THE SCOTS COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

YEAR 12

EXTENSION 2 MATHEMATICS

AUGUST 2001

TIME ALLOWED: THREE HOURS
[PLUS 5 MINUTES READING TIME]

OUTCOMES:

- Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections. [E3]
- Uses efficient techniques for algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials. [E4]
- Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions. [E6]
- Uses the techniques of slicing and cylindrical shells to determine volumes. Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems. [E7]
- Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems. [E8]
- Uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion. [E5]

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QUESTION ONE [START A NEW ANSWER BOOKLET]

a. Find: [4 MARKS]

(i) $\int x^3 \log_e x \, dx$

(ii) $\int \sin^3 \theta \, d\theta$

b. Find the exact value of: [3 MARKS]

$$\int_4^7 \frac{dx}{x^2 - 8x + 25}$$

c. Using the substitution $u = \cos x$ to evaluate: [3 MARKS]

$$\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x} dx$$

d. [5 MARKS]

(i) Show that $\left(1-\sqrt{x}\right)^{n-1}\sqrt{x} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$

(ii) If $I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$ for $n \geq 0$ show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$

(iii) Deduce that $\dfrac{1}{I_n} = \dbinom{n+2}{n}$ for $n \geq 0$

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a. Let $z = 3 - 2i$ and $u = -5 + 6i$ [4 MARKS]

(i) Find $\text{Im}(uz)$

(ii) Find $|u - z|$

(iii) Find $\overline{-2iz}$

(iv) Express $\dfrac{u}{z}$ in the form $a + ib$, where a and b are real numbers.

b. On separate Argand diagrams sketch: [4 MARKS]

(i) $\{z : |z - 2i| < 2\}$

(ii) $\{z : \arg(z - (1+i)) = -\frac{3\pi}{4}\}$

c. z_1 and z_2 are two complex numbers such that $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$ [7 MARKS]

(i) On an Argand diagram show vectors representing: z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$.

(ii) Show that $|z_1| = |z_2|$

(iii) If α is the angle between the vectors representing z_1 and z_2, show that $\tan\dfrac{\alpha}{2} = \dfrac{1}{2}$

(vi) Show that $z_2 = \dfrac{1}{5}(3 + 4i)z_1$

QUESTION THREE [START A NEW ANSWER BOOKLET]

a. The base of a solid is the region between the lines $y = 3x$ and $y = -x$ from $x = 0$ to $x = 2$. Each cross section by planes perpendicular to the x axis is a square with its side determined by the base. Calculate the volume of the solid. [3 MARKS]

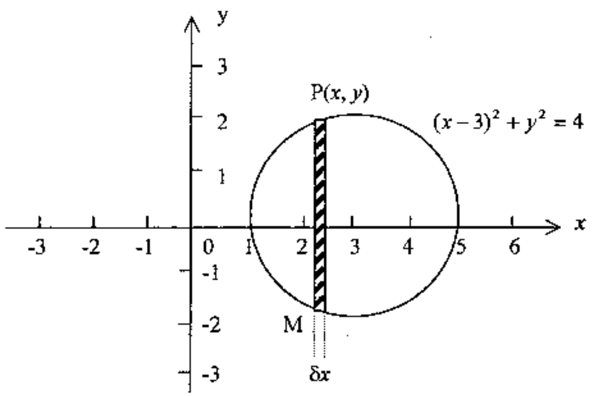
b. The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis. [4 MARKS]

(i) Sketch the curve and the line clearly showing and labelling all the points of intersection.

(ii) By considering slices perpendicular to the x-axis, find the volume of the solid formed.

c. The graph below is of the circle $(x-3)^2 + y^2 = 4$. [8 MARKS]

$P(x, y)$ is a point on the circumference of the circle. PM is the left-hand end of a strip of width δx which is parallel to the y-axis.



(i) Show, using the method of cylindrical shells, that the volume V of the doughnut-shaped solid formed when the region inside the circle is rotated about the y-axis is given by:

$$V = 4\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx$$

(ii) Hence, by using the substitution $u = x - 3$ or otherwise find the volume of the doughnut.

QUESTION FOUR [START A NEW ANSWER BOOKLET]

Consider the function $f(x) = x - 2\sqrt{x}$ [15 MARKS]

a. Determine the domain of $f(x)$.

b. Find the x intercepts of the graph of $y = f(x)$.

c. Show that the curve $y = f(x)$ is concave upwards for all positive values of x.

d. Find the coordinates of the turning point and determine its nature.

e. Sketch the graph of $y = f(x)$ clearly showing all essential details.

f. Hence, sketch on separate diagrams:

(i) $y = |f(x)|$

(ii) $y = f(x-1)$

(iii) $y = f(|x|)$

(iv) $|y| = f(x)$

(v) $y = \dfrac{1}{f(x)}$

a. Given that $z = -1 + \sqrt{3}i$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$, find the other roots. [4 MARKS]

b. Given that α, β and γ are the roots of the cubic equation $x^3 - x^2 + 5x - 3 = 0$, find: [5 MARKS]

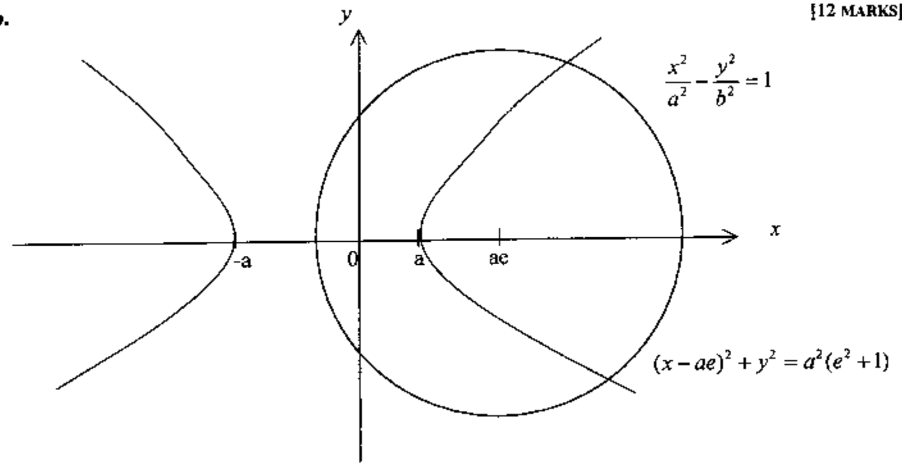
(i) the equation whose roots are $-\alpha, -\beta, -\gamma$.

(ii) the equation whose roots are αβ, αγ, βγ.

c. For what values of m does the equation $x^3 - 12x^2 + 45x - m = 0$ have three distinct solutions? [6 MARKS]

a. A hyperbola has asymptotes $y = x$ and $y = -x$. It passes through the point (3, 2). Find the equation of the hyperbola and determine its eccentricity and foci. [3 MARKS]

b. [12 MARKS]



(i) Show that the tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has equation

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$$

(ii) Show that if the tangent at P is also tangent to the circle with centre $(ae, 0)$ and radius $a\sqrt{e^2 + 1}$, then show $\sec\theta = -e$.

(iii) Given that $\sec\theta = -e$, deduce that the points of contact P, Q on the hyperbola of the common tangents to the circle and hyperbola are the extremities of a latus rectum of the hyperbola, and state the coordinates of P and Q.

(iv) Find the equations of the common tangents to the circle and hyperbola, and find the coordinates of their points of contact with the circle.

QUESTION SEVEN [START A NEW ANSWER BOOKLET]

a. A mass of 10kg falls freely from rest through 10 metres and then comes to rest again after penetrating 0.2 metres of sand.

Find the resistance of the sand, assumed constant. **[4 MARKS]**

b. A particle moving in a straight line experiences a force numerically equal to $\left(x+\frac{1}{x}\right)$ newtons per unit mass, towards the origin. The particle starts from rest, d units from the origin. **[4 MARKS]**

(i) Find an expression for its speed in terms of x.

(ii) Hence or otherwise, deduce its speed when it is half way from the origin.

c. An object of irregular shape and of mass 100kg is found to experience a resistive force, in newtons, of magnitude one-tenth the square of its velocity in metres per second when it moves through air $\left[\text{use } g = 9.8ms^{-2}\right]$. **[7 MARKS]**

If the object falls from rest under gravity:

(i) show that acceleration is given by $a = g - \frac{v^2}{1000}$.

(ii) calculate its terminal velocity.

(iii) calculate the maximum height, to the nearest metre, of the release point above the ground, if the object attains a speed of 80% of its terminal velocity before striking the ground.

QUESTION EIGHT [START A NEW ANSWER BOOKLET]

a. Let α, β and γ be the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A, and B are real. Furthermore $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$. **[5 MARKS]**

(i) Explain why β is real and α and γ are not real.

(ii) Show that α and γ are purely imaginary.

(iii) Find A and B.

b. It is given that if $J_n = \int \cos^{n-1} x \sin nx \, dx$ and $n \geq 1$ then: **[5 MARKS]**

$$J_n = \frac{1}{2n-1}\left[(n-1)J_{n-1} - \cos^{n-1} x \cos nx\right]$$

Use this reduction formula to show that:

$$\int_0^{\frac{\pi}{4}} \cos^2 x \sin 3x \, dx = \frac{1}{60}(28 - \sqrt{2})$$

c. **[5 MARKS]**

(i) Prove that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = 2\sec^n\theta\cos n\theta$

(ii) Hence prove that $\text{Re}(1 + i\tan\frac{\pi}{8})^8 = 64(17 - 12\sqrt{2})$.