WESTERN REGION
4 Unit Mathematics
1998 Trial HSC Examination

Question One.

(a) Evaluate $\int_0^4 \frac{1}{\sqrt{x^2+9}}\, dx$.

(b) Find the volume of the solid of revolution formed when the area underneath the curve $y = \frac{1}{\sqrt{x^2+9}}$ between the ordinates $x = 0$ and $x = 3$ is rotated about the x-axis.

(c) Evaluate $\int_{-\pi/2}^{\pi/2} x \cos x \; dx$

(d) Show that $\int_0^1 \frac{dx}{4-x^2} = \frac{1}{4} \ln 3$.

(e) Show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ equal to πab units^2.

(f) Evaluate $\int_0^{\pi/4} x \cos^2 x \; dx$.

Question Two.

(a) **(i)** Express $-1 + \sqrt{3}i$ in modulus-argument form.

(ii) Hence evaluate $(-1 + \sqrt{3}i)^9$.

(b) **(i)** Find the value of the product $(-1 + \sqrt{3}i)(1 + i)$.

(ii) Hence, or otherwise, find the exact value of $\cos \frac{11\pi}{12}$.

(c) In an Argand diagram, the point A represents the complex number z, the point C represents the complex number w and the point B represents the complex number $z + w$ with O being the origin. Describe the geometric properties of the quadrilateral $OABC$, providing full reasoning for your answer, given that $z - w = 2i(z + w)$.

(d) If $1 - 2i$ is a root of the equation $z^2 - (3 + i)z + c = 0$.

(i) Explain why the conjugate $1 + 2i$ cannot be a root to the equation.

(ii) Show that the other root is $2 + 3i$.

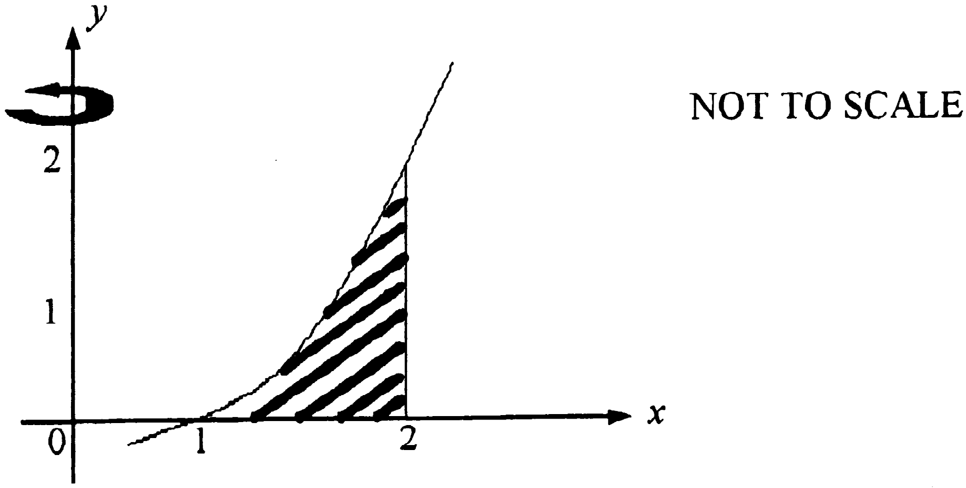
(iii) Find the value of c.

(iv) Hence, or otherwise, find the two square roots of $-24 + 10i$.

(e) The locus of a point $P(x, y)$, which moves in the complex plane is represented by the equation $|z - 3i| = 2$. Show that the minimum value of $\arg z$ is $\cos^{-1}\left(\frac{2}{3}\right)$ and find the modulus of z when P is in the position of the minimum argument.

Question Three.

(a)



The diagram shows the shaded region bounded by the parabola $y = x^2 - x$, the x axis and the line $x = 2$. Find the volume of the solid formed when this region is rotated about the y axis. Use the method of cylindrical shells.

(b) Find the volume of the solid of revolution formed when the region enclosed between the circle $x^2 + y^2 = 16$ and the ellipse $4x^2 + y^2 = 16$ is rotated about the x-axis.

(c) (i) Factorise $f(x) = x^6 + x^4 + x^2 + 1$ by grouping in pairs.

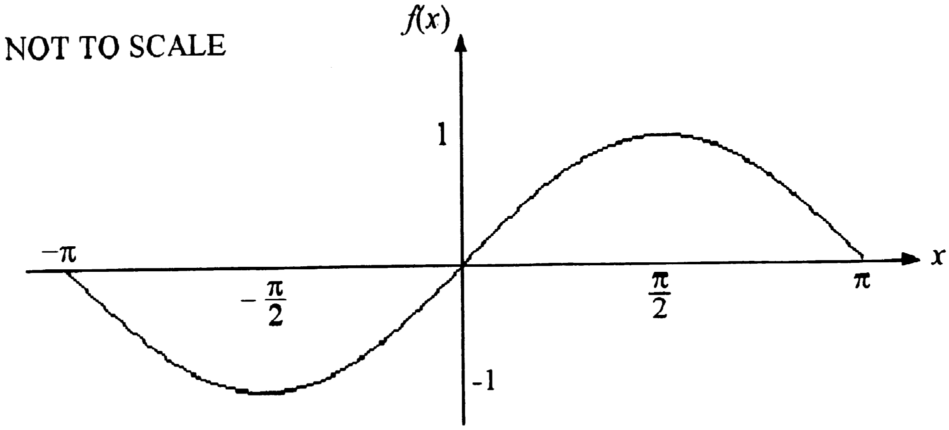
(ii) Hence find all the roots of $f(x) = 0$.

(d) (i) Find all the values of c for which the polynomial equation $3x^4 - 4x^3 + c = 0$ has no real roots.

(ii) Determine the real roots of the polynomial when $c = 1$.

Question Four.

(a) The graph of $f(x) = \sin x$ is shown below for the interval $-\pi \leq x \leq \pi$



On separate axes, draw neat sketches of the following functions:

(i) $y = [f(x)]^2$ **(ii)** $y = \frac{1}{f(x-\frac{\pi}{2})}$ **(iii)** $y^3 = f(x)$ **(iv)** $y = f(\sqrt{|x|}\ )$

(b) **(i)** Show that the line $y = x$ is a tangent to the curve $y = x \sin x$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

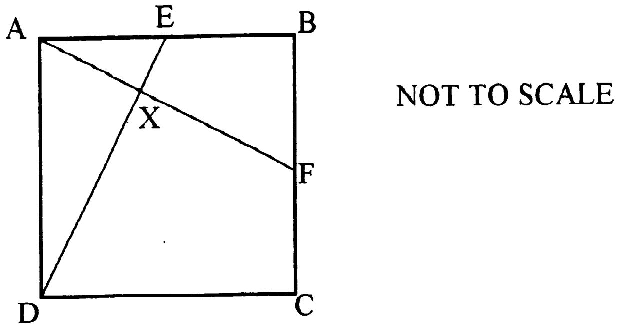
(ii) Using the sketch provided in part **(a)** of this question, or otherwise, sketch the graph of $y = x \sin x$, for the interval $-\pi \leq x \leq \pi$.

(iii) Find the area enclosed by the line $y = x$ and the curve $y = x \sin x$ for the interval $0 \leq x \leq \pi$.

(c) Show that the function $f(x) = e^{-x} + x - 1$ never crosses the x axis.

Question Five.

(a)



In the diagram, $ABCD$ is a square. E and F are the mid-points of the sides AB and BC respectively and X is the point of intersection of the lines AF and ED.

(i) Prove that $DXFC$ is a cyclic quadrilateral.

(ii) Hence, or otherwise, prove that $CX = CD$.

(b) The point $P\left(cp, \frac{c}{p}\right)$ lies on the rectangular hyperbola $xy = c^2$ in the first quadrant. The tangent to the hyperbola at the point P, crosses the x axis at the point A and the y axis at the point B.

(i) Find the equation of the tangent to the hyperbola at the point P.

(ii) Show that the equation of the normal to the hyperbola at the point P is

$$p^3 - py = cp^4 - c.$$

(iii) If the normal at P meets the other branch of the hyperbola at the point Q, determine the coordinates of Q.

(iv) Show that the area of the triangle ABQ is $c^2\left(p^2 + \frac{1}{p^2}\right)^2$

(v) Prove that the area of this triangle is a minimum when $p = 1$.

Question Six.

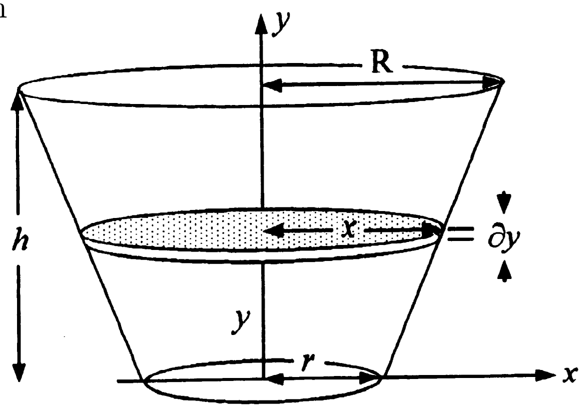
(a) A polynomial of degree n is given by $P(x) = x^n + ax - b$. It is given that the polynomial has a double root at $x = \alpha$.

(i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1} = -\frac{\alpha}{n}$.

(ii) Show that $\left(\frac{a}{n}\right) + \left(\frac{b}{n-1}\right)^{n-1} = 0$.

(iii) Hence deduce that the double root is $\frac{bn}{a(n-1)}$.

(b) A bucket, in the shape of an inverted frustrum of a cone has a base radius of r units, a top radius of R units and a height of h units, as shown in the diagram. A typical slice of thickness ∂y units, at a height of y units above the base of the bucket has been drawn on the diagram.



(i) Write down an expression for ∂V, the volume of the slice in terms of x and ∂y.

(ii) By using similar triangles, or otherwise, show that $x = \frac{(R-r)y}{h} + r$.

(iii) Hence show that the volume of the bucket is $\frac{\pi h}{3}\left[R^2 + rR + r^2\right]$ unit^3.

(c) Show that $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta} = \sqrt{3} - 1$.

Question Seven.

(a) The point $P(2\cos\theta, \sqrt{3}\sin\theta)$ lies on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$. The point N is where the normal at P crosses the x axis.

(i) Determine the coordinates of the foci S and S' of the ellipse.

(ii) Show that the equation of the normal to the ellipse at point P is

$$y - \sqrt{3}\sin\theta = \frac{4\sqrt{3}\sin\theta}{6\cos\theta}(x - 2\cos\theta)$$

(iii) Hence show that N has coordinates $\left(\frac{1}{2}\cos\theta, 0\right)$.

(iv) By considering the ration $S'P : SP$, or otherwise, prove that the normal PN bisects the angle $S'PS$.

(b) By rearranging $I_n = \int \tan^n hx\ dx$ as $\int \tan^{n-2} x \cdot \tan^2 x\ dx$

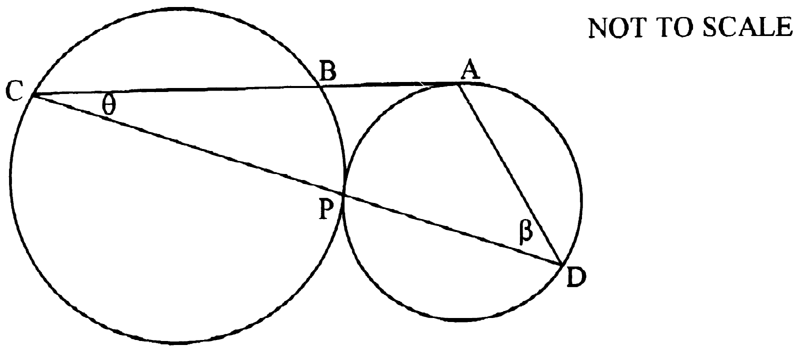
(i) Show that $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

(ii) Hence show that $\int_0^{\frac{\pi}{4}} \tan^4 x\ dx = \frac{\pi}{4} - \frac{2}{3}$.

(c) For two positive real numbers a, b prove that their arithmetic mean $\frac{a+b}{2}$ is always greater than or equal to their geometric mean $\sqrt{ab}$.

Question Eight.

(a) Two circles are touching externally at point P. From point A on the smaller circle a tangent is drawn to cut the larger circle at points B and C. The points C, P and D are collinear. If $\angle ACP = \theta$ and $\angle ADP = \beta$,

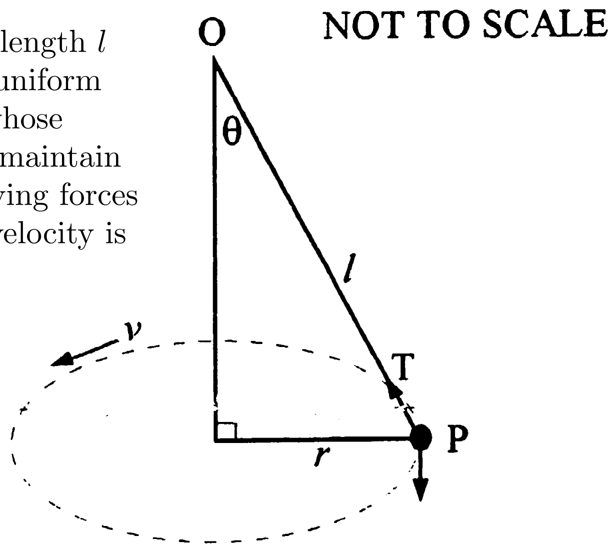


(i) Prove that $\angle APD = \theta + \beta$.

(ii) Prove that $\angle APB = \theta + \beta$.

(iii) Hence, deduce that the point A is equidistant from the lines PB and PD.

(b) A particle of mass m is tied by a string of length l to a fixed point O. The particle moves with a uniform speed v m/s in a horizontal circle of radius r whose centre is directly below O. If the particle is to maintain its motion in a horizontal circle, show by resolving forces vertically and horizontally, that the particle's velocity is given by $V = \sqrt{rg\tan\theta}$.



(c) A singles tennis tournament has 8 players entered to compete. The tournament is organised on a knockout basis, with the defeated player of each match being eliminated and the winning player progressing to the next round.

(i) Write down the number of possible ways that the 8 players could be paired to play in any match of the tournament.

(ii) Assuming the players are of equal ability, what is the probability that two specified players, Alex and Ben, would meet to play each other in this tournament?

(iii) If Alex and Ben are entered in another tournament with a total of 2^n players competing. Find the probability, in its simplest form, that they meet to play each other somewhere in the tournament.