the quantile indicator (denoted as $\sigma_L^2(Q; \tau)$), as in the KPSS test of Kwiatkowski et al. (1992) that generalizes the basic setup of Nyblom and Mäkeläinen. The test statistic for the null hypothesis of constant τ -th quantile is then

$$\eta_\tau(Q) = \frac{1}{T^2 \hat{\sigma}_L^2(Q; \tau)} \sum_{t=1}^T \left(\sum_{i=1}^t IQ(y_i - \tilde{\xi}(\tau)) \right)^2, \quad (5)$$

where $\hat{\sigma}_L^2(Q; \tau) = \hat{\gamma}_\tau(Q; 0) + 2 \sum_{j=1}^m k(j/m) \hat{\gamma}_\tau(Q; j)$, $k(j/m)$ is a weighting function, such as the Bartlett window, $1 - |j|/(m+1)$, and $\hat{\gamma}_\tau(Q; j)$ is the sample autocovariance of the quantics at lag j . The case of $\tau = 0.5$ was dealt with in De Jong, Amsler and Schmidt (2007) - hereafter DAS - where the aim was to obtain a stationarity test robust to heavy tail distributions. The statistic (5), on the other hand, can be applied to any quantile and so it allows us to detect time variation in different parts of the distribution.

Proposition 2 *Under the assumptions of DAS (2007, p.314) with median replace by τ -quantile, $\eta_\tau(Q) \xrightarrow{d} \int_0^1 (W(r) - rW(1))^2 dr \equiv CvM$, where $W(r)$ is a standard Brownian motion.*

The main assumptions required to prove the proposition are strict stationarity, weak dependence in the form of α -mixing, and continuity of the

density function at the τ -quantile. The proof is an extension of the one provided in DAS for the case $\tau = 0$; see the working paper, Busetti and Harvey (2007), for details. Note that the invariance principle for the quantic holds without moment assumptions on y_t . This is important for applying the tests to financial time series where distributions are typically heavy tailed.

Remark 3 *If, in addition, the y_t are independent and identically distributed, the same limiting distribution applies when the test statistic (5) is modified by replacing $\widehat{\sigma}_L^2(Q; \tau)$ with $\tau(1 - \tau)$, the variance of the quantic.*

A joint test to see if a group of N quantiles show evidence of changing over time can be based on a generalisation of (5), namely

$$\eta_\tau(Q; N) = T^{-2} \sum_{t=1}^T \left[\sum_{i=1}^t \mathbf{IQ}_i \right]' \widehat{\Omega}^{-1} \left[\sum_{i=1}^t \mathbf{IQ}_i \right], \quad (6)$$

where the j -th element of the $N \times 1$ vector \mathbf{IQ}_t is $IQ(y_t - \xi(\tau_j))$ and $\widehat{\Omega}$ is the corresponding non parametric estimator of the $N \times N$ long-run covariance matrix of \mathbf{IQ}_t . Under IID observations $\widehat{\Omega}$ can be replaced by the matrix with jk -th element equal to $\tau_j(1 - \tau_k)$ for $\tau_k > \tau_j$; see appendix B. Under the null hypothesis of strict stationarity (with the regularity conditions of assumption 1 holding for all quantiles $\xi(\tau_N), \dots, \xi(\tau_1)$), the limiting distribution of $\eta_\tau(Q; N)$ is Cramér-von Mises with N degrees of freedom, denoted $CvM(N)$. The proof is straightforward given the result for tests based on a single quantile. For example, if the median is assumed to be constant, as might be the case with stock returns, a test on the 5,25,75 and 95 % quantiles might be regarded as a general test for a change in the distribution over time.

3 Tests based on expectiles

Expectiles are measures of location similar to quantiles but they are determined by tail expectations rather than tail probabilities. If $F(y)$ is the distribution function of a random variable y with finite mean, setting

$$(1 - \omega) \int_{-\infty}^{\mu(\omega)} (y - \mu(\omega)) dF(y) + \omega \int_{\mu(\omega)}^{\infty} (y - \mu(\omega)) dF(y), \quad 0 < \omega < 1 \quad (7)$$

to zero gives a unique solution for the ω -expectile, denoted $\mu(\omega)$; see Newey and Powell (1987, theorem 1). The sample expectile, $\tilde{\mu}(\omega)$, is obtained by minimizing the asymmetric least squares function

$$S_\omega = \sum_t |\omega - I(y_t - \mu(\omega) < 0)| (y_t - \mu(\omega))^2, \quad 0 < \omega < 1$$

with respect to $\mu(\omega)$. Differentiating S_ω and dividing by minus two gives

$$\sum_{t=1}^T IE(y_t - \mu(\omega)) \tag{8}$$

where

$$IE(y_t - \mu(\omega)) = |\omega - I(y_t - \mu(\omega) < 0)| (y_t - \mu(\omega)), \quad t = 1, \dots, T. \tag{9}$$

There is no problem with defining $IE(0)$: since (9) is continuous, $IE(0) = 0$. The sample expectile, $\tilde{\mu}(\omega)$, is the value of $\mu(\omega)$ that makes (8) equal to zero. Setting $\omega = 0.5$ gives the sample mean; for other ω , $\tilde{\mu}(\omega)$ can be obtained by successive iterations.

For symmetry with quantiles, we refer to (9) as the ω -th *expectile indicator*, even though it is only partly based on an indicator function. When evaluated at the sample expectile we call it the ω -*expectic*, i.e. $IE(y_t - \tilde{\mu}(\omega))$, $t = 1, \dots, T$. The expectics sum to zero by construction. As regards the properties of the expectile indicators, it can be easily seen that (i) they have zero mean and finite variance if the variance of y_t finite, (ii) they are covariance stationary under covariance stationarity of y_t . The right hand panel of table 1 shows that expectile indicators exhibit weaker serial correlation than the original series, although less weak than for quantile indicators.

DAS show that for most distributions with finite variance the indicator-based stationarity test is less powerful than the KPSS test (based on residuals from the mean). This leads us to investigate stationarity tests based on expectics. The statistic is

$$\eta_\omega(E) = \frac{\sum_{t=1}^T \left(\sum_{j=1}^t IE(y_j - \tilde{\mu}(\omega)) \right)^2}{T^2 \hat{\sigma}_L^2(E; \tau)}, \tag{10}$$

where $\hat{\sigma}_L^2(E; \tau) = \hat{\gamma}_\omega(E; 0) + 2 \sum_{j=1}^m k(j/m) \hat{\gamma}_\omega(E; j)$, $k(j/m)$ is a weighting function, such as the Bartlett window, and $\hat{\gamma}_\omega(E; j)$ is the sample autocovariance of the expectic at lag j . The case of $\omega = 0.5$ corresponds to the standard KPSS test.

Under the null hypothesis of independently identically distributed observations one would replace $\hat{\sigma}_L^2(E; \tau)$ by $\hat{\gamma}_\tau(E; 0)$. Using the results of Newey and Powell (1987) on the distribution of the sample expectile, one can show that the null distribution of (10) is CvM , as for the quantic based test (and the KPSS). The proof is in appendix C. A joint test on a group of N expectile can be constructed by adapting the statistic (6) appropriately.

As expectiles can be mapped into quantiles, one can use expectile based statistics to test for constant quantiles. As noted by Efron (1991), counting the number of observations smaller than an estimated expectile gives the value of τ needed for it to be interpreted as a quantile. Conversely, to find the value of ω for a (sample) expectile that makes it equal to the τ -quantile, we note that a simple re-arrangement of (8) when it is set to zero and $\tilde{\mu}(\omega)$ is set to $\tilde{\xi}(\tau)$ gives

$$\tilde{\omega} = \frac{\sum_{y_t < \tilde{\xi}(\tau)} (y_t - \tilde{\xi}(\tau))}{\sum_{y_t < \tilde{\xi}(\tau)} (y_t - \tilde{\xi}(\tau)) - \sum_{y_t \geq \tilde{\xi}(\tau)} (y_t - \tilde{\xi}(\tau))} \quad (11)$$

Thus an expectic may also be based on the τ -th quantile, in which case we will call it a τ -*expectic* and write it as

$$IE(y_t - \tilde{\xi}(\tau)) = \left| \tilde{\omega} - I(y_t - \tilde{\xi}(\tau)) \right| (y_t - \tilde{\xi}(\tau)), \quad t = 1, \dots, T, \quad (12)$$

where $\tilde{\omega}$ is defined by (11). The constancy of a particular quantile can be tested by using the τ -expectic to construct a statistic analogous to (10), denoted $\eta_\tau(E)$. In section 5 we compare the power properties of this test with the one based on quantics for various distributions.

Finally, expectile tests could be made robust by Winsorizing the observations, that is observations greater in absolute value than a tail quantile are set equal to that quantile. A simple procedure is as follows: (i) estimate the τ -quantile; (ii) Winsorize the observations using a robust measure of range; (iii) find $\tilde{\omega}$ and carry out the expectic test. The precise way in which Winsorizing is carried out may vary with τ .

4 Properties of the tests

We now compare the performance of quantic and expectic tests for the model

$$y_t = \mu_t + \varepsilon_t, \quad \mu_t = \mu_{t-1} + \eta_t, \quad t = 1, \dots, T, \quad (13)$$

where ε_t is strictly stationary with zero median, while $\eta_t \sim NID(0, \sigma_\eta^2)$. Since the distribution of ε_t is time invariant, the (time-varying) quantiles of y_t , defined as the corresponding quantiles of ε_t plus μ_t , are a constant distance apart and each corresponds to an expectile. The null hypothesis for all tests corresponds to $\sigma_\eta^2 = 0$.

For the quantic tests, the model can be regarded as the same as (3) with μ_t being the median, $\xi(0.5)$, and $\xi_t(\tau)$ being the median plus a constant, while $\varepsilon_t(\tau)$ is equal to ε_t , minus the same constant. Thus in terms of (4), $\sigma_{\eta(\tau)}^2 = \sigma_\eta^2$, and, for a given τ , we could think of the stationarity test as testing $H_0 : \sigma_{\eta(\tau)}^2 = 0$ against $H_1 : \sigma_{\eta(\tau)}^2 > 0$. The probability of rejection (possibly size corrected) can then be formally interpreted as power.

For a Gaussian series the quantic test for $\tau = 0.5$ is less powerful than the standard stationarity test. This is hardly surprising since the latter is LBI and it is shown in DAS that the asymptotic distributions under the alternative are different. The extent of the difference can be seen in the simulations in DAS. On the other hand heavy-tails can favour the indicator test and the interesting question is then whether some kind of robustification can swing the balance back to a residual based test. We seek to investigate this matter for a range of τ .

Linton and Whang (2007) study the quantilogram, the plot of the autocorrelations calculated from the quantics, and investigate the performance of Box-Ljung tests for a range of τ . Assuming symmetry, they show that the statistic constructed from τ -quantics, which we denote as $BL_\tau(P)$, has the usual asymptotic χ_P^2 distribution, where P is the number of lags in the autocorrelations. They find a fall in power towards the tails, that is as τ approaches zero and one. One could extend their ideas by constructing the ‘expectilogram’ and Box-Ljung statistics from the expectics. Our main interest, though, is in seeing how the expectic stationarity tests behave as we move towards the tails. Note that, for the model under consideration, the probability of rejection with the $BL_\tau(P)$ test is likely to be somewhat below that of the corresponding stationarity test. From Harvey and Streibel (1998), the stationarity test statistic can be written $\eta = -T^{-1} \sum_k kr_k$, where

r_k is the lag k sample autocorrelation, showing the importance of high order autocorrelations.

4.1 Monte Carlo simulations under the random walk plus noise data generating process

A limiting representation of the statistics $\eta_\tau(Q)$ and $\eta_\tau(E)$ under the I(1) data generating process (13) is given in the working paper version Busetti and Harvey (2007): in particular, it is showed that both statistics are of $O_p(T)$ which demonstrate the consistency of the tests against the I(1) alternative. Here we present finite sample Monte-Carlo evidence on the size and power properties of the quantic and expectic tests under the same data generating process (13). The statistics $\eta_\tau(Q)$ and $\eta_\tau(E)$ are constructed for the case of IID errors under the null, i.e. the long-run variance estimator is replaced by the sample variance of the quantic/expectic. To shed light on the behaviour of the tests with respect to fat tails, three cases are considered for the noise process³, ε_t : (i) $N(0, 1)$, (ii) t -distribution with three degrees of freedom, (iii) Cauchy distribution. In all cases the random walk component is driven by a Gaussian disturbance with variance equal to $\sigma_\eta^2 = c^2/T^2$, for $c = 0, 2.5, 5, 10, 25, 50$. The quantic and expectic tests are compared with the quantic Box-Ljung statistic, introduced by Linton and Whang (2007). Tables 3,4 and 5 contain the rejection frequencies of the tests for the case $T = 200$ (and 50,000 replications); additional results for other sample sizes are available upon request. Because of symmetry, we only present results for $\tau \leq 0.5$.

The main conclusions for the Gaussian case, reported in Table 3, are as follows.

- (a) Except when $\tau = 0.01$, the empirical sizes of the quantic and expectic tests are very close to the nominal 5%; on the other hand a test based only on the first autocorrelation coefficient, denoted $BL(1)$, does not appear to control size very well in the tails.
- (b) As expected, power⁴ increases with c for all tests.
- (c) Power falls off as we move towards the tails.

³DAS put the same distribution on the disturbance driving the random walk.

⁴Power here means the proportion of rejections; no attempt has been made to correct for size.

(d) The IE test is more powerful than IQ, except when $\tau = 0.01$ (when the IE test is very undersized).

(e) The IQ stationarity test is generally far more powerful than the Box-Ljung test of Linton-Whang with 5 autocorrelations, $BL(5)$, except when $c = 50$ when the rejection probabilities are close to one (using 10 autocorrelations would give a little more power, with a gain of the order of 1% to 5%). $BL(1)$ is clearly dominated.

The IKPSS test (with $m = 0$) of DAS corresponds to our IQ test for $\tau = 0.5$ and the simulation results in their table 3 for $T = 200$ are consistent with the rejection frequencies of our table 1 (note that their $\lambda = 0.01$ would imply $c = 20$).

The results for $t(3)$ errors are reported in Table 4. The IQ and IE tests still appear to control size well, except in the tails when they are somewhat undersized, particularly IE. The main difference⁵ with table 1 is that IQ is now more powerful than IE . (This remains true even with size correction). Finally, when errors are taken from a Cauchy distribution the IQ test still appears to control size well and display good power properties at least in the central quantiles; see table 5. The expectic test is not valid in this case, and is undersized. (The extent of the undersizing for $\tau = 0.5$ is the same as reported by DAS). The rejection frequencies under the alternative are very low.

4.2 Serial correlation

Correcting the IQ and IE tests to allow for weakly dependent errors implies the usual trade-off between size and power, as shown in Kwiatkowski *et al.* (1992) for the standard KPSS test. Table 6 presents Monte Carlo evidence for a data generating process⁶ made up of a random walk plus an AR(1) process with parameter $\phi = 0.5$. The disturbances are either $N(0, 1)$ or scaled⁷ Cauchy, with the random walk component scaled by a factor $\sigma_\eta = q = 0, .0001^{.5}, .001^{.5}, .01^{.5}, .1^{.5}, 1$. The sample size is $T = 200$ with

⁵Power is slightly lower than in Gaussian case, but the variance of the t_3 distribution is three and so the signal-noise ratios are smaller.

⁶The experimental design is as in DAS and our results for $\tau = 0.5$ correspond to those contained in their table 1 (lower part) and in their table 4 (except that they present size-adjusted power, while we give rejection frequencies).

⁷"Scaled Cauchy" means a Cauchy random variable multiplied by 0.1, so that the noise is not too small or large and hence rejection frequencies take meaningful values.

the lag truncation parameter $m = \text{int}\left(4(T/100)^{1/4}\right) = 5$. With this choice of m , both tests are generally oversized, the worst case being the *IQ* test with Cauchy errors. Under Gaussianity, the rejection frequencies under the alternative largely mirror the *IID* case, with the *IE* test being more powerful. The results for the Cauchy distribution are rather surprising, however, since although the *IE* test is not valid, it is less oversized than the *IQ* test and the rejection frequencies are much higher than in the *IID* case.

Even the standard KPSS test is oversized with the above DGP. The oversizing of all tests becomes greater as ϕ moves towards one, though since the aim is to detect slowly near persistent movements this is not necessarily a bad thing.

Table 7 shows result for the stationary component following an MA(1) with negative first-order correlation of -0.4 (the parameter θ is -0.8). In contrast to the previous table the tests are now undersized, particularly for expectics. Nevertheless, in the Gaussian case, the *IE* test dominates the *IQ* test even for the smallest q considered. With (scaled) Cauchy disturbances, the rejection frequencies are well below those of the *IQ* test except when q is high.

In section 2 we showed that serial correlation tends to fall as τ moves towards zero or one. This suggests that it might be useful to adopt a rule that gives lower values of m for such values.

4.3 Multivariate tests

Simulation evidence (not reported here) shows that there is no power gain from considering many quantiles (with respect to just considering the median) when the data is generated from a random walk plus noise. This is to be expected since all the quantiles and expectiles follow exactly the same path, differing only by a constant. On the other hand the power does not appear to fall as more tests are combined. Thus a failure to reject can be confidently interpreted as time invariance rather than low power against a time series that is nonstationary in the level.

To illustrate, we generated 500 observations from a scale mixture of two Gaussian distributions with time-varying weights and variances chosen so as to keep the overall variance constant over time. Figure 2 shows the absolute values of the series and 98%, 90% and 50% quantiles, corresponding to 1%, 5% and 25% in the original series. The 98% and 90% quantiles diverge,

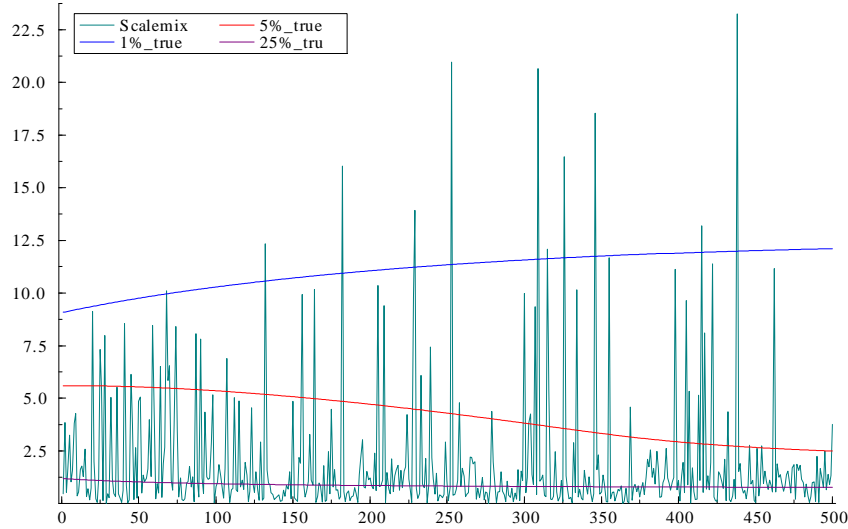


Figure 2: A tail of two quantics

reflecting the changing behaviour in the tails. The IQ test statistics for $\tau = .98$ and 0.9 were $.214$ and $.594$ respectively, so only the second rejects at the 5% level, though not at the 1% level. The joint test statistic is 1.265 , which clearly rejects at the 1% level (the 5% critical value is $.748$, while the 1% is 1.074). The reason is that the correlation between the two quantics under the null is 0.44 but in this series the quantiles head off in different directions.

Adding the 50% quantics gives a joint test of 1.633 . The individual test statistic for the 50% quantics is 0.311 ; this is not significant at the 10% level, which is hardly surprising given that the 50% quantile shows little movement. However, incorporating the 50% quantics into the joint test appears to have little adverse effect since the 1% critical value is 1.359 .

5 Empirical illustration: GM stock returns

Test statistics for the daily GM returns of figure 1 are shown in table 2 for a range of quantiles and various values of m in the nonparametric estimator of the long-run variance. All the quantic stationarity tests, including the

median, reject the null hypothesis of time invariance at the 1% level of significance when $m = 0$. However, the test statistic for $\tau = 0.01$ (not reported in the table) is only 0.289 which is not significant at the 10% level. Similarly the statistic is 0.354 for $\tau = 0.99$ which is only just significant at the 10% level. The values of the test statistics fall as m increases, but they all still reject at the 5% level of significance. In a situation where independence is a reasonable assumption under the null hypothesis, this fall is to be expected given what is known about the estimator of the long-run variance in other contexts; see, in particular, the Monte Carlo results for the median indicator test in DAS.

The expectic tests, $\eta_\tau(E)$, do not reject for $\tau \leq 0.5$, except in one case where the test just rejects at the 10% level. There are more rejections for $\tau > 0.5$, but rarely at the 1% level. The outlying values around the time of the stock market crash of 1987 probably account for the low power. Robustifying the tests, by setting the observations less than the 0.01 quantile or greater than the 0.99 quantile equal to the corresponding quantile (Winsorizing), leads to far more rejections, but in most cases the test statistics are still less than the corresponding quantile statistics. This is perhaps a little surprising.

The Box-Ljung statistics are based on quantiles and here m is the number of lags in the autocorrelations, except that the column headed 0 refers to one lag. There are clear rejections in the tails, particularly $\tau = 0.05$ and 0.1, but not at the quartiles. The 5% critical values for 1,8,17 and 25 lags are 3.84, 15.5, 27.6 and 37.7 respectively.

6 Contrasts

A quantile contrast is a linear combination of quantiles. The contrasts between complementary quantiles, that is

$$D_\xi(\tau) = \xi_t(1 - \tau) - \xi_t(\tau), \quad \tau < 0.5, \quad t = 1, \dots, T \quad (14)$$

yield measures of dispersion. For a symmetric distribution

$$S_\xi(\tau) = \xi_t(\tau) + \xi_t(1 - \tau) - 2\xi_t(0.5) \quad (15)$$

is zero for all $t = 1, \dots, T$, so a plot of this contrast shows how the asymmetry captured by the complementary quantiles changes over time.

A test based on a *quantile contrast*,

$$a.IQ(y_t - \tilde{\xi}(\tau_1)) + b.IQ(y_t - \tilde{\xi}(\tau_2)), \quad t = 1, \dots, T,$$

where a and b are constants, can be useful in pointing to specific departures from a time-invariant distribution. The corresponding population quantile indicator contrast has a mean of zero and a variance⁸ that can be obtained by using the formula

$$\text{cov}(IQ(y_t - \xi(\tau_1)), IQ(y_t - \xi(\tau_2))) = \tau_1(1 - \tau_2), \quad \tau_2 > \tau_1, \quad (16)$$

derived in appendix B. Tests statistics analogous to $\eta_\tau(Q)$ and $\eta_\tau(Q; m)$ can be constructed from the sample quantic contrasts. Under the conditions of proposition 1 applied to each quantile, they have the *CvM* distribution when the observations are, respectively, *IID* or strictly stationary. Similar tests can be constructed using expectics.

6.1 Dispersion

A test of constant dispersion can be based on the complementary quantic contrast

$$DIQ_t(\tau) = IQ(y_t - \tilde{\xi}(1 - \tau)) - IQ(y_t - \tilde{\xi}(\tau)), \quad \tau < 0.5 \quad (17)$$

This mirrors the quantile contrast, $\tilde{\xi}(1 - \tau) - \tilde{\xi}(\tau)$, used as a measure of dispersion. It follows from (16) that the variance of the population quantic contrast is $2\tau(1 - 2\tau)$. The test statistic will be denoted $\eta_\tau(DQ)$.

A complementary expectic contrast, denoted $DIE_t(\omega)$, can be constructed in a similar way. The contrast can be matched to a given τ and hence denoted $DIE_t(\tau)$. Values of ω and $1 - \omega$ are obtained from (11); they could be constrained to add to one if desired. The statistic will be denoted $\eta_\tau(DE)$.

A test of constant dispersion can be constructed using the quantics from the $(1 - 2\tau)$ th quantile for the absolute values, $|y_t|$, when the median is zero. For known quantiles, it is shown in appendix D that $DIQ_t(\tau)$ is equal to $IQ(|y_t| - \xi(1 - 2\tau))$ when there is symmetry. Thus any difference in performance comes from enforcing the restriction that $\tilde{\xi}(\tau) + \tilde{\xi}(1 - \tau) = 0$.

For known expectiles, $DIE_t(\omega)$ is not, in general, equal to $IE(|y_t| - \xi(1 - 2\tau))$. Forming expectics from the mean of absolute values gives the stationarity test of Nyblom and Mäkeläinen (1983) which Harvey and Streibel (1998) find to be more powerful than a test based on squared observations

⁸If neither $\tilde{\xi}(\tau)$ nor $\tilde{\xi}(1 - \tau)$ coincides with an observation or observations, the sample and population covariances are exactly the same, as are the variances.

for detecting stochastic volatility. Note that for a Gaussian distribution, the mean of absolute values, $\sqrt{2/\pi}$, corresponds to the complementary quantiles at $\tau = 0.288$ in the original series.

6.1.1 Monte Carlo experiments for the stochastic volatility model

Here we consider the stochastic volatility model

$$y_t = \varepsilon_t \exp(0.5h_t), \quad \varepsilon_t \sim NID(0, 1) \quad (18)$$

$$h_t = \phi h_{t-1} + \kappa_t, \quad \kappa_t \sim NID(0, \sigma_\kappa^2) \quad (19)$$

with $h_0 = 0$. Table 8 reports Monte Carlo simulated percentage rejections for tests, all with $m = 0$, based on the following statistics: (i) single quantile tests, $\eta_\tau(Q)$ and $\eta_\tau(E)$; dispersion tests $\eta_\tau(DQ)$ and $\eta_\tau(DE)$; $\eta_{1-2\tau}(Q)$ and $\eta_{1-2\tau}(E)$ computed from $|y_t|$.

The figures are for a sample size $T = 1000$, $\phi = 1$ (integrated volatility) and σ_κ set to 0, .005, .01, .02, .03, .05. Because of symmetry we only report results for $\tau \leq 0.5$. As in the earlier tables, the number of replications is 50,000. The main findings are as follows.

(a) The $\eta_\tau(Q)$ tests reject the null hypothesis of constancy with increasing frequency as τ moves towards zero or one. The expectic tests, $\eta_\tau(E)$, display similar behaviour, except that the rejection frequencies are essentially the same for $\tau = 0.15$ and 0.05.

Although y_t is not strictly stationary when there is integrated stochastic volatility, the sizes of the quantic and expectic tests for $\tau = 0.5$ are affected very little; even with $\sigma_\kappa = 0.05$, they are still below 0.06. This is consistent with the theoretical analysis in Cavaliere and Taylor (2005).

(b) Power is monotonically increasing with σ_κ .

(c) The expectic test, $\eta_\tau(E)$, appears to be more powerful than the quantic test, $\eta_\tau(Q)$, except in the tails.

(d) As expected, the dispersion tests, $\eta_\tau(DQ)$ and $\eta_\tau(DE)$, reject more frequently than the corresponding single quantile tests, $\eta_\tau(Q)$ and $\eta_\tau(E)$. The expectic dispersion tests show a clearer domination over the dispersion quantic tests than do the corresponding single quantile tests. Furthermore, for the range of τ 's that we consider, the power of $\eta_\tau(DQ)$ continues to increase as we move towards the tails, while that of $\eta_\tau(DE)$ falls off slightly. The best choice for the expectic test seems to be a value of ω corresponding to $\tau = 0.25$.

(e) The rejection probabilities for the quantic tests based on absolute values are very close to those of the $\eta_\tau(DQ)$ tests. The earlier theoretical analysis indicated that the statistics are identical for known quantiles and it appears that enforcing the symmetry restriction when estimating quantiles makes little or no difference. On the other hand, the expectic tests based on absolute values do behave somewhat differently from the $\eta_\tau(DE)$ tests. They are more powerful for $\tau = 0.15$ and $\tau = 0.05$, but less powerful for $\tau = 0.25$ and $\tau = 0.33$. The best tests against stochastic volatility appear to be the ones based on $\eta_\tau(DE)$ with τ equal to 0.25 and the absolute value expectic test with $\tau = 0.15$.

(f) We confirmed (in experiments not reported here) that all the tests display a tendency to reject the null hypothesis for very persistent but stationary volatility, for example, $\phi = 0.99$; see Harvey and Streibel (1998). However for $\phi = 0.95$, we found rejection frequencies close to the nominal size of 0.05.

The results for heavy-tails disturbances, generated by ε_t having a t -distribution with three degrees of freedom, are presented in Table 9. As with Gaussian disturbances, the test sizes are all close to the nominal. The main differences are as follows.

(a) The power of the single quantile test is now maximized at $\tau = 0.15$ rather than $\tau = 0.05$. The power of the expectic test is also maximized at $\tau = 0.15$.

(b) The quantic tests are more powerful than the corresponding expectic tests for all τ . Even though $\sigma_\varepsilon^2 = 3$, rather than one, the rejection frequencies for quantic tests are actually higher than those for Gaussian disturbances. On the other hand the rejection frequencies for expectic tests are lower.

(c) As regards the dispersion tests, the rejection frequencies for expectics are higher than those of quantics for $\tau = 0.33$, but smaller for $\tau = 0.15$ and $\tau = 0.05$.

(d) In contrast to the Gaussian case, the absolute value expectic test is now more powerful than the $\eta_\tau(DE)$ test for all quantiles. The best choice seems to be $\tau = 0.25$. However, the absolute value quantic test - and the very similar $\eta_\tau(DQ)$ test - has higher rejection frequencies for $\tau = 0.15$ and $\tau = 0.05$. If heavy tails are a possibility, a quantic dispersion test based on $\tau = 0.15$ might be the best option.

6.1.2 Financial time series

For GM, the test statistics $\eta_\tau(DQ)$ for the interquartile range and the 5%/95% range are 3.589 and 3.210 respectively. Thus both decisively reject. The corresponding expectic statistics, $\eta_\tau(DE)$, are 2.567 and 0.506. The failure to reject for $\tau = 0.05$ is consistent with the Monte Carlo results of the previous sub-section.

6.2 Asymmetry

A test of changing asymmetry may be carried out using the quantic contrast

$$SIQ_t(\tau) = IQ(y_t - \tilde{\xi}(1 - \tau)) + IQ(y_t - \tilde{\xi}(\tau)), \quad \tau < 0.5 \quad (20)$$

The variance of the population contrast is 2τ and it is uncorrelated with the population contrast corresponding to $DIQ(\tau)$. The sample values are -1 when $y_t < \tilde{\xi}(\tau)$, 1 when $y_t > \tilde{\xi}(1 - \tau)$ and zero otherwise. The test statistic will be denoted $\eta_\tau(SQ)$.

For GM returns the asymmetry test statistics are 0.133 and 0.039 for $\tau = 0.25$ and $\tau = 0.05$ respectively. Thus neither rejects at the 10% level of significance. The same is true of the expectic tests where the test statistics are 0.061 and 0.069. However, changing skewness may be a feature of some financial time series, as shown in Harvey and Siddique (1999).

6.3 Combined tests

A multivariate test statistic for a pair of complementary quantics, denoted $\eta_{2(\tau)}(Q)$, contains information on changing dispersion and asymmetry. In fact since the population quantile indicator contrasts corresponding to $DIQ(\tau)$ and $SIQ(\tau)$ are mutually uncorrelated, it follows that, if the variances and covariances in the test statistics are set equal to population values, then

$$\eta_{2(\tau)}(Q) = \eta_\tau(DQ) + \eta_\tau(SQ), \quad \tau < 0.5. \quad (21)$$

This will be approximately true when the long-run variance is estimated. For expectiles, the population variables corresponding to $DIQ(\tau)$ and $SIQ(\tau)$ are only uncorrelated if the distribution is symmetric.

For GM, the multivariate test statistic, $\eta_{2(\tau)}(Q)$, based on the 0.25 and 0.75 quantics is 3.722, which is exactly equal to the sum of the dispersion

and asymmetry test statistics, 3.589 and 0.133. This clearly rejects as the 1% critical value is 1.074. The corresponding multivariate statistic for expectiles, $\eta_{2(\tau)}(E)$, is 2.669 which, although not equal to the sum of the dispersion and asymmetry test statistics, 2.567 and 0.061 respectively, is not too far away.

7 Relative time invariance

In looking at quantile and expectile tests so far it has been assumed that the location, that is the median and/or mean, is constant. Our main concern in this section is when the median and/or mean might be changing over time. The aim is to test time invariance of aspects of the distribution apart from its location. An example is provided by monthly U.S. inflation, as measured by the first difference of the logarithm of the personal consumer expenditure deflator (all) from 1959(1) to 2005(6); see Stock and Watson (2007). The level was obtained by fitting a random walk plus noise using the STAMP 7 package of Koopman *et al* (2006). The signal-noise ratio was estimated by maximum likelihood as $q = 0.074$. The smoothed level is clearly not constant (indeed stationarity tests on all quantiles reject as one would expect), but there may be issues of changing dispersion, as highlighted recently by Stock and Watson (2007), and asymmetry.

7.1 Residuals and tests

We first consider testing the null hypothesis of the time-invariance of a given quantile with respect to the median, that is

$$\xi_t(\tau) = \alpha_\tau + \xi_t(0.5), \quad t = 1, \dots, T \quad (22)$$

where α_τ is a constant. Thus in (3) $\varepsilon_t(\tau) = \varepsilon_t(0.5) - \alpha_\tau$. The model therefore corresponds to the alternative hypothesis used for the investigation in section 5.

Quantile tests may be carried out by estimating the time-varying median, $\tilde{\xi}_t(0.5)$, as in De Rossi and Harvey (2006), and then constructing the $IQ(y_t - \tilde{\xi}_t(\tau))$'s, where $\tilde{\xi}_t(\tau) = \tilde{\alpha}_\tau + \tilde{\xi}_t(0.5)$ and $\tilde{\alpha}_\tau$ is the τ -th quantile of the residuals from the median, $y_t - \tilde{\xi}_t(0.5)$. Expectiles, $IE(y_t - \tilde{\xi}_t(\tau))$ in the notation of (12), could be computed for given quantiles and the corresponding tests carried out as usual.

Quantic tests, and the corresponding τ -expectic tests, can also be based on the quantiles computed from the residuals from the time-varying mean, that is $y_t - \tilde{\mu}_t(0.5), t = 1, \dots, T$. This raises an immediate issue, which is that when a signal plus noise model with a unit root is fitted, the residuals are strictly non-invertible. As a consequence, the long-run variance is zero. This non-invertibility does not carry over to the median indicators because as was shown in section 2 the correlations are all closer to zero. As τ moves towards the tails, the correlations become even smaller in absolute value. The same is true for the expectics, but the reductions are smaller. This weakening of the autocorrelations is more pronounced for negative autocorrelations and for a strictly non-invertible process, with spectrum at zero frequency, $\sum_{k=1}^{\infty} \rho(k) = -0.5$, and so negative values dominate. For example, (in a doubly infinite sample) the residuals from a (correctly specified) random walk plus noise follow a strictly non-invertible ARMA(1,1) process in which all the autocorrelations are negative. The non-invertibility of the residuals from the mean may therefore be of little practical importance. Note that because the operation used to estimate a time-varying median is a nonlinear one, it is difficult to determine the properties of the corresponding residuals. Indeed, the fact that they do not apparently follow a linear process when the original process is linear (after differencing) may be a cause for concern. But, of course, the linearity of the original is itself a convenient fiction.

Tests can also be based on the standardized innovations (one-step ahead prediction errors) from the Kalman filter. If the model is Gaussian and correctly specified, the standardized innovations will be normally distributed and serially independent with mean zero and constant variance. Hence the tests may be applied with no correction for serial correlation ($m = 0$). However, given model uncertainty, it might be better to correct for serial correlation, particularly if the procedure is robustified in the way suggested in sub-section 7.3. Tests constructed from innovations may have better size properties, but the question is whether a price is paid in terms of power.

7.2 Simulation evidence

For a Gaussian random walk plus noise model with stochastic volatility in the noise, as in sub-section 6.1, we computed Monte Carlo rejection frequencies of the dispersion tests for relative invariance, that is $\eta_{\tau}(DQ)$ and $\eta_{\tau}(DE)$, based on the quantics and expectics constructed as $IQ(y_t - \hat{\xi}_t(\tau))$

and $IE(y_t - \widehat{\xi}_t(\tau))$, where $\widehat{\xi}_t(\tau) = \widehat{\alpha}_\tau + \widetilde{\mu}_t(0.5)$ and $\widehat{\alpha}_\tau$ is the τ -th quantile obtained from the residuals, $y_t - \widetilde{\mu}_t(0.5)$, from the time-varying mean. The data generating process is the same as the one in Table 8, except that a random walk component is included with innovation standard deviation σ_η equal to either 0.1 or 1. The estimated mean is obtained by Kalman smoothing, with signal-noise ratio fixed at its value under the null. Table 10 provides the percentage rejections for $\tau = 0.05, 0.25$, $T = 1000$ and $m = 7$, based on 10,000 replications.

The empirical size of the $\eta_\tau(DQ)$ and $\eta_\tau(DE)$ tests is still close to the nominal 5% (although unreported results for the single quantile tests $\eta_\tau(Q)$ and $\eta_\tau(E)$ would display some undersizing for small values of σ_η). As regards power, in the case of small random walk component ($\sigma_\eta = 0.1$) the rejection frequencies for both tests are not far from those reported of table 8, just a few percentage points smaller (e.g. 67.7% against 71.7% for the $\eta_\tau(DQ)$ test when $\sigma_\kappa = 0.03$). The power loss from estimating the underlying time-varying mean turns out to be increasing with the size of the random walk: for $\sigma_\eta = 1$ the tests rejects the null hypothesis with significantly lower frequency, around 20% lower when $\sigma_\kappa = 0.03$.

7.3 Robustness

Fitting a (time-varying) mean requires finite second moments for the disturbances in both the level and noise. A (time-varying) median requires finite second moments only for the disturbances in the level, together with assumption 1 on the noise. However, expectic tests computed from the residuals from a fitted median still need finite second moments for the noise.

In order to make the procedure robust we suggest proceeding as follows:

- i) estimate the time-varying median;
- ii) Winsorize the residuals (using a robust estimator of range);
- iii) add the Winsorized residuals to the median and estimate the time-varying mean;
- iv) compute quantiles from the residuals and carry out quantic and τ -expectic tests.

Step (iii) is effectively a time series M -estimator; see Huber (1981). It could be iterated, with the range re-estimated at each iteration.

7.4 Misspecified detrending

The need to identify a suitable model for detrending prior to testing for relative time invariance may be source of unease. However, misspecification of such a model need not have serious consequences as the following proposition makes clear.

Proposition 4 *If the true model is a linear process after differencing d times, and the fitted model is a linear process after differencing \bar{d} times, the modified test statistics, $\eta_\tau(Q)$ and $\eta_\tau(E)$, constructed from the residuals will still have an asymptotic CvM distribution under the null hypothesis provided that $\bar{d} \geq d$.*

The result may be shown by using the Wiener-Kolmogorov formula to find the properties of the residuals. Suppose that the true process is such that $\Delta^d y_t = \psi(L)\nu_t$, where $\psi(L)$ is a polynomial in the lag operator with all roots outside the unit circle and ν_t is white noise with variance σ^2 . If the fitted UC model is of the form $y_t = \mu_t + \varepsilon_t$, where the estimated variance of the white noise process ε_t is σ_ε^2 , and its reduced form is $\Delta^{\bar{d}} y_t = \bar{\psi}(L)\bar{\nu}_t$, where the variance of $\bar{\nu}_t$ is $\bar{\sigma}^2$, the process followed by the extracted residuals in a doubly infinite sample is

$$\tilde{\varepsilon}_t = \frac{\sigma_\varepsilon^2(1-L)^{\bar{d}}(1-L^{-1})^{\bar{d}}}{\bar{\sigma}^2\bar{\psi}(L)\bar{\psi}(L^{-1})}y_t = \frac{\sigma_\varepsilon^2(1-L)^{\bar{d}-d}(1-L^{-1})^{\bar{d}}}{\bar{\sigma}^2\bar{\psi}(L)\bar{\psi}(L^{-1})}\psi(L)\nu_t$$

(Note the simplification if the model is correctly specified.) Clearly $\bar{d} \geq d$ is needed if the process is to be stationary. Although the process is strictly non-invertible, the weakening of the autocorrelation structure for quantics and expectics means that the nonparametric correction in the test statistics can be legitimately employed. More generally, if ε_t is modeled as a stationary process, the same conclusions hold.

7.5 US inflation

Fitting a random walk plus noise model to US inflation results in some residual serial correlation in the innovation residuals, in particular the first-order autocorrelation is 0.14. The residual autocorrelation can be removed by fitting a stochastic cycle or low order autoregressive model. However, the result on misspecified trends in the previous sub-section indicates that this is not

necessary. Indeed the first-order autocorrelation for the smoothed residuals from the random walk plus noise model is not strongly negative, presumably because of the misspecification.

A full set of results for the test statistics, as in table 2, is available on request. The main features are that the $\eta_\tau(DQ)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 are 0.444, 0.636 and 0.273 respectively while the corresponding figures⁹ for $\eta_\tau(DE)$ are 0.247, 0.469 and 0.612; the statistics are computed with $m = 8$. There is clear evidence for changing dispersion and the statistics are consistent with the results of table 8 in that the quantic statistics are higher in the tails and lower for the interquartile range. There is no indication of changing asymmetry.

The tests based on innovations tell a similar story. The $\eta_\tau(DQ)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 are 0.538, 0.393 and 0.417 respectively while the corresponding figures for $\eta_\tau(DE)$ are 0.247, 0.352 and 0.447. Again the asymmetry test statistics are small and statistically insignificant at the 10% level.

Finally we fitted an integrated random walk plus noise model and found the maximum likelihood estimate of the signal-noise ratio to be 0.0007. The $\eta_\tau(DQ)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 were 0.598, 0.288 and 0.200 respectively while the corresponding figures for $\eta_\tau(DE)$ were 0.240, 0.425 and 0.550. These test statistics are not too different from those obtained with the random walk plus noise model.

8 Conclusion

We have shown how tests of time invariance may be constructed from what we call quantics or expectics. The former are based on weighted indicators obtained from an estimated quantile while the latter are based on residuals weighted differently according to whether the observation is above or below the quantile. The serial correlation structure is weakened by transforming to quantics or expectics with autocorrelations moving closer to zero as the quantile moves towards either of the tails. Stationarity test statistics constructed from quantics or expectics are shown to have the usual Cramér von Mises distribution, asymptotically, under the null hypothesis of time invariance. Our simulation results confirm certain prior expectations such as the

⁹As expected from table 8 these figures are bigger than the corresponding figures for test statistics on individual quantics and expectics.

higher power of expectic tests for Gaussian series for most quantiles, coupled with a tendency for the quantic tests to do better when there are heavy tails (where power is generally lower). A compromise is to robustify the expectic tests, but this does not always produce an improvement over the quantic tests. Multivariate tests, combining the information in a set of quantics or expectics, appear to be useful as general tests of time invariance.

Tests based on contrasts can be used to detect changing dispersion and asymmetry. Dispersion tests are based on complementary quantics or expectics. Alternatively tests can be based on absolute values (about the mean or median), though for quantics the tests are essentially the same when the distribution is symmetric. With expectics, a good test against changing volatility is the one based on the quartile contrast, though with heavy tails (but still finite variance) the corresponding test based on the mean of absolute values seems preferable. If there are concerns about the existence of moments, a quantic dispersion test constructed using the 15% and 85% quantiles is good choice.

Testing for relative time invariance introduces some new issues. Our recommendation is to estimate a stochastic trend or level as for a Gaussian model, possibly with prior treatment of potential outliers, and then to proceed to compute quantics and expectics by fitting quantiles to the residuals. Even if the fitted model is misspecified, the smoothed residuals are stationary and serial correlation not accounted for by the model will be accommodated by the nonparametric estimator of the long run variance; all that is required is that the number of unit roots in the fitted model is at least as great as in the true model.

There are a number of other open problems. For example, more work needs to be done on the effectiveness of methods for dealing with relative time invariance, including the treatment of time trends, and on robustification. Tests for movements in the tails, reflecting changing kurtosis, can be based on the outer quantiles, but an allowance needs to be made for changing dispersion and this requires some investigation.

APPENDICES

A Autocorrelations of quantile indicators

The autocorrelation of the τ -th quantile indicator corresponding to ρ_y , the autocorrelation of the covariance stationary process, y_t , at any lag $k = 1, 2, \dots$, is

$$\rho_\tau(\rho_y) = \frac{E[IQ(y_t - \xi(\tau)) IQ(y_{t-k} - \xi(\tau))]}{Var(IQ(y_t - \xi(\tau)))}.$$

The denominator is $\tau(1 - \tau)$, while the numerator is

$$\begin{aligned} & \tau^2 \Pr(y_t > \xi(\tau), y_{t-k} > \xi(\tau)) + (\tau - 1)^2 \Pr(y_t < \xi(\tau), y_{t-k} < \xi(\tau)) \\ & + \tau(\tau - 1) \Pr(y_t > \xi(\tau), y_{t-k} < \xi(\tau)) + \tau(\tau - 1) \Pr(y_t < \xi(\tau), y_{t-k} > \xi(\tau)) \end{aligned}$$

But

$$\begin{aligned} & \Pr(y_t > \xi(\tau), y_{t-k} < \xi(\tau)) + \Pr(y_t < \xi(\tau), y_{t-k} > \xi(\tau)) \\ = & 1 - \Pr(y_t > \xi(\tau), y_{t-k} > \xi(\tau)) - \Pr(y_t < \xi(\tau), y_{t-k} < \xi(\tau)) \end{aligned}$$

So the numerator becomes

$$\tau \Pr(y_t > \xi(\tau), y_{t-k} > \xi(\tau)) + (1 - \tau) \Pr(y_t > -\xi(\tau), y_{t-k} > -\xi(\tau)) + \tau^2 - \tau$$

Hence (2) is obtained.

B Correlations between quantics

Tests involving more than one quantic need to take account of the correlation between them. To find the covariance of the τ_1 and τ_2 population quantics with $\tau_2 > \tau_1$, their product must be evaluated and weighted by (i) τ_1 when $y_t < \xi(\tau_1)$, (ii) $\tau_2 - \tau_1$ when $y_t > \xi(\tau_1)$ but $y_t < \xi(\tau_2)$, (iii) $1 - \tau_2$ when $y_t > \xi(\tau_2)$. This gives

$$(\tau_2 - 1)(\tau_1 - 1)\tau_1 + (\tau_2 - 1)\tau_1(\tau_2 - \tau_1) + \tau_2\tau_1(1 - \tau_2) = \tau_1(1 - \tau_2)$$

and on collecting terms we find that

$$cov(IQ(y_t - \xi(\tau_1)), IQ(y_t - \xi(\tau_2))) = \tau_1(1 - \tau_2), \quad \tau_2 > \tau_1. \quad (23)$$

It follows that the correlation between the population quantics for τ_1 and τ_2 is

$$\frac{\tau_1(1 - \tau_2)}{\sqrt{\tau_1(1 - \tau_1)\tau_2(1 - \tau_2)}}, \quad \tau_2 > \tau_1$$

The correlation between the complementary quantile indicators, $IQ(y_t - \xi(\tau))$ and $IQ(y_t - \xi(1 - \tau))$, is simply $\tau/(1 - \tau)$. This is $1/3$ for the quartiles and $1/9$ for the first and last deciles.

C Distribution of the expectic test

Newey and Powell (1987) establish consistency and asymptotic normality of the sample expectile $\tilde{\mu}(\omega)$ for iid random variables with continuous density and finite moments of order slightly higher than four. Given their results (and recalling that $\tilde{\mu}(\omega)$ is obtained by setting (8) equal to zero) it follows that

$$IE(y_t - \tilde{\mu}(\omega)) = w_t(\omega)(y_t - \mu(\omega)) - T^{-1} \sum_{t=1}^T w_t(\omega)(y_t - \mu(\omega)) + o_p(1),$$

where $w_t(\omega) = |\omega - I(y_t - \mu(\omega) < 0)|$. The first term in the expression above is just the expectile indicator that has zero mean and finite variance $\gamma_\omega(E; 0)$. The invariance principle clearly holds and thus $T^{-\frac{1}{2}} \sum_{t=1}^{\lfloor rT \rfloor} IE(y_t - \tilde{\mu}(\omega)) \Rightarrow \gamma_\omega(E; 0)^{1/2} (W(r) - rW(1))$, $r \in [0, 1]$, where \Rightarrow denotes weak convergence. By consistency of $\hat{\gamma}_\omega(E; 0)$ and an application of the continuous mapping theorem, we then obtain that the limiting distribution of the test statistic is *CvM*. For serially correlated observations the previous arguments need to be modified replacing $\gamma_\omega(E; 0)$ by the long run variance of the expectile indicator.

D Dispersion and absolute values

Assume the quantiles are known. Let n_A be the number of observations greater than or equal to $\xi(1 - \tau)$, n_B the number of observations less than or equal to $\xi(\tau)$ and n_C the number of observations less than $\xi(1 - \tau)$ or greater than $\xi(\tau)$. Then

$$DIQ(\tau) = (1 - \tau)n_A + (1 - \tau)n_B - \tau(n_B + n_C) - \tau(n_A + n_C)$$

For a symmetric distribution and $\xi(1 - \tau) - \xi(0.5) = -\xi(\tau) + \xi(0.5)$ and these are equal to the quantile at $1 - 2\tau$ for $|y_t - \xi(0.5)|$. Hence

$$IQ(1 - 2\tau) = (1 - 2\tau)(n_A + n_B) - 2\tau n_C$$

which on re-arrangement is seen to be the same as the previous formula.

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REFERENCES

- Abramowitz, M. and I.A. Stegun (1964). *Handbook of Mathematical Functions*. Washington: U.S. Bureau of Commerce.
- Busetti, F. and Harvey, A.C. (2007). Tests of time-invariance. Cambridge, Faculty of Economics, CWPE 0657.
- Cavaliere, G., and A.M.R. Taylor (2005). Stationarity tests under time varying second moments. *Econometric Theory*, 21, 1112-1129.
- De Jong, R.M., Amsler, C. and P. Schmidt (2007). A Robust Version of the KPSS Test based on Indicators, *Journal of Econometrics*, 137, 311-33.
- De Rossi, G. and Harvey, A C (2006). Time-varying quantiles. Cambridge, Faculty of Economics, CWPE 0649.
- De Rossi, G. and Harvey, A C (2007). Quantiles, expectiles and splines, Cambridge, Faculty of Economics, CWPE 0660.
- Doornik, J. A. (1999). *Ox: An Object-Oriented Matrix Language*. 3rd edition. Timberlake Consultants Press: London.
- Efron, B. (1991). Regression percentiles using asymmetric squared error loss. *Statistica Sinica*, 1, 93-125.
- Engle, R.F., and Manganelli, S. (2004). CAViaR: conditional autoregressive value at risk by regression quantiles, *Journal of Business and Economic Statistics*, 22, 367–381.
- Harvey, A. C., and M. Streibel (1998). Testing for a slowly changing level with special reference to stochastic volatility. *Journal of Econometrics* 87, 167-89.
- Harvey, C., and A. Siddique (1999). Autoregressive conditional skewness. *Journal of Financial and Quantitative Analysis*, 34, 465-87.
- Huber, P.J. (1981). *Robust statistics*. New York: Wiley.
- Koenker, R. (2005). *Quantile regression*. Cambridge University Press.
- Koopman, S.J., N. Shephard and J. Doornik (1999). Statistical algorithms for models in state space using SsfPack 2.2. *Econometrics Journal*, 2, 113-66.
- Koopman, S.J., A.C. Harvey, J.A. Doornik and N. Shephard (2006). *STAMP 7.0: Structural Time Series Analyser, Modeller and Predictor*, London: Timberlake Consultants Ltd.
- Kwiatkowski, D., Phillips, P.C.B., Schmidt P. and Y. Shin (1992). Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root? *Journal of Econometrics* 44, 159-78.

Linton, O. and Y.-J.Whang (2007). The quantilogram: with an application to evaluating directional predictability. *Journal of Econometrics*, 141, 250-82.

Newey, W K and J.L. Powell (1987). Asymmetric least squares estimation and testing, *Econometrica*, 55, 819-847.

Nyblom, J.(1989). Testing for the constancy of parameters over time. *Journal of the American Statistical Association* 84, 223-30.

Nyblom, J., and A.C. Harvey (2000). Tests of common stochastic trends, *Econometric Theory*, 16, 176-99.

Nyblom, J., and A.C. Harvey (2001). Testing against smooth stochastic trends, *Journal of Applied Econometrics*, 16, 415-29.

Nyblom, J. and T. Mäkeläinen (1983). Comparison of tests for the presence of random walk coefficients in a simple linear model, *Journal of the American Statistical Association*, 78, 856-64.

Stock, J.H. and M.W.Watson (2007). Why has U.S. inflation become harder to forecast? *Journal of Money, Credit and Banking*, 39, supplement F, 3-33.

Table 1. Autocorrelations for population quantics and expectics from a Gaussian time series.

ρ_y	τ for quantics					$\omega(\tau)$ for expectics			
	0.50	0.33	0.25	0.10	0.05	0.248(0.33)	0.153(0.25)	0.034(0.10)	0.012(0.05)
0.80	0.59	0.58	0.56	0.50	0.45	0.77	0.74	0.63	0.55
0.50	0.33	0.32	0.30	0.24	0.20	0.47	0.43	0.31	0.23
0.25	0.16	0.15	0.14	0.10	0.07	0.23	0.20	0.12	0.08
-0.25	-0.16	-0.15	-0.13	-0.07	-0.04	-0.22	-0.18	-0.08	-0.04
-0.50	-0.33	-0.29	-0.24	-0.10	-0.05	-0.42	-0.34	-0.14	-0.07
-0.80	-0.59	-0.45	-0.32	-0.11	-0.05	-0.66	-0.52	-0.20	-0.10

Table 2. Quantic, Expectic and Box-Ljung Indicator test for GM returns (2000 observations)

τ	m=0	m=8	m=17	m=25
Quantic test				
0.05	1.823	1.349	1.138	1.025
0.10	1.425	1.115	0.871	0.741
0.25	1.026	0.949	0.823	0.743
0.33	0.799	0.782	0.705	0.648
0.50	2.532	2.667	2.621	2.437
0.67	0.660	0.649	0.601	0.579
0.75	1.544	1.347	1.206	1.142
0.90	1.389	1.088	0.918	0.832
0.95	1.350	1.136	0.983	0.902
Expectic test				
0.05	0.116	0.081	0.077	0.077
0.10	0.301	0.209	0.187	0.180
0.25	0.350	0.295	0.251	0.232
0.33	0.203	0.197	0.171	0.160
0.50	0.074	0.090	0.085	0.083
0.67	0.462	0.533	0.501	0.487
0.75	0.715	0.751	0.682	0.649
0.90	0.906	0.811	0.698	0.648
0.95	0.578	0.509	0.440	0.410
Expectic-robust test				
0.05	1.047	0.877	0.787	0.769
0.10	1.189	0.968	0.813	0.747
0.25	0.749	0.718	0.604	0.544
0.33	0.394	0.416	0.366	0.337
0.50	0.064	0.075	0.072	0.070
0.67	0.489	0.523	0.490	0.475
0.75	0.819	0.802	0.727	0.690
0.90	1.412	1.262	1.091	1.007
0.95	1.241	1.137	0.997	0.922
Box-Ljung test				
0.05	3.547	40.694	54.174	79.925
0.10	1.539	38.747	64.821	92.909
0.25	0.123	7.927	22.389	35.324
0.33	0.045	6.865	17.686	33.029
0.50	0.526	8.776	13.888	24.229
0.67	0.785	10.523	20.706	24.217
0.75	2.116	11.809	19.042	24.523
0.90	4.843	19.019	35.425	39.768
0.95	8.117	16.496	35.239	41.228

Table 3. Percentage rejection frequencies. Errors $\mathbf{N(0,1)}$. The random walk under the alternative is gaussian with innovation standard deviation equal to c/T . $T=200$.

τ	c=0				c=2.5				c=5				c=10				c=25				c=50			
	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.9	4.9	7.0	6.3	6.2	8.6	7.3	6.5	11.2	19.0	8.4	8.0	25.8	43.4	12.1	13.8	57.8	78.9	29.9	43.0	77.0	93.4	58.7	77.1
0.15	4.9	4.9	2.9	4.5	7.6	10.6	2.8	4.9	17.2	25.8	3.5	6.2	40.0	54.2	7.8	14.7	76.3	87.2	40.8	62.3	91.4	96.9	82.6	94.0
0.25	5.2	5.1	5.1	5.1	9.4	11.6	5.3	5.0	19.9	28.5	6.0	6.8	45.6	57.8	11.3	17.0	81.3	89.1	55.5	71.3	94.4	97.8	92.7	97.2
0.33	5.1	5.2	4.4	5.0	9.4	12.2	4.4	5.0	21.3	29.9	5.3	6.6	48.1	59.2	11.2	18.8	83.2	89.8	58.7	74.8	95.3	98.1	94.5	98.2
0.50	5.0	5.3	4.5	5.2	9.6	12.6	4.8	5.4	22.3	30.7	5.3	6.5	49.4	60.2	12.8	20.8	84.3	90.5	63.5	78.1	95.9	98.3	96.1	98.7

Table 4. Percentage rejection frequencies. Errors $t(3)$. The random walk under the alternative is gaussian with innovation standard deviation equal to c/T . $T=200$

τ	c=0				c=2.5				c=5				c=10				c=25				c=50			
	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.6	3.9	6.7	5.6	4.8	4.4	6.8	5.6	5.9	6.1	7.0	5.9	9.6	12.2	7.5	7.0	28.9	38.0	12.7	15.4	53.8	65.3	30.3	42.5
0.15	5.0	4.3	2.4	4.2	6.5	5.8	2.5	4.6	11.2	10.3	2.7	4.9	25.6	25.1	4.4	8.2	60.9	60.0	22.6	37.7	83.5	84.3	62.1	79.5
0.25	4.9	4.5	5.5	4.9	7.4	6.7	5.4	4.9	15.0	13.2	5.6	5.3	36.6	31.8	8.6	10.8	72.5	68.9	39.8	55.4	90.3	89.6	82.2	91.4
0.33	5.3	4.6	4.4	5.3	8.5	7.4	4.3	5.5	17.8	15.1	4.5	5.9	41.3	35.6	8.7	14.4	77.2	73.0	45.6	63.0	92.5	91.6	86.9	94.8
0.50	5.0	4.7	4.6	5.2	9.5	7.8	4.6	5.2	20.5	16.2	5.2	6.1	45.0	38.4	10.1	16.4	80.2	75.3	52.2	68.4	93.9	92.8	90.8	96.5

Table 5. Percentage rejection frequencies. Errors **Cauchy**. The random walk under the alternative is gaussian with innovation standard deviation equal to c/T . $T=200$

τ	c=0				c=2.5				c=5				c=10				c=25				c=50			
	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.5	2.0	6.7	6.1	4.6	2.0	6.5	6.0	4.5	2.0	6.5	6.0	4.7	2.1	6.5	6.0	6.0	2.6	6.8	6.3	9.6	4.1	7.7	7.6
0.15	5.1	2.3	2.6	4.7	5.3	2.3	2.6	4.8	6.2	2.4	2.7	4.9	9.8	2.7	2.8	5.3	27.7	4.6	5.5	9.9	53.8	10.1	20.0	32.4
0.25	5.2	2.4	5.5	5.1	6.2	2.5	5.7	5.4	9.5	2.6	5.6	5.4	21.0	3.1	6.1	6.6	53.1	6.5	17.5	27.0	77.6	14.6	52.5	67.0
0.33	5.2	2.5	4.5	5.2	7.0	2.6	4.5	5.2	12.8	2.8	4.7	5.6	29.6	3.5	6.2	8.7	64.5	7.8	25.8	40.0	84.9	17.3	65.7	81.1
0.50	5.2	2.7	4.7	5.3	8.2	2.8	4.8	5.8	16.8	3.1	4.9	6.2	38.1	4.0	7.3	11.3	72.7	9.2	35.6	52.6	89.4	19.4	76.3	89.4

Table 6. T=200. The d.g.p. is Gaussian random walk plus AR(1) with $\phi=0.5$. Disturbances driving the AR(1) are either N(0,1) or (scaled) Cauchy; q is the standard deviation of the AR(1) innovations.

τ	q=0				q=.0005 ^{.5}				q=.001 ^{.5}				q=.01 ^{.5}				q=.1 ^{.5}				q=1			
	Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy	
	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE
0.05	5.4	4.5	7.3	3.8	7.1	7.2	7.9	4.5	9.2	9.9	8.6	5.0	32.6	34.7	17.5	11.5	55.7	61.5	34.1	27.4	56.6	65.4	46.0	45.4
0.15	6.8	6.8	8.8	6.2	10.6	11.5	18.4	9.0	14.6	16.3	24.1	11.2	48.0	51.5	52.5	27.3	73.9	77.4	68.7	50.1	76.9	82.0	73.7	68.1
0.25	7.6	7.9	9.5	7.2	12.3	13.8	32.0	12.2	17.0	19.4	41.7	15.3	52.3	56.9	69.3	36.6	79.8	83.1	80.1	61.7	84.9	88.6	83.3	78.4
0.33	7.7	8.3	9.7	7.5	13.3	14.7	40.2	13.6	18.3	20.8	50.7	17.6	55.0	58.8	75.6	41.3	82.1	85.2	83.8	67.2	86.5	90.3	85.9	81.9
0.50	7.8	8.7	10.1	7.9	13.5	15.6	47.6	15.0	19.0	21.8	57.5	19.6	56.1	61.0	79.3	45.3	83.6	86.7	86.0	71.2	88.1	91.4	87.4	84.5

Table 7. T=200. The d.g.p. is Gaussian random walk plus MA(1) with $\theta=-0.8$. Disturbances driving the MA(1) are either $N(0,1)$ or (scaled) Cauchy; q is the standard deviation of the MA(1) innovations.

τ	q=0				q=.0005 ^{.5}				q=.001 ^{.5}				q=.01 ^{.5}				q=.1 ^{.5}				q=1			
	Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy	
	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE
0.05	4.2	3.6	4.7	1.7	7.4	7.1	5.5	2.3	10.6	10.7	6.6	3.0	40.2	43.1	21.3	10.1	65.6	69.2	45.0	22.8	62.0	69.5	51.7	40.2
0.15	4.1	3.2	4.2	0.8	11.7	14.8	27.6	6.3	18.4	24.0	39.8	9.9	59.4	66.2	69.6	28.2	81.4	85.2	75.7	49.4	80.0	84.2	75.9	67.5
0.25	3.3	1.7	3.8	0.2	14.6	22.7	55.3	12.3	24.1	35.9	65.4	18.3	66.2	76.6	82.5	44.2	86.0	89.2	84.9	66.8	86.3	89.5	84.9	80.1
0.33	2.7	0.5	2.7	0.1	16.3	30.0	66.1	16.7	27.2	44.8	74.3	24.5	69.7	81.3	85.9	54.4	87.1	90.8	86.8	75.3	88.0	90.9	86.8	84.8
0.50	1.8	0.0	2.0	0.0	18.4	39.1	74.0	21.3	30.6	54.4	80.1	31.6	72.1	85.2	87.6	62.9	88.2	91.5	88.4	80.8	89.0	92.1	88.2	88.2

Table 8. Percentage rejections against Stochastic Volatility data generating process with Gaussian innovations. Sample size T=1000.

	q:	0	0.005	0.01	0.02	0.03	0.05
Quantic test							
τ :	0.05	4.64	9.39	21.71	48.02	65.08	82.76
	0.15	5.13	8.34	18.60	42.56	60.66	81.03
	0.25	4.94	6.58	12.52	30.74	46.98	68.93
	0.33	5.09	5.67	8.37	19.15	31.59	52.70
	0.50	5.08	5.09	5.11	5.18	5.24	5.74
Expectic test							
τ :	0.05	4.69	9.19	21.65	47.26	63.34	80.08
	0.15	4.67	9.26	21.27	47.20	63.90	81.81
	0.25	4.57	7.43	14.91	36.16	52.43	72.21
	0.33	4.60	6.08	9.93	23.05	36.74	55.76
	0.50	4.82	4.85	4.87	5.00	5.01	4.89
Quantic dispersion test (DQ)							
τ :	0.05	5.02	14.60	35.65	65.31	79.99	92.64
	0.15	5.07	13.87	33.15	63.29	79.16	92.58
	0.25	5.01	10.91	25.75	54.33	71.74	88.51
	0.33	5.08	8.77	19.08	43.17	61.82	81.69
Expectic dispersion test (DE)							
τ :	0.05	4.97	15.59	36.83	66.69	80.86	92.25
	0.15	4.98	19.20	43.94	73.45	86.36	96.01
	0.25	5.13	19.29	44.43	74.01	87.23	96.31
	0.33	5.02	19.23	43.96	73.63	86.88	96.12
Dispersion test (IQ absolute deviations from median)							
τ :	0.05	4.94	14.58	35.43	64.94	80.10	92.56
	0.15	5.14	13.96	33.02	63.28	79.54	92.52
	0.25	4.79	10.64	25.65	54.43	71.50	88.53
	0.33	5.26	8.92	19.22	43.38	61.44	81.71
Dispersion test (IE absolute deviations from mean)							
τ :	0.05	5.04	17.12	39.75	69.93	83.41	94.07
	0.15	5.01	19.62	44.79	74.29	87.21	96.39
	0.25	4.85	17.08	40.06	70.22	84.69	95.22
	0.33	4.77	13.73	32.71	63.08	79.06	92.32

Table 9. Percentage rejections against Stochastic Volatility data generating process with t(3) innovations. Sample size T=1000.

	q:	0	0.005	0.01	0.02	0.03	0.05
Quantic test							
τ :	0.05	4.92	9.86	23.86	51.22	68.11	83.84
	0.15	4.65	11.29	28.94	58.01	73.89	88.82
	0.25	4.33	8.98	21.87	48.75	66.21	83.58
	0.33	4.99	7.15	14.31	35.10	51.92	72.73
	0.50	5.01	5.03	5.03	5.21	5.52	6.87
Expectic test							
		:					
τ :	0.05	4.19	6.73	13.43	32.44	47.73	66.90
	0.15	4.33	7.87	17.63	40.15	57.07	74.65
	0.25	4.43	6.91	14.00	33.48	49.09	66.37
	0.33	4.54	6.04	9.67	21.95	32.68	43.89
	0.50	4.94	4.87	4.93	4.59	4.15	3.59
Quantic dispersion test (DQ)							
τ :	0.05	4.66	16.30	38.01	67.90	82.47	93.41
	0.15	5.05	20.57	46.71	75.84	88.05	96.53
	0.25	4.89	18.24	42.09	72.62	85.91	95.56
	0.33	5.07	14.16	34.75	64.87	80.29	92.92
Expectic dispersion test (DE)							
τ :	0.05	4.50	9.86	23.46	49.18	65.72	82.64
	0.15	4.22	15.67	37.36	66.49	81.33	92.82
	0.25	4.46	18.96	42.58	72.15	85.28	94.98
	0.33	4.48	20.09	44.53	74.39	86.79	95.74
Dispersion test (IQ absolute deviations from median)							
τ :	0.05	4.55	16.37	38.16	68.26	82.52	93.40
	0.15	4.84	20.81	46.74	75.94	88.11	96.52
	0.25	5.01	18.22	42.16	72.71	85.81	95.49
	0.33	4.90	14.56	34.95	64.71	80.40	92.78
Dispersion test (IE absolute deviations from mean)							
τ :	0.05	4.39	11.56	28.74	56.04	72.25	87.51
	0.15	4.44	19.77	44.19	74.12	86.63	95.76
	0.25	4.81	22.02	48.27	78.02	89.06	96.90
	0.33	4.75	20.60	45.45	75.45	87.81	96.24

Table 10: Percentage rejections of the dispersion tests of relative invariance (T=1000, m=7)

		σ_{ξ} :	0	0.005	0.01	0.02	0.03	0.05
Quantic dispersion test (DQ)								
$\sigma_{\eta}=.1$	$\tau=.05$		5.1	14.0	33.7	61.7	76.2	89.5
	$\tau=.25$		4.8	10.1	23.9	51.0	67.7	84.3
$\sigma_{\eta}=1$	$\tau=.05$		5.2	9.0	19.9	43.4	59.9	78.0
	$\tau=.25$		5.5	7.7	15.1	34.0	49.6	69.3
Expectic dispersion test (DE)								
$\sigma_{\eta}=.1$	$\tau=.05$		4.8	14.7	35.2	62.4	76.7	88.5
	$\tau=.25$		5.2	18.1	41.7	70.2	83.3	93.2
$\sigma_{\eta}=1$	$\tau=.05$		5.1	9.1	20.5	44.6	60.7	77.9
	$\tau=.25$		5.5	10.9	25.2	51.3	67.7	84.0