Hurstville Boys High School
Trial Higher School Certificate Examination 1989
4 Unit Mathematics

Question 1.

(i) Find these indefinite integrals:

(a) $\int \frac{\sin x \, dx}{\cos^4 x}$

(b) $\int \frac{x^2+4x+1}{(x-1)(x+2)} \, dx$

(c) $\int \frac{dx}{\sqrt{5-8x-4x^2}}$

(ii) Evaluate the following:

(a) $\int_0^5 \frac{x \, dx}{\sqrt{x+4}}$

(b) $\int_0^{3/4} \sqrt{9-4x^2} \, dx$

(c) $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

Question 2.

(i) **(a)** Find the stationary point(s) and point(s) of inflexion of the curve $f(x) = x^2 \log\left(\frac{1}{x^3}\right), \ x > 0$.

Sketch the curve $f(x)$ and state the range of the function.

(b) Without further calculus, sketch $y = \frac{1}{f(x)}$

(ii) The curve $y = ax^3 + bx^2 + cx + d$ has a minimum at $(-1, -2)$ and a maximum at $(1, 2)$. Find the values of a, b, c, and d.

Question 3.

(i) Redeuce to the form $a + ib$ where a and b are real numbers.

(a) $\frac{(1+i)(4-2i)}{(2+i)}$

(b) $\frac{1}{(1-i\sqrt{3})^{14}}$

(ii) Skecth on the argand diagram the locus of z if:

(a) $1 \leq |z-1| \leq 2$

(b) $\Re(z^2) = 4$

(c) $\frac{z-i}{z-2}$ is purely imaginary.

(iii) Find the roots of $z^7 - 1 = 0$ expressing them in mod-arg form.

Question 4.

An ellipse E has equation $4x^2 + 9y^2 = 36$.

(a) Write down its eccentricity, the co-ordinates of its foci S and S' and the equation of each directrix.
Sketch the curve and indicate on your diagram the foci and directrices.

(b) $P(3\cos\theta, 2\sin\theta)$ is an arbitrary point on E. Show that the equation of the tangent to E at P is given by

$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$

and the equation of the normal at P is given by

$$3x\sec\theta - 2y\operatorname{cosec}\theta = 5$$

(c) If the normal meets the x axis at G and OV is the distance of the tangent from the origin, show that

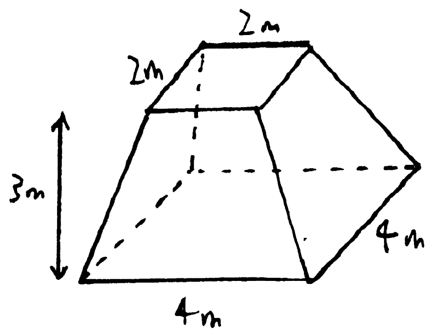
$$OV.PG = 4$$

Question 5.

(i) Sketch the curve $x^2 + y^2 + 1 = 2(x+y)$

(ii) The region bounded by the curve $y = \frac{1}{2}$ and $y = \cos x$ and the y axis is rotated about the y axis.
Find the volume of the solid formed using the method of cylindrical shells.

(iii) The shape in the diagram has a square as a base and all cross-sections parallel to the base are squares. If the block has perpendicular height 3m, find its volume.



Question 6.

(i) Prove by the method of mathematical induction

$$2^n > n^2 \quad \text{for} \quad n > 4.$$

(ii) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad c^2 = a^2m^2 + b^2$$

(iii) The cubic equation $x^3 + px + q = 0$ has 3 non-zero roots α, β, γ

(a) Find in terms of the constants p, q the values of

$$\alpha^{-1} + \beta^{-1} + \gamma^{-1} \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2$$

(b) Form the cubic equation with roots

$$\frac{\alpha\beta}{\gamma}, \quad \frac{\beta\gamma}{\alpha}, \quad \frac{\gamma\alpha}{\beta}$$

Question 7.

(i) It is given that a particle moves in simple harmonic motion according to the equation

$$\ddot{x} = -4x$$

If the motion has amplitude 3m, find

(a) an equation for velocity in terms of displacement (x).

(b) the maximum velocity

(c) the maximum acceleration

(d) the period of the motion

(ii) A gun fires a projectile with initial velocity V m/s at an angle of elevation of θ.
A target moving at a constant speed of a m/s is level with and b metres away from the gun at the instant the gun is fired.
If the target moves in a horizontal line directly away from the gun, show that if the projectile is to hit the target then V and θ must satisfy

$$V^2 \sin 2\theta - 2aV \sin\theta - bg = 0.$$

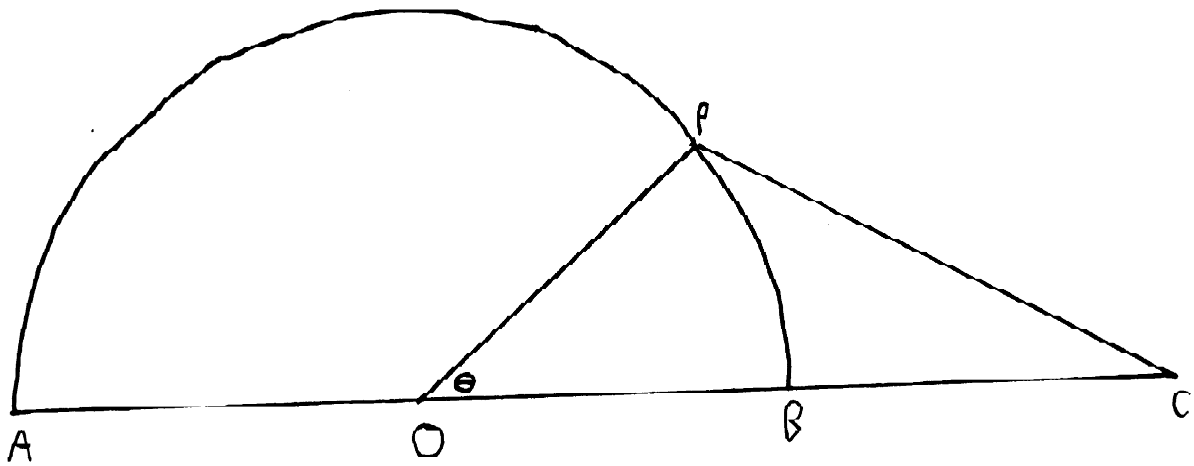
Question 8.

(i) If $I_n = \int \sec^n x \, dx$ show that

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

Evaluate $\int_0^{\pi/3} \sec^6 \theta \, d\theta$

(ii)



In the diagram above the fixed points A, O, B and C are on a straight line such that $AO = OB = BC = 1$ unit. The points A and B are also joined by a semicircle and P is a variable point on this semicircle such that the angle POC is θ. R is the region bounded by the arc AP of the semicircle and the straight lines AC and PC.

(a) Show that the area S of R is given by

$$S = \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$$

Find the value of θ for which S is a maximum.

(b) Show that the perimeter L of R is given by

$$L = 3 + \pi - \theta + \sqrt{5 - 4\cos\theta}$$

Show the L has just one stationary point and that accurs at the same value of θ for which S is a maximum.

Find the least value of L and the greatest value of L in the interval $0 \leq \theta \leq \pi$.